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REAL SUBMANIFOLDS OF MAXIMUM COMPLEX TANGENT SPACE AT A CR SINGULAR POINT, II

XIANGHONG GONG & LAURENT STOLOVITCH

Abstract

We study germs of real analytic n -dimensional submanifold of \mathbf{C}^n that has a complex tangent space of maximal dimension at a CR singularity. Under some assumptions, we first classify holomorphically the quadrics having this property. We then study higher order perturbations of these quadrics and their transformations to a normal form under the action of local (possibly formal) biholomorphisms at the singularity. We are led to study formal Poincaré-Dulac normal forms (non-unique) of reversible biholomorphisms. We exhibit a reversible map of which the normal forms are all divergent at the singularity. We then construct a unique formal normal form of the submanifolds under a non degeneracy condition.

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1. Introduction and main results

1.1. Introduction. We say that a point x_0 in a real submanifold M in \mathbf{C}^n is a CR singularity, if the complex tangent spaces $T_x M \cap J_x T_x M$ do not have a constant dimension in any neighborhood of x_0 . The study of real submanifolds with CR singularities was initiated by E. Bishop in his pioneering work [4], when the complex tangent space of M at a CR singularity is minimal, that is exactly one-dimensional. The very elementary models of this kind of manifolds are classified as the Bishop quadrics in \mathbf{C}^2 , given by

$$(1.1) \quad Q: z_2 = |z_1|^2 + \gamma(z_1^2 + \bar{z}_1^2), \quad 0 \leq \gamma < \infty; \quad Q: z_2 = z_1^2 + \bar{z}_1^2, \quad \gamma = \infty$$

with Bishop invariant γ . The origin is a complex tangent which is said to be *elliptic* if $0 \leq \gamma < 1/2$, *parabolic* if $\gamma = 1/2$, or *hyperbolic* if $\gamma > 1/2$.

In [19], Moser and Webster studied the normal form problem of a real analytic surface M in \mathbf{C}^2 which is the higher order perturbation of Q . They showed that when $0 < \gamma < 1/2$, M is holomorphically equivalent to a normal form which is an algebraic surface that depends only on γ and two discrete invariants. They also constructed a formal normal form of M when the origin is a non-exceptional hyperbolic complex tangent point; although the normal form is still convergent, they showed that the normalization is divergent in general for the hyperbolic case. In fact, Moser-Webster dealt with an n -dimensional real submanifold M in \mathbf{C}^n , of which the complex tangent space has (minimum) dimension 1 at a CR singularity. When $n > 2$, they also found normal forms under suitable non-degeneracy condition.

In this paper we continue our previous investigation on an n dimensional real analytic submanifold M in \mathbf{C}^n of which the complex tangent space has the *largest* possible dimension at a given CR singularity [12]. The dimension must be $p = n/2$. Therefore, $n = 2p$ is even. As shown in [22] and [12], there is yet another basic quadratic model

$$(1.2) \quad Q_{\gamma_s} \subset \mathbf{C}^4: z_3 = (z_1 + 2\gamma_s \bar{z}_2)^2, \quad z_4 = (z_2 + 2(1 - \bar{\gamma}_s) \bar{z}_1)^2$$

with γ_s an invariant satisfying $\operatorname{Re} \gamma_s \leq 1/2$, $\operatorname{Im} \gamma_s \geq 0$, and $\gamma_s \neq 0$. The complex tangent at the origin is said of complex type. In [12], we obtained convergence of normalization for abelian CR singularity. In this paper, we study systematically the normal forms of the manifolds M under the condition that M admit the maximum number of deck transformations, condition D, introduced in [12].

In suitable holomorphic coordinates, a $2p$ -dimensional real analytic submanifold in \mathbf{C}^{2p} that has a complex tangent space of maximum dimension at the origin is given by

$$M: z_{p+j} = E_j(z', \bar{z}'), \quad 1 \leq j \leq p,$$

$$E_j(z', \bar{z}') = h_j(z', \bar{z}') + q_j(\bar{z}') + O(|(z', \bar{z}')|^3),$$

where $z' = (z_1, \dots, z_p)$, each $h_j(z', \bar{z}')$ is a homogeneous quadratic polynomial in z', \bar{z}' without holomorphic or anti-holomorphic terms, and each $q_j(\bar{z}')$ is a homogeneous quadratic polynomial in \bar{z}' . We call M a *quadratic manifold* in \mathbf{C}^{2p} if E_j are homogeneous quadratic polynomials. If M is a product of Bishop quadrics (1.1) and quadrics of the form (1.2), it is called a *product quadric*.

1.2. Basic invariants. We first describe some basic invariants of real analytic submanifolds, which are essential to the normal forms. To study M , we consider its complexification in $\mathbf{C}^{2p} \times \mathbf{C}^{2p}$ defined by

$$\mathcal{M}: \begin{cases} z_{p+i} = E_i(z', w'), & i = 1, \dots, p, \\ w_{p+i} = \bar{E}_i(w', z'), & i = 1, \dots, p. \end{cases}$$

It is a complex submanifold of complex dimension $2p$ with coordinates $(z', w') \in \mathbf{C}^{2p}$. Let π_1, π_2 be the restrictions of the projections $(z, w) \rightarrow z$ and $(z, w) \rightarrow w$ to \mathcal{M} , respectively. Note that $\pi_2 = C\pi_1\rho_0$, where ρ_0 is the restriction to \mathcal{M} of the anti-holomorphic involution $(z, w) \rightarrow (\bar{w}, \bar{z})$ and C is the complex conjugate. It is proved in [12] that when M satisfies *condition B*, i.e. $q^{-1}(0) = 0$, the deck transformations of π_1 are involutions that commute pairwise, while the number of deck transformations can be 2^ℓ for $1 \leq \ell \leq p$. Throughout the paper, we assume that *all* manifolds M satisfy the following condition introduced in [12]:

Condition D. *M satisfies condition B, i.e. $q^{-1}(0) = 0$, and π_1 admits the maximum number, 2^p , of deck transformations.*

Then it is proved in [12] that the group of deck transformations of π_1 is generated uniquely by p involutions $\tau_{11}, \dots, \tau_{1p}$ such that each τ_{1j} fixes a hypersurface in \mathcal{M} . Furthermore,

$$\tau_1 := \tau_{11} \cdots \tau_{1p}$$

is the unique deck transformation of which the set of the fixed-points has the smallest dimension p . We call $\{\tau_{11}, \dots, \tau_{1p}, \rho_0\}$ the set of *Moser-Webster involutions*. Let $\tau_2 = \rho_0\tau_1\rho_0$ and

$$\sigma = \tau_1\tau_2.$$

Then σ is *reversible* by τ_j and ρ_0 , i.e. $\sigma^{-1} = \tau_j\sigma\tau_j^{-1}$ and $\sigma^{-1} = \rho_0\sigma\rho_0$.

In this paper for classification purposes, we will impose the following condition:

Condition E. M has distinct eigenvalues, i.e. σ has $2p$ distinct eigenvalues.

We now introduce our main results.

Our first step is to normalize $\{\tau_1, \tau_2, \rho_0\}$. When $p = 1$, this normalization is the main step in order to obtain the Moser-Webster normal form; in fact a simple further normalization allows Moser and Webster to achieve a convergent normal form under a suitable non-resonance condition even for the non-exceptional hyperbolic complex tangent.

When $p > 1$, we need a further normalization for $\{\tau_{11}, \dots, \tau_{1p}, \rho_0\}$; this is our second step. Here the normalization has a large degree of freedom as shown by our formal and convergence results.

1.3. A normal form of quadrics. In section 3, we study all quadrics which admit the maximum number of deck transformations. For such quadrics, all deck transformations are linear. Under condition E, we will first normalize σ, τ_1, τ_2 and ρ_0 into $\hat{S}, \hat{T}_1, \hat{T}_2$ and ρ where

$$\begin{aligned} \hat{T}_1: \quad \xi'_j &= \lambda_j^{-1} \eta_j, & \eta'_j &= \lambda_j \xi_j, \\ \hat{T}_2: \quad \xi'_j &= \lambda_j \eta_j, & \eta'_j &= \lambda_j^{-1} \xi_j, \\ \hat{S}: \quad \xi'_j &= \mu_j \xi_j, & \eta'_j &= \mu_j^{-1} \eta_j \end{aligned}$$

with

$$\lambda_e > 1, \quad |\lambda_h| = 1, \quad |\lambda_s| > 1, \quad \lambda_{s+s_*} = \bar{\lambda}_s^{-1}, \quad \mu_j = \lambda_j^2.$$

Here $1 \leq j \leq p$.

Notation on indices. Throughout the paper, the indices e, h, s have the ranges: $1 \leq e \leq e_*$, $e_* < h \leq e_* + h_*$, $e_* + h_* < s \leq p - s_*$. Thus $e_* + h_* + 2s_* = p$. We will call e_*, h_*, s_* the numbers of *elliptic*, *hyperbolic* and *complex* components of a product quadric, respectively.

As in the Moser-Webster theory, at the complex tangent (the origin) an *elliptic* component of a product quadric corresponds a *hyperbolic* component of \hat{S} , while a *hyperbolic* component of the quadric corresponds an *elliptic* component of \hat{S} . On the other hand, a *complex* component of the quadric behaves like an elliptic component when the CR singularity is abelian, and it also behaves like a hyperbolic components for the existence of attached complex manifolds; see [12] for details.

For the above normal form of \hat{T}_1, \hat{T}_2 and \hat{S} , we always normalize the anti-holomorphic involution ρ_0 as

$$(1.3) \quad \rho: \begin{cases} \xi'_e &= \bar{\eta}_e, & \eta'_e &= \bar{\xi}_e, \\ \xi'_h &= \bar{\xi}_h, & \eta'_h &= \bar{\eta}_h, \\ \xi'_s &= \bar{\xi}_{s+s_*}, & \eta'_s &= \bar{\eta}_{s+s_*}, \\ \xi'_{s+s_*} &= \bar{\xi}_s, & \eta'_{s+s_*} &= \bar{\eta}_s. \end{cases}$$

With the above normal forms $\hat{T}_1, \hat{T}_2, \hat{S}, \rho$ with $\hat{S} = \hat{T}_1 \hat{T}_2$, we will then normalize the $\tau_{11}, \dots, \tau_{1p}$ under linear transformations that commute

with \hat{T}_1, \hat{T}_2 , and ρ , i.e. the linear transformations belonging to the *centralizer* of \hat{T}_1, \hat{T}_2 and ρ . This is a subtle step. Instead of normalizing the involutions directly, we will use the pairwise commutativity of $\tau_{11}, \dots, \tau_{1p}$ to associate to these p involutions a non-singular $p \times p$ matrix \mathbf{B} . The normalization of $\{\tau_{11}, \dots, \tau_{1p}, \rho\}$ is then identified with the normalization of the matrices \mathbf{B} under a suitable equivalence relation. The latter is easy to solve. Our normal form of $\{\tau_{11}, \dots, \tau_{1p}, \rho\}$ is then constructed from the normal forms of T_1, T_2, ρ , and the matrix \mathbf{B} . Following Moser-Webster [19], we will construct the normal form of the quadrics from the normal form of involutions. Let us first state a Bishop type holomorphic classification for quadratic real manifolds.

Theorem 1.1. *Let M be a quadratic submanifold defined by*

$$z_{p+j} = h_j(z', \bar{z}') + q_j(\bar{z}'), \quad 1 \leq j \leq p.$$

Suppose that M satisfies condition E , i.e. the branched covering of π_1 of complexification \mathcal{M} has 2^p deck transformations and $2p$ distinct eigenvalues. Then M is holomorphically equivalent to

$$Q_{\mathbf{B}, \gamma}: z_{p+j} = L_j^2(z', \bar{z}'), \quad 1 \leq j \leq p$$

where $(L_1(z', \bar{z}'), \dots, L_p(z', \bar{z}'))^t = \mathbf{B}(z' + 2\gamma\bar{z}')$, $\mathbf{B} \in GL_p(\mathbf{C})$ and

$$\gamma := \begin{pmatrix} \gamma_{e_*} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \gamma_{h_*} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \gamma_{s_*} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_{s_*} - \bar{\gamma}_{s_*} & \mathbf{0} \end{pmatrix}.$$

Here $p = e_* + h_* + 2s_*$, \mathbf{I}_{s_*} denotes the $s_* \times s_*$ identity matrix, and

$$\begin{aligned} \gamma_{e_*} &= \text{diag}(\gamma_1, \dots, \gamma_{e_*}), & \gamma_{h_*} &= \text{diag}(\gamma_{e_*+1}, \dots, \gamma_{e_*+h_*}), \\ \gamma_{s_*} &= \text{diag}(\gamma_{e_*+h_*+1}, \dots, \gamma_{p-s_*}) \end{aligned}$$

with γ_e, γ_h , and γ_s satisfying

$$0 < \gamma_e < 1/2, \quad 1/2 < \gamma_h < \infty, \quad \text{Re } \gamma_s < 1/2, \quad \text{Im } \gamma_s > 0.$$

Moreover, \mathbf{B} is uniquely determined by an equivalence relation $\mathbf{B} \sim \mathbf{C}\mathbf{B}\mathbf{R}$ for suitable non-singular matrices \mathbf{C}, \mathbf{R} which have exactly p non-zero entries.

When \mathbf{B} is the identity matrix, we get a product quadric or its equivalent form. See Theorem 3.7 for detail of the equivalence relation. The scheme of finding quadratic normal forms turns out to be useful. It will be applied to the study of normal forms of the general real submanifolds.

1.4. Formal submanifolds, formal involutions, and formal centralizers. The normal forms of σ turn out to be in the centralizer of \hat{S} , the normal form of the linear part of σ . The family is subject to a second step of normalization under mappings which again turn out to be in the centralizer of \hat{S} . Thus, before we introduce normalization, we

will first study various centralizers. We will discuss the centralizer of \hat{S} as well as the centralizer of $\{\hat{T}_1, \hat{T}_2\}$ in section 4.

1.5. Normalization of σ . As mentioned earlier, we will divide the normalization for the families of non-linear involutions into two steps. This division will serve two purposes: first, it helps us to find the formal normal forms of the family of involutions $\{\tau_{11}, \dots, \tau_{1p}, \rho\}$; second, it helps us understand the convergence of normalization of the original normal form problem for the real submanifolds. For purpose of normalization, we will assume that M is *non-resonant*, i.e. σ is *non-resonant*, that is that its eigenvalues $\mu_1, \dots, \mu_p, \mu_1^{-1}, \dots, \mu_p^{-1}$ satisfy

$$(1.4) \quad \mu^Q \neq 1, \quad \forall Q \in \mathbf{Z}^p, \quad |Q| \neq 0.$$

Before stating next result, we introduce the following.

Condition L. $\boldsymbol{\mu} = (\boldsymbol{\mu}_1, \dots, \boldsymbol{\mu}_p)$ is a formal map from \mathbf{C}^p to \mathbf{C}^p and satisfies

$$\boldsymbol{\mu}_j(\zeta) = \begin{cases} \mu_j \exp(\zeta_j + O(|\zeta|^2)), & j = e; \\ \mu_j \exp(\sqrt{-1}\zeta_j + O(|\zeta|^2)), & j \neq e. \end{cases}$$

In section 5, we obtain the normalization of σ by proving the following.

Theorem 1.2. *Let σ be a holomorphic map with linear part \hat{S} . Assume that \hat{S} has eigenvalues $\mu_1, \dots, \mu_p, \mu_1^{-1}, \dots, \mu_p^{-1}$ satisfying the non-resonant condition (1.4). Suppose that $\sigma = \tau_1 \tau_2$ where τ_1 is a holomorphic involution, ρ is an anti-holomorphic involution, and $\tau_2 = \rho \tau_1 \rho$. Then there exists a formal map Ψ such that $\rho := \Psi^{-1} \rho \Psi$ is given by (1.3), $\sigma^* = \Psi^{-1} \sigma \Psi$ and $\tau_i^* = \Psi^{-1} \tau_i \Psi$ have the form*

$$(1.5) \quad \begin{aligned} \sigma^*: \xi'_j &= \boldsymbol{\mu}_j(\xi \eta) \xi_j, & \eta'_j &= \boldsymbol{\mu}_j^{-1}(\xi \eta) \eta_j, & \boldsymbol{\mu}_j(0) &= \mu_j, & 1 \leq j \leq p, \\ \tau_i^*: \xi'_j &= \Lambda_{ij}(\xi \eta) \eta_j, & \eta'_j &= \Lambda_{ij}^{-1}(\xi \eta) \xi_j, & i &= 1, 2; & 1 \leq j \leq p. \end{aligned}$$

Here, $\xi \eta = (\xi_1 \eta_1, \dots, \xi_p \eta_p)$, $\boldsymbol{\mu}_j = \Lambda_{1j}^2$ and $\Lambda_{1j} = \Lambda_{2j}^{-1}$. Assume further that $\boldsymbol{\mu}$ satisfies condition L. By a further holomorphic (resp. formal) change of coordinates that preserves ρ , we can transform convergent (resp. formal) σ^* and τ_i^* into

$$(1.6) \quad \begin{aligned} \hat{\sigma}: \xi'_j &= \hat{\boldsymbol{\mu}}_j(\xi \eta) \xi_j, & \eta'_j &= \hat{\boldsymbol{\mu}}_j^{-1}(\xi \eta) \eta_j, & 1 \leq j \leq p, \\ \hat{\tau}_i: \xi'_j &= \hat{\Lambda}_{ij}(\xi \eta) \eta_j, & \eta'_j &= \hat{\Lambda}_{ij}^{-1}(\xi \eta) \xi_j, \end{aligned}$$

with $\hat{\Lambda}_{2j} = \hat{\Lambda}_{1j}^{-1}$ and $\hat{\boldsymbol{\mu}}_j = \hat{\Lambda}_{1j}^2$, while

$$(1.7) \quad \hat{\boldsymbol{\mu}}_j(\zeta) = \begin{cases} \mu_j \exp(\zeta_j + O_j(|\zeta|^2)), & j = e; \\ \mu_j \exp(\sqrt{-1}\zeta_j + O_j(|\zeta|^2)), & j \neq e. \end{cases}$$

Here $O_j(|\zeta|^2)$ indicates terms of order at least two and independent of ζ_j , and $\hat{\mu}$ and hence $\hat{\tau}_i, \hat{\tau}$ are uniquely determined with $\hat{\Lambda}_{1j}(0) = \lambda_j$ and $\mu_j = \lambda_j^2$.

Remark 1.3. Condition L has to be understood as a non-degeneracy condition of the simplest form. To avoid confusion in this paper, the $\mu_i, \hat{\mu}_j$ in this paper replace M_i, \hat{M}_j in [12]. We however keep other notation from [12].

We will conclude in section 5 with an example showing that although σ, τ_1, τ_2 are linear, $\{\tau_{11}, \dots, \tau_{1p}, \rho\}$ are not necessarily linearizable, provided $p > 1$.

Section 6 is devoted to the proof of the following divergence result.

Theorem 1.4. *There exists a non-resonant real analytic submanifold M with pure elliptic complex tangent in \mathbf{C}^6 such that if its associated σ is transformed into a map σ^* that commutes with the linear part of σ at the origin, then σ^* must diverge.*

Note that the theorem says that all normal forms of σ (by definition, they belong to the centralizer of its linear part, i.e. they are in the Poincaré-Dulac normal forms) are divergent. It implies that any transformation for M that transforms σ into a Poincaré-Dulac normal form must diverge. This is in contrast with the Moser-Webster theory: For $p = 1$, a convergent normal form can always be achieved even if the associated transformation is divergent (in the case of hyperbolic complex tangent), and furthermore in case of $p = 1$ and elliptic complex tangent with a non-variant Bishop invariant, the normal form can be achieved by a convergent transformation. A divergent Birkhoff normal form for the classical Hamiltonian systems was obtained in [10]. See Yin [24] for the existence of divergent Birkhoff normal forms for real analytic area-preserving mappings.

