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Olivier Piltant, Ana J Reguera

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Local uniformization and arc spaces.

Olivier Piltant, Ana J. Reguera

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Abstract. Let X be a variety over a perfect field k and let X_∞ be its space of arcs. We prove that Local Uniformization on X implies that, for every closed subset Z of X , the closed subset of X_∞ consisting of all arcs centered at some point of Z has a finite number of irreducible components. This strengthens a theorem due to J. Nash.

1 Introduction

In 1968, J. Nash initiated the study of the space of arcs X_∞ of a (singular) algebraic variety X with the purpose of understanding the resolutions of singularities of X . His work [Na] was done shortly after the proof of resolution of singularities in characteristic zero by H. Hironaka [Hi].

Nash's starting point is the following: let X be a variety over a perfect field k and suppose that there exists a resolution of singularities $\pi : Y \rightarrow X$. For every irreducible component E of the exceptional locus of π , let us consider the *Nash family of arcs* N_E consisting of the Zariski closure of the image of the set of arcs on Y which are centered in some point of E . He observes that each N_E is irreducible and, moreover, N_E only depends on the divisorial valuation defined by E . Besides, due to the properness of π , every arc in $X_\infty \setminus (\text{Sing } X)_\infty$ which is centered in some point of the singular locus

of X belongs to some of the N_E 's. That is, the space of arcs X_∞^{Sing} centered in $\text{Sing } X$ decomposes as

$$X_\infty^{Sing} = \cup_E N_E \cup (\text{Sing } X)_\infty.$$

From this, and arguing by induction on $\dim X$, one deduces that the number of irreducible components of X_∞^{Sing} is finite (see [Na] or [IK], [Re]).

No other proof of the previous result is known until now. In particular, to know whether the number of irreducible components of X_∞^{Sing} is finite remains an open problem if $\text{char } k = p > 0$ and $\dim X \geq 4$, since resolution of singularities is still an open problem in this situation.

The purpose of this note is to prove that Hironaka's resolution of singularities can be substituted by the weaker notion of Zariski's generalized Local Uniformization along valuations in Nash's argument. More precisely, given a variety X over a perfect field k and $Z \subseteq X$ a Zariski closed subset, let X_∞^Z be the space of arcs on X centered at some point of Z . Say that $LU(X, \zeta)$ holds for $\zeta \in X$ an arbitrary point if any valuation of X centered at ζ admits a local uniformization dominating $\mathcal{O}_{X, \zeta}$. We prove the following result:

Corollary. *Let $X|k$ be a k -variety, $Z \subseteq X$ a closed subset of X . Assume that, for every subvariety $F \subseteq X$ and for every $\zeta \in F \cap Z$, Local Uniformization holds on F at ζ . Then X_∞^Z has a finite number of irreducible components.*

2 Preliminaries

In this section we will set the notation and recall some known properties of the space of arcs that will be used in this work.

2.1. Let k be a perfect field and let X be a *variety over k* , i.e. X is a reduced separated k -scheme of finite type. Let X_∞ denote the *space of arcs of X* : It is a k -scheme whose K -rational points are the K -arcs on X , i.e. the k -morphisms $\text{Spec } K[[t]] \rightarrow X$, for any field extension $k \subseteq K$. More precisely, $X_\infty := \lim_{\leftarrow} X_n$ where, for $n \in \mathbb{N}$, X_n is the k -scheme of finite type whose K -rational points are the K -arcs of order n on X , i.e. the k -morphisms $\text{Spec } K[[t]]/(t)^{n+1} \rightarrow X$.

We denote by $j_n : X_\infty \rightarrow X_n$, $n \geq 0$, the natural projections. We also denote by $j : X_\infty \rightarrow X$ the projection j_0 . Given a closed subset Z of X , we denote by X_∞^Z the subscheme $j^{-1}(Z)$ of X_∞ .

