GENERICITY OF WEAK-MIXING MEASURES ON GEOMETRICALLY FINITE MANIFOLDS

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Abstract. Let M be a manifold with pinched negative sectional curvature. We show that when M is geometrically finite and the geodesic flow on T^1M is topologically mixing then the set of mixing invariant measures is dense in the set $\mathscr{M}^1(T^1M)$ of invariant probability measures. This implies that the set of weak-mixing measures which are invariant by the geodesic flow is a dense G_δ subset of $\mathscr{M}^1(T^1M)$. We also show how to extend these results to manifolds with cusps or with constant negative curvature.

1. Introduction

Let $\widetilde{M}$ be a complete, simply connected manifold with pinched sectional curvature (*i.e* there exists $b > a > 0$ such that $-b^2 \leq \kappa \leq -a^2$) and Γ a non elementary group of isometries of $\widetilde{M}$. We will denote by M the quotient manifold $\widetilde{M}/\Gamma$ and ϕ_t the geodesic flow on the unit tangent bundle T^1M. The interesting behavior of this flow occurs on its non-wandering set Ω.
Let us recall a few definitions related to the mixing property of the geodesic flow.

(1) ϕ_t is topologically mixing if for all open sets $\mathscr{U}, \mathscr{V} \subset \Omega$ there exists $T > 0$ such that
$$\phi_t(\mathscr{U}) \cap \mathscr{V} \neq \emptyset \text{ for all } |t| > T,$$
(2) Given a finite measure μ the geodesic flow is mixing with respect to μ if for all $f \in L^2(T^1M, \mu)$,
$$\lim_{t\to\infty} \int_{T^1M} f \circ \phi_t \cdot f d\mu = \left(\int_{T^1M} f d\mu \right)^2,$$
(3) it is weakly-mixing if for all continuous function f with compact support we have
$$\lim_{T\to\infty} \tfrac{1}{T} \int_0^T \left| \int_{T^1M} f \circ \phi_t(x) f(x) d\mu(x) - \left(\int_{T^1M} f d\mu \right)^2 \right| dt = 0.$$

Finally, let us recall what is the weak topology: a sequence μ_n of probability measures converges to a probability measure μ if for all bounded continuous functions f,
$$\int_{T^1M} f d\mu_n \xrightarrow{n\to\infty} \int_{T^1M} f d\mu.$$

In this article, we will show that the weak-mixing property is generic in the set $\mathscr{M}^1(T^1M)$ of probability measures invariant by the geodesic flow and supported on Ω. We endow $\mathscr{M}^1(T^1M)$ with the weak topology.
In [Sig72], K. Sigmund studies this question for Anosov flows defined on compact manifolds and shows that the set of ergodic probability measures is a dense G_δ set (*i.e* a countable intersection of dense sets) in $\mathscr{M}^1(T^1M)$.
On non-compact manifolds, the question has been studied by Y. Coudène and B. Schapira in [CS10] and [CS14]. It appears that ergodicity and zero entropy are typical properties for the geodesic flow on negatively curved manifolds.
Since the set of mixing measures with respect to the geodesic flow is contained in a meager set (see [CS14]) it is natural to consider the set of weak-mixing measures from the generic point of view.

Let us recall that a manifold M is geometrically finite if it is negatively curved, complete and has finitely many ends, all of which are cusps of finite volume and funnels.

Here is our main result.

Theorem 1. *Let M be a geometrically finite manifold with pinched curvature and ϕ_t the geodesic flow defined on its unit tangent bundle T^1M.*
If ϕ_t is topologically mixing on Ω, then the set of mixing probability measures invariant by the geodesic flow is dense in $\mathscr{M}^1(T^1M)$ for the weak topology.

Corollary 1.1. *Let M be a geometrically finite manifold with pinched negative curvature and ϕ_t the geodesic flow defined on its unit tangent bundle T^1M.*
If ϕ_t is topologically mixing on the non-wandering set of T^1M, then the set of invariant weak-mixing probability measures with full support on Ω is a dense G_δ subset of $\mathscr{M}^1(T^1M)$.

To prove theorem 1, we will use the fact that Dirac measures supported on periodic orbits are dense in $\mathscr{M}^1(T^1M)$. This comes from [CS10] where the result is shown for any metric space X admitting a local product structure and satisfying the closing lemma.
The rest of the proof relies on the approximation of a single Dirac measure supported on a periodic orbit $\mathscr{O}(p)$ using a sequence of Gibbs measures associated to (Γ, F_n) where $F_n : T^1M \to \mathbb{R}$ is a Hölder-continuous potential.
The notion of Gibbs measures which is related to the construction of Patterson-Sullivan densities on the boundary at infinity of $\widetilde{M}$ will be recalled in §.2 .
In §.4 we will prove a criterion connecting the divergence of some subgroups of Γ with the finiteness of the Gibbs measures which comes from [DOP00] for the potential $F = 0$.
After this, we will construct in §.5 a sequence of bounded potentials satisfying the desired property. The main step of this paragraph builds on a result of [Cou03] which claims that there exists a bounded potential such that the Gibbs measure is finite.
Finally, we will prove in §.6 the convergence of the Gibbs measures using

the variational principle which is recalled in §.2.
Now, assume that Theorem 1 is true. The proof of corollary 1.1 is a consequence of [CS14] where it is shown that the set of weak-mixing measures with full support is a G_δ subset of the set of invariant Borel probability measures supported on Ω.

In the previous theorem, we restricted ourselves to the case of geometrically finite manifolds but we make the following conjecture: the result is still true for non geometrically finite manifolds as soon as the geodesic flow is topologically mixing on its non wandering set.
The conjecture is supported by the following two results.