We do not know if there exists a non-resonant real analytic submanifold with pure elliptic eigenvalues in \mathbf{C}^4 of which all Poincaré-Dulac normal forms are divergent.

1.6. A unique normalization for the family $\{\tau_{ij}, \rho\}$. In section 7, we will follow the normalization scheme developed for the quadric normal forms in order to normalize $\{\tau_{11}, \dots, \tau_{1p}, \rho\}$. Let $\hat{\sigma}$ be given by (1.6). We define

$$\hat{\tau}_{1j}: \xi'_j = \hat{\Lambda}_{1j}(\xi\eta)\eta_j, \quad \eta'_j = \hat{\Lambda}_{1j}^{-1}(\xi\eta)\xi_j, \quad \xi'_k = \xi_k, \quad \eta'_k = \eta_k, \quad k \neq j,$$

where $\hat{\Lambda}_{1j}(0) = \lambda_j$ and $\hat{\mu}_j = \hat{\Lambda}_{1j}^2$. We now state the main result of this paper:

Theorem 1.5. *Let M be a real analytic submanifold that is a third order perturbation of a non-resonant product quadric. Suppose that its*

associated σ is formally equivalent to $\hat{\sigma}$ given by (1.6). Suppose that $\hat{\mu}$ is given by (1.7). Then the formal normal form \hat{M} of the submanifold M is completely determined by

$$\hat{\mu}, \quad \Phi(\xi, \eta).$$

Here the formal mapping Φ is in $\mathcal{C}^c(\hat{\tau}_{11}, \dots, \hat{\tau}_{1p}) \cap \mathcal{C}(\hat{\tau}_1)$ and tangent to the identity. Moreover, Φ is uniquely determined up to the equivalence relation $\Phi \sim R_\epsilon \Phi R_\epsilon^{-1}$ with $R_\epsilon: \xi_j = \epsilon_j \xi, \eta'_j = \epsilon_j \eta_j$ ($1 \leq j \leq p$), $\epsilon_j^2 = 1$ and $\epsilon_{s+s^*} = \epsilon_s$. Furthermore, if the normal form (1.5) of σ can be achieved by a convergent transformation, so does the normal form \hat{M} of M .

Here the set $\mathcal{C}(\hat{\tau}_1) \cap \mathcal{C}^c(\hat{\tau}_{11}, \dots, \hat{\tau}_{1p})$ is defined in Lemma 7.2 for an invertible matrix \mathbf{B}_1 , while the \mathbf{B}_1 in the above theorem needs to be the identity matrix. See also Theorem 7.7 for an expanded form of Theorem 1.5, including the expression of the normal form \hat{M} .

We should mention some very recent works related to the study of CR-singularities [13, 9, 11, 17]. We now mention related normal form problems. The normal form problem, that is the equivalence to a model manifold, of analytic real hypersurfaces in \mathbf{C}^n with a non-degenerate Levi-form has a complete theory achieved through the works of E. Cartan [5], [6], Tanaka [23], and Chern-Moser [7]. In another direction, the relations between formal and holomorphic equivalences of real analytic hypersurfaces (thus there is no CR singularity) have been investigated by Baouendi-Ebenfelt-Rothschild [1], [2], Baouendi-Mir-Rothschild [3], and Juhlin-Lamel [14], where positive (i.e. convergent) results were obtained. In a recent paper, Kossovskiy and Shafikov [16] showed that there are real analytic real hypersurfaces which are formally but not holomorphically equivalent. In the presence of CR singularity, the problems and techniques required are however different from those used in the CR case. See [12] for further references therein. The reader is also referred to a recent work of Kossovskiy-Lamel [15] for convergence results and a recent survey by Mir [18] on the interplay between formal and holomorphic equivalence in CR-geometry.

1.7. Notation. We briefly introduce notation used in the paper. The identity map is denoted by I . The matrix of a linear map $y = Ax$ is denoted by a bold-faced \mathbf{A} . We denote by LF the linear part at the origin of a mapping $F: \mathbf{C}^m \rightarrow \mathbf{C}^n$ with $F(0) = 0$. Let $F'(0)$ or $DF(0)$ denote the Jacobian matrix of the F at the origin. Then $LF(z) = F'(0)z$. We also denote by $DF(z)$ or simply DF , the Jacobian matrix of F at z , when there is no ambiguity. If \mathcal{F} is a family of mappings fixing the origin, let $L\mathcal{F}$ denote the family of linear parts of mappings in \mathcal{F} . By an analytic (or holomorphic) function, we shall mean a *germ* of analytic function at a point (which will be defined by the context)

otherwise stated. We shall denote by \mathcal{O}_n (resp. $\widehat{\mathcal{O}}_n, \mathfrak{M}_n, \widehat{\mathfrak{M}}_n$) the space of germs of holomorphic functions of \mathbf{C}^n at the origin (resp. of formal power series in \mathbf{C}^n , holomorphic germs, and formal germs vanishing at the origin).

2. Moser-Webster involutions and product quadrics

In this section we will first recall a formal and convergent result from [12] that will be used to classify real submanifolds admitting the maximum number of deck transformations. We will then derive the family of deck transformations for the product quadrics.

We consider a formal real submanifold of dimension $2p$ in \mathbf{C}^{2p} defined by

$$(2.1) \quad M: z_{p+j} = E_j(z', \bar{z}'), \quad 1 \leq j \leq p.$$

Here E_j are formal power series in z', \bar{z}' . We assume that

$$(2.2) \quad E_j(z', \bar{z}') = h_j(z', \bar{z}') + q_j(\bar{z}') + O(|(z', \bar{z}')|^3)$$

and h_j, q_j are homogeneous quadratic polynomials. The formal complexification of M is defined by

$$\mathcal{M}: \begin{cases} z_{p+i} = E_i(z', w'), & i = 1, \dots, p, \\ w_{p+i} = \bar{E}_i(w', z'), & i = 1, \dots, p. \end{cases}$$

We define a *formal deck transformation* of π_1 to be a formal biholomorphic map

$$\tau: (z', w') \rightarrow (z', f(z', w')), \quad \tau(0) = 0$$

such that $\pi_1 \tau = \pi_1$, i.e. $E \circ \tau = E$. Assume that $q^{-1}(0) = 0$ and that the formal manifold defined by (2.1)-(2.2) satisfies condition D that its formal branched covering π_1 admits 2^p formal deck transformations. Then π admits a unique set of p deck transformations $\{\tau_{11}, \dots, \tau_{1p}\}$ such that each τ_{1j} fixes a hypersurface in \mathcal{M} .

As in the Moser-Webster theory, the significance of the two sets of involutions is the following proposition that transforms the normalization of the real manifolds into that of two families $\{\tau_{i1}, \dots, \tau_{ip}\}$ ($i = 1, 2$) of commuting involutions satisfying $\tau_{2j} = \rho \tau_{1j} \rho$ for an antiholomorphic involution ρ . Let us recall the anti-holomorphic involution

$$(2.3) \quad \rho_0: (z', w') \rightarrow (\bar{w}', \bar{z}').$$

Proposition 2.1. *Let M, \tilde{M} be formal (resp. real analytic) real submanifolds of dimension $2p$ in \mathbf{C}^{2p} of the form (2.1)-(2.2). Suppose that M, \tilde{M} satisfy condition D. Then the following hold :*

- (i) *M and \tilde{M} are formally (resp. holomorphically) equivalent if and only if their associated families of involutions $\{\tau_{11}, \dots, \tau_{1p}, \rho_0\}$ and $\{\tilde{\tau}_{11}, \dots, \tilde{\tau}_{1p}, \rho_0\}$ are formally (resp. holomorphically) equivalent.*

(ii) Let $\mathcal{T}_1 = \{\tau_{11}, \dots, \tau_{1p}\}$ be a family of formal holomorphic (resp. holomorphic) commuting involutions such that the tangent spaces of $\text{Fix}(\tau_{11}), \dots, \text{Fix}(\tau_{1p})$ are hyperplanes intersecting transversally 0. Let ρ be an anti-holomorphic formal (resp. holomorphic) involution and let $\mathcal{T}_2 = \{\tau_{21}, \dots, \tau_{2p}\}$ with $\tau_{2j} = \rho\tau_{1j}\rho$. Let $[\mathfrak{M}_{2p}_1]^{L\mathcal{T}_i}$ be the set of linear functions without constant terms that are invariant by $L\mathcal{T}_i$. Suppose that

$$(2.4) \quad [\mathfrak{M}_{2p}_1]^{L\mathcal{T}_1} \cap [\mathfrak{M}_{2p}_1]^{L\mathcal{T}_2} = \{0\}.$$

There exists a formal (resp. real analytic) submanifold defined by

$$(2.5) \quad z'' = (B_1^2, \dots, B_p^2)(z', \bar{z}')$$

for some formal (resp. convergent) power series B_1, \dots, B_p such that M satisfies condition D. The set $\{\tilde{\tau}_{11}, \dots, \tilde{\tau}_{1p}, \rho\}$ of involutions of M is formally (resp. holomorphically) equivalent to $\{\tau_{11}, \dots, \tau_{1p}, \rho\}$.

The above proposition is proved in [12, Propositions 2.8 and 3.2]. Since we need to apply the realization several times, let us recall how (2.5) is constructed. Using the fact that $\tau_{11}, \dots, \tau_{1p}$ are commuting involutions of which the sets of fixed points are hypersurfaces intersecting transversally, we ignore ρ and linearize them simultaneously as

$$Z_j: z_{p+j} \rightarrow -z_{p+i}, \quad z_i \rightarrow z_i, \quad i \neq j$$

for $1 \leq j \leq p$. Thus in z coordinates, invariant functions of $\tau_{11}, \dots, \tau_{1p}$ are generated by z_1, \dots, z_p and $z_{p+1}^2, \dots, z_{2p}^2$. In the original coordinates, $z_j = A_j(\xi, \eta)$, $1 \leq j \leq p$, are invariant by the involutions, while $z_{p+j} = \tilde{B}_j(\xi, \eta)$ is skew-invariant by τ_{1j} . Then $\overline{A_j \circ \rho(\xi, \eta)}$ are invariant by the second family $\{\tau_{2i}\}$. Condition (2.4) ensures that $\varphi: (z', w') = (A(\xi, \eta), \overline{A \circ \rho(\xi, \eta)})$ is a germ of formal (biholomorphic) mapping at the origin. Then

$$M: z_{p+j} = \tilde{B}_j^2 \circ \varphi^{-1}(z', \bar{z}'), \quad 1 \leq j \leq p$$

is a realization for $\{\tau_{11}, \dots, \tau_{1p}, \rho\}$ in the sense stated in the above proposition.

Next we recall the deck transformations for a product quadric from [12].

Let us first recall involutions in [19] where the complex tangents are elliptic (with non-vanishing Bishop invariant) or hyperbolic. When $\gamma_1 \neq 0$, the non-trivial deck transformations of

$$Q_{\gamma_1}: z_2 = |z_1|^2 + \gamma_1(z_1^2 + \bar{z}_1^2)$$

for π_1, π_2 are τ_1 and τ_2 , respectively. They are

$$\tau_1: z'_1 = z_1, \quad w'_1 = -w_1 - \gamma_1^{-1}z_1; \quad \tau_2 = \rho\tau_1\rho$$

with ρ being defined by (2.3). Here the formula is valid for $\gamma_1 = \infty$ (i.e. $\gamma_1^{-1} = 0$). Note that τ_1 and τ_2 do not commute and $\sigma = \tau_1\tau_2$ satisfies

$$\sigma^{-1} = \tau_i\sigma\tau_i = \rho\sigma\rho, \quad \tau_i^2 = I, \quad \rho^2 = I.$$

When the complex tangent is not parabolic, the eigenvalues of σ are μ, μ^{-1} with $\mu = \lambda^2$ and $\gamma\lambda^2 - \lambda + \gamma = 0$. For the elliptic complex tangent, we can choose a solution $\lambda > 1$, and in suitable coordinates we obtain

$$\begin{aligned} \tau_1: \xi' &= \lambda\eta + O(|(\xi, \eta)|^2), & \eta' &= \lambda^{-1}\xi + O(|(\xi, \eta)|^2), \\ \tau_2 &= \rho\tau_1\rho, & \rho(\xi, \eta) &= (\bar{\eta}, \bar{\xi}), \\ \sigma: \xi' &= \mu\xi + O(|(\xi, \eta)|^2), & \eta' &= \mu^{-1}\eta + O(|(\xi, \eta)|^2), \quad \mu = \lambda^2. \end{aligned}$$

When the complex tangent is hyperbolic, i.e. $1/2 < \gamma \leq \infty$, τ_i and σ still have the above form, while $|\mu| = 1 = |\lambda|$ and

$$\rho(\xi, \eta) = (\bar{\xi}, \bar{\eta}).$$

We recall from [19] that

$$\gamma_1 = \frac{1}{\lambda + \lambda^{-1}}.$$

Note that for a parabolic Bishop surface, the linear part of σ is not diagonalizable.

Consider a quadric of the complex type of CR singularity

$$(2.6) \quad Q_{\gamma_s}: z_3 = z_1\bar{z}_2 + \gamma_s\bar{z}_2^2 + (1 - \gamma_s)z_1^2, \quad z_4 = \bar{z}_3.$$

Here γ_s is a complex number.

By condition B, we know that $\gamma_s \neq 0, 1$. Recall from [12] that the deck transformations for π_1 are generated by two involutions

$$\tau_{11}: \begin{cases} z'_1 = z_1, \\ z'_2 = z_2, \\ w'_1 = -w_1 - (1 - \bar{\gamma}_s)^{-1}z_2, \\ w'_2 = w_2; \end{cases} \quad \tau_{12}: \begin{cases} z'_1 = z_1, \\ z'_2 = z_2, \\ w'_1 = w_1, \\ w'_2 = -w_2 - \gamma_s^{-1}z_1. \end{cases}$$

We still have ρ defined by (2.3). Then $\tau_{2j} = \rho\tau_{1j}\rho$, $j = 1, 2$, are given by

$$\tau_{21}: \begin{cases} z'_1 = -z_1 - (1 - \gamma_s)^{-1}w_2, \\ z'_2 = z_2, \\ w'_1 = w_1, \\ w'_2 = w_2; \end{cases} \quad \tau_{22}: \begin{cases} z'_1 = z_1, \\ z'_2 = -z_2 - \bar{\gamma}_s^{-1}w_1, \\ w'_1 = w_1, \\ w'_2 = w_2. \end{cases}$$

Thus $\tau_i = \tau_{i1}\tau_{i2}$ is the unique deck transformation of π_i that has the smallest dimension of the fixed-point set among all deck transformations. They are

$$\tau_1: \begin{cases} z'_1 = z_1, \\ z'_2 = z_2, \\ w'_1 = -w_1 - (1 - \bar{\gamma}_s)^{-1}z_2, \\ w'_2 = -w_2 - \gamma_s^{-1}z_1; \end{cases} \quad \tau_2: \begin{cases} z'_1 = -z_1 - (1 - \gamma_s)^{-1}w_2, \\ z'_2 = -z_2 - \bar{\gamma}_s^{-1}w_1, \\ w'_1 = w_1, \\ w'_2 = w_2. \end{cases}$$

Also $\sigma_{s1} := \tau_{11}\tau_{22}$ and $\sigma_{s2} := \tau_{12}\tau_{21}$ are given by

$$\sigma_{s1}: \begin{cases} z'_1 = z_1, \\ z'_2 = -z_2 - \bar{\gamma}_s^{-1}w_1, \\ w'_1 = (1 - \bar{\gamma}_s)^{-1}z_2 + ((\bar{\gamma}_s - \bar{\gamma}_s^2)^{-1} - 1)w_1, \\ w'_2 = w_2; \end{cases} \quad \sigma_{s2}: \begin{cases} z'_1 = -z_1 - (1 - \gamma_s)^{-1}w_2, \\ z'_2 = z_2, \\ w'_1 = w_1, \\ w'_2 = \gamma_s^{-1}z_1 + ((\gamma_s - \gamma_s^2)^{-1} - 1)w_2. \end{cases}$$

And $\sigma_s := \tau_1\tau_2 = \sigma_{s1}\sigma_{s2}$ is given by

$$\sigma_s: \begin{cases} z'_1 = -z_1 - (1 - \gamma_s)^{-1}w_2, \\ z'_2 = -z_2 - \bar{\gamma}_s^{-1}w_1, \\ w'_1 = (1 - \bar{\gamma}_s)^{-1}z_2 + ((\bar{\gamma}_s - \bar{\gamma}_s^2)^{-1} - 1)w_1, \\ w'_2 = \gamma_s^{-1}z_1 + ((\gamma_s - \gamma_s^2)^{-1} - 1)w_2. \end{cases}$$

Suppose that $\gamma_s \neq 1/2$. The eigenvalues of σ_s are

$$(2.7) \quad \mu_s, \quad \mu_s^{-1}, \quad \bar{\mu}_s^{-1}, \quad \bar{\mu}_s,$$

$$(2.8) \quad \mu_s = \bar{\gamma}_s^{-1} - 1.$$

Here if $\mu_s = \bar{\mu}_s$ and $\mu_s^{-1} = \bar{\mu}_s^{-1}$ then each eigenspace has dimension 2. Under suitable linear coordinates, the involution ρ , defined by (2.3), takes the form

$$(2.9) \quad \rho(\xi_1, \xi_2, \eta_1, \eta_2) = (\bar{\xi}_2, \bar{\xi}_1, \bar{\eta}_2, \bar{\eta}_1).$$

Moreover, for $j = 1, 2$, we have $\tau_{2j} = \rho\tau_{1j}\rho$ and

$$\tau_{1j}: \xi'_j = \lambda_j \eta_j, \quad \eta'_j = \lambda_j^{-1} \xi_j; \quad \xi'_i = \xi_i, \quad \eta'_i = \eta_i, \quad i \neq j; \\ \lambda_1 = \lambda_s, \quad \lambda_2 = \bar{\lambda}_s^{-1}, \quad \mu_s = \lambda_s^2.$$

By a permutation of coordinates that preserves ρ , we obtain a unique holomorphic invariant μ_s satisfying

$$(2.10) \quad |\mu_s| \geq 1, \quad \text{Im } \mu_s \geq 0, \quad 0 \leq \arg \lambda_s \leq \pi/2, \quad \mu_s \neq -1.$$

By condition E, we have $|\mu_s| \neq 1$.

Although the case $\gamma_s = 1/2$ is not studied in this paper, we remark that when $\gamma_s = 1/2$ the only eigenvalue of σ_{s1} is 1. We can choose suitable linear coordinates such that ρ is given by (2.9), while

$$(2.11) \quad \begin{aligned} \sigma_{s1}: \quad & \xi'_1 = \xi_1, & \eta'_1 &= \eta_1 + \xi_1, & \xi'_2 &= \xi_2, & \eta'_2 &= \eta_2 \\ \sigma_{s2}: \quad & \xi'_1 = \xi_1, & \eta'_1 &= \eta_1, & \xi'_2 &= \xi_2, & \eta'_2 &= -\xi_2 + \eta_2, \\ \sigma_s: \quad & \xi'_1 = \xi_1, & \eta'_1 &= \xi_1 + \eta_1, & \xi'_2 &= \xi_2, & \eta'_2 &= -\xi_2 + \eta_2. \end{aligned}$$

Note that eigenvalue formulae (2.7) and the Jordan normal form (2.11) tell us that τ_1 and τ_2 do not commute, while σ_{s1} and σ_{s2} commute and they are diagonalizable if and only if $\gamma_s \neq 1/2$. We further remark that when μ_s satisfies (2.10), we have

$$(2.12) \quad \operatorname{Re} \gamma_s \leq 1/2, \operatorname{Im} \gamma_s \geq 0, \quad \text{if } |\mu_s| \geq 1, \operatorname{Im} \mu_s \geq 0;$$

$$(2.13) \quad \operatorname{Re} \gamma_s = 1/2, \operatorname{Im} \gamma_s \geq 0, \gamma_s \neq 1/2, \quad \text{if } |\mu_s| = 1, \operatorname{Im} \mu_s \geq 0, \mu_s \neq 1;$$

$$(2.14) \quad \gamma_s < 1/2, \gamma_s \neq 0, \quad \text{if } \mu_s^2 > 1; \quad \gamma_s = 1/2, \quad \text{if } \mu_s = 1.$$

We have therefore proved the following.