Given $P \in X_\infty$, with residue field $\kappa(P)$, we denote by $h_P : \text{Spec } \kappa(P)[[t]] \rightarrow X$ the induced $\kappa(P)$ -arc on X . The image x_0 in X of the closed point of $\text{Spec } \kappa(P)[[t]]$, i.e. $x_0 = j(P)$, is called the *center* of P . The image x_η of the generic point of $\text{Spec } \kappa(P)[[t]]$ is the generic point of $\text{Im } h_P$. Then, h_P induces a morphism of k -algebras $h_P^\# : \mathcal{O}_{X, x_0} \rightarrow \kappa(P)[[t]]$, or an injective morphism $h_P^\# : \mathcal{O}_{\overline{\{x_\eta\}}, x_0} \rightarrow \kappa(P)[[t]]$. We denote by ν_P the function $\text{ord}_t h_P^\# : \mathcal{O}_{X, x_0} \rightarrow \mathbb{N} \cup \{\infty\}$, which defines a valuation of $\mathcal{O}_{\overline{\{x_\eta\}}, x_0}$.

The scheme X_∞ is not in general finitely generated. However it satisfies several finiteness properties. The following result refers to one of them. It is proved in [Ko] chap. IV, prop. 10 if $\text{char } k = 0$ and in [Gi] cor. 1.28, [Re] th. 2.9 for any perfect field k .

Proposition 2.2. (*[Ko] and [Gi], [Re]*) *Let X be a variety over a perfect field k and let $Z \subseteq X$ be a nowhere dense subvariety such that $\text{Sing } X \subseteq Z$. The restriction of j :*

$$X_\infty \setminus Z_\infty \longrightarrow X$$

induces a bijection on irreducible components.

We conclude that X_∞ has finitely many irreducible components.

Dealing with irreducible components of X_∞^Z , where Z is a subvariety of X , in the case of a regular variety we have:

Lemma 2.3. *Let Y be a regular k -variety. The following holds:*

- (1) *For every subvariety $Z \subseteq Y$, the induced map $Y_\infty^Z \rightarrow Z$ induces a bijection on irreducible components. More precisely, if $Z = E_1 \cup \dots \cup E_r$ is the decomposition of Z into irreducible components, then*

$$Y_\infty^Z = Y_\infty^{E_1} \cup \dots \cup Y_\infty^{E_r}$$

is the decomposition of Y_∞^Z into irreducible components.

- (2) *If furthermore $Z \subseteq Y$ is nowhere dense, then $Y_\infty \setminus Y_\infty^Z$ is Zariski dense in Y_∞ .*

Proof. We may suppose that Y is affine, let $Y \subseteq \mathbb{A}^m$. There exists an étale morphism from Y to a subset of \mathbb{A}_k^d where $d = \dim Y$. Then we have

$$\mathcal{O}_{Y_\infty} \cong \mathcal{O}_Y[\underline{X}_1, \underline{X}_2, \dots, \underline{X}_n, \dots]$$

where, for $n \geq 0$, $\underline{X}_n = (X_{1,n}, \dots, X_{d,n})$ is a d -uple of variables. Then, (1) follows from the induced isomorphisms

$$\mathcal{O}_{Y_\infty^Z} \cong \mathcal{O}_Z[\underline{X}_1, \dots, \underline{X}_n, \dots] \quad \text{and} \quad \mathcal{O}_{Y_\infty^{E_i}} \cong \mathcal{O}_{E_i}[\underline{X}_1, \dots, \underline{X}_n, \dots] \quad 1 \leq i \leq r$$

and the fact that, for $1 \leq i \leq r$, E_i is irreducible hence \mathcal{O}_{E_i} is a domain. Analogously, (2) follows from the fact that $Y \setminus Z$ is Zariski dense in Y . \square

From this and Nash's argument (see the introduction), it follows that:

Proposition 2.4. (*[Na]*) *Let X be a variety over a perfect field k and let $Z \subseteq X$ be a nowhere dense subvariety such that $\text{Sing } X \subseteq Z$. Assume the existence of a resolution of singularities of X . Then X_∞^Z has a finite number of irreducible components.*

3 Local uniformization.

Let $X|k$ be a k -variety with irreducible components X_1, \dots, X_c . By a k -valuation of X , we mean a valuation of the fraction field $k(X_i)$ of some X_i , trivial on k , with a center on X_i . By a *birational chart over X* , we mean a dominant morphism of k -varieties $Y \rightarrow X$ such that every irreducible component of Y maps birationally to an irreducible component of X . A *regular birational chart* is a birational chart $Y \rightarrow X$ with Y a regular k -variety.

Definition 3.1. Let $X|k$ be a k -variety and v be a k -valuation of X . A *local uniformization of v* is a birational chart $Y_v \rightarrow X$ such that v has a regular center on Y_v . We say that *Local Uniformization holds on X* at some $x \in X$ (*LU(X, x)* holds for short) if every k -valuation of X with center x has a local uniformization.