Corollary 1.2. *Let M be a connected, complete pinched negatively curved manifold with a cusp then the set of probability measures fully supported on Ω that are weakly mixing with respect to the geodesic flow is a dense G_δ set of $\mathscr{M}^1(T^1M)$.*

Corollary 1.3. *Let S be a pinched negatively curved surface or a manifold with constant negative curvature then the set of probability measures fully supported on Ω that are weakly mixing with respect to the geodesic flow is a dense G_δ set of $\mathscr{M}^1(T^1M)$.*

Let M be a manifold such that $dim(M) = 2$, $\kappa_M = -1$ or M possesses a cusp and denote by ϕ_t the geodesic flow on T^1M. We will show in §.7 that we can find a geometrically finite manifold $\hat{M}$ on which theorem 1 applies and for which ϕ_t is a factor of the geodesic flow $\hat{\phi}_t$ on $T^1\hat{M}$.

One way to confirm the conjecture is to find a positive answer to the following question:
Let M be a connected, complete manifold with pinched negative curvature. We will suppose that M is not geometrically finite and has no cusp. Does there exist a potential $F : T^1M \to \mathbb{R}$ such that the Gibbs measure associated with (Γ, F) is finite?

2. Preliminaries

2.1. **Geometry on $T^1\widetilde{M}$.** We first recall a few notations and results related to the geometry of negatively curved manifolds.
Let $\partial_\infty\widetilde{M}$ be the boundary at infinity of $\widetilde{M}$, we define the limit set of Γ by

$$\Lambda\Gamma = \overline{\Gamma x} \cap \partial_\infty\widetilde{M},$$

where x is any point of $\widetilde{M}$.
An element $\xi_p \in \Lambda\Gamma$ is **parabolic** if there exists a parabolic isometry $\gamma \in \Gamma$ satisfying $\gamma\xi = \xi$.
A parabolic point $\xi \in \Lambda\Gamma$ is **bounded** if $\Lambda\Gamma/\Gamma_{\xi_p}$ is compact where Γ_{ξ_p} is the maximal subgroup of Γ fixing ξ_p. In this case, let $\mathscr{H}_\xi$ be a horoball centered at ξ then,

$$\mathscr{C}_\xi = \mathscr{H}_\xi/\Gamma_{\xi_p}$$

is called the cusp associated with ξ.
We say that $\Lambda_c\Gamma \subset \Lambda\Gamma$ is the conical limit set if for all $\xi \in \Lambda_c\Gamma$ for some $\widetilde{x} \in \widetilde{M}$ there exists $\epsilon > 0$ and a sequence $(\gamma_n)_{n\in\mathbb{N}}$ such that $(\gamma_n(\widetilde{x}))_{n\in\mathbb{N}}$

converges to ξ and stays at distance at most ϵ from the geodesic ray $(\widetilde{x}\xi)$. We define the parabolic limit set as follows.

$$\Lambda_p\Gamma = \{\eta \in \Lambda\Gamma : \exists \gamma \in \Gamma \text{ parabolic } , \gamma \cdot \eta = \eta\}$$

Let us choose an origin $\widetilde{x}_0$ in $\widetilde{M}$ once and for all. We define the Dirichlet domain of Γ, centered on $\widetilde{x}_0$ as follows.

$$\mathscr{D} = \bigcap_{\gamma\in\Gamma,\gamma\neq Id} \{\widetilde{x} \in \widetilde{M} : d(\widetilde{x}, \widetilde{x}_0) \leq d(\widetilde{x}, \gamma\widetilde{x}_0)\}.$$

It is a convex domain having the following properties.

- $\bigcup_{\gamma\in\Gamma} \gamma\mathscr{D} = \widetilde{M}$,
- for all $\gamma \in \Gamma\backslash\{Id\}$, $\mathring{\mathscr{D}} \cap \gamma\mathring{\mathscr{D}} = \emptyset$.

We define the diagonal of $\Lambda\Gamma \times \Lambda\Gamma$ the set of points $(x, y) \in \Lambda\Gamma \times \Lambda\Gamma$ such that $x = y$. We denote by Δ this set.
For all $\xi, \eta \in \partial_\infty\widetilde{M}$, we denote by $(\xi\eta)$ the geodesic joining ξ to η. We define the lift of the non wandering set on $T^1\widetilde{M}$ by

$$\widetilde{\Omega} = \{\widetilde{x} \in T^1\widetilde{M} : \exists(\xi, \eta) \in \Lambda\Gamma \times \Lambda\Gamma\backslash\Delta, \quad \widetilde{x} \in (\xi\eta)\}.$$

Let $\pi : T^1\widetilde{M} \to \widetilde{M}$ be the natural projection of the unit tangent bundle to the associated manifold. We denote by $\mathscr{C}\Lambda\Gamma$ the smallest convex set in $\widetilde{M}$ containing $\pi(\widetilde{\Omega})$.

M is geometrically finite if one of the following equivalent conditions is satisfied

(1) $\Lambda\Gamma = \Lambda_c\Gamma \cup \Lambda_p\Gamma = \Lambda_c\Gamma \cup \{$ bounded parabolic fixed points $\}$,
(2) For some $\epsilon > 0$, the $\epsilon-$neighborhood of $\mathscr{C}\Lambda\Gamma/\Gamma$ has finite volume,
(3) M has finitely many ends, all of which are funnels and cusps of finite volume.

We define a map

$$C_F : \partial_\infty\widetilde{M} \times \widetilde{M}^2 \to \mathbb{R}$$

called the Gibbs cocycle of (Γ, F) by

$$C_{F,\xi}(x, y) = C_F(\xi, x, y) = \lim_{t\to\infty} \int_y^{\xi(t)} \widetilde{F} - \int_x^{\xi(t)} \widetilde{F}$$

where $t \mapsto \xi(t)$ is any geodesic ending at ξ.

Here is a technical lemma of [PPS15] giving estimates for the Gibbs cocycle.