Proposition 2.2. *Quadratic surfaces in \mathbf{C}^4 of complex type CR singularity at the origin are classified by (2.6) with γ_s uniquely determined by (2.12)-(2.14).*

The region of eigenvalue μ , restricted to $E := \{|\mu| \geq 1, \operatorname{Im} \mu \geq 0\}$, can be described as follows: For a Bishop quadric, μ is precisely located in $\omega := \{\mu \in \mathbf{C} : |\mu| = 1\} \cup [1, \infty)$. The value of μ of a quadric of complex type, is precisely located in $\Omega := E \setminus \{-1\}$, while $\omega = \partial E \setminus (-\infty, -1)$.

In summary, under the condition that no component is a Bishop parabolic quadric or a complex quadric with $\gamma_s = 1/2$, we have found linear coordinates for the product quadrics such that the normal forms of S, T_{ij}, ρ of the corresponding $\sigma, \sigma_j, \tau_{ij}, \rho_0$ are given by

$$\begin{aligned} S: \quad & \xi'_j = \mu_j \xi_j, \quad \eta'_j = \mu_j^{-1} \eta_j; \\ T_{ij}: \quad & \xi'_j = \lambda_{ij} \eta_j, \quad \eta'_j = \lambda_{ij}^{-1} \xi_j, \quad \xi'_k = \xi_k, \quad \eta'_k = \eta_k, \quad k \neq j; \\ \rho: \quad & \begin{cases} (\xi'_e, \eta'_e, \xi'_h, \eta'_h) = (\bar{\eta}_e, \bar{\xi}_e, \bar{\xi}_h, \bar{\eta}_h), \\ (\xi'_s, \xi'_{s+s_*}, \eta'_s, \eta'_{s+s_*}) = (\bar{\xi}_{s+s_*}, \bar{\xi}_s, \bar{\eta}_{s+s_*}, \bar{\eta}_s). \end{cases} \end{aligned}$$

Notice that we can always normalize ρ_0 into the above normal form ρ .

For various reversible mappings and their relations with general mappings, the reader is referred to [20] for recent results and references therein.

To derive our normal forms, we shall transform $\{\tau_1, \tau_2, \rho\}$ into a normal form first. We will further normalize $\{\tau_{1j}, \rho\}$ by using the group of biholomorphic maps that preserve the normal form of $\{\tau_1, \tau_2, \rho\}$, i.e. the centralizer of the normal form of $\{\tau_1, \tau_2, \rho\}$.

3. Quadrics with the maximum number of deck transformations

In Proposition 2.1, we describe the basic relation between the classification of real manifolds and that of two families of involutions intertwined by an antiholomorphic involution, which is established in [12]. As an application, we obtain in this section a normal form for two families of linear involutions and use it to construct the normal form for their associated quadrics. This section also serves as an introduction to our approach to find the normal forms of the real submanifolds at least at the formal level. At the end of the section, we will also introduce examples of quadrics of which S is given by Jordan matrices. The perturbation of such quadrics will not be studied in this paper.

3.1. Normal form of two families of linear involutions. To formulate our results, we first discuss the normal forms which we are seeking for the involutions. We are given two families of commuting linear involutions $\mathcal{T}_1 = \{T_{11}, \dots, T_{1p}\}$ and $\mathcal{T}_2 = \{T_{21}, \dots, T_{2p}\}$ with $T_{2j} = \rho T_{1j} \rho$. Here ρ is a linear anti-holomorphic involution. We set

$$T_1 = T_{11} \cdots T_{1p}, \quad T_2 = \rho T_1 \rho.$$

We also assume that each $\text{Fix}(T_{1j})$ is a hyperplane and $\cap \text{Fix}(T_{1j})$ has dimension p . By [12, Lemma 2.4], in suitable linear coordinates, each T_{1j} has the form

$$Z_j: \xi' = \xi, \quad \eta'_i = \eta_i \ (i \neq j), \quad \eta'_j = -\eta_j.$$

Thus combining with (2.4) gives us

$$(3.1) \quad \dim[\mathfrak{M}_{2p}]_1^{T_i} = p, \quad [\mathfrak{M}_{2p}]_1^{T_i} = [\mathfrak{M}_{2p}]_1^{T_i},$$

$$(3.2) \quad \dim[\mathfrak{M}_{2p}]_1^{T_i} = p, \quad [\mathfrak{M}_{2p}]_1^{T_1} \cap [\mathfrak{M}_{2p}]_1^{T_2} = \{0\}.$$

Recall that $[\mathfrak{M}_{2p}]_1$ denotes the linear holomorphic functions without constant terms. We would like to find a change of coordinates φ such that $\varphi^{-1} T_{1j} \varphi$ and $\varphi^{-1} \rho \varphi$ have a simpler form. We would like to show that two such families of involutions $\{\mathcal{T}_1, \rho\}$ and $\{\tilde{\mathcal{T}}_1, \tilde{\rho}\}$ are holomorphically equivalent, if there are normal forms are equivalent under a much smaller set of changes of coordinates, or if they are identical in the ideal situation.

Next, we describe our scheme to derive the normal forms for linear involutions. The scheme to derive the linear normal forms turns out to be essential to derive normal forms for non-linear involutions and the perturbed quadrics. We define

$$S = T_1 T_2.$$

Besides conditions (3.1)-(3.2), we will soon impose condition E that S has $2p$ distinct eigenvalues.

We first use a linear map ψ to diagonalize S to its normal form

$$\hat{S}: \xi'_j = \mu_j \xi_j, \quad \eta'_j = \mu_j^{-1} \eta_j, \quad 1 \leq j \leq p.$$

The choice of ψ is not unique. We further normalize T_1, T_2, ρ under linear transformations commuting with \hat{S} , i.e. the invertible mappings in the *linear centralizer* of \hat{S} . We use a linear map that commutes with \hat{S} to transform ρ into a normal form too, which is still denoted by ρ . We then use a transformation ψ_0 in the linear centralizer of \hat{S} and ρ to normalize the T_1, T_2 into the normal form

$$\hat{T}_i: \xi'_j = \lambda_{ij} \eta_j, \quad \eta'_j = \lambda_{ij}^{-1} \xi_j, \quad 1 \leq j \leq p.$$

Here we require $\lambda_{2j} = \lambda_{1j}^{-1}$. Thus $\mu_j = \lambda_{1j}^2$ for $1 \leq j \leq p$, and $\lambda_{11}, \dots, \lambda_{1p}$ form a complete set of invariants of T_1, T_2, ρ , provided the normalization satisfies

$$\lambda_{1e} > 1, \quad \text{Im } \lambda_{1h} > 0, \quad \arg \lambda_{1s} \in (0, \pi/2), \quad |\lambda_s| > 1.$$

This normalization will be verified under condition E.

Next we normalize the family \mathcal{T}_1 of linear involutions under mappings in the linear centralizer of \hat{T}_1, ρ . Let us assume that T_1, ρ are in the normal forms \hat{T}_1, ρ . To further normalize the family $\{\mathcal{T}_1, \rho\}$, we use the crucial property that T_{11}, \dots, T_{1p} commute pairwise and each T_{1j} fixes a hyperplane. This allows us to express the family of involutions via a single linear mapping ϕ_1 :

$$T_{1j} = \varphi_1 \phi_1 Z_j \phi_1^{-1} \varphi_1^{-1}.$$

Here the linear mapping φ_1 depends only on $\lambda_1, \dots, \lambda_p$. Expressing ϕ_1 in a non-singular $p \times p$ constant matrix \mathbf{B} , the normal form for $\{T_{11}, \dots, T_{1p}, \rho\}$ consists of invariants $\lambda_1, \dots, \lambda_p$ and a normal form of \mathbf{B} . After we obtain the normal form for \mathbf{B} , we will construct the normal form of the quadrics by using the realization procedure in Proposition 2.1 (see the paragraph after that proposition or the proof in [12]).

We now carry out the details.

Let $T_1 = T_{11} \cdots T_{1p}$, $T_2 = \rho T_1 \rho$ and $S = T_1 T_2$. Since T_i and ρ are involutions, then S is reversible with respect to T_i and ρ , i.e.

$$S^{-1} = T_i^{-1} S T_i, \quad S^{-1} = \rho^{-1} S \rho, \quad T_i^2 = I, \quad \rho^2 = I.$$

Therefore, if κ is an eigenvalue of S with a (non-zero) eigenvector u , then

$$S u = \kappa u, \quad S(T_i u) = \kappa^{-1} T_i u, \quad S(\rho u) = \bar{\kappa}^{-1} \rho u, \quad S(\rho T_i u) = \bar{\kappa} \rho T_i u.$$

Following [19] and [St07], we will divide eigenvalues of product quadrics that satisfy condition E into 3 types: μ is *elliptic* if $\mu \neq \pm 1$ and μ is real, μ is *hyperbolic* if $|\mu| = 1$ and $\mu \neq 1$, and μ is *complex* otherwise. The classification of σ into the types corresponds to the classification of the types of complex tangents described in section 2; namely, an

elliptic (resp. hyperbolic) complex tangent is tied to a hyperbolic (resp. elliptic) mapping σ .

We first characterize the linear family $\{T_1, T_2, \rho\}$ that can be realized by a product quadric with S being diagonal.

Lemma 3.1. *Let $\{T_1, T_2\}$ be a pair of linear involutions on \mathbf{C}^{2p} satisfying (3.2). Suppose that $T_2 = \rho T_1 \rho$ for a linear anti-holomorphic involution and $S = T_1 T_2$ is diagonalizable. Then $\{T_1, T_1, \rho\}$ is realized by the product of quadrics of type elliptic, hyperbolic, or complex. In particular, if S has $2p$ distinct eigenvalues, then 1 and -1 are not eigenvalues of S .*

Proof. The last assertion follows from the first part of the lemma immediately. Thus the following entire proof does not assume that S has distinct eigenvalues. Let $E_i(\nu_i)$ with $i = 1, \dots, 2p$ be eigenspaces of $S = T_1 T_2$ with eigenvalues ν_i . Thus

$$\mathbf{C}^{2p} = \bigoplus_{i=1}^{2p} E_i(\nu_i), \quad \mathbf{C}^{2p} \ominus E_i(\nu_i) := \bigoplus_{j \neq i} E_j(\nu_j).$$

Fix an i and denote the corresponding space by $E(\nu)$. Since $\sigma^{-1} = T_1 \sigma T_1$, then $T_1 E(\nu) = T_2 E(\nu)$, which is equal to some invariant space $E(\nu^{-1})$. Take an eigenvector $e \in E(\nu)$ and set $e' = T_1 e$.

Let us first show that 1 is not an eigenvalue. Assume for the sake of contradiction that $E(1)$ is spanned by a (non-zero) eigenvector e . Then T_1 preserves $E(1)$. Otherwise, e' and e are independent. Now $T_2 e = T_1 e = e'$ and $T_i(e + e') = e' + e$, which contradicts $\text{Fix}(T_1) \cap \text{Fix}(T_2) = \{0\}$. With $E(1)$ being preserved by T_i , we have $T_i e = \epsilon e$ and $\epsilon = \pm 1$, since T_i are involutions. We have $\epsilon \neq 1$ since $\text{Fix}(T_1) \cap \text{Fix}(T_2) = \{0\}$. Thus $T_1 e = -e = T_2 e$. Then $\text{Fix}(T_1)$ and $\text{Fix}(T_2)$ are subspaces of $\mathbf{C}^{2p} \ominus E(1)$ and both are of dimension p . Hence $\text{Fix}(T_1) \cap \text{Fix}(T_2) \neq \{0\}$, a contradiction.

Since $S^{-1} = \rho^{-1} S \rho$ and $S^{-1} = T_i^{-1} S T_i$ then T_1 sends $E(\nu)$ to some $E(\nu^{-1})$ as mentioned earlier, while ρ sends $E(\nu)$ to some $E(\bar{\nu}^{-1})$. Thus, each of T_i, ρ yields an involution on the set $\{E(\nu_1), \dots, E(\nu_{2p})\}$.

Let $E_1(-1), \dots, E_k(-1)$ be all spaces invariant by T_1 . Since $T_2 = T_1 S$, they are also invariant by T_2 . Then none of the k spaces is invariant by ρ . Indeed, if one of them, say E_j generated by e_j , is invariant by ρ , we have $T_1 e_j = \epsilon e_j$ and $\rho e_j = b e_j$ with $\epsilon^2 = 1 = |b|$. We get $T_2 e_j = (\rho T_1 \rho) e_j = \epsilon e_j$ and $\sigma e_j = e_j$, which contracts that σ has eigenvalue -1 on E_j . Furthermore, if $E(-1)$ is invariant by T_1 , then $\rho E(-1)$ is also invariant by T_1 as $T_1 = \rho T_2 \rho$. Thus we may assume that $\rho E_j = E_{\ell+j}$ for $1 \leq j \leq \ell := k/2$. For each j with $1 \leq j \leq \ell$, either $T_1 = I = -T_2$ on E_j and $T_1 = \rho T_2 \rho = -I$ on $E_{\ell+j}$, or $T_1 = -I$ on E_j and $T_1 = I$ on $E_{\ell+j}$. Interchanging $E_j, E_{\ell+j}$ if necessary, we may assume that $T_1 = I = -T_2$ on E_j and $T_1 = -I = -T_2$ on $E_{\ell+j}$. We can restrict the involutions

T_1, T_2, ρ on $\mathbf{C}^2 := E_j \oplus E_{\ell+j}$ as it is invariant by the three involutions. By the realization in [19], $\{T_1, T_2, \rho\}$ is realized by a Bishop quadric; in fact, it is Q_∞ . Assume now that $E(-1)$ is not invariant by T_1 . Thus T_i sends $E(-1)$ into a different $\tilde{E}(-1)$. Assume first that $E(-1)$ is invariant by ρ . Then $\tilde{E}(-1)$ is also invariant by ρ as $\rho = T_2\rho T_1$. Thus as the previous case $\{T_1, T_2, \rho\}$, restricted to $E(-1) \oplus \tilde{E}(-1)$ is realized by Q_∞ .

Suppose now that ρ does not preserve $E(-1)$. Recall that we already assume that $T_1(E(-1)) = \tilde{E}(-1)$ is different from $E(-1)$. Let us show that $\tilde{E}(-1) \neq \rho E(-1)$. Otherwise, we let $\tilde{e} = \rho e$ with e being an eigenvector in $E(-1)$. Then $T_1 e = a\tilde{e}$. So $T_2 e = \rho T_1 \rho e = \bar{a}^{-1}\tilde{e}$ and $T_1 T_2 e = |a|^{-2}e$. This contracts $Se = -e$. We now realize $E(-1) \oplus \rho E(-1) \oplus \tilde{E}(-1) \oplus \rho \tilde{E}(-1)$ by a product of two copies of Q_∞ as follows. Take a non-zero vector $e \in E(-1)$. Define $e_1 = e + T_1 e$. So $T_1 e_1 = e_1$, $Se_1 = -e_1$, and $T_2 e_1 = T_1 S e_1 = -e_1$. Define $\tilde{e}_1 = \rho e_1$; then $T_1 \tilde{e}_1 = \rho T_2 \rho \tilde{e}_1 = -\tilde{e}_1$. Define $\tilde{e}_2 = e_1 - T_1 e_1$; then $T_1 \tilde{e}_2 = -\tilde{e}_2$ and $T_2 \tilde{e}_2 = \tilde{e}_2$. Define $e_2 = \rho \tilde{e}_2$; then $T_1 e_2 = \rho T_2 \rho e_2 = e_2$. In coordinates $z_1 e_1 + w_1 \tilde{e}_1 + z_2 e_2 + w_2 \tilde{e}_2$, we have $T_1(z_j) = z_j$ and $T_1(w_j) = -w_j$ and $\rho(z_j) = \bar{w}_j$. Therefore, $\{T_1, T_2, \rho\}$ is realized by the product of two copies of Q_∞ .

Consider now the case ν , denoting some ν_i , is positive and $\nu \neq 1$. We have

$$(3.3) \quad T_i: E(\nu) \rightarrow E(\nu^{-1}), \quad i = 1, 2.$$

There are two cases: $\rho E(\nu) = E(\nu^{-1})$ or $\rho E(\nu) := \tilde{E}(\nu^{-1}) \neq E(\nu^{-1})$. For the first case, the family $\{T_1, T_2, \rho\}$, restricted to $E(\nu) \oplus E(\nu^{-1})$, is realized by an elliptic Bishop quadric Q_γ with $\gamma \neq 0$. For the second case, we want to verify that $\{T_1, T_2, \rho\}$, restricted to $E(\nu) \oplus \rho E(\nu) \oplus E(\nu^{-1}) \oplus \rho E(\nu^{-1})$, is realized by a quadric of complex type singularity. Write $\nu_1 := \nu = \lambda_1^2$, $\lambda_2 := \bar{\lambda}_1^{-1}$, and $\nu_2 := \bar{\nu}_1^{-1}$. (The redundant complex conjugate is for the rest of cases.) Let u_1 be an eigenvector in $E(\nu)$. Define $v_1 = \lambda_1 T_1 u_1 \in E(\nu^{-1})$. Then $T_j u_1 = \lambda_j^{-1} v_1$. Define $u_2 = \rho u_1$ and $v_2 = \rho v_1$. Then $T_1 u_2 = \rho T_2 \rho u_2 = \rho T_2 u_1 = \lambda_2^{-1} v_2$. Thus $\sigma u_j = \nu_j u_j$ and $\sigma v_j = \nu_j^{-1} v_j$. We now realize the family of involutions by a quadratic submanifold. For the convenience of the reader, we repeat part of argument in [12]; see the paragraph after Proposition 2.1. In coordinates $\xi_1 u_1 + \xi_2 u_2 + \eta_1 v_1 + \eta_2 v_2$, we have $T_i(\xi, \eta) = (\lambda_i \eta, \lambda_i^{-1} \xi)$ and $\rho(\xi, \eta) = (\bar{\xi}_2, \bar{\xi}_1, \bar{\eta}_2, \bar{\eta}_1)$. Let

$$\begin{aligned} z_j &= \xi_j + \lambda_j \eta_j, & w_j &= \overline{z_j \circ \rho}, & j &= 1, 2; \\ z_3 &= (\eta_1 - \lambda_1^{-1} \xi_1)^2, & z_4 &= (\eta_2 - \lambda_2^{-1} \xi_2)^2. \end{aligned}$$

Expressing ξ_j, η_j via (z_1, z_2, w_1, w_2) , we obtain

$$z_3 = L_1^2(z_1, z_2, w_1, w_2), \quad z_4 = L_2^2(z_1, z_2, w_1, w_2).$$

Setting $w_1 = \bar{z}_1$ and $w_2 = \bar{z}_2$, we obtain the defining equations of $M \subset \mathbf{C}^4$ that is a realization of $\{T_1, T_2, \rho\}$.