Remark 3.2. The definition of Local Uniformization in 3.1 is precisely the one in Zariski's generalized theorem of Local Uniformization, in its stronger form ([Za] A.III theorem U_3 , p. 858).

3.3. Local Uniformization results ([Za] A.III th. U_3 and [CP] theorem on p.1839): Let $X|k$ be a k -variety and $x \in X$. Local Uniformization holds at x if we assume that $\text{char } k = 0$ or that $\text{char } k = p > 0$ and $\dim \mathcal{O}_{X,x} \leq 3$.

Suppose that X is affine and irreducible. Recall that the Riemann-Zariski space $\mathcal{V}(X)$, consisting of all the valuations of $k(X)$ which are nonnegative in \mathcal{O}_X , is provided with the following topology: given a subring S of $k(X)$ which is a \mathcal{O}_X -algebra of finite type, let $\mathcal{E}(S)$ be the set of $v \in \mathcal{V}(X)$ such that v is nonnegative on S , i.e. S is contained in the valuation ring R_v . Then the sets $\mathcal{E}(S)$, where S runs on all \mathcal{O}_X -algebras of finite type contained in $k(X)$, define a basis of a topology on $\mathcal{V}(X)$. With this topology, $\mathcal{V}(X)$ is quasi-compact and the map $\pi_X : \mathcal{V}(X) \rightarrow X$, sending v to its center on X , is continuous ([ZS] chap. VI, sec. 17)). The following is a consequence of the quasi-compactness of the Riemann-Zariski space.

Proposition 3.4. *Let $X|k$ be a k -variety and $Z \subseteq X$ an irreducible closed subset of X . Let ζ be the generic point of Z and assume that Local Uniformization holds on X at ζ . Then there exists a nonempty Zariski open subset $U \subseteq Z$ and a regular birational chart $Y \rightarrow X$ which is a local uniformization for every k -valuation of X with center any point z in U .*

Proof. It can be assumed that X is affine and irreducible. Since Z is a closed subset of X , by the quasi-compactness of $\mathcal{V}(X)$, there exists a regular birational chart $Y \rightarrow X$ which is a local uniformization for every k -valuation of X with center ζ . Let $\{Y_i\}_{i=1}^n$ be the irreducible components of Y . For $1 \leq i \leq n$, let $\bar{Y}_i \hookrightarrow \mathbb{P}_X^N$ be a projective closure of Y_i and let us denote by $\eta_i : \bar{Y}_i \rightarrow X$ the corresponding projective birational morphism.

Note that, since η_i is birational and proper, we have $\mathcal{V}(\bar{Y}_i) = \mathcal{V}(X)$. Let \mathcal{Z}_i be the space of valuations whose center in \bar{Y}_i belongs to $\text{Sing}(\bar{Y}_i)$, which is a closed subset of $\mathcal{V}(X)$, precisely $\mathcal{Z}_i = \pi_{\bar{Y}_i}^{-1}(\text{Sing}(\bar{Y}_i))$. Then $\mathcal{Z} := \mathcal{Z}_1 \cap \dots \cap \mathcal{Z}_n$ is a closed subset of $\mathcal{V}(X)$. Let $F \subset X$ be the image of \mathcal{Z} in X , that is, $F = \bigcap_{i=1}^n \eta_i(\text{Sing}(\bar{Y}_i))$, which is a closed subset of X . Note that $x \notin F$ by construction, because every k -valuation of X with center x has a center in $\text{Reg}(\bar{Y}_i)$ for some i . We define

$$U := Z \cap (X \setminus F) \subseteq Z,$$

which is a nonempty Zariski open subset. Let v be a k -valuation of X with center $z \in U$ and let $y_i \in \bar{Y}_i$ denote its center in \bar{Y}_i . Since $v \notin \mathcal{Z}$, we have

$y_i \in \text{Reg}(\overline{Y}_i)$ for some i . This proves that $LU(X, z)$ holds for every $z \in U$ and this concludes the proof. \square

Corollary 3.5. *Let $X|k$ be a k -variety. The set*

$$LU(X) := \{x \in X, LU(X, x) \text{ holds}\}$$

is a dense Zariski open subset of X .