Proposition 2.1. *(*[PPS15]*) For every $r_0 > 0$, there exists $c_1, c_2, c_3, c_4 > 0$ with $c_2, c_4 \leq 1$ such that the following assertions hold.*

(1) For all $x, y \in \widetilde{M}$ and $\xi \in \partial_\infty\widetilde{M}$,

$$| C_{F,\xi}(x, y) | \leq c_1 e^{d(x,y)} + d(x, y) \max_{\pi^{-1}(B(x,d(x,y)))} |\widetilde{F}| ,$$

and if furthermore $d(x, y) \leq r_0$, then

$$| C_{F,\xi}(x, y) | \leq c_1 d(x, y)^{c_2} + d(x, y) \max_{\pi^{-1}(B(x,d(x,y)))} |\widetilde{F}| .$$

(2) For every $r \in [0, r_0]$, for all x, y' in $\widetilde{M}$, for every ξ in the shadow $\mathscr{O}_x B(y', r)$ of the ball $B(y', r)$ seen from x, we have

$$\Big| C_{F,\xi}(x, y') + \int_x^{y'} \widetilde{F} \Big| \leq c_3\, r^{c_4} + 2r \max_{\pi^{-1}(B(y',r))} |\widetilde{F}| \,.$$

2.2. **Thermodynamic formalism for negatively curved manifolds.** We start by recalling a few facts on Gibbs measures on negatively curved manifolds. The results of this paragraph come from [PPS15].

Let $\widetilde{F} : T^1\widetilde{M} \to \mathbb{R}$ be a $\Gamma-$invariant Hölder function. We will say that the induced function F on $T^1M = T^1\widetilde{M}/\Gamma$ is a potential.

The Poincaré series associated with (Γ, F) is defined by

$$P_{x,\Gamma,F}(s) = \sum_{\gamma\in\Gamma} e^{\int_x^{\gamma x} \widetilde{F} - s}.$$

Its critical exponent is given by

$$\delta_{\Gamma,F} = \limsup_{n\to\infty} \tfrac{1}{n} \log\Big(\sum_{\gamma\in\Gamma, n-1\leq d(x,\gamma x)\leq n} e^{\int_x^{\gamma x} \widetilde{F}}\Big).$$

We say that (Γ, F) is of divergence type if
$P_{x,\Gamma,F}(\delta_{\Gamma,F})$ diverges.
When $F = 0$ on T^1M, we will denote by δ_Γ the critical exponent associated to (Γ, F).

Proposition 2.2. *Let F be the potential on $T^1M = (T^1\widetilde{M})/\Gamma$ induced by the $\Gamma-$invariant potential $\widetilde{F} : T^1\widetilde{M} \to \mathbb{R}$.*

1. *The Poincaré series associated with (Γ, F) converges if $s > \delta_{\Gamma,F}$ and diverges if $s < \delta_{\Gamma,F}$,*
2. *We have the upper bound*

$$\delta_{\Gamma,F} \leq \delta_\Gamma + \sup_{\pi^{-1}(\mathscr{C}\Lambda\Gamma)} \widetilde{F},$$

3. *For every $c > 0$, we have*

$$\delta_{\Gamma,F} = \limsup_{n\to\infty} \tfrac{1}{n} \log\Big(\sum_{\gamma\in\Gamma, n-c\leq d(x,\gamma x)\leq n} e^{\int_x^{\gamma x} \widetilde{F}}\Big).$$

We define a set of measures on $\partial_\infty\widetilde{M}$ as the limit points when $s \to \delta^+_{\Gamma,F}$ of

$$\tfrac{1}{P_{x,\Gamma,F}(s)} \sum_{\gamma\in\Gamma} e^{\int_x^{\gamma x} \widetilde{F} - s} h(d(x, \gamma x)) D_{\gamma x} = \mu^F_{x,s}.$$

where $h : \mathbb{R}_+ \to \mathbb{R}^*_+$ is a well chosen non-decreasing map and $D_{\gamma x}$ is the Dirac measure supported on γx.

Proposition 2.3. *If $\delta_{\Gamma,F} < \infty$ then*

1. *$\{\mu^F_{x,s}\}$ has at least one limit point when $s \to \delta^+_{\Gamma,F}$ with support $\Lambda\Gamma$,*
2. *If μ^F_x is a limit point then it is a Patterson density i.e*
   $\forall \gamma \in \Gamma, x, y \in \widetilde{M}, \xi \in \partial_\infty\widetilde{M}$

$$\gamma_* \mu^F_x = \mu^F_{\gamma x},$$

$$d\mu_x^F(\xi) = e^{-C_{F-\delta_{\Gamma,F},\xi}(x,y)} d\mu_y^F(\xi).$$

Using the Hopf parametrization on $T^1\widetilde{M}$, each unit tangent vector v can be written as $v = (v_+, v_-, t) \in \partial_\infty \widetilde{M} \times \partial_\infty \widetilde{M} \times \mathbb{R}$. We define a measure on $T^1\widetilde{M}$ by

$$d\widetilde{m}_{\widetilde{F}}(v) = \frac{d\widetilde{\mu}_x^{\widetilde{F}\circ\iota}(v_-) d\widetilde{\mu}_x^{\widetilde{F}}(v) dt}{D_{F,x}(v_+,v_-)}$$

where

$$D_{F,x}(v_+, v_-) = e^{-\frac{1}{2}(C_{F,v_-}(x,\pi(v))+(C_{F\circ\iota,v_+}(x,\pi(v)))}$$

is the potential gap and

$$\iota \begin{cases} T^1\widetilde{M} \to & T^1\widetilde{M} \\ v \mapsto & -v \end{cases}$$

is the antipodal map.
This measure is called the Gibbs measure associated to (Γ, F).
It is a measure independent of x, invariant under the action of Γ and invariant by the geodesic flow. Hence it defines a measure m^F on T^1M invariant by the geodesic flow.