Assume now that $\nu < 0$ and $\nu \neq -1$. We still have (3.3). We want to show that $\rho(E(\nu)) \neq E(\nu^{-1})$ where $E(\nu^{-1})$ is in (3.3), i.e. the above second case in which $\nu > 0$ occurs and the above argument shows that $\{T_1, T_2, \rho\}$, restricted to $E(\nu) \oplus \rho E(\nu) \oplus E(\nu^{-1}) \oplus \rho E(\nu^{-1})$, is realized by a quadric of complex type singularity. Suppose that $\rho E(\nu) = E(\nu^{-1})$. Take $e \in E(\nu)$. We can write $\tilde{e} = \rho e \in E(\nu^{-1})$. Then $T_1 e = a\tilde{e}$. We have $T_2 e = T_1 S e = \nu a\tilde{e}$ and $T_2 e = \rho T_1 \rho e = \rho(a^{-1}e) = \bar{a}^{-1}\tilde{e}$. We obtain $\nu = |a|^{-2} > 0$, a contradiction.

Analogously, if ν has modulus 1 and is different from ± 1 , we have two cases: $\rho E(\nu) = E(\nu^{-1})$ or $\rho E(\nu) := \tilde{E}(\nu^{-1}) \neq E(\nu^{-1})$. In the first case, $\{T_1, T_2, \rho\}$ restricted to the two dimensional subspace is realized by a hyperbolic quadric Q_γ with $\gamma \neq \infty$. In the second case its restriction to the 4-dimensional subspace is realized by a quadric of complex CR singularity with $|\nu| = 1$. In fact the same argument is valid. Namely, let $\lambda_1^2 = \nu = \nu_1$. Let $\lambda_2 = \bar{\lambda}_1^{-1}$ and $\nu_2 = \bar{\nu}_1^{-1}$. Take an eigenvector $e_1 \in E(\nu)$. Define $\tilde{e}_1 = \lambda_1 T_1 e_1$, $e_2 = \rho e_1$ and $\tilde{e}_2 = \rho \tilde{e}_1$. Then define z_j, w_j and L_j as above, which gives us a realization. We leave the details to the reader. Finally, if $\nu, \bar{\nu}^{-1}, \nu^{-1}, \bar{\nu}$ are distinct, then we have a realization proved in Theorem 3.7 for a general case where all eigenvalues are distinct. q.e.d.

Of course, there are non-product quadrics that realize $\{T_1, T_2, \rho\}$ in Lemma 3.1 and the main purpose of this section is to classify them under condition E. We now assume conditions E and (3.1)-(3.2) for the rest of the section to derive a normal form for T_{1j} and ρ .

We need to choose the eigenvectors of S and their eigenvalues in such a way that T_1, T_2 and ρ are in a normal form. We will first choose eigenvectors to put ρ into a normal form. After normalizing ρ , we will then choose eigenvectors to normalize T_1 and T_2 .

First, let us consider an elliptic eigenvalue μ_e . Let u be an eigenvector of μ_e . Then u and $v = \rho(u)$ satisfy

$$(3.4) \quad S(v) = \mu_e^{-1}v, \quad T_j(u) = \lambda_j^{-1}v, \quad \mu_e = \lambda_1 \lambda_2^{-1}.$$

Now $T_2(u) = \rho T_1 \rho(u)$ implies that

$$\lambda_2 = \bar{\lambda}_1^{-1}, \quad \mu_e = |\lambda_1|^2.$$

Replacing (u, v) by $(cu, \bar{c}v)$, we may assume that $\lambda_1 > 0$ and $\lambda_2 = \lambda_1^{-1}$. Replacing (u, v) by (v, u) if necessary, we may further achieve

$$\rho(u) = v, \quad \lambda_1 = \lambda_e > 1, \quad \mu_e = \lambda_e^2 > 1.$$

We still have the freedom to replace (u, v) by (ru, rv) for $r \in \mathbf{R}^*$, while preserving the above conditions.

Next, let μ_h be a hyperbolic eigenvalue of S and $S(u) = \mu_h u$. Then u and $v = T_1(u)$ satisfy

$$\rho(u) = au, \quad \rho(v) = bv, \quad |a| = |b| = 1.$$

Replacing (u, v) by (cu, v) , we may assume that $a = 1$. Now $T_2(v) = \rho T_1 \rho(v) = \bar{b}u$. To obtain $b = 1$, we replace (u, v) by $(u, b^{-1/2}v)$. This give us (3.4) with $|\lambda_j| = 1$. Replacing (u, v) by (v, u) if necessary, we may further achieve

$$\rho(u) = u, \quad \rho(v) = v, \quad \lambda_1 = \lambda_h, \quad \mu_h = \lambda_h^2, \quad \arg \lambda_h \in (0, \pi/2).$$

Again, we have the freedom to replace (u, v) by (ru, rv) for $r \in \mathbf{R}^*$, while preserving the above conditions.

Finally, we consider a complex eigenvalue μ_s . Let $S(u) = \mu_s u$. Then $\tilde{u} = \rho(u)$ satisfies $S(\tilde{u}) = \bar{\mu}_s^{-1} \tilde{u}$. Let $u^* = T_1(u)$ and $\tilde{u}^* = \rho(u^*)$. Then $S(u^*) = \mu_s^{-1} u^*$ and $S(\tilde{u}^*) = \bar{\mu}_s \tilde{u}^*$. We change eigenvectors by

$$(u, \tilde{u}, u^*, \tilde{u}^*) \rightarrow (u, \tilde{u}, cu^*, \bar{c}\tilde{u}^*)$$

so that

$$\begin{aligned} \rho(u) &= \tilde{u}, \quad \rho(u^*) = \tilde{u}^*, \\ T_j(u) &= \lambda_j^{-1} u^*, \quad T_j(\tilde{u}) = \bar{\lambda}_j \tilde{u}^*, \quad \lambda_2 = \lambda_1^{-1}. \end{aligned}$$

Note that $S(u) = \lambda_1^2 u$, $S(u^*) = \lambda_1^{-2} u^*$, $S(\tilde{u}) = \bar{\lambda}_1^{-2} \tilde{u}$, and $S(\tilde{u}^*) = \bar{\lambda}_1^2 \tilde{u}^*$. Replacing $(u, \tilde{u}, u^*, \tilde{u}^*)$ by $(u^*, \tilde{u}^*, u, \tilde{u})$ changes the argument and the modulus of λ_1 as λ_1^{-1} becomes λ_1 . Replacing them by $(\tilde{u}, u, \tilde{u}^*, u^*)$ changes only the modulus as λ_1 becomes $\bar{\lambda}_1^{-1}$ and then replacing them by $(u^*, \tilde{u}^*, -u, -\tilde{u})$ changes the sign of λ_1 . Therefore, we may achieve

$$\mu_s = \lambda_s^2, \quad \lambda_1 = \lambda_s, \quad \arg \lambda_s \in (0, \pi/2), \quad |\lambda_s| > 1.$$

We still have the freedom to replace $(u, u^*, \tilde{u}, \tilde{u}^*)$ by $(cu, cu^*, \bar{c}\tilde{u}, \bar{c}\tilde{u}^*)$.

We summarize the above choice of eigenvectors and their corresponding coordinates. First, S has distinct eigenvalues

$$\lambda_e^2 = \bar{\lambda}_e^2, \quad \lambda_e^{-2}; \quad \lambda_h^2, \quad \bar{\lambda}_h^2 = \lambda_h^{-2}; \quad \lambda_s^2, \quad \lambda_s^{-2}, \quad \bar{\lambda}_s^{-2}, \quad \bar{\lambda}_s^2.$$

Also, S has linearly independent eigenvectors satisfying

$$\begin{aligned} Su_e &= \lambda_e^2 u_e, & Su_e^* &= \lambda_e^{-2} u_e^*, \\ Sv_h &= \lambda_h^2 v_h, & Sv_h^* &= \lambda_h^{-2} v_h^*, \\ Sw_s &= \lambda_s^2 w_s, & Sw_s^* &= \lambda_s^{-2} w_s^*, & S\tilde{w}_s &= \bar{\lambda}_s^{-2} \tilde{w}_s, & S\tilde{w}_s^* &= \bar{\lambda}_s^2 \tilde{w}_s^*. \end{aligned}$$

Furthermore, the ρ , T_1 , and the chosen eigenvectors of S satisfy

$$\begin{aligned} \rho u_e &= u_e^*, & T_1 u_e &= \lambda_e^{-1} u_e^*; \\ \rho v_h &= v_h, & \rho v_h^* &= v_h^*, & T_1 v_h &= \lambda_h^{-1} v_h^*; \\ \rho w_s &= \tilde{w}_s, & \rho w_s^* &= \tilde{w}_s^*, & T_1 w_s &= \lambda_s^{-1} w_s^*, & T_1 \tilde{w}_s &= \bar{\lambda}_s \tilde{w}_s^*. \end{aligned}$$

For normalization, we collect elliptic eigenvalues μ_e and μ_e^{-1} , hyperbolic eigenvalues μ_h and μ_h^{-1} , and complex eigenvalues in $\mu_s, \mu_s^{-1}, \bar{\mu}_s^{-1}$ and $\bar{\mu}_s$. We put them in the order

$$\begin{aligned} \mu_e &= \bar{\mu}_e, & \mu_{p+e} &= \mu_e^{-1}, \\ \mu_h, & & \mu_{p+h_*+h} &= \bar{\mu}_h, \\ \mu_s, & \mu_{s+s_*} &= \bar{\mu}_s^{-1}, & \mu_{p+s} &= \mu_s^{-1}, & \mu_{p+s_*+s} &= \bar{\mu}_s. \end{aligned}$$

Here and throughout the paper the ranges of subscripts e, h, s are restricted to

$$1 \leq e \leq e_*, \quad e_* < h \leq e_* + h_*, \quad e_* + h_* < s \leq p - s_*.$$

Thus $e_* + h_* + 2s_* = p$. Using the new coordinates

$$\sum (\xi_e u_e + \eta_e u_e^*) + \sum (\xi_h v_h + \eta_h v_h^*) + \sum (\xi_s w_s + \xi_{s+s_*} \tilde{w}_s + \eta_s w_s^* + \eta_{s+s_*} \tilde{w}_s^*),$$

we have normalized σ, T_1, T_2 and ρ . In summary, we have the following normal form.

Lemma 3.2. *Let T_1, T_2 be linear holomorphic involutions on \mathbf{C}^{2p} that satisfy (3.2). Suppose that $T_2 = \rho_0 T_1 \rho_0$ for some anti-holomorphic linear involution ρ_0 . Assume that $S = T_1 T_2$ has n distinct eigenvalues. There exists a linear change of holomorphic coordinates that transforms T_1, T_2, S, ρ_0 simultaneously into the normal forms $\hat{T}_1, \hat{T}_2, \hat{S}, \rho$:*

$$(3.5) \quad \hat{T}_1: \xi'_j = \lambda_j \eta_j, \quad \eta'_j = \lambda_j^{-1} \xi_j, \quad 1 \leq j \leq p;$$

$$(3.6) \quad \hat{T}_2: \xi'_j = \lambda_j^{-1} \eta_j, \quad \eta'_j = \lambda_j \xi_j, \quad 1 \leq j \leq p;$$

$$(3.7) \quad \hat{S}: \xi'_j = \mu_j \xi_j, \quad \eta'_j = \mu_j^{-1} \eta_j, \quad 1 \leq j \leq p;$$

$$(3.8) \quad \rho: \begin{cases} \xi'_e = \bar{\eta}_e, & \eta'_e = \bar{\xi}_e, \\ \xi'_h = \bar{\xi}_h, & \eta'_h = \bar{\eta}_h, \\ \xi'_s = \bar{\xi}_{s+s_*}, & \xi'_{s+s_*} = \bar{\xi}_s, \\ \eta'_s = \bar{\eta}_{s+s_*}, & \eta'_{s+s_*} = \bar{\eta}_s. \end{cases}$$

Moreover, the eigenvalues μ_1, \dots, μ_p satisfy

$$(3.9) \quad \mu_j = \lambda_j^2, \quad 1 \leq j \leq p;$$

$$(3.10) \quad \lambda_e > 1, \quad |\lambda_h| = 1, \quad |\lambda_s| > 1, \quad \lambda_{s+s_*} = \bar{\lambda}_s^{-1};$$

$$(3.11) \quad \arg \lambda_h \in (0, \pi/2), \quad \arg \lambda_s \in (0, \pi/2);$$

$$(3.12) \quad \lambda_{e'} < \lambda_{e'+1}, \quad 0 < \arg \lambda_{h'} < \arg \lambda_{h'+1} < \pi/2;$$

$$(3.13) \quad \arg \lambda_{s'} < \arg \lambda_{s'+1}, \text{ or } \arg \lambda_{s'} = \arg \lambda_{s'+1} \text{ and } |\lambda_{s'}| < |\lambda_{s'+1}|.$$

Here $1 \leq e' < e_*$, $e_* < h' < e_* + h_*$, and $e_* + h_* < s' < p - s_*$. And $1 \leq e \leq e_*$, $e_* < h \leq e_* + h_*$, and $e_* + h_* < s \leq p - s_*$. If \tilde{S} is also in the normal form (3.7) for possible different eigenvalues $\tilde{\mu}_1, \dots, \tilde{\mu}_p$ satisfying (3.9)-(3.13), then S and \tilde{S} are equivalent if and only if their eigenvalues are identical.

The above normal form of ρ will be fixed for the rest of paper. Note that in case of non-linear involutions $\{\tau_{11}, \dots, \tau_{1p}, \rho\}$ of which the linear part are given by $\{T_{11}, \dots, T_{1p}, \rho\}$ we can always linearize ρ first under a holomorphic map of which the linear part at the origin is described in above normalization for the linear part of $\{\tau_{11}, \dots, \tau_{1p}, \rho\}$. Indeed, we may assume that the linear part of the latter family is already in the normal form. Then $\psi = \frac{1}{2}(I + (L\rho) \circ \rho)$ is tangent to the identity and $(L\rho) \circ \psi \circ \rho = \psi$, i.e. ψ transforms ρ into $L\rho$ while preserving the linear parts of $\tau_{11}, \dots, \tau_{1p}$. Therefore in the non-linear case, we can assume that ρ is given by the above normal form. The above lemma tells us the ranges of eigenvalues μ_e, μ_h and μ_s that can be realized by quadrics that satisfy conditions E and (3.1)-(3.2).

Having normalized T_1 and ρ , we want to further normalize the family $\{T_{11}, \dots, T_{1p}\}$ under linear maps that preserve the normal forms of \hat{T}_1 and ρ . We know that the composition of T_{1j} is in the normal form, i.e.

$$(3.14) \quad T_{11} \cdots T_{1p} = \hat{T}_1$$

is given in Lemma 3.2. We first find an expression for all T_{1j} that commute pairwise and satisfy (3.14), by using invariant and skew-invariant functions of \hat{T}_1 . Let

$$(\xi, \eta) = \varphi_1(z^+, z^-)$$

be defined by

$$(3.15) \quad z_e^+ = \xi_e + \lambda_e \eta_e, \quad z_e^- = \eta_e - \lambda_e^{-1} \xi_e,$$

$$(3.16) \quad z_h^+ = \xi_h + \lambda_h \eta_h, \quad z_h^- = \eta_h - \bar{\lambda}_h \xi_h,$$

$$(3.17) \quad z_s^+ = \xi_s + \lambda_s \eta_s, \quad z_s^- = \eta_s - \lambda_s^{-1} \xi_s,$$

$$(3.18) \quad z_{s+s_*}^+ = \xi_{s+s_*} + \bar{\lambda}_s^{-1} \eta_{s+s_*}, \quad z_{s+s_*}^- = \eta_{s+s_*} - \bar{\lambda}_s \xi_{s+s_*}.$$

In (z^+, z^-) coordinates, $\varphi_1^{-1} \hat{T}_1 \varphi_1$ becomes

$$Z: z^+ \rightarrow z^+, \quad z^- \rightarrow -z^-.$$

We decompose $Z = Z_1 \cdots Z_p$ by using

$$Z_j: (z^+, z^-) \rightarrow (z^+, z_1^-, \dots, z_{j-1}^-, -z_j^-, z_{j+1}^-, \dots, z_p^-).$$

To keep simple notation, let us use the same notions x, y for a linear transformation $y = A(x)$ and its matrix representation:

$$A: x \rightarrow \mathbf{A}x.$$

The following lemma, which can be verified immediately, shows the advantages of coordinates z^+, z^- .

Lemma 3.3. *The linear centralizer of Z is the set of mappings of the form*

$$(3.19) \quad \phi: (z^+, z^-) \rightarrow (\mathbf{A}z^+, \mathbf{B}z^-),$$

where \mathbf{A}, \mathbf{B} are constant and possibly singular matrices. Let ν be a permutation of $\{1, \dots, p\}$. Then $Z_j \phi = \phi Z_{\nu(j)}$ for all j if and only if ϕ has the above form with $\mathbf{B} = \text{diag}_\nu \mathbf{d}$. Here

$$(3.20) \quad \text{diag}_\nu(d_1, \dots, d_p) := (b_{ij})_{p \times p}, \quad b_{j\nu(j)} = d_j, \quad b_{jk} = 0 \text{ if } k \neq \nu(j).$$

In particular, the linear centralizer of $\{Z_1, \dots, Z_p\}$ is the set of mappings (3.19) in which \mathbf{B} are diagonal.

To continue our normalization for the family $\{T_{1j}\}$, we note that $\varphi_1^{-1}T_{11}\varphi_1, \dots, \varphi_1^{-1}T_{1p}\varphi_1$ generate an abelian group of 2^p involutions and each of these p generators fixes a hyperplane. By [12, Lemma 2.4], there is a linear transformation ϕ_1 such that

$$\varphi_1^{-1}T_{1j}\varphi_1 = \phi_1 Z_j \phi_1^{-1}, \quad 1 \leq j \leq p.$$

Computing two compositions on both sides, we see that ϕ_1 must be in the linear centralizer of Z . Thus, it is in the form (3.19). Of course, ϕ_1 is not unique; $\tilde{\phi}_1$ is another such linear map for the same T_{1j} if and only if $\tilde{\phi}_1 = \phi_1 \psi_1$ with $\psi_1 \in \mathcal{C}(Z_1, \dots, Z_p)$. By (3.19), we may restrict ourselves to ϕ_1 given by

$$(3.21) \quad \phi_1: (z^+, z^-) \rightarrow (z^+, \mathbf{B}z^-).$$

Then $\tilde{\phi}_1$ yields the same family $\{T_{1j}\}$ if and only if its corresponding matrix $\tilde{\mathbf{B}} = \mathbf{B}\mathbf{D}$ for a diagonal matrix \mathbf{D} .

In the above we have expressed all T_{11}, \dots, T_{1p} via equivalence classes of matrices. It will be convenient to restate them via matrices.