Proof. The set $LU(X)$ contains the generic point of every irreducible component of X . To check openness, we use Nagata's criterion for openness (see [Ma] th. 24.2).

First, let $x \rightsquigarrow x_0$ be a specialization on X and assume that $LU(X, x_0)$ holds. Let v be a k -valuation of X with center x . Let us denote by (R_v, m_v) the valuation ring, so that $\mathcal{O}_{X,x} \prec R_v$, i.e. R_v dominates $\mathcal{O}_{X,x}$. Since x_0 is a specialization of x , the residue field $k(x)$ of x contains the ring $\mathcal{O}_{X,x_0}/\wp_x$, where \wp_x is the prime ideal defining x . Hence there exists a valuation \bar{v} of R_v/m_v such that $\mathcal{O}_{X,x_0}/\wp_x \prec R_{\bar{v}}$. Let us consider the composed valuation $v_0 = v \circ \bar{v}$, whose valuation ring is

$$R_{v_0} = \{g \in R_v / \bar{g} \in R_{\bar{v}}\}$$

\bar{g} denoting the class of g in R_v/m_v ([ZS] chap. VI, sec. 10). Then $\mathcal{O}_{X,x_0} \prec R_{v_0}$, i.e. v_0 has center in x_0 . Let $Y_{v_0} \rightarrow X$ be a local uniformization for v_0 with center $y_0 \in \text{Reg}(Y_{v_0})$ and let $y \in Y_{v_0}$ be the center of v on Y_{v_0} . Then y_0 is a specialization of y , and hence $y \in \text{Reg}(Y_{v_0})$. This proves that $LU(X, x)$ holds.

Finally note that from prop. 3.4 it follows that, if $\zeta \in LU(X)$ then there exists a nonempty open subset $U \subseteq Z := \overline{\{\zeta\}}$ such that U is contained in $LU(X)$. Therefore we have proved that $LU(X)$ is open and this concludes the proof. \square

Theorem 3.6. *Let $X|k$ be a k -variety, $Z \subseteq X$ an irreducible closed subset of X and let ζ be the generic point of Z . Assume that, for every subvariety $F \subseteq X$ with $\zeta \in F$, Local Uniformization holds on F at ζ , i.e. $LU(F, \zeta)$ holds. Then there exist finitely many irreducible closed subsets $\mathcal{N}_1, \dots, \mathcal{N}_n \subset X_\infty$ such that the following holds:*

- (1) *we have $j(\mathcal{N}_i) = Z$ for each i , $1 \leq i \leq n$;*

(2) there exists a closed subset $B \subset Z$, $B \neq Z$ such that

$$X_\infty^Z \setminus X_\infty^B = \bigcup_{i=1}^n \mathcal{N}_i \cap (X_\infty \setminus X_\infty^B).$$

Proof. If X is regular at ζ , the result follows from proposition 2.3(1). Let us suppose that $\zeta \in \text{Sing } X$. By prop. 3.4 there exists a nonempty Zariski open subset $U \subseteq Z$ and a regular birational chart $\phi : Y \rightarrow X$ which is a local uniformization for every k -valuation of X with center any point z in U . Let $E \subset Y$ be the exceptional locus, i.e.

$$E := \{y \in Y \mid \phi \text{ does not induce an isomorphism } \mathcal{O}_{Y,y} \simeq \mathcal{O}_{X,f(y)}\}. \quad (3.1)$$

Note that E is Zariski closed and nowhere dense in Y . Let F be the closure of $\phi(E)$, which is a nowhere dense closed subset of X containing $\text{Sing } X$ (recall that Y is regular). Therefore ζ belongs to F . After a possible replacement of X by an open subset, it can be assumed that X is affine, $Z = U$ and that every irreducible component of F contains Z . Similarly, by generic flatness ([Ma] th. 24.3) it can be assumed that $\phi^{-1}(Z)_{\text{red}} \rightarrow Z$ is flat. Moreover, if $\phi^{-1}(Z) = E_1 \cup \dots \cup E_m$ is a decomposition into irreducible components then it can be assumed that ϕ maps E_i onto Z for $1 \leq i \leq m$.