Let $m \in \mathscr{M}^1(T^1M)$ be a measure with finite entropy $h_m(\phi_t)$. We define the **metric pressure** of a potential F with respect to the measure m as the quantity

$$P_{\Gamma,F}(m) = h_m(\phi_t) + \int_{T^1M} F dm.$$

We say that the supremum

$$P(\Gamma, F) = \sup_{m\in\mathscr{M}(T^1M)} P_{\Gamma,F}(m)$$

is the **topological pressure** of the potential F. An element realizing this upper bound is called an **equilibrium state** for (Γ, F).

Theorem 2. [OP04, PPS15] *Let $\widetilde{M}$ be a complete, simply connected Riemannian manifold with pinched negative curvature, Γ a non-elementary discrete group of isometries of $\widetilde{M}$ and $\widetilde{F} : T^1\widetilde{M} \to \mathbb{R}$ a Hölder-continuous Γ-invariant map with $\delta_{\Gamma,F} < \infty$.*

(1) *We have*
$$P(\Gamma, F) = \delta_{\Gamma,F}.$$
(2) *If there exists a finite Gibbs measure m_F for (Γ, F) such that the negative part of F is m_F-integrable, then $m^F = \frac{m_F}{\|m_F\|}$ is the unique equilibrium state for (Γ, F). Otherwise, there exists no equilibrium state for (Γ, F).*

3. Mixing property for the geodesic flow

The question of the topological mixing of the geodesic flow on a negatively curved manifold is still open in full generality. This question is closely related to mixing with respect to a Gibbs measure (see [PPS15]).
We define the length of an element $\gamma \in \Gamma$ by $\ell(\gamma) = \inf_{z\in M} d(z, \gamma z)$.

Theorem 3. *If $\delta_{\Gamma,F} < \infty$ and m^F is finite then the following propositions are equivalent.*

(1) *The geodesic flow is topologically mixing on Ω,*
(2) *The geodesic flow is mixing with respect to m^F,*
(3) *$L(\Gamma) = \{\ell(\gamma); \gamma \in \Gamma\}$ is not contained in a discrete subgroup of $\mathbb{R}$.*

Here are some cases where the geodesic flow is known to be topologically mixing [DP98],[Dal99],[Dal00].

Lemma 3.1. *Let Γ be a non elementary group of isometries of a Hadamard manifold $\widetilde{M}$ with pinched negative curvature. If $M = \widetilde{M}/\Gamma$ satisfies one of the following properties then the restriction of the geodesic flow to its non-wandering set is topologically mixing.*

(1) *The curvature of M is constant,*
(2) *$\dim M = 2$,*
(3) *There exists a parabolic isometry in Γ,*
(4) *$\Omega = T^1M$.*

To conclude this section, let us recall the Hopf-Tsuji-Sullivan criterion for the ergodicity of the geodesic flow with respect to the Gibbs measure (see [PPS15] for a proof)

Theorem 4. *The following assertions are equivalent:*

(1) *(Γ, F) is of divergence type,*
(2) *$\forall x \in \widetilde{M}, \mu_x^F(\partial_\infty \widetilde{M} \backslash \Lambda_c\Gamma) = 0$,*
(3) *The dynamical system $(T^1M, (\phi_t)_{t\in\mathbb{R}}, m_F)$ is ergodic.*

As a consequence of this theorem, one can show that if $\delta_{\Gamma,F} < \infty$, the Patterson density $(\mu_x^F)_{x\in\widetilde{M}}$ associated with (Γ, F) is non-atomic (see [PPS15] Proposition 5.13).

4. A finiteness criterion

First, let us give a criterion for the finiteness of the Gibbs measure. This result comes from [DOP00] for a potential $F = 0$. For the general case where F is an Hölder potential, the proof is given in [PPS15].

Theorem 5. *Suppose that Γ is a geometrically finite group with (Γ, F) of divergence type and $\delta_{\Gamma,F} < \infty$. The Gibbs measure m_F is finite if and only if for every parabolic fixed point ξ_p*

$$\sum_{\gamma\in\Gamma_{\xi_p}} d(x,\gamma x) e^{\int_x^{\gamma x}(\widetilde{F}-\delta_{\Gamma,F})}$$

converges.

Definition 4.1. *(Γ, F) satisfies the spectral gap property if for all parabolic points $\xi_p \in \partial_\infty\widetilde{M}$,*

$$\delta_{\Gamma_{\xi_p},F} < \delta_{\Gamma,F}.$$

Proposition 2 of [DOP00] gives a criterion for this property for the zero potential. The following proposition is more general and applies to all Hölderian potentials.

Proposition 4.1. *Let $\widetilde{M}$ be a Hadamard manifold with pinched negative curvature and Γ a geometrically finite discrete group acting on it. Suppose there exists a bounded Hölderian potential $\widetilde{F} : T^1\widetilde{M} \to \mathbb{R}$.*
If for all parabolic fixed point ξ_p the couple (Γ_{ξ_p}, F) is of divergence type, then (Γ, F) satisfies the spectral gap property. Moreover, the Gibbs measure m^F associated to (Γ, F) is finite.

Proof. To prove the first claim, we follow the ideas of [DOP00] when $F = 0$. Since the action of Γ_{ξ_p} on $\partial_\infty\widetilde{M}$ has a fundamental domain $\mathscr{G}$ in $\partial_\infty\widetilde{M}$, we have

$$\mu_x^F(\partial_\infty\widetilde{M}) = \sum_{g\in\Gamma_{\xi_p}} \mu_x^F(g\mathscr{G}) + \mu_x^F(\Lambda\Gamma_{\xi_p}).$$

Moreover, since there exists $K \in \mathbb{R}$ such that

$$\begin{cases} \mu_x^F(g\mathscr{G}) = \displaystyle\int_{\mathscr{G}} e^{-C_{F-\delta_{\Gamma,F},\xi}(x,gx)} d\mu_{gx}(\xi) \\ \qquad\qquad |C_{F,\xi}(x,gx) + \displaystyle\int_x^{gx} \widetilde{F}| \leq K, \end{cases}$$

we have

$$\mu_x^F(g\mathscr{G}) \geq (e^{\int_x^{gx}\widetilde{F}-\delta_{\Gamma,F}})(e^{-K}\mu_x^F(\mathscr{G}))$$

and

$$\infty > \mu_x^F(\partial_\infty\widetilde{X}) \geq \sum_{g\in\Gamma_{\xi_p}} \mu_x^F(g\mathscr{G}) \geq C_0 \cdot P_{x,\Gamma_{\xi_p},F}(\delta_{\Gamma,F}).$$

So, $\delta_{\Gamma,F} > \delta_{\Gamma_{\xi_p},F}$.