For simplicity, T_i and S denote \hat{T}_i, \hat{S} , respectively. In matrices, we write

$$T_1: \begin{pmatrix} \xi \\ \eta \end{pmatrix} \rightarrow \mathbf{T}_1 \begin{pmatrix} \xi \\ \eta \end{pmatrix}, \rho: \begin{pmatrix} \xi \\ \eta \end{pmatrix} \rightarrow \boldsymbol{\rho} \begin{pmatrix} \bar{\xi} \\ \bar{\eta} \end{pmatrix}, S: \begin{pmatrix} \xi \\ \eta \end{pmatrix} \rightarrow \mathbf{S} \begin{pmatrix} \xi \\ \eta \end{pmatrix}.$$

Recall that the bold faced \mathbf{A} represents a linear map A . Then

$$\mathbf{T}_1 = \begin{pmatrix} \mathbf{0} & \mathbf{\Lambda}_1 \\ \mathbf{\Lambda}_1^{-1} & \mathbf{0} \end{pmatrix}_{2p \times 2p}, \quad \mathbf{S} = \begin{pmatrix} \mathbf{\Lambda}_1^2 & \mathbf{0} \\ \mathbf{0} & \mathbf{\Lambda}_1^{-2} \end{pmatrix}_{2p \times 2p}.$$

We will abbreviate

$$\boldsymbol{\xi}_{e_*} = (\xi_1, \dots, \xi_{e_*}), \quad \boldsymbol{\xi}_{h_*} = (\xi_{e_*+1}, \dots, \xi_{e_*+h_*}), \quad \boldsymbol{\xi}_{2s_*} = (\xi_{e_*+h_*+1}, \dots, \xi_p).$$

We use the same abbreviation for η . Then $(\boldsymbol{\xi}_{e_*}, \boldsymbol{\eta}_{e_*})$, $(\boldsymbol{\xi}_{h_*}, \boldsymbol{\eta}_{h_*})$, and $(\boldsymbol{\xi}_{2s_*}, \boldsymbol{\eta}_{2s_*})$ subspaces are invariant under T_{1j} , T_1 , and ρ . We also denote by $T_1^{e_*}, T_1^{h_*}, T_1^{s_*}$ the restrictions of T_1 to these subspaces. Define analogously for the restrictions of ρ, S to these subspaces. Define diagonal matrices $\mathbf{\Lambda}_{1e_*}, \mathbf{\Lambda}_{1h_*}, \mathbf{\Lambda}_{1s_*}$, of size $e_* \times e_*$, $h_* \times h_*$ and $s_* \times s_*$

respectively, by

$$\Lambda_1 = \begin{pmatrix} \Lambda_{1e_*} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \Lambda_{1h_*} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \Lambda_{1s_*} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \overline{\Lambda_{1s_*}^{-1}} \end{pmatrix}, \quad \overline{\Lambda}_1 = \begin{pmatrix} \Lambda_{1e_*} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \Lambda_{1h_*}^{-1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \overline{\Lambda_{1s_*}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \Lambda_{1s_*}^{-1} \end{pmatrix}.$$

Thus, we can express $T_1^{s_*}$ and S^{s_*} in $(2s_*) \times (2s_*)$ matrices

$$\mathbf{T}_1^{s_*} = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \Lambda_{1s_*} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \overline{\Lambda_{1s_*}^{-1}} \\ \Lambda_{1s_*}^{-1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \overline{\Lambda_{1s_*}} & \mathbf{0} & \mathbf{0} \end{pmatrix}, \quad \mathbf{S}^{s_*} = \begin{pmatrix} \Lambda_{1s_*}^2 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \overline{\Lambda_{1s_*}^{-2}} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \Lambda_{1s_*}^{-2} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \overline{\Lambda_{1s_*}^2} \end{pmatrix}.$$

Let \mathbf{I}_k denote the $k \times k$ identity matrix. With the abbreviation, we can express ρ as

$$\rho^{e_*} = \begin{pmatrix} \mathbf{0} & \mathbf{I}_{e_*} \\ \mathbf{I}_{e_*} & \mathbf{0} \end{pmatrix}, \quad \rho^{h_*} = \mathbf{I}_{2h_*},$$

$$\rho^{s_*} = \begin{pmatrix} \mathbf{0} & \mathbf{I}_{s_*} & \mathbf{0} & \mathbf{0} \\ \mathbf{I}_{s_*} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_{s_*} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_{s_*} & \mathbf{0} \end{pmatrix}.$$

Note that ρ is anti-holomorphic linear transformation. If A is a complex linear transformation, in (ξ, η) coordinates the matrix of ρA is $\rho \overline{\mathbf{A}}$, i.e.

$$\rho A: \begin{pmatrix} \xi \\ \eta \end{pmatrix} \rightarrow \rho \overline{\mathbf{A}} \begin{pmatrix} \overline{\xi} \\ \overline{\eta} \end{pmatrix}$$

with

$$\rho = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_{e_*} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{h_*} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_{s_*} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_{s_*} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{I}_{e_*} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_{h_*} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_{s_*} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_{s_*} & \mathbf{0} \end{pmatrix}.$$

For an invertible $p \times p$ matrix \mathbf{A} , define an $n \times n$ matrix $\mathbf{E}_{\mathbf{A}}$ by

$$(3.22) \quad \mathbf{E}_{\mathbf{A}} := \frac{1}{2} \begin{pmatrix} \mathbf{I}_p & -\mathbf{A} \\ \mathbf{A}^{-1} & \mathbf{I}_p \end{pmatrix}, \quad \mathbf{E}_{\mathbf{A}}^{-1} = \begin{pmatrix} \mathbf{I}_p & \mathbf{A} \\ -\mathbf{A}^{-1} & \mathbf{I}_p \end{pmatrix}.$$

For a $p \times p$ matrix \mathbf{B} , we define

$$\mathbf{B}_* := \begin{pmatrix} \mathbf{I}_p & \mathbf{0} \\ \mathbf{0} & \mathbf{B} \end{pmatrix}.$$

Therefore, we can express

$$(3.23) \quad \mathbf{T}_{1j} = \mathbf{E}_{\Lambda_1} \mathbf{B}_* \mathbf{Z}_j \mathbf{B}_*^{-1} \mathbf{E}_{\Lambda_1}^{-1}, \quad \mathbf{T}_{2j} = \rho \overline{\mathbf{T}_{1j}} \rho,$$

$$(3.24) \quad \mathbf{Z}_j = \text{diag}(1, \dots, 1, -1, 1, \dots, 1).$$

Here -1 is at the $(p+j)$ -th place. By Lemma 3.3, \mathbf{B} is uniquely determined up to equivalence relation via diagonal matrices \mathbf{D} :

$$(3.25) \quad \mathbf{B} \sim \mathbf{B}\mathbf{D}.$$

We have expressed all $\{T_{11}, \dots, T_{1p}, \rho\}$ for which $\hat{T}_1 = T_{11} \cdots T_{1p}$ and ρ are in the normal forms in Lemma 3.2 and we have found an equivalence relation to classify the involutions. Let us summarize the results in a lemma.

Lemma 3.4. *Let $\{T_{11}, \dots, T_{1p}, \rho\}$ be the involutions of a quadric manifold M . Assume that $S = T_{1p} \rho T_{1p}$ has distinct eigenvalues. Then in suitable linear (ξ, η) coordinates, T_{11}, \dots, T_{1p} are given by (3.23), while $T_{11} \cdots T_{1p} = \hat{T}_1$ and ρ are given by (3.5) and (3.8), respectively. Moreover, \mathbf{B} in (3.23) is uniquely determined by the equivalence relation (3.25) for diagonal matrices \mathbf{D} .*

Recall that we divide the classification for $\{T_{11}, \dots, T_{1p}, \rho\}$ into two steps. We have obtained the classification for $T_{11} \circ \cdots \circ T_{1p} = \hat{T}_1$ and ρ in Lemma 3.2. Having found all $\{T_{11}, \dots, T_{1p}, \rho\}$ and an equivalence relation, we are ready to reduce their classification to an equivalence problem that involves two dilatations and a coordinate permutation.

Lemma 3.5. *Let $\{T_{i1}, \dots, T_{ip}, \rho\}$ be given by (3.23). Suppose that $\hat{T}_1 = T_{11} \cdots T_{1p}$, ρ , $\hat{T}_2 = \rho \hat{T}_1 \rho$, and $\hat{S} = \hat{T}_1 \hat{T}_2$ have the forms in Lemma 3.2. Suppose that \hat{S} has distinct eigenvalues. Let $\{\hat{T}_{11}, \dots, \hat{T}_{1p}, \rho\}$ be given by (3.23) where λ_j are unchanged and \mathbf{B} is replaced by $\hat{\mathbf{B}}$. Suppose that $R^{-1} T_{1j} R = \hat{T}_{1\nu(j)}$ for all j and $R\rho = \rho R$. Then the matrix of R is $\mathbf{R} = \text{diag}(\mathbf{a}, \mathbf{a})$ with $\mathbf{a} = (\mathbf{a}_{e_*}, \mathbf{a}_{h_*}, \mathbf{a}_{s_*}, \mathbf{a}'_{s_*})$, while \mathbf{a} satisfies the reality condition*

$$(3.26) \quad \mathbf{a}_{e_*} \in (\mathbf{R}^*)^{e_*}, \quad \mathbf{a}_{h_*} \in (\mathbf{R}^*)^{h_*}, \quad \overline{\mathbf{a}_{s_*}} = \mathbf{a}'_{s_*} \in (\mathbf{C}^*)^{s_*}.$$

Moreover, there exists $\mathbf{d} \in (\mathbf{C}^*)^p$ such that for $1 \leq i, j \leq p$

$$(3.27) \quad \hat{\mathbf{B}} = (\text{diag } \mathbf{a})^{-1} \mathbf{B} (\text{diag}_\nu \mathbf{d}), \quad \text{i.e., } a_i^{-1} b_{i\nu^{-1}(j)} d_{\nu^{-1}(j)} = \hat{b}_{ij}.$$

Conversely, if \mathbf{a}, \mathbf{d} satisfy (3.26) and (3.27), then $R^{-1} T_{1j} R = \hat{T}_{1\nu(j)}$ and $R\rho = \rho R$.

Proof. Suppose $R^{-1} T_{1j} R = \hat{T}_{1\nu(j)}$ and $R\rho = \rho R$. Then $R^{-1} \hat{T}_1 R = \hat{T}_1$ and $R^{-1} \hat{S} R = \hat{S}$. The latter implies that the matrix of R is diagonal. The former implies that

$$R: \xi'_j = a_j \xi_j, \quad \eta'_j = a_j \eta_j$$

with $a_j \in \mathbf{C}^*$. Now $R\rho = \rho R$ implies (3.26). We express $R^{-1}T_{1j}R = \widehat{T}_{1\nu(j)}$ via matrices:

$$(3.28) \quad \mathbf{E}_{\Lambda_1} \widehat{\mathbf{B}}_* \mathbf{Z}_{\nu(j)} \widehat{\mathbf{B}}_*^{-1} \mathbf{E}_{\Lambda_1}^{-1} = \mathbf{R}^{-1} \mathbf{E}_{\Lambda_1} \mathbf{B}_* \mathbf{Z}_j \mathbf{B}_*^{-1} \mathbf{E}_{\Lambda_1}^{-1} \mathbf{R}.$$

In view of formula (3.22), we see that \mathbf{E}_{Λ_1} commutes with $\mathbf{R} = \text{diag}(\mathbf{a}, \mathbf{a})$. The above is equivalent to that $\psi := \mathbf{B}_*^{-1} \mathbf{R} \widehat{\mathbf{B}}_*$ satisfies $\mathbf{Z}_{\nu(j)} = \psi^{-1} \mathbf{Z}_j \psi$. By Lemma 3.3 we obtain $\psi = \text{diag}(\mathbf{A}, \text{diag}_\nu \mathbf{d})$. This shows that

$$\begin{pmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \text{diag}_\nu \mathbf{d} \end{pmatrix} = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{B} \end{pmatrix}^{-1} \begin{pmatrix} \text{diag } \mathbf{a} & \mathbf{0} \\ \mathbf{0} & \text{diag } \mathbf{a} \end{pmatrix} \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \widehat{\mathbf{B}} \end{pmatrix}.$$

The matrices on diagonal yield $\mathbf{A} = \text{diag } \mathbf{a}$ and (3.27). The lemma is proved. q.e.d.

Lemma 3.5 does not give us an explicit description of the normal form for the families of involutions $\{T_{11}, \dots, T_{1p}, \rho\}$. Nevertheless by the lemma, we can always choose a ν and $\text{diag } \mathbf{d}$ such that the diagonal elements of $\widehat{\mathbf{B}}$, corresponding to $\{\widehat{T}_{1\nu(1)}, \dots, \widehat{T}_{1\nu(p)}, \rho\}$, are 1.

Remark 3.6. In what follows, we will fix a \mathbf{B} and its associated $\{\mathcal{T}_1, \rho\}$ to further study our normal form problems.

3.2. Normal form of the quadrics. We now use the matrices \mathbf{B} to express the normal form for the quadratic submanifolds. Here we follow the realization procedure in Proposition 2.1. We will use the coordinates z^+, z^- again to express invariant functions of T_{1j} and use them to construct the corresponding quadric. We will then pull back the quadric to the (ξ, η) coordinates and then to the z, \bar{z} coordinates to achieve the final normal form of the quadrics.

We return to the construction of invariant and skew-invariant functions z^+, z^- in (3.15)-(3.18) when \mathbf{B} is the identity matrix. For a general \mathbf{B} , we define Φ_1 and the matrix Φ_1^{-1} by

$$\Phi_1(Z^+, Z^-) = (\xi, \eta), \quad \Phi_1^{-1} := \mathbf{B}_*^{-1} \mathbf{E}_{\Lambda_1}^{-1} = \begin{pmatrix} \mathbf{I} & \Lambda_1 \\ -\mathbf{B}^{-1} \Lambda_1^{-1} & \mathbf{B}^{-1} \end{pmatrix}.$$

Note that $Z^+ = z^+$ and $\Phi_1^{-1} T_{1j} \Phi_1 = Z_j$, by (3.23). The Z^+, Z_j^- with $i \neq j$ are invariant functions of T_{1j} , while Z_j^- is a skew-invariant function of T_{1j} . They can be written as

$$(3.29) \quad Z^+ = \xi + \Lambda_1 \eta, \quad Z^- = \mathbf{B}^{-1} (-\Lambda_1^{-1} \xi + \eta).$$

Therefore, the invariant functions of \mathcal{T}_1 are generated by

$$Z_j^+ = \xi_j + \lambda_j \eta_j, \quad (Z_j^-)^2 = (\widetilde{\mathbf{B}}_j (-\Lambda_1^{-1} \xi + \eta))^2, \quad 1 \leq j \leq p.$$

Here $\widetilde{\mathbf{B}}_j$ is the j th row of \mathbf{B}^{-1} . The invariant (holomorphic) functions of \mathcal{T}_2 are generated by

$$(3.30) \quad W_j^+ = \overline{Z_j^+ \circ \rho}, \quad (W_j^-)^2 = \overline{(Z_j^- \circ \rho)^2}, \quad 1 \leq j \leq p.$$

Here $W_j^- = \overline{Z_j^-} \circ \rho$. We will soon verify that

$$m: (\xi, \eta) \rightarrow (z', w') = (Z^+(\xi, \eta), W^+(\xi, \eta))$$

is biholomorphic. A straightforward computation shows that $m\rho m^{-1}$ equals

$$\rho_0: (z', w') \rightarrow (\overline{w'}, \overline{z'}).$$

We define

$$M: z''_{p+j} = (Z_j^- \circ m^{-1}(z', \overline{z'}))^2.$$

We want to find a simpler expression for M . We first separate B from Z^- by writing

$$(3.31) \quad \hat{\mathbf{Z}}^- := (-\Lambda_1^{-1} \mathbf{I}), \quad \mathbf{Z}^- = \mathbf{B}^{-1} \hat{\mathbf{Z}}^-.$$

Note that m does not depend on \mathbf{B} . To compute $\hat{\mathbf{Z}}^- \circ m^{-1}$, we will use matrix expressions for (ξ_{e_*}, η_{e_*}) , (ξ_{h_*}, η_{h_*}) and $(\xi_{2s_*}, \eta_{2s_*})$ subspaces. Let $m_{e_*}, m_{h_*}, m_{s_*}$ be the restrictions m to these subspaces. In the matrix form, we have by (3.30)

$$\mathbf{W}^+ = \overline{\mathbf{Z}^+ \rho}, \quad \mathbf{W}^- = \overline{\mathbf{Z}^- \rho}.$$

Recall that $\Lambda_1 = \text{diag}(\Lambda_{e_*}, \Lambda_{h_*}, \Lambda_{1s_*}, \overline{\Lambda_{1s_*}}^{-1})$. Thus

$$\begin{aligned} \mathbf{m}_{e_*} &= \begin{bmatrix} \mathbf{I} & \Lambda_{1e_*} \\ \Lambda_{1e_*} & \mathbf{I} \end{bmatrix}, & \mathbf{m}_{h_*} &= \begin{bmatrix} \mathbf{I} & \Lambda_{1h_*} \\ \mathbf{I} & \Lambda_{1h_*}^{-1} \end{bmatrix}, \\ \mathbf{m}_{s_*} &= \begin{bmatrix} \mathbf{I} & \mathbf{0} & \Lambda_{1s_*} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & \overline{\Lambda_{1s_*}}^{-1} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & \overline{\Lambda_{1s_*}} \\ \mathbf{I} & \mathbf{0} & \Lambda_{1s_*}^{-1} & \mathbf{0} \end{bmatrix}, \\ \mathbf{m}_{e_*}^{-1} &= \begin{bmatrix} \mathbf{I} & -\Lambda_{1e_*} \\ -\Lambda_{1e_*} & \mathbf{I} \end{bmatrix} \begin{bmatrix} (\mathbf{I} - \Lambda_{1e_*}^2)^{-1} & \mathbf{0} \\ \mathbf{0} & (\mathbf{I} - \Lambda_{1e_*}^2)^{-1} \end{bmatrix}, \\ \mathbf{m}_{h_*}^{-1} &= \begin{bmatrix} \mathbf{I} & -\Lambda_{1h_*}^2 \\ -\Lambda_{1h_*} & \Lambda_{1h_*} \end{bmatrix} \begin{bmatrix} (\mathbf{I} - \Lambda_{1h_*}^2)^{-1} & \mathbf{0} \\ \mathbf{0} & (\mathbf{I} - \Lambda_{1h_*}^2)^{-1} \end{bmatrix}, \\ \mathbf{m}_{s_*}^{-1} &= \begin{bmatrix} \Lambda_{1s_*}^{-1} & \mathbf{0} & \mathbf{0} & -\Lambda_{1s_*} \\ \mathbf{0} & \overline{\Lambda_{1s_*}} & -\overline{\Lambda_{1s_*}}^{-1} & \mathbf{0} \\ -\mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{I} \\ \mathbf{0} & -\mathbf{I} & \mathbf{I} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{L}_{s_*} & \mathbf{0} \\ \mathbf{0} & -\overline{\mathbf{L}}_{s_*} \end{bmatrix}, \\ \mathbf{L}_{s_*} &= \begin{bmatrix} (\Lambda_{1s_*}^{-1} - \Lambda_{1s_*})^{-1} & \mathbf{0} \\ \mathbf{0} & (\overline{\Lambda_{1s_*}} - \overline{\Lambda_{1s_*}}^{-1})^{-1} \end{bmatrix}. \end{aligned}$$