We claim that there exists finitely many irreducible subsets $\mathcal{N}_1, \dots, \mathcal{N}_r \subset X_\infty$ such that the following holds:

- (1) we have $j(\mathcal{N}_i) = Z$ for each i , $1 \leq i \leq r$;
- (2)' we have

$$X_\infty^Z = \left(\bigcup_{i=1}^r \mathcal{N}_i \right) \cup F_\infty^Z$$

Note that the pair $Z \subseteq F$ satisfies again the assumptions of the theorem. Since $\dim F < \dim X$, the proof then concludes by induction on the dimension once the claim has been proved.

We now prove the claim. Let $P \in X_\infty^Z \setminus F_\infty^Z$. Let $h_P : \text{Spec } \kappa(P)[[t]] \rightarrow X$ be the corresponding arc and let x_0 (resp. x_η) be the image by h_P of the closed (resp. generic) point of $\text{Spec } \kappa(P)[[t]]$, that is $x_0 = j(P)$ and x_η is the generic point of $\text{Im } h_P$. We are assuming that

$$x_0 \in Z \text{ and } x_\eta \notin F. \quad (3.2)$$

Let v be any valuation on X with center x_η . Let y be the center of v on Y , then $\phi(y) = x_\eta$ and hence y does not belong to E . Therefore there is an isomorphism $\mathcal{O}_{Y,y} \simeq \mathcal{O}_{X,x_\eta}$ and thus $x_\eta \in X_{\text{reg}}$. We may therefore choose v as above and such that $k(v) = k(x_\eta)$ (for instance v being the multiplicity at x_η). Let ν_P be the valuation of $\mathcal{O}_{\overline{x_\eta}, h(0)}$ defined by P (see 2.21). Then the morphism

$$\mathcal{O}_{\overline{\{x_\eta\}}, x_0} \rightarrow \kappa(P)[[t]]$$

induced by the arc h_P extends to R_{ν_P} . Since $k(v) = k(x_\eta)$, we may consider the composed valuation $v_0 := v \circ \nu_P$. If $y_0 \in Y$ is the center of v_0 then $\mathcal{O}_{Y,y_0} \prec R_{v_0}$ hence there is a morphism

$$\mathcal{O}_{Y,y_0} \rightarrow R_{v_0}/m_{v_0} = R_{\nu_P}.$$

So h_P lifts to Y by the previous argument.

Summing up, we have just proved that:

$$X_\infty^Z \setminus F_\infty^Z \subseteq \phi_\infty(Y_\infty^{\phi^{-1}(Z)}). \quad (3.3)$$

On the other hand, there is an obvious inclusion

$$\phi_\infty(Y_\infty^{\phi^{-1}(Z)}) \subseteq X_\infty^Z. \quad (3.4)$$

Recall that $\phi^{-1}(Z) = E_1 \cup \dots \cup E_r$ is the decomposition of $\phi^{-1}(Z)$ into irreducible components, hence by proposition 2.3(1), there is a decomposition

$$Y_\infty^{\phi^{-1}(Z)} = Y_\infty^{E_1} \cup \dots \cup Y_\infty^{E_r}$$

into irreducible components. For $1 \leq i \leq r$, let \mathcal{N}_i be the closure of $\phi_\infty(Y_\infty^{E_i})$. Then (1) holds because $\phi(E_i) = Z$. On the other hand, we get (2)' from (3.3) and (3.4). This concludes the proof. \square

Corollary 3.7. *Let $X|k$ be a k -variety, $Z \subseteq X$ a closed subset of X . Assume that, for every subvariety $F \subseteq X$ and every $\zeta \in F \cap Z$, Local Uniformization holds on F at ζ , i.e. $LU(F, \zeta)$ holds. Then X_∞^Z has a finite number of irreducible components.*

Proof. This follows from the previous theorem by Noetherian induction on Z . \square

Remark 3.8. We donot know how to deduce corollary 3.7 from De Jong’s theorem ([dJ], theorem 4.1) or its valuative version ([KK], theorem 1.1). An even simpler question is the following one:

Question: Suppose that there exists a finite separable and dominant morphism of irreducible varieties $Y \rightarrow X$ with Y regular. Does X_∞^{Sing} have a finite number of irreducible components?

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O. Piltant. LMV-UMR 8100 UVSQ. 45 avenue des États-Unis, 78035 Versailles Cedex, France. E-mail: piltant@math.uvsq.fr

Ana J. Reguera. Dpto de Álgebra, Análisis Matemático, Geometría y Topología. Univ. Valladolid, 47011 Valladolid, Spain. E-mail: areguera@agt.uva.es.