For the second claim, since (Γ, F) satisfies the spectral gap property for $M = \widetilde{M}/\Gamma$ and

$$\forall\epsilon > 0, \exists C_\epsilon > 0, d(x,\gamma x) \geq C_\epsilon \Rightarrow e^{\epsilon d(x,\gamma x)} > d(x,\gamma x),$$

we have

$$\sum_{\gamma\in\Gamma_{\xi_p}} d(x,\gamma x) e^{\int_x^{\gamma x}\widetilde{F}-\delta_{\Gamma,F}} \leq \sum_{\gamma\in\Gamma_{\xi_p}} e^{\int_x^{\gamma x}\widetilde{F}-(\delta_{\Gamma,F}-\epsilon)}$$

Choosing ϵ small enough such that $\frac{\delta_{\Gamma,F}-\delta_{\Gamma_{\xi_p},F}}{2} > \epsilon$, the series converges and the Gibbs measure is finite. □

5. Construction of the potentials

We now construct a $\Gamma-$invariant potential $\widetilde{H} : T^1\widetilde{M} \to \mathbb{R}$ such that the associated Gibbs measure is finite and which critical exponent associated to (Γ_{ξ_p}, H) is of divergence type.

Since M is geometrically finite, the set Par_Γ of parabolic points $\xi_p \in \partial_\infty\widetilde{M}$ intersecting the boundary of the Dirichlet domain is finite.
We define for those parabolic points a family of disjoint horoballs $\{\mathscr{H}_{\xi_p}(u_{0,\xi_p})\}_{\xi_p}$ on $\widetilde{M}$ passing through a well chosen point $\widetilde{u}_{0,\xi_p}$ of the cusp.

For any $\widetilde{u}$ in $\mathscr{H}_{\xi_p}(\widetilde{u}_{0,\xi_p})$, we define a height function $\rho : \widetilde{M} \to \mathbb{R}$ by the Buseman cocycle at $\widetilde{u}_{0,\xi_p}$:

$$\rho(\widetilde{u}) = \beta_{\xi_p}(\widetilde{u}, \widetilde{u}_{0,\xi_p}).$$

This cocycle coincides with the Gibbs cocycle for the potential $F = -1$. The curve levels of this function are the horocycles based on ξ_p passing through $\widetilde{u}$.

Let t_n be a decreasing sequence of positive numbers converging to 0. One can construct a sequence Y_n of positive numbers such that

$$\begin{cases} Y_{n+1} \geq Y_n + t_n - t_{n+1}, \\ \sum_{p \in \rho^{-1}(]Y_n, Y_{n+1}])} e^{d(x_0, px_0)(t_n - \delta_{\Gamma_{\xi_p}})} \geq 1. \end{cases}$$

for all $\xi_p \in Par_\Gamma$, we define a Γ-invariant map on $\mathscr{D} \cap \mathscr{H}_{\xi_p}(u_{0,\xi_p})$ as follows: for every $\widetilde{u}$ in $\mathscr{H}_{\xi_p}(u_{0,\xi_p})$,

$$\widetilde{H}_1(\widetilde{u}) = \begin{cases} t_n + Y_n - \rho(\widetilde{u}) \text{ on } \ \rho^{-1}(]Y_n, Y_n + t_n - t_{n+1}]), \\ t_{n+1} \text{ on } \ \rho^{-1}(]Y_n + t_n - t_{n+1}, Y_{n+1}]). \end{cases}$$

We extend $\widetilde{H}_1$ to $\widetilde{M}$ as follows.
First, we extend it to $\mathscr{D}$ by a constant function such that

$$\widetilde{H}_1 : \mathscr{D} \to \mathbb{R}$$

is Hölder-continuous. Next, we extend it to $\widetilde{M}$ as follows. For all $\gamma \in \Gamma$, if $\widetilde{x} \in \gamma\mathscr{D}$ then

$$\widetilde{H}_1(\widetilde{x}) = \widetilde{H}_1(\gamma^{-1}\widetilde{x}).$$

Let $\widetilde{H}$ be the $\Gamma-$invariant potential obtained by pulling back H_1 on $T^1\widetilde{M}$. We denote by $H : T^1M \to \mathbb{R}$ the map induced by $\widetilde{H}$ on T^1M.

Lemma 5.1. *(Coudène* [Cou03]*) $\widetilde{H}$ is a $\Gamma-$invariant Hölderian potential such that for all parabolic fixed point ξ_p, the critical exponent associated with (Γ_{ξ_p}, H) is of divergence type.*

We have therefore constructed a Γ-invariant Hölderian potential on T^1M such that the associated Gibbs measure is finite and is supported on Ω.