Note that $\mathbf{I} - \Lambda_1^2$ is diagonal. Using (3.31) and the above formulae, the matrices of $\hat{\mathbf{Z}}_{e_*}^{-1} \circ m^{-1}$, $\hat{\mathbf{Z}}_{h_*}^{-1} \circ m^{-1}$, and $\hat{\mathbf{Z}}_{s_*}^{-1} \circ m^{-1}$ are respectively given

by

$$\begin{aligned}
 \hat{\mathbf{Z}}_{e_*}^- \mathbf{m}_{e_*}^{-1} &= \mathbf{L}_{e_*} \left[\mathbf{I} \quad -2(\boldsymbol{\Lambda}_{1e_*} + \boldsymbol{\Lambda}_{1e_*}^{-1})^{-1} \right], \\
 \mathbf{L}_{e_*} &= (\mathbf{I} - \boldsymbol{\Lambda}_{1e_*}^2)^{-1} (-\boldsymbol{\Lambda}_{1e_*} - \boldsymbol{\Lambda}_{1e_*}^{-1}), \\
 \hat{\mathbf{Z}}_{h_*}^- \mathbf{m}_{h_*}^{-1} &= \mathbf{L}_{h_*} \left[\mathbf{I} \quad -2\boldsymbol{\Lambda}_{1h_*} (\boldsymbol{\Lambda}_{1h_*} + \boldsymbol{\Lambda}_{1h_*}^{-1})^{-1} \right], \\
 \mathbf{L}_{h_*} &= (\mathbf{I} - \boldsymbol{\Lambda}_{1h_*}^2)^{-1} (-\boldsymbol{\Lambda}_{1h_*} - \boldsymbol{\Lambda}_{1h_*}^{-1}), \\
 \hat{\mathbf{Z}}_{s_*}^- \mathbf{m}_{s_*}^{-1} &= \begin{bmatrix} -\mathbf{I} - \boldsymbol{\Lambda}_{1s_*}^{-2} & \mathbf{0} & \mathbf{0} & 2\mathbf{I} \\ \mathbf{0} & -\mathbf{I} - \bar{\boldsymbol{\Lambda}}_{1s_*}^{-2} & 2\mathbf{I} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{L}_{s_*} & \mathbf{0} \\ \mathbf{0} & -\bar{\mathbf{L}}_{s_*} \end{bmatrix} \\
 &= \tilde{\mathbf{L}}_{s_*} \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} & -2(\mathbf{I} + \boldsymbol{\Lambda}_{1s_*}^{-2})^{-1} \\ \mathbf{0} & \mathbf{I} & -2(\mathbf{I} + \bar{\boldsymbol{\Lambda}}_{1s_*}^{-2})^{-1} & \mathbf{0} \end{bmatrix}, \\
 \tilde{\mathbf{L}}_{s_*} &= \begin{bmatrix} (\mathbf{I} + \boldsymbol{\Lambda}_{1s_*}^{-2})(\boldsymbol{\Lambda}_{1s_*} - \boldsymbol{\Lambda}_{1s_*}^{-1})^{-1} & \mathbf{0} \\ \mathbf{0} & (\mathbf{I} + \bar{\boldsymbol{\Lambda}}_{1s_*}^{-2})(\bar{\boldsymbol{\Lambda}}_{1s_*}^{-1} - \bar{\boldsymbol{\Lambda}}_{1s_*})^{-1} \end{bmatrix}.
 \end{aligned}$$

Combining the above identities, we obtain

$$\hat{\mathbf{Z}}^{-1} \mathbf{m}^{-1} = \text{diag}(\mathbf{L}_{e_*}, \mathbf{L}_{h_*}, \tilde{\mathbf{L}}_{s_*}) \left(\mathbf{I}_p, -2 \text{diag} \left(\boldsymbol{\Gamma}_{e_*}, \boldsymbol{\Lambda}_{1h_*} \boldsymbol{\Gamma}_{h_*}, \begin{bmatrix} \mathbf{0} & \tilde{\boldsymbol{\Gamma}}_{s_*} \\ \boldsymbol{\Gamma}_{s_*} & \mathbf{0} \end{bmatrix} \right) \right)$$

with $\tilde{\boldsymbol{\Gamma}}_{s_*} = \mathbf{I} - \bar{\boldsymbol{\Gamma}}_{s_*}$ and

$$(3.32) \quad \begin{aligned}
 \boldsymbol{\Gamma}_{e_*} &= (\boldsymbol{\Lambda}_{1e_*} + \boldsymbol{\Lambda}_{1e_*}^{-1})^{-1}, & \boldsymbol{\Gamma}_{h_*} &= (\boldsymbol{\Lambda}_{1h_*} + \boldsymbol{\Lambda}_{1h_*}^{-1})^{-1}, \\
 \boldsymbol{\Gamma}_{s_*} &= (\mathbf{I} + \bar{\boldsymbol{\Lambda}}_{1s_*}^{-2})^{-1}.
 \end{aligned}$$

We define $\tilde{\mathbf{B}}_j$ to be the j -th row of

$$(3.33) \quad \tilde{\mathbf{B}} := \mathbf{B}^{-1} \text{diag}(\mathbf{L}_{e_*}, \mathbf{L}_{h_*}, \tilde{\mathbf{L}}_{s_*}).$$

With $\mathbf{z}'_{s_*} = (z_{p-s_*+1}, \dots, z_p)$, the defining equations of M are given by

$$\begin{aligned}
 z''_{p+j} &= \left\{ \tilde{\mathbf{B}}_j \cdot (\mathbf{z}_{e_*} - 2\boldsymbol{\Gamma}_{e_*} \bar{\mathbf{z}}_{e_*}, \mathbf{z}_{h_*} - 2\boldsymbol{\Gamma}_{h_*} \boldsymbol{\Lambda}_{1h_*} \bar{\mathbf{z}}_{h_*}, \right. \\
 &\quad \left. \mathbf{z}_{s_*} - 2\boldsymbol{\Gamma}_{s_*} \bar{\mathbf{z}}'_{s_*}, \mathbf{z}'_{s_*} - 2(\mathbf{I} - \bar{\boldsymbol{\Gamma}}_{s_*}) \bar{\mathbf{z}}_{s_*} \right\}^2.
 \end{aligned}$$

Let us replace z_j with $j \neq h$ and z_h by iz_j and $i\sqrt{\lambda_h} z_h$, respectively for $1 \leq j \leq p$. Replace z_{p+j} by $-z_{p+j}$. In the new coordinates, M is given by

$$\begin{aligned}
 z''_{p+j} &= \left\{ \hat{\mathbf{B}}_j \cdot (\mathbf{z}_{e_*} + 2\boldsymbol{\Gamma}_{e_*} \bar{\mathbf{z}}_{e_*}, \mathbf{z}_{h_*} + 2\boldsymbol{\Gamma}_{h_*} \bar{\mathbf{z}}_{h_*}, \right. \\
 &\quad \left. \mathbf{z}_{s_*} + 2\boldsymbol{\Gamma}_{s_*} \bar{\mathbf{z}}'_{s_*}, \mathbf{z}'_{s_*} + 2(1 - \bar{\boldsymbol{\Gamma}}_{s_*}) \bar{\mathbf{z}}_{s_*} \right\}^2.
 \end{aligned}$$

Explicitly, we have

$$(3.34) \quad \mathcal{Q}_{\mathbf{B}, \gamma}: z_{p+j} = \left(\sum_{\ell=1}^{e_*+h_*} \hat{b}_{j\ell}(z_\ell + 2\gamma_\ell \bar{z}_\ell) \right. \\ \left. + \sum_{s=e_*+h_*+1}^{p-s_*} \hat{b}_{js}(z_s + 2\gamma_s \bar{z}_{s+s_*}) + \hat{b}_{j(s+s_*)}(z_{s+s_*} + 2(1 - \bar{\gamma}_s) \bar{z}_s) \right)^2$$

for $1 \leq j \leq p$. Here

$$\bar{\gamma}_s = \frac{1}{1 + \Lambda_{1s}^2} = \bar{\Gamma}_s.$$

By (3.33), we also obtain the following identity

$$\hat{\mathbf{B}} = \mathbf{B}^{-1} \text{diag}(\mathbf{L}_{e_*}, \mathbf{L}_{h_*} \Lambda_{1h_*}^{1/2}, \tilde{\mathbf{L}}_{s_*}).$$

The equivalence relation (3.27) on the set of non-singular matrices \mathbf{B} now takes the form

$$(3.35) \quad \widehat{\mathbf{B}} = (\text{diag}_\nu \mathbf{d})^{-1} \hat{\mathbf{B}} \text{diag } \mathbf{a},$$

where \mathbf{a} satisfies (3.26) and $\text{diag}_\nu \mathbf{d}$ is defined in (3.20).

Therefore, by Proposition 2.1 we obtain the following classification for the quadrics.

Theorem 3.7. *Let M be a quadratic submanifold defined by (2.1)-(2.2) satisfying condition D. Let $T_1, T_2 = \rho T_1 \rho$ be the pair of Moser-Webster involutions of M . Suppose that $S = T_1 T_2$ has $2p$ distinct eigenvalues. Then M is holomorphically equivalent to (3.34) with $\hat{\mathbf{B}} \in GL(p, \mathbf{C})$ being uniquely determined by the equivalence relation (3.35).*

Note that the holomorphic classification of $\{T_1, \rho\}$ agrees with its linear classification, while the holomorphic classification of the quadratic manifolds cannot be restricted to linear transformations.

When $\hat{\mathbf{B}}$ is the identity, we obtain the product of 3 types of quadrics

$$(3.36) \quad \mathcal{Q}_{\gamma_e}: z_{p+e} = (z_e + 2\gamma_e \bar{z}_e)^2; \\ \mathcal{Q}_{\gamma_h}: z_{p+h} = (z_h + 2\gamma_h \bar{z}_h)^2; \\ \mathcal{Q}_{\gamma_s}: z_{p+s} = (z_s + 2\gamma_s \bar{z}_{s+s_*})^2, \quad z_{p+s+s_*} = (z_{s+s_*} + 2(1 - \bar{\gamma}_s) \bar{z}_s)^2$$

with

$$(3.37) \quad \gamma_e = \frac{1}{\lambda_e + \lambda_e^{-1}}, \quad \gamma_h = \frac{1}{\lambda_h + \bar{\lambda}_h}, \quad \gamma_s = \frac{1}{1 + \bar{\lambda}_s^2}.$$

Note that $\arg \lambda_s \in (0, \pi/2)$ and $|\lambda_s| > 1$. Thus

$$(3.38) \quad 0 < \gamma_e < 1/2, \quad \gamma_h > 1/2, \quad \gamma_s \in \{z \in \mathbf{C}: \text{Re } z < 1/2, \text{Im } z > 0\}.$$

Remark 3.8. By seeking simple formulae (3.29) for invariant functions Z^+ of $\{T_{1j}\}$ and (3.30) for invariant functions W^+ of $\{T_{2j}\} = \{\rho T_{1j}\rho\}$, we have mismatched the indices so that $W_{s+s_*}^+(\xi, \eta)$, instead of W_s^+ , is invariant by T_{2s} . In (3.36) for $p = 2$ and $h_* = e_* = 0$, by interchanging (z_s, z_{p+s}) with (z_{s+s_*}, z_{p+s+s_*}) we get the quadric (1.2), an equivalent form of (3.36).

We define the following invariants.

Definition 3.9. We call $(\gamma_1, \dots, \gamma_{p-s_*})$, given by formulae (3.37)-(3.38), the *Bishop invariants* of the quadrics. The equivalence classes $\tilde{\mathbf{B}}$ of non-singular matrices \mathbf{B} under the equivalence relation (3.27) are called the *extended Bishop invariants* for the quadrics.

Note that $\mathbf{\Gamma}_{e_*}$ has diagonal elements in $(0, 1/2)$, and $\mathbf{\Gamma}_{h_*}$ has diagonal elements in $(1/2, \infty)$, and $\mathbf{\Gamma}_{s_*}$ has diagonal elements in $(-\infty, 1/2) + i(0, \infty)$.

We remark that Z_j^- is skew-invariant by T_{1i} for $i \neq j$ and invariant by τ_{1j} . Therefore, the square of a linear combination of Z_1^-, \dots, Z_p^- might not be invariant by all T_{1j} . This explains the presence of \mathbf{B} as invariants in the normal form.

It is worthy stating the following normal form for two families of linear holomorphic involutions which may not satisfy the reality condition.

Proposition 3.10. *Let $\mathcal{T}_i = \{T_{i1}, \dots, T_{ip}\}$, $i = 1, 2$ be two families of distinct and commuting linear holomorphic involutions on \mathbf{C}^{2p} . Let $T_i = T_{i1} \cdots T_{ip}$. Suppose that for each i , $\text{Fix}(T_{i1}), \dots, \text{Fix}(T_{ip})$ are hyperplanes intersecting transversally. Suppose that T_1, T_2 satisfy (3.2) and $S = T_1 T_2$ has $2p$ distinct eigenvalues. In suitable linear coordinates, the matrices of T_i, S are*

$$\mathbf{T}_i = \begin{pmatrix} \mathbf{0} & \mathbf{\Lambda}_i \\ \mathbf{\Lambda}_i^{-1} & \mathbf{0} \end{pmatrix}, \quad \mathbf{S} = \begin{pmatrix} \mathbf{\Lambda}_1^2 & \mathbf{0} \\ \mathbf{0} & \mathbf{\Lambda}_1^{-2} \end{pmatrix}$$

with $\mathbf{\Lambda}_2 = \mathbf{\Lambda}_1^{-1}$ being diagonal matrix whose entries do not contain $\pm 1, \pm i$. The $\mathbf{\Lambda}_1^2$ is uniquely determined up to a permutation in diagonal entries. Moreover, the matrices of T_{ij} are

$$(3.39) \quad \mathbf{T}_{ij} = \mathbf{E}_{\mathbf{\Lambda}_i}(\mathbf{B}_i)_* \mathbf{Z}_j(\mathbf{B}_i)_*^{-1} \mathbf{E}_{\mathbf{\Lambda}_i}^{-1}$$

for some non-singular complex matrices $\mathbf{B}_1, \mathbf{B}_2$ uniquely determined by the equivalence relation

$$(3.40) \quad (\mathbf{B}_1, \mathbf{B}_2) \sim (\tilde{\mathbf{B}}_1, \tilde{\mathbf{B}}_2) \\ := ((\text{diag } \mathbf{a})^{-1} \mathbf{B}_1 \text{diag}_{\nu_1} \mathbf{d}_1, (\text{diag } \mathbf{a})^{-1} \mathbf{B}_2 \text{diag}_{\nu_2} \mathbf{d}_2),$$

where $\text{diag}_{\mathfrak{S}_{\nu_1}} \mathbf{d}_1, \text{diag}_{\mathfrak{S}_{\nu_2}} \mathbf{d}_2$ are defined as in (3.20), and $\mathbf{R} = \text{diag}(\mathbf{a}, \mathbf{a})$ is a non-singular diagonal complex matrix representing the linear transformation φ such that

$$\varphi^{-1}T_{ij}\varphi = \tilde{T}_{i\nu_i(j)}, \quad i = 1, 2, j = 1, \dots, p.$$

Here \tilde{T}_i is the family of the involutions associated to the matrices $\tilde{\mathbf{B}}_i$, and \mathbf{E}_{Λ_i} and \mathbf{B}_* are defined by (3.22)-(3.23).

Proof. Let κ be an eigenvalue of S with (non-zero) eigenvector u . Since $T_i S T_i = S^{-1}$. Then $S(T_i(u)) = \kappa^{-1}T_i(u)$. This shows that κ^{-1} is also an eigenvalue of S . By Lemma 3.1, 1 and -1 are not eigenvalues of S . Thus, we can list the eigenvalues of S as $\mu_1, \dots, \mu_p, \mu_1^{-1}, \dots, \mu_p^{-1}$. Let u_j be an eigenvector of S with eigenvalue μ_j . Fix λ_j such that $\lambda_j^2 = \mu_j$. Then $v_j := \lambda_j T_1(u_j)$ is an eigenvector of S with eigenvalue μ_j^{-1} . The $\sum \xi_j u_j + \eta_j v_j$ defines a coordinate system on \mathbf{C}^{2p} such that T_i, S have the above matrices \mathbf{A}_i and \mathbf{S} , respectively. By (3.21) and (3.23), T_{ij} can be expressed in (3.39), where each \mathbf{B}_i is uniquely determined up to $\mathbf{B}_i \text{diag } \mathbf{d}_i$. Suppose that $\{\tilde{T}_{1j}\}, \{\tilde{T}_{2j}\}$ are another pair of families of linear involutions of which the corresponding matrices are $\tilde{\mathbf{B}}_1, \tilde{\mathbf{B}}_2$. If there is a linear change of coordinates φ such that $\varphi^{-1}T_{ij}\varphi = \tilde{T}_{i\nu_i(j)}$, then in the matrix \mathbf{R} of φ , we obtain (3.40); see a similar computation for (3.27) by using (3.28). Conversely, (3.28) implies that the corresponding pairs of families of involutions are equivalent. q.e.d.

Finally, we conclude the section with examples of quadratic manifolds of maximum deck transformations for which the corresponding σ is not diagonalizable.

Example 3.11. Let \mathbf{K} be a $p \times p$ invertible matrix. Let $T_1, \rho, T_2 = \rho T_1 \rho, S$ have matrices

$$\mathbf{T}_1 = \begin{pmatrix} \mathbf{0} & \mathbf{K} \\ \mathbf{K}^{-1} & \mathbf{0} \end{pmatrix}, \quad \mathbf{T}_2 = \begin{pmatrix} \mathbf{0} & \overline{\mathbf{K}}^{-1} \\ \overline{\mathbf{K}} & \mathbf{0} \end{pmatrix},$$

$$\rho = \begin{pmatrix} \mathbf{0} & \mathbf{I}_p \\ \mathbf{I}_p & \mathbf{0} \end{pmatrix}, \quad \mathbf{S} = \begin{pmatrix} \mathbf{K}\overline{\mathbf{K}} & \mathbf{0} \\ \mathbf{0} & \mathbf{K}^{-1}\overline{\mathbf{K}}^{-1} \end{pmatrix}.$$

One can verify that the sets of fixed points of T_1, T_2 intersect transversally if

$$(3.41) \quad \det(\mathbf{K} - \overline{\mathbf{K}}^{-1}) \neq 0.$$

We can decompose $T_1 = T_{11} \cdots T_{1p}$ where T_{11}, \dots, T_{1p} are commuting involutions and each of them fixes a hyperplane by using

$$\begin{pmatrix} \mathbf{0} & \mathbf{K} \\ \mathbf{K}^{-1} & \mathbf{0} \end{pmatrix} = \begin{pmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{pmatrix}^{-1}.$$

In coordinates, we have $T_1: (\xi, \eta)^t \rightarrow \mathbf{T}_1(\xi, \eta)^t$. Thus the linear invariant functions of $\{T_{11}, \dots, T_{1p}\}$ are precisely generated by linear invariant functions of T_1 , and they are linear combinations of the entries of the column vector $\xi^t + \mathbf{K}\eta^t$. On the other hand, the linear invariant functions of $\{T_{21}, \dots, T_{2p}\}$ are linear combinations of the entries of the vector $\xi^t + \overline{\mathbf{K}}^{-1}\eta^t$. The two sets of entries are linearly independent functions; indeed if there are row vectors \mathbf{a}, \mathbf{b} such that

$$\mathbf{a}(\xi^t + \mathbf{K}\eta^t) + \mathbf{b}(\xi^t + \overline{\mathbf{K}}^{-1}\eta^t) = 0$$

then $\mathbf{a} = \mathbf{b}$ and $\mathbf{a}(\mathbf{K} - \overline{\mathbf{K}}^{-1}) = \mathbf{0}$. Thus $\mathbf{a} = \mathbf{0}$ if (3.41) holds. Thus condition (3.41) also implies (3.1)-(3.2). By Proposition 2.1, the family of $\{T_{11}, \dots, T_{1p}, \rho\}$, in particular the matrix S , can be realized by a quadratic manifold.

For a more explicit example, let \mathbf{J}_p be the $p \times p$ Jordan matrix with entries 1 or 0. Then $\mathbf{K} = \lambda\mathbf{J}_p$ satisfies (3.41) if λ is positive and $\lambda \neq 1$, as $\overline{\mathbf{K}}^{-1} = \overline{\lambda}^{-1}\mathbf{J}_p^{-1}$. For another example, set

$$\mathbf{K}_\lambda = \begin{pmatrix} \mathbf{0} & \lambda\mathbf{J}_q \\ \overline{\lambda}\mathbf{J}_q & \mathbf{0} \end{pmatrix}, \quad \mathbf{K}_\lambda\overline{\mathbf{K}}_\lambda = \begin{pmatrix} \lambda^2\mathbf{J}_q^2 & \mathbf{0} \\ \mathbf{0} & \overline{\lambda}^2\mathbf{J}_q^2 \end{pmatrix}$$

with $q = p/2$ and p even. If $\lambda \in \mathbf{C}$ and $\lambda \neq 0, \pm 1$ then K satisfies (3.41) as

$$\overline{\mathbf{K}}_\lambda^{-1} = \begin{pmatrix} \mathbf{0} & \lambda^{-1}\mathbf{J}_q^{-1} \\ \overline{\lambda}^{-1}\mathbf{J}_q^{-1} & \mathbf{0} \end{pmatrix}.$$

When $\lambda = 1$ and $q = 2$, we obtain \mathbf{S} in (2.11) if the above \mathbf{J}_2 is replaced by $\mathbf{J}_2^{1/2}$, the Jordan matrix with eigenvalue 1 and off-diagonal entries $1/2$.