Remark that the divergence of the parabolic subgroups only depends on the value of the potential in the cusp. In the previous construction, we made the assumption that the potential was constant on

$$T^1M_0 = T^1(\mathscr{C}\Lambda\Gamma \bigcup_{\xi_p \in Par_\Gamma} \mathscr{H}_{\xi_p})/\Gamma.$$

However, taking any bounded potential such that the resulting function on T^1M_0 is Hölder-continuous, the associated Gibbs measure will still be finite. We now choose $p \in T^1M = T^1\widetilde{M}/\Gamma$ such that $\phi_t(p)$ is periodic. We will denote by $\mathscr{O}(p)$ the closed curve $\phi_{\mathbb{R}}(p)$ and assume that $\mathscr{O}(p) \subset T^1M_0$.
For all $n \in \mathbb{N}$, we define a Lipschitz-continuous potential by

$$F_n(x) = \max\{c_n - c_n d(\mathscr{O}(p), x); H(x)\}.$$

Proposition 5.1. *For all $n \in \mathbb{N}$, the critical exponent δ_{Γ,F_n} is finite. Moreover, we have*

$$c_n \leq \delta_{\Gamma,F_n} \leq \delta_\Gamma + c_n.$$

Proof. Since $c_k = \sup_{x\in T^1M} F_k(x)$, the upper bound $\delta_{\Gamma,F_k} \leq \delta_\Gamma + c_k$ is evident by the second claim of Proposition 2.2.
Let p be the periodic point of T^1M defined above and $h \in \Gamma$ the generator of the isometry group fixing the periodic orbit $\phi_{\mathbb{R}}(p)$. Let $H =< h >$ and denote by ℓ the length of h. By the very definition of critical exponents, we have

$$\delta_{\Gamma,F_k} = \limsup_{n\to\infty} \tfrac{1}{n} \log \sum_{\gamma\in\Gamma,\ n-\ell<d(x,\gamma x)\leq n} e^{\int_x^{\gamma x} \widetilde{F}_k} \tag{1}$$

$$\geq \limsup_{n\to\infty} \tfrac{1}{n} \log \sum_{\gamma\in H,\ n-\ell<d(x,\gamma x)\leq n} e^{\int_x^{\gamma x} \widetilde{F}_k}. \tag{2}$$

Since the critical exponent does not depend on the choice of a base point, one can choose $x = \widetilde{p}$ where $\widetilde{p}$ is a lift of p on $T^1\widetilde{M}$. Therefore, the fact that the value of the potential on $\mathscr{O}(p)$ is constant, equal to c_k implies that

$$\int_{\widetilde{p}}^{\gamma\widetilde{p}} \widetilde{F}_k = d(x,\gamma x)c_k \geq (n-\ell)c_k$$

which gives

(3)
$$\delta_{\Gamma,F_k} \geq \limsup_{n\to\infty} \tfrac{1}{n} \log \sum_{\gamma\in H,\ n-\ell<d(x,\gamma x)\leq n} e^{(n-\ell)c_k} \geq \limsup_{n\to\infty} \tfrac{1}{n} \log \left(A_x e^{(n-\ell)c_k}\right),$$

where $A_x = \sharp\{\gamma \in H : n-\ell < d(x,\gamma x) \leq n\}$.
Since the group H is generated by a hyperbolic isometry h, for all $\gamma \in H$, there exists $i \in \mathbb{N}$ such that $\gamma = h^i$ and $\ell(\gamma) = \ell(h^i) = i\ell(h)$. Therefore the quantity A_x does not depend on n and

$$\delta_{\Gamma,F_k} \geq c_k.$$

which concludes the proof. □

Therefore, the Gibbs measure m_{F_n} associated with (Γ, F_n) of dimension δ_{Γ,F_n} exists for all $n \in \mathbb{N}$.

6. Proof of the main theorem

Let $D_{\mathscr{O}(p)}$ be the Dirac measure supported on $\mathscr{O}(p)$. We prove Theorem 1 which states the following.
Let M be a geometrically finite, negatively curved manifold and ϕ_t its geodesic flow. If ϕ_t is topologically mixing on Ω, then the set of probability measures that are mixing with respect to the geodesic flow is dense in $\mathscr{M}^1(T^1M)$ for the weak topology.

Here is our strategy: since the geodesic flow on a manifold with pinched negative curvature satisfies the closing lemma (see [Ebe96]) and admits a local product structure, we use the following result.

Proposition 6.1. *(Coudène-Schapira* [CS10]*) Let M be a complete, connected Riemannian manifold with pinched negative curvature and ϕ_t its geodesic flow. Then the set of normalized Dirac measures on periodic orbits is dense in the set of all invariant measures $\mathscr{M}^1(T^1M)$.*

It is therefore clear that the following proposition implies Theorem 1.

Proposition 6.2. *For all $p \in T^1M$ such that $\mathscr{O}(p) = \phi_{\mathbb{R}}(p)$ is periodic, there exists a sequence $\{m_k\}_{k\in\mathbb{N}}$ of measures satisfying the following properties*

1. *m_k is a probability measure which is mixing with respect to the geodesic flow,*
2. *$m_k \rightharpoonup D_{\mathscr{O}(p)}$.*

Let $\mathrm{h_{top}}(\phi_t)$ be the topological entropy of the geodesic flow on T^1M and $h_\mu(\phi_t)$ the measure theoretic entropy of the geodesic flow with respect to μ. D. Sullivan proved in [Sul] that if the Bowen-Margulis measure m_{BM} (*i.e* the Gibbs measure associated with the potential $F = 0$) is finite then

$$\delta_\Gamma = h_{m_{BM}}(\phi_t).$$

Using a result of C.J.Bishop and P.W.Jones [BJ] connecting the critical exponent δ_Γ with the Hausdorff dimension of the conical limit set of Γ, J.P.Otal and M.Peigné proved that for all ϕ_t-invariant probability measures $m \in \mathscr{M}^1(T^1\widetilde{M})$ which are not the Bowen-Margulis measure, we have the strict inequality,

$$h_\mu(\phi_t) < \delta_\Gamma.$$

We refer to F. Ledrappier [Led13] for a survey of these results.

Theorem 6. [OP04] *Let $\widetilde{M}$ be a simply connected, complete Riemannian manifold with pinched negative curvature and Γ be a non-elementary discrete group of isometries of $\widetilde{M}$, then*

$$\mathrm{h_{top}}(\phi_t) = \delta_\Gamma.$$

Moreover, there exists a probability measure μ maximizing the entropy if and only if the Bowen-Margulis measure is finite and $\mu = m_{BM}$.