4. Formal deck transformations and centralizers

In section 2 we describe the equivalence of the classification of real analytic submanifolds M that admit the maximum number of deck transformations and the classification of the families $\{\tau_{11}, \dots, \tau_{1p}, \rho\}$ of involutions that satisfy some mild conditions (see Proposition 2.1). To classify the families of involutions and to find their normal forms, we will also study the centralizers of various linear maps to deal with resonance. This is relevant as the normal form of σ will belong to the centralizer of its linear part and any further normalization will also be performed by transformations that are in the centralizer.

In this subsection, we describe centralizers regarding \hat{S}, \hat{T}_1 and $\hat{\mathcal{T}}_1$. We will also describe the complement sets of the centralizers, i.e. the sets of mappings which satisfy suitable normalizing conditions. Roughly speaking, our normal forms are in the centralizers and coordinate transformations that achieve the normal forms are normalized, while an arbitrary

formal transformation admits a unique decomposition of a mapping in a centralizer and a mapping in the complement of the centralizer.

Recall that

$$(4.1) \quad \hat{S}: \xi'_j = \mu_j \xi_j, \quad \eta'_j = \mu_j^{-1} \eta_j, \quad 1 \leq j \leq p,$$

$$(4.2) \quad \hat{T}_i: \xi'_j = \lambda_{ij} \eta_j, \quad \eta'_j = \lambda_{ij}^{-1} \xi_j, \quad 1 \leq j \leq p$$

with $\mu_j = \lambda_{1j}^2$ and $\lambda_{2j}^{-1} = \lambda_{1j} = \lambda_j$.

Definition 4.1. Let \mathcal{F} be a family of formal mappings on \mathbf{C}^n fixing the origin. Let $\mathcal{C}(\mathcal{F})$ be the *centralizer* of \mathcal{F} , i.e. the set of formal holomorphic mappings g that fix the origin and commute with each element f of \mathcal{F} , i.e., $f \circ g = g \circ f$.

Note that we do not require that elements in $\mathcal{C}(\mathcal{F})$ be invertible or convergent.

We first compute the centralizers.

Lemma 4.2. *Let \hat{S} be given by (4.1) with μ_1, \dots, μ_p being non-resonant. Then $\mathcal{C}(\hat{S})$ consists of mappings of the form*

$$(4.3) \quad \psi: \xi'_j = a_j(\xi, \eta) \xi_j, \quad \eta'_j = b_j(\xi, \eta) \eta_j, \quad 1 \leq j \leq p.$$

Let τ_1, τ_2 be formal holomorphic involutions such that $\hat{S} = \tau_1 \tau_2$. Then

$$\tau_i: \xi'_j = \Lambda_{ij}(\xi, \eta) \eta_j, \quad \eta'_j = \Lambda_{ij}^{-1}(\xi, \eta) \xi_j, \quad 1 \leq j \leq p$$

with $\Lambda_{1j} \Lambda_{2j}^{-1} = \mu_j$. The centralizer of $\{\hat{T}_1, \hat{T}_2\}$ consists of the above transformations satisfying

$$(4.4) \quad b_j = a_j, \quad 1 \leq j \leq p.$$

Proof. Let $e_j = (0, \dots, 1, \dots, 0) \in \mathbf{N}^p$, where 1 is at the j th place. Let ψ be given by

$$\xi'_j = \sum a_{j,PQ} \xi^P \eta^Q, \quad \eta'_j = \sum b_{j,PQ} \xi^P \eta^Q.$$

By the non-resonance condition, it is straightforward that if $\psi \hat{S} = \hat{S} \psi$, then $a_{j,PQ} = b_{j,QP} = 0$ if $P - Q \neq e_j$. Note that $\hat{S}^{-1} = T_0 \hat{S} T_0$ for $T_0: (\xi, \eta) \rightarrow (\eta, \xi)$. Thus $\tau_1 T_0$ commutes with \hat{S} . So $\tau_1 T_0$ has the form (4.3) in which we rename a_j, b_j by $\Lambda_{1j}, \tilde{\Lambda}_{1j}$, respectively. Now $\tau_1^2 = I$ implies that

$$\Lambda_{1j}((\Lambda_{11} \tilde{\Lambda}_{11})(\zeta) \zeta_1, \dots, (\Lambda_{1p} \tilde{\Lambda}_{1p})(\zeta) \zeta_p) \tilde{\Lambda}_{1j}(\zeta) = 1, \quad 1 \leq j \leq p.$$

Then $\Lambda_{1j}(0) \tilde{\Lambda}_{1j}(0) = 1$. Applying induction on d , we verify that for all j

$$\Lambda_{1j}(\zeta) \tilde{\Lambda}_{1j}(\zeta) = 1 + O(|\zeta|^d), \quad d > 1.$$

Having found the formula for $\tau_1 T_0$, we obtain the desired formula of τ_1 via composition $(\tau_1 T_0) T_0$. q.e.d.

Let $\mathbf{D}_1 := \text{diag}(\mu_{11}, \dots, \mu_{1n}), \dots, \mathbf{D}_\ell := \text{diag}(\mu_{\ell 1}, \dots, \mu_{\ell n})$ be diagonal invertible matrices with complex entries. Set $D := \{\mathbf{D}_i z\}_{i=1, \dots, \ell}$.

Definition 4.3. Let F be a formal mapping of \mathbf{C}^n that is tangent to the identity.

- (i) Let $n = 2p$. F is *normalized* with respect to \hat{S} , if $F = (f, g)$ is tangent to the identity and F contains no resonant terms, i.e.

$$f_{j, (A+e_j)A} = 0 = g_{j, A(A+e_j)}, \quad |A| > 0.$$

- (ii) Let $n = 2p$. F is *normalized* with respect to $\{\hat{T}_1, \hat{T}_2\}$, if $F = (f, g)$ is tangent to the identity and

$$f_{j, (A+e_j)A} = -g_{j, A(A+e_j)}, \quad |A| > 0.$$

- (iii) F is *normalized* with respect to D if it does not have components along the centralizer of D , i.e. for each Q with $|Q| \geq 2$,

$$f_{j, Q} = 0, \quad \text{if } \mu_i^Q = \mu_{ij} \text{ for all } i.$$

Let $\mathcal{C}^c(\hat{S})$ (resp. $\mathcal{C}^c(\hat{T}_1, \hat{T}_2)$, $\mathcal{C}^c(D)$) denote the set of formal mappings normalized with respect to \hat{S} (resp. $\{\hat{T}_1, \hat{T}_2\}$, the family D). For convenience, we let $\mathcal{C}_2^c(\hat{S})$ (resp. $\mathcal{C}_2^c(\hat{T}_1, \hat{T}_2)$, $\mathcal{C}_2^c(D)$) denote the set of formal mappings $F - I$ with $F \in \mathcal{C}^c(\hat{S})$ (resp. $\mathcal{C}^c(\hat{T}_1, \hat{T}_2)$, $\mathcal{C}^c(D)$).

Let $F = (F_1, \dots, F_n): \mathbf{C}^n \rightarrow \mathbf{C}^n$ be a formal mapping. Define a formal mapping $F_{sym}: \mathbf{C}^n \rightarrow \mathbf{C}^n$ by

$$(F_{sym})_{i, P} = \max_{1 \leq j \leq n, \nu \in S_n} |\{F_j \circ \nu\}_P|,$$

where S_n is the set of permutations ν of coordinates $z_i \rightarrow z_{\nu(i)}$. Let us recall the following lemma from [12, Lemma 4.3].

Lemma 4.4. *Let $\hat{\mathcal{H}}$ be a real subspace of $(\widehat{\mathfrak{M}}_n^2)^n$. Let $\pi: (\widehat{\mathfrak{M}}_n^2)^n \rightarrow \hat{\mathcal{H}}$ be a \mathbf{R} linear projection (i.e. $\pi^2 = \pi$) that preserves the degrees of the mappings and let $\hat{\mathcal{G}} := (I - \pi)(\widehat{\mathfrak{M}}_n^2)^n$. Suppose that there is a positive constant C such that $\pi(E) \prec CE_{sym}$ for any $E \in (\widehat{\mathfrak{M}}_n^2)^n$. Let F be a formal map tangent to the identity. There exists a unique decomposition $F = HG^{-1}$ with $G - I \in \hat{\mathcal{G}}$ and $H - I \in \hat{\mathcal{H}}$. If F is convergent, then G and H are also convergent.*

Lemma 4.5. *Let ψ be a mapping that is tangent to the identity. There exist unique $\psi_0 \in \mathcal{C}(\hat{T}_1, \hat{T}_2)$ and $\psi_1 \in \mathcal{C}^c(\hat{T}_1, \hat{T}_2)$ such that $\psi = \psi_1 \psi_0^{-1}$. Moreover, if ψ is convergent, then ψ_0 and ψ_1 are convergent.*

Proof. Let $\hat{\mathcal{G}} = \mathcal{C}_2(\hat{T}_1, \hat{T}_2)$ and $\hat{\mathcal{H}} = \mathcal{C}_2^c(\hat{T}_1, \hat{T}_2)$. We need to find a \mathbf{R} -linear projection such that $\hat{\mathcal{H}} = \pi(\widehat{\mathfrak{M}}_n^2)^n$, $\hat{\mathcal{G}} = (I - \pi)(\widehat{\mathfrak{M}}_n^2)^n$, and $\pi(E) \prec$

CE_{sym} . Note that $g \in \mathcal{C}_2(\hat{T}_1, \hat{T}_2)$ and $h \in \mathcal{C}_2^c(\hat{T}_1, \hat{T}_2)$ are determined by conditions

$$\begin{aligned} g_{j,(\gamma+e_j)\gamma} &= g_{(j+p),\gamma(\gamma+e_j)}, & h_{j,(\gamma+e_j)\gamma} &= -h_{(j+p),\gamma(\gamma+e_j)}, & 1 \leq j \leq p, \\ g_{j,PQ} &= g_{(j+p),QP} = 0, & P - Q &\neq e_j. \end{aligned}$$

Thus, if $h - g = K$, we determine g uniquely by combining the above identities with

$$\begin{aligned} g_{j,(\gamma+e_j)\gamma} &= \frac{-1}{2} \left\{ K_{j,(\gamma+e_j)\gamma} + K_{(j+p),\gamma(\gamma+e_j)} \right\}, \\ h_{j,(\gamma+e_j)\gamma} &= \frac{1}{2} \left\{ K_{j,(\gamma+e_j)\gamma} - K_{(j+p),\gamma(\gamma+e_j)} \right\} \end{aligned}$$

for $1 \leq j \leq p$. For the remaining coefficients of h , set $h_{i,PQ} = K_{i,PQ}$. Therefore, $\pi(K) := h \prec K_{sym}$ and the lemma follows from Lemma 4.4. q.e.d.

Analogously, for any formal mapping ψ that is tangent to the identity, there is a unique decomposition $\psi = \psi_1 \psi_0^{-1}$ with $\psi_1 \in \mathcal{C}^c(\hat{S})$ and $\psi_0 \in \mathcal{C}(\hat{S})$. If ψ is convergent, then ψ_0, ψ_1 are convergent.

Recall that for $j = 1, \dots, p$, we define

$$Z_j: \xi' = \xi, \quad \eta'_k = \eta_k, \quad k \neq j, \quad \eta'_j = -\eta_j.$$

We have seen in section 3 how invariant functions of Z_j play a role in constructing normal form of quadrics. In section 7, we will also need a centralizer for non linear maps (see Lemma 7.2) to obtain normal forms for two families of involutions. Therefore, let us first recall the following lemma on the centralizer of Z_1, \dots, Z_p .

Lemma 4.6. *The centralizer, $\mathcal{C}(Z_1, \dots, Z_p)$, consists of formal mappings*

$$(\xi, \eta) \rightarrow (U(\xi, \eta), \eta_1 V_1(\xi, \eta), \dots, \eta_p V_p(\xi, \eta))$$

such that $U(\xi, \eta), V(\xi, \eta)$ are even in each η_j . Let $\mathcal{C}^c(Z_1, \dots, Z_p)$ denote the set of mappings $I + (U, V)$ which are tangent to the identity such that for $Q, Q' \in 2\mathbf{N}^p$, $|P| + |Q| > 1$, and $|P| + |Q'| > 0$,

$$(4.5) \quad U_{j,PQ} = V_{j,P(e_j+Q')} = 0.$$

Let ψ be a mapping that is tangent to the identity. There exist unique $\psi_0 \in \mathcal{C}(Z_1, \dots, Z_p)$ and $\psi_1 \in \mathcal{C}^c(Z_1, \dots, Z_p)$ such that $\psi = \psi_1 \psi_0^{-1}$. Moreover, if ψ is convergent, then ψ_0 and ψ_1 are convergent.

Proof. The lemma follows immediately from Lemma 4.4 in which $\hat{\mathcal{H}}$ is the \mathbf{R} linear space of mappings (U, V) without constant or linear terms, which satisfy (4.5). The projection π is the unique projection onto $\hat{\mathcal{H}}$ (i.e. $\pi^2 = \pi$, and π is the identity on $\hat{\mathcal{H}}$) such that π is linear and preserves degrees, and $\pi(E) = 0$ if $E(\xi, \eta) = O(|(\xi, \eta)|^2)$ and $E \in \mathcal{C}_2(Z_1, \dots, Z_p)$. q.e.d.

5. Formal normal forms of the reversible map σ

Let us first describe our plan to derive the normal forms of M . We would like to show that two families of involutions $\{\tau_{1j}, \tau_{2j}, \rho\}$ and $\{\tilde{\tau}_{1j}, \tilde{\tau}_{2j}, \tilde{\rho}\}$ are holomorphically equivalent, if their corresponding normal forms are equivalent under a much smaller set of changes of coordinates. Ideally, we would like to conclude that $\{\tilde{\tau}_{1j}, \tilde{\tau}_{2j}, \tilde{\rho}\}$ are holomorphically equivalent if and only if their corresponding normal forms are the same, or if they are the same under a change of coordinates with finitely many parameters. For instance the Moser-Webster normal form for real analytic surfaces ($p = 1$) with non-vanishing elliptic Bishop invariant falls into the former situation, while the Chern-Moser theory [7] for real analytic hypersurfaces with non-degenerate Levi-form is an example for the latter. Such a normal form will tell us if the real manifolds have infinitely many invariants or not. One of our goals is to understand if the normal form so achieved can be realized by a convergent normalizing transformation. We will see soon that we can achieve our last goal under some assumptions on the family of involutions. Alternatively and perhaps for simplicity of the normal form theory, we would like to seek normal forms which are dynamically or geometrically significant.

Recall that for each real analytic manifold that has 2^p , the maximum number of, commuting deck transformations $\{\tau_{1j}\}$, we have found a unique set of generators $\tau_{11}, \dots, \tau_{1p}$ so that each $\text{Fix}(\tau_{1j})$ has codimension 1. More importantly $\tau_1 = \tau_{11} \cdots \tau_{1p}$ is the unique deck transformation of which the set of fixed points has dimension p . Let $\tau_2 = \rho\tau_1\rho$ and $\sigma = \tau_1\tau_2$. To normalize $\{\tau_{1j}, \tau_{2j}, \rho\}$, we will choose ρ to be the standard anti-holomorphic involution determined by the linear parts of σ . Then we normalize $\sigma = \tau_1\tau_2$ under formal mapping commuting with ρ . This will determine a normal form $\{\tau_1^*, \tau_2^*, \rho\}$ and $\sigma^* = \tau_1^*\tau_2^*$. This part of normalization is analogous to the Moser-Webster normalization. When $p = 1$, Moser and Webster obtained a unique normal form by a simple argument. However, this last step of simple normalization is not available when $p > 1$. By assuming μ associated to σ^* satisfies condition L, we will obtain a **unique** formal normal form $\hat{\sigma}, \hat{\tau}_1, \hat{\tau}_2$ for σ, τ_1, τ_2 . Next, we need to construct the normal form for the families of involutions. We first ignore the reality condition, by finding Φ which transforms $\{\tau_{1j}\}$ into a set of involutions $\{\hat{\tau}_{1j}\}$ which is decomposed canonically according to $\hat{\tau}_1$. This allows us to express $\{\tau_{11}, \dots, \tau_{1p}, \rho\}$ via $\{\hat{\tau}_1, \hat{\tau}_2, \Phi, \rho\}$, as in the classification of the families of linear involutions. Finally, we further normalize $\{\hat{\tau}_1, \hat{\tau}_2, \Phi, \rho\}$ to get our normal form.

Definition 5.1. Throughout this section and next, we denote $\{h\}_d$ the set of coefficients of h_P with multiindices P satisfying $|P| \leq d$ if $h(x)$ is a map or function in x as power series. We denote by $\mathcal{A}_P(t), \mathcal{A}(y; t)$, etc., a universal *polynomial* whose coefficients and degree depend on a

multiindex. The variables in these polynomials will involve a collection of Taylor coefficients of various mappings. The collection will also depend on $|P|$. As such dependency (or independency to coefficients of higher degrees) is crucial to our computation, we will remind the reader the dependency when emphasis is necessary.

For instance, let us take two formal mappings F, G from \mathbf{C}^n into itself. Suppose that $F = I + f$ with $f(x) = O(|x|^2)$ and $G = LG + g$ with $g(x) = O(|x|^2)$ and LG being linear. For $P \in \mathbf{N}^n$ with $|P| > 1$, we can express

$$(5.1) \quad (F^{-1})_P = -f_P + \mathcal{F}_P(\{f\}_{|P|-1}),$$

$$(5.2) \quad (G \circ F)_P = g_P + ((LG) \circ f)_P + \mathcal{G}_P(LG; \{f, g\}_{|P|-1}),$$

$$(5.3) \quad (F^{-1} \circ G \circ F)_P = g_P - (f \circ (LG))_P + ((LG) \circ f)_P \\ + \mathcal{H}_P(LG; \{f, g\}_{|P|-1}).$$

5.1. Formal normal forms of pair of involutions $\{\tau_1, \tau_2\}$. We first find a normal form for σ in $\mathcal{C}(S)$.