We begin the proof of the main theorem. First, we state a general result which holds true for any metric space X satisfying a variational principle. In the next claim, we suppose that $F : X \to \mathbb{R}$ is a measurable function

such that there exist an invariant compact set $K \subset X$ and a neighborhood V of K satisfying the following assumptions:

$$\begin{cases} \forall x \in K, F(x) = c = \sup\limits_{x \in X} F(x) \\ \sup\limits_{x \in X \backslash V} F(x) = c' < c. \end{cases}$$

We say that a probability measure μ with finite entropy is an equilibrium state for a potential $F : X \to \mathbb{R}$ if it achieves the supremum of

$$m \mapsto h_m(\phi_t) + \int_X F dm$$

over all invariant probability measures with finite entropy.

Lemma 6.1. *Let X be a metric space, ϕ_t a flow defined on X and $F : X \to \mathbb{R}$, K, V defined as above.*
Suppose there exists an equilibrium state μ for F, then

$$\mu(X \backslash V) \leq \tfrac{h_\mu(\phi_t)}{c - c'}.$$

Proof. Let m_K be a probability measure supported on K. Since μ realises the supremum of

$$m \in \mathscr{M}^1(X) \mapsto h_m(\phi_t) + \int_X F dm$$

we have

$$h_\mu(\phi_t) + \int_X F d\mu \geq h_{m_K}(\phi_t) + \int_X F dm_K$$

which implies

$$h_\mu(\phi_t) + \int_X F d\mu \geq \int_X F dm_K \tag{4}$$

since $h_{m_K}(\phi_t) \geq 0$. Moreover, since the potential F is constant on K we have

$$\int_X F dm_K = c$$

which can be written as

$$\int_X F dm_K = c(\mu(V) + \mu(X \backslash V)). \tag{5}$$

Combining equations 4 and 5, we obtain

$$\begin{array}{rcl} h_\mu(\phi_t) & \geq & \int_X F dm_K - \int_X F d\mu \\ & \geq & c(\mu(V) + \mu(X \backslash V)) - c\mu(V) - c'\mu(X \backslash V)) \end{array}$$

which finally gives

$$\tfrac{h_\mu(\phi_t)}{c - c'} \geq \mu(X \backslash V).$$

□

Recall that a sequence $\{\mu_n\}_{n \in \mathbb{N}}$ of probability measures on a Polish space X is tight if for all $\epsilon > 0$ there exists a compact set $K \subset X$ such that

$$\forall n \in \mathbb{N}, \quad \mu_n(X \backslash K) < \epsilon.$$

We give a criterion for the convergence of a sequence of probability measures to the Dirac measure supported on the periodic orbit $\mathscr{O}(p)$. We denote by V_ϵ the subset of T^1M defined by

$$V_\epsilon = \{x \in T^1M : d(x, \mathscr{O}(p)) \leq \epsilon\}.$$

Lemma 6.2. *The following assertions are equivalent:*

(1) *The sequence of probability measures* $\{m^{F_n}\}_{n\in\mathbb{N}}$ *converges to the Dirac measure supported on* $\mathscr{O}(p)$,
(2) *for all* $\epsilon > 0$,
$$\lim_{n\to\infty} m^{F_n}(T^1M\backslash V_\epsilon) = 0.$$

Proof. It is clear that (1) $\Rightarrow$ (2). Let us show that (2) $\Rightarrow$ (1).
We first notice that $\{m^{F_n}\}_{n\in\mathbb{N}}$ is tight. Indeed, let V be a compact subset of T^1M containing $\mathscr{O}(p)$. Since condition (2) is satisfied, for all $\epsilon > 0$, there exists $N_0 > 0$ such that for all $n \geq N_0$,
$$m^{F_n}(T^1M\backslash V) < \epsilon.$$
For all $n \in \{1, .., N_0 - 1\}$, we can also find a compact set K_n such that
$$m^{F_n}(T^1M\backslash K_n) < \epsilon.$$
Define the compact set $K = (\bigcup_{n=1}^{N_0-1} K_n) \cup V$.
For all $n \in \mathbb{N}$, K satisfies
$$m^{F_n}(T^1M\backslash K) \leq \epsilon.$$
Therefore, the sequence $\{m^{F_n}\}_{n\in\mathbb{N}}$ is tight.
Since condition (2) is satisfied and using the fact that the unique invariant probability measure supported on $\mathscr{O}(p)$ is $D_{\mathscr{O}(p)}$, each converging subsequence of $\{m^{F_n}\}$ converges to $D_{\mathscr{O}(p)}$.
Therefore, by Prokhorov's Theorem [Pro56], each sub sequence of $\{m^{F_n}\}$ possesses a further subsequence converging weakly to $D_{\mathscr{O}(p)}$ so the sequence m^{F_n} converges weakly to $D_{\mathscr{O}(p)}$. □

We are now able to prove our main result of convergence.

Theorem 7. *The sequence* $\{m^{F_n}\}_{n\in\mathbb{N}}$ *of probability measures converges to the Dirac measure supported on* $\mathscr{O}(p)$.

Proof. Let $D_{\mathscr{O}(p)}$ be the Dirac measure supported on the periodic orbit $\mathscr{O}(p)$.
Recall that $c_n = \sup_{x\in T^1M} F_n(x)$.
By lemma 6.2, we have to show that
$$\lim_{n\to\infty} m^{F_n}(T^1M\backslash V_\epsilon) = 0.$$
Using the variational principle described in Theorem 2, we have
$$\delta_{\Gamma,F_n} = \sup_{\mu\in\mathscr{M}^1(T^1M)} P_{\Gamma,F_n}(\mu) = P_{\Gamma,F_n}(m^{F_n})$$
and
$$\delta_{\Gamma,F_n} \geq P_{\Gamma,F_n}(D_{\mathscr{O}(p)}) = h_{D_{\mathscr{O}(p)}}(\phi_t) + \int_{T^1M} F_n dD_{\mathscr{O}(p)}.$$
Since $\mathscr{O}(p)$ is invariant by the action of the geodesic flow and
$$h_{m_{F_n}}(\phi_t) \leq \delta_\Gamma < \infty,$$

one can use the claim of lemma 6.1 to obtain the following inequality

$$m^{F_n}(T^1M\backslash V_\epsilon) \leq \frac{\delta_\Gamma}{c_n - c'_n}$$

where

$$c'_n = \sup_{x \in T^1M\backslash V_\epsilon} F_n(x).$$

By the definition of the potential F_n, we know that

$$c'_n \leq \max\{c_0, c_n(1-\epsilon)\}.$$

Therefore, for all $\epsilon > 0$ and n large enough,

$$m^{F_n}(T^1M\backslash V_\epsilon) \leq \frac{\delta_\Gamma}{c_n \epsilon}$$

implies that

$$\lim_{n\to\infty} m^{F_n}(T^1M\backslash V_\epsilon) = 0.$$

Which concludes the proof. □

Finally, we are able to prove corollary 1.1 which states that the set of weak-mixing measures is a dense G_δ subset of $\mathscr{M}^1(T^1M)$.
Our proof relies on the following theorem.

Theorem 8. *(Coudène-Schapira* [CS14]*) Let $(\varphi^t)_{t\in\mathbb{R}}$ be a continuous flow on a Polish space. The set of weak-mixing measures on X is a G_δ subset of the set of Borel invariant probability measures on X.*

Proof. (of corollary 1.1)
Since mixing measures are weak-mixing, Theorem 1 implies that weak-mixing measures are dense in the set of probability measures on Ω. The previous theorem insures us that it is a G_δ set.

Since negatively curved manifolds with pinched curvature satisfy the closing lemma, it is shown in [CS10] that the set of invariant measures with full support on Ω is a dense G_δ subset of the set of invariant probability measures on Ω.

The intersection of those two dense G_δ sets is a dense G_δ set by the Baire Category Theorem. □

7. Non-geometrically finite manifolds with cusps

We now prove corollaries 1.2 and 1.3. First of all, remark that since the manifold possesses a cusp then from lemma 3.1, the geodesic flow is topologically mixing on $T^1M = T^1\widetilde{M}/\Gamma$.
Let $\gamma \in \Gamma$ and $D(\gamma)$ the subset of $\widetilde{M}$ bounded by

$$\{\widetilde{x} \in \widetilde{M} : d(\widetilde{x}, \widetilde{x}_0) = d(\widetilde{x}, \gamma\widetilde{x}_0)\}$$

and containing $\gamma\widetilde{x_0}$. We define the subset C_γ on $\partial_\infty\widetilde{M}$ as follows.

$$C_\gamma = \partial_\infty\widetilde{M} \cap \overline{D(\gamma)}.$$

The proof of corollary 1.2 is deduced from the following result.

Lemma 7.1. *Let M be a connected, complete pinched negatively curved manifold with a cusp and ϕ_t the geodesic flow defined on T^1M. There exists a geometrically finite manifold $\hat{M}$ such that its geodesic flow $\hat{\phi}_t$ is topologically mixing and a covering map $\rho : T^1\hat{M} \to T^1M$ such that the diagram*

$$\begin{array}{ccc} T^1\hat{M} & \xrightarrow{\hat{\phi}_t} & T^1\hat{M} \\ \rho\downarrow & & \downarrow\rho \\ T^1M & \xrightarrow{\phi_t} & T^1M \end{array}$$

commutes.

Proof. Let $\xi_p \in \partial_\infty\widetilde{M}$ be a bounded parabolic point fixed by a parabolic isometry $\gamma_p \in \Gamma$ and $h \in \Gamma$ be a hyperbolic transformation.
Let $N > 0$ be defined such that the sets $C_{\gamma_p^N}, C_{h^N}, C_{\gamma_p^{-N}}, C_{h^{-N}}$ have disjoint interiors. We define $\Gamma_0 = < \gamma_p^N, h^N >$, a subgroup of Γ. The ping-pong lemma shows that Γ_0 is a discrete group which acts freely discontinuously on $\widetilde{M}$. So, the quotient $\widetilde{M}/\Gamma_0$ is a geometrically finite manifold. □

Proof. (of corollary 1.2) Let $\hat{M}$ be a geometrically finite manifold constructed as in lemma 7.1 and ρ its associated covering map. We can use the proof of Theorem 1 on $\hat{M}$ and construct a sequence $\hat{m}_k$ of invariant mixing measures for $\hat{\phi}_t$ such that $\hat{m}_k \rightharpoonup D_{\mathcal{O}(p)}$.
Since the geodesic flow ϕ_t is a factor of $\hat{\phi}_t$, we can define ν_k to be the push-forward by ρ of $\hat{m}_k$, then it is an invariant mixing measure on T^1M and for all bounded continuous function g on T^1M,

$$\begin{array}{rcl} \lim_{k\to\infty} \int_{T^1M} g d\nu_k & = & \lim_{k\to\infty} \int_{T^1\hat{M}} g\circ\rho dm_k \\ & = & \int_{T^1\hat{M}} g\circ\rho dD_{\mathcal{O}(p)} \\ & = & \int_{T^1M} g d(\rho_* D_{\mathcal{O}(p)}). \end{array}$$

□

We end up by the proof of corollary 1.3. In the case of a surface S (or a constant negatively curved manifold), we don't need to ask for the existence of a bounded parabolic point. Since the geodesic flow is always topologically mixing in restriction to its non wandering set, we can choose two hyperbolic isometries h_1, h_2 in Γ such that the subgroup $\Gamma_0 =< h_1, h_2 >$ is convex-cocompact.
The same proof as lemma 7.1 shows us that the geodesic flow ϕ_t on S is a factor of the geodesic flow $\hat{\phi}_t$ on the convex-cocompact manifold $T^1S_0 = T^1\widetilde{S}/\Gamma_0$. Therefore, the previous proof gives us the density result.

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