Proposition 5.2. *Let σ be a holomorphic map. Suppose that σ has a non-resonant linear part*

$$\hat{S}: \xi'_j = \mu_j \xi_j, \quad \eta_j = \mu_j^{-1} \eta_j, \quad 1 \leq j \leq p.$$

Then there exists a unique normalized formal map $\Psi \in \mathcal{C}^c(\hat{S})$ such that $\sigma^ = \Psi^{-1} \sigma \Psi \in \mathcal{C}(\hat{S})$. Moreover, $\tilde{\sigma} = \psi_0^{-1} \sigma^* \psi_0 \in \mathcal{C}(\hat{S})$, if and only if $\psi_0 \in \mathcal{C}(\hat{S})$ and it is invertible. Let*

$$\sigma^*: \xi'_j = \hat{\mu}_j(\xi\eta) \xi_j, \quad \eta'_j = \hat{\nu}_j(\xi\eta) \eta_j,$$

$$\tilde{\sigma}: \xi'_j = \tilde{\mu}_j(\xi\eta) \xi_j, \quad \eta'_j = \tilde{\nu}_j(\xi\eta) \eta_j,$$

$$\psi_0: \xi'_j = a_j(\xi\eta) \xi_j, \quad \eta'_j = b_j(\xi\eta) \eta_j.$$

(i) *Assume that τ_1, τ_2 are holomorphic involutions and $\sigma = \tau_1 \tau_2$. Then $\sigma^* = \tau_1^* \tau_2^*$ with*

$$(5.4) \quad \tau_i^*: \xi'_j = \Psi^{-1} \tau_i \Psi: \xi'_j = \Lambda_{ij}(\xi\eta) \eta_j, \quad \eta'_j = \Lambda_{ij}^{-1}(\xi\eta) \xi_j; \\ \mathbf{v}_j = \boldsymbol{\mu}_j^{-1}, \quad \boldsymbol{\mu}_j = \Lambda_{1j} \Lambda_{2j}^{-1}.$$

Let the linear part of τ_i be given by

$$\hat{T}_i: \xi'_j = \lambda_{ij} \eta_j, \quad \eta'_j = \lambda_{ij}^{-1} \xi_j.$$

Suppose that $\lambda_{2j}^{-1} = \lambda_{1j}$. There exists a unique $\psi_0 \in \mathcal{C}^c(\hat{T}_1, \hat{T}_2)$ such that

$$(5.5) \quad \tilde{\tau}_i = \psi_0^{-1} \tau_i^* \psi_0: \xi'_j = \tilde{\Lambda}_{ij}(\xi\eta) \eta_j, \quad \eta'_j = \tilde{\Lambda}_{ij}^{-1}(\xi\eta) \xi_j; \\ \tilde{\boldsymbol{\mu}}_j = \tilde{\Lambda}_{1j}^2 = \tilde{\boldsymbol{\nu}}_j^{-1}, \quad \tilde{\Lambda}_{2j} = \tilde{\Lambda}_{1j}^{-1}.$$

Let ψ_1 be a formal biholomorphic map. Then $\{\psi_1^{-1}\tilde{\tau}_1\psi_1, \psi_1^{-1}\tilde{\tau}_2\psi_1\}$ has the same form as of $\{\tilde{\tau}_1, \tilde{\tau}_2\}$ if and only if $\psi_1 \in \mathcal{C}(\hat{T}_1, \hat{T}_2)$; moreover, $\tilde{\Lambda}_{ij}(\xi\eta)$, $\tilde{\mu}_j(\xi\eta)$ are transformed into

$$(5.6) \quad \tilde{\Lambda}_{ij} \circ \tilde{\psi}_1, \quad \tilde{\mu}_j \circ \tilde{\psi}_1,$$

for $\tilde{\psi}_1(\zeta) = (\text{diag } c(\zeta))^2 \zeta$, $\psi_1(\xi, \eta) = ((\text{diag } c(\xi\eta))\xi, (\text{diag } c(\xi\eta))\eta)$.

(ii) Assume further that $\tau_2 = \rho\tau_1\rho$, where ρ is defined by (3.8). Let $\rho_z: \zeta_j \rightarrow \bar{\zeta}_j$, $1 \leq j \leq e_* + h_*$; $\zeta_s \rightarrow \bar{\zeta}_{s+s_*}$, $e_* + h_* < s \leq p - s_*$.

Then $\rho\Psi = \Psi\rho$, $\tau_2^* = \rho\tau_1^*\rho$, and $(\sigma^*)^{-1} = \rho\sigma^*\rho$. The last two identities are equivalent to

$$(5.7) \quad \Lambda_{2e}^{-1} = \overline{\Lambda_{1e} \circ \rho_z}, \quad \overline{\mu_e \circ \rho_z} = \mu_e, \quad 1 \leq e \leq e_*;$$

$$(5.8) \quad \Lambda_{2h} = \overline{\Lambda_{1h} \circ \rho_z}, \quad \overline{\mu_h \circ \rho_z} = \mu_h^{-1}, \quad e_* < h \leq h_* + e_*;$$

$$(5.9) \quad \Lambda_{2(s)} = \overline{\Lambda_{1(s_*+s)} \circ \rho_z},$$

$$(5.10) \quad \Lambda_{2(s_*+s)} = \overline{\Lambda_{1s} \circ \rho_z}, \quad \overline{\mu_s^{-1} \circ \rho_z} = \mu_{s_*+s}, \quad h_* + e_* < s \leq p - s_*.$$

Let ψ_0 and $\tilde{\tau}_i = \psi_0^{-1}\tau_i^*\psi_0$ be as in (i). Then $\rho\psi_0 = \psi_0\rho$, and $\hat{\tau}_1, \hat{\tau}_2$ satisfy

$$(5.11) \quad \tilde{\Lambda}_{ie} = \overline{\tilde{\Lambda}_{ie} \circ \rho_z}, \quad \tilde{\Lambda}_{ih}^{-1} = \overline{\tilde{\Lambda}_{ih} \circ \rho_z}, \quad \tilde{\Lambda}_{is+s_*} = \overline{\tilde{\Lambda}_{is}^{-1} \circ \rho_z}.$$

Proof. We will use the Taylor formula

$$f(x+y) = f(x) + \sum_{k=1}^m \frac{1}{k!} D_k f(x; y) + R_{m+1} f(x; y)$$

with $D_k f(x; y) = \{\partial_t^k f(x+ty)\}|_{t=0}$ and

$$(5.12) \quad R_{m+1} f(x; y) = (m+1) \int_0^1 (1-t)^m \sum_{|\alpha|=m+1} \frac{1}{\alpha!} \partial^\alpha f(x+ty) y^\alpha dt.$$

Set $D = D_1$. Let σ be given by

$$\xi'_j = \mu_j^0(\xi\eta)\xi_j + f_j(\xi, \eta), \quad \eta'_j = \nu_j^0(\xi\eta)\eta_j + g_j(\xi, \eta)$$

with

$$(5.13) \quad (f, g) \in \mathcal{C}_2^c(\hat{S}), \quad \text{ord}(f, g) = d \geq 2.$$

We need to find $\Phi \in \mathcal{C}^c(S)$ such that $\Psi^{-1}\sigma\Psi = \sigma^*$ is given by

$$\xi'_j = \mu_j(\xi\eta)\xi_j, \quad \eta'_j = \nu_j(\xi\eta)\eta_j.$$

By definition, Ψ has the form

$$\xi'_j = \xi_j + U_j(\xi, \eta), \quad \eta'_j = \eta_j + V_j(\xi, \eta), \quad U_{j, (P+e_j)P} = V_{j, P(P+e_j)} = 0.$$

The components of $\Psi\sigma^*$ are

$$(5.14) \quad \xi'_j = \mu_j(\xi\eta)\xi_j + U_j(\mu(\xi\eta)\xi, \nu(\xi\eta)\eta),$$

$$(5.15) \quad \eta'_j = \nu_j(\xi\eta)\eta_j + V_j(\mu(\xi\eta)\xi, \nu(\xi\eta)\eta).$$

To derive the normal form, we only need Taylor theorem in order one. This can also demonstrate small divisors in the normalizing transformation; however, one cannot see the small divisors in the normal forms. Later we will show the existence of divergent normal forms. This requires us to use Taylor formula whose remainder has order two. By the Taylor theorem, we write the components of $\sigma\Psi$ as

$$(5.16) \quad \xi'_j = (\boldsymbol{\mu}_j^0(\xi\eta) + D\boldsymbol{\mu}_j^0(\xi\eta)(\eta U + \xi V + UV))(\xi_j + U_j) \\ + f_j(\xi, \eta) + Df_j(\xi, \eta)(U, V) + A_j(\xi, \eta),$$

$$(5.17) \quad \eta'_j = (\mathbf{v}_j^0(\xi\eta) + D\mathbf{v}_j^0(\xi\eta)(\eta U + \xi V + UV))(\eta_j + V_j) \\ + g_j(\xi, \eta) + Dg_j(\xi, \eta)(U, V) + B_j(\xi, \eta).$$

Recall our notation that $UV = (U_1(\xi, \eta)V_1(\xi, \eta), \dots, U_p(\xi, \eta)V_p(\xi, \eta))$. The second order remainders are

$$(5.18) \quad A_j(\xi, \eta) = R_2\boldsymbol{\mu}_j^0(\xi\eta; \xi U + \eta V + UV)(\xi_j + U_j) \\ + R_2f_j(\xi, \eta; U, V),$$

$$(5.19) \quad B_j(\xi, \eta) = R_2\mathbf{v}_j^0(\xi\eta; \xi U + \eta V + UV)(\eta_j + V_j) \\ + R_2g_j(\xi, \eta; U, V).$$

Note that the remainder $R_2\boldsymbol{\mu}^0$ is independent of the linear part of $\boldsymbol{\mu}^0$. Thus

$$R_2\boldsymbol{\mu}_j^0 = R_2(\boldsymbol{\mu}_j^0 - L\boldsymbol{\mu}_j^0), \quad R_2\mathbf{v}_j^0 = R_2(\mathbf{v}_j^0 - L\mathbf{v}_j^0).$$

Calculate the largest degrees w, d' of coefficients of $\boldsymbol{\mu}^0 - L\boldsymbol{\mu}^0, (U, V, f, g)$ on which $A_{j,PQ}$ depend: It is easy to see that $d' \geq d \geq 2$ and $w \geq 2$. We have

$$2(w-2) + 2(d+1) + 1 \leq |P| + |Q|; \\ 3 + d + d' \leq |P| + |Q| \quad \text{or} \quad 2d + d' - 2 \leq |P| + |Q|,$$

where the first two inequalities are obtained from the first term on the right-hand side of (5.18) and its second term yields the last inequality. Thus, we have crude bounds

$$w \leq \frac{|P| + |Q| + 1 - 2d}{2}, \quad d' \leq |P| + |Q| - d.$$

Analogously, we can estimate the degrees of coefficients of \mathbf{v}^0 . We obtain

$$(5.20) \quad A_{j,PQ} = \mathcal{A}_{j,PQ}(\{\boldsymbol{\mu}^0 - L\boldsymbol{\mu}^0\}_{\lfloor \frac{|P|+|Q|+1-2d}{2} \rfloor}; \{f, U, V\}_{|P|+|Q|-d}),$$

$$(5.21) \quad B_{j,QP} = \mathcal{B}_{j,QP}(\{\mathbf{v}^0 - L\mathbf{v}^0\}_{\lfloor \frac{|P|+|Q|+1-2d}{2} \rfloor}; \{g, U, V\}_{|P|+|Q|-d}).$$

Recall that $\{f, U, V\}_d$ is the set of coefficients of f_{PQ}, U_{PQ}, V_{PQ} with $|P| + |Q| \leq d$. Here $\mathcal{A}_{j,PQ}(t'; t''), \mathcal{B}_{j,QP}(t'; t'')$ are polynomials of which each has coefficients that depend only on j, P, Q and they vanish at $t'' = 0$.

To finish the proof of the proposition, we will not need the explicit expressions involving $D\boldsymbol{\mu}_j^0, D\boldsymbol{\nu}_j^0, Df_j, Dg_j$. We will use these derivatives in the proof of Lemma 6.1. So we derive these expression in this proof too.

We apply the projection (5.14)-(5.15) and (5.16)-(5.17) onto $\mathcal{C}_2^{\xi}(S)$, via monomials in each component of both sides of the identities. The images of the mappings

$$\begin{aligned} (\xi, \eta) &\mapsto (U(\boldsymbol{\mu}(\xi\eta)\xi, \boldsymbol{\nu}(\xi\eta)\eta), V(\boldsymbol{\mu}(\xi\eta)\xi, \boldsymbol{\nu}(\xi\eta)\eta)), \\ (\xi, \eta) &\mapsto (\boldsymbol{\mu}^0(\xi\eta)U(\xi, \eta), \boldsymbol{\nu}^0(\xi\eta)V(\xi, \eta)) \end{aligned}$$

under the projection are 0. We obtain from (5.14)-(5.17) and (5.18)-(5.19) that $d_0 = d$. Next, we project (5.14)-(5.15) and (5.16)-(5.17) onto $\mathcal{C}_2(\hat{S})$, via monomials in each component of both sides of the identities. Using (5.13) and (5.20) we obtain

$$(5.22) \quad \begin{aligned} \boldsymbol{\mu}_{j,P} &= \boldsymbol{\mu}_{j,P}^0 + \{Df_j(U, V)\}_{P+e_j, P} \\ &\quad + \mathcal{M}_P(\{\boldsymbol{\mu}^0\}_{\frac{2|P|+1-2d}{2}}; \{f, U, V\}_{P(d)}), \end{aligned}$$

$$(5.23) \quad \begin{aligned} \boldsymbol{\nu}_{j,P} &= \boldsymbol{\nu}_{j,P}^0 + \{Dg_j(U, V)\}_{P, P+e_j} \\ &\quad + \mathcal{N}_P(\{\boldsymbol{\nu}^0\}_{\frac{2|P|+1-2d}{2}}; \{g, U, V\}_{P(d)}) \end{aligned}$$

with

$$P(d) = 2|P| + 1 - d.$$

Here $\mathcal{M}_P, \mathcal{N}_P$ are polynomials of which each has coefficients that depend only on P , and $\{\boldsymbol{\mu}^0\}_a$ stands for the set of coefficients $\boldsymbol{\mu}_Q^0$ with $|Q| \leq a$ for a real number $a \geq 0$. Note that $\mathcal{U}_{j,PQ} = \mathcal{V}_{j,QP} = 0$ when $|P| + |Q| = 2$, or $\text{ord}(f, g) > |P| + |Q|$. And $\mathcal{M}_P = \mathcal{N}_P = 0$ when $\text{ord}(f, g) > P(d)$, by (5.13). We have

$$\begin{aligned} \{U_j(\boldsymbol{\mu}(\xi\eta)\xi, \boldsymbol{\nu}(\xi\eta)\eta)\}_{PQ} &= \\ &\mu^{P-Q}U_{j,PQ} + \mathcal{U}_{j,PQ}(\{\boldsymbol{\mu}, \boldsymbol{\nu}\}_{\frac{|P|+|Q|-d}{2}}, \{U\}_{|P|+|Q|-2}). \end{aligned}$$

Comparing coefficients in (5.14), (5.16), and using (5.20), we get for $\ell = |P| + |Q|$

$$\begin{aligned} (\mu^{P-Q} - \mu_j)U_{j,PQ} &= \{f_j + Df_j(U, V)\}_{PQ} \\ &\quad + \mathcal{U}_{j,PQ}(\{\boldsymbol{\mu}^0\}_{\frac{\ell+1-2d}{2}}, \{\boldsymbol{\mu}, \boldsymbol{\nu}\}_{\frac{\ell-d}{2}}; \{f, U, V\}_{\ell-2}). \end{aligned}$$

We have analogous formulae for $V_{j,QP}$. Using (5.22), we obtain with $|P| + |Q| = \ell$

$$(5.24) \quad (\mu^{P-Q} - \mu_j)U_{j,PQ} = \{f_j + Df_j(U, V)\}_{PQ} \\ + \mathcal{U}_{j,PQ}(\{\boldsymbol{\mu}^0, \boldsymbol{\nu}^0\}_{\frac{\ell-d}{2}}; \{f, g, U, V\}_{\ell-2}),$$

$$(5.25) \quad (\mu^{Q-P} - \mu_j^{-1})V_{j,QP} = \{g_j + Dg_j(U, V)\}_{PQ} \\ + \mathcal{V}_{j,QP}(\{\boldsymbol{\mu}^0, \boldsymbol{\nu}^0\}_{\frac{\ell-d}{2}}; \{f, g, U, V\}_{\ell-2}).$$

for $\mu^{P-Q} \neq \mu_j$, which are always solvable. Inductively, by using (5.25)-(5.26) and (5.22)-(5.23), we obtain unique solutions $U, V, \boldsymbol{\mu}, \boldsymbol{\nu}$. Moreover, the solutions and their dependence on the coefficients of f, g and small divisors have the form

$$(5.26) \quad (\mu^{P-Q} - \mu_j)U_{j,PQ} = \{f_j + Df_j(U, V)\}_{PQ} \\ + \mathcal{U}_{j,PQ}^*(\delta_{\ell-2}, \{\boldsymbol{\mu}^0, \boldsymbol{\nu}^0\}_{\frac{\ell-d}{2}}; \{f, g\}_{\ell-2}),$$

$$(5.27) \quad (\mu^{Q-P} - \mu_j^{-1})V_{j,QP} = \{g_j + Dg_j(U, V)\}_{PQ} \\ + \mathcal{V}_{j,QP}^*(\delta_{\ell-2}, \{\boldsymbol{\mu}^0, \boldsymbol{\nu}^0\}_{\frac{\ell-d}{2}}; \{f, g\}_{\ell-2}),$$

where $\ell = |P| + |Q|$ and $\mu^{P-Q} \neq \mu_j$, and

$$(5.28) \quad \delta_\ell = \bigcup_{j=1}^p \left\{ \mu_j, \mu_j^{-1}, \frac{1}{\mu^P - \mu_j} : P \in \mathbf{Z}^p, P \neq e_j, |P| \leq \ell \right\}.$$

This shows that for any $\boldsymbol{\mu}^0, \boldsymbol{\nu}^0$ there exists a unique mapping Ψ transforms σ into σ^* . Furthermore, $\mathcal{U}_{j,PQ}^*(t'; t''), \mathcal{V}_{j,QP}^*(t'; t'')$ are polynomials of which each has coefficients that depend only on j, P, Q , and they vanish at $t'' = 0$.

For later purpose, let us express $\boldsymbol{\mu}, \boldsymbol{\nu}$ in terms of f, g . We substitute expressions (5.27)-(5.28) for U, V in (5.22)-(5.23) to obtain

$$(5.29) \quad \boldsymbol{\mu}_{j,P} = \boldsymbol{\mu}_{j,P}^0 + \{Df_j(U, V)\}_{P+e_jP} \\ + \mathcal{M}_{j,P}^*(\delta_{P(d)}, \{\boldsymbol{\mu}^0, \boldsymbol{\nu}^0\}_{\frac{P(d)}{2}}; \{f, g\}_{P(d)}),$$

$$(5.30) \quad \boldsymbol{\nu}_{j,P} = \boldsymbol{\nu}_{j,P}^0 + \{Dg_j(U, V)\}_{PP+e_j} \\ + \mathcal{N}_{j,P}^*(\delta_{P(d)}, \{\boldsymbol{\mu}^0, \boldsymbol{\nu}^0\}_{\frac{P(d)}{2}}; \{f, g\}_{P(d)}).$$

with f, g satisfying (5.13).

Assume that $\tilde{\sigma} = \psi_0^{-1}\sigma^*\psi_0$ commutes with \hat{S} . By Corollary 4.5, we can decompose $\psi_0 = HG^{-1}$ with $G \in \mathcal{C}(\hat{S})$ and $H \in \mathcal{C}^c(\hat{S})$. Furthermore, $G^{-1}\tilde{\sigma}G$ commutes with \hat{S} and $H^{-1}\sigma^*H$. By the uniqueness conclusion for the above ψ_0 , H must be the identity. This shows that $\psi_0 \in \mathcal{C}(\hat{S})$.

(i). Assume that we have normalized σ . We now use it to normalize the pair of involutions. Assume that $\sigma = \tau_1\tau_2$ and $\tau_j^2 = I$. Then

$\sigma^* = \tau_1^* \tau_2^*$. Let $T_0(\xi, \eta) := (\eta, \xi)$. We have $T_0(\sigma^*)^{-1}T_0 = T_0\tau_1^*\sigma^*\tau_1^*T_0$. By the above normalization, $T_0(\sigma^*)^{-1}T_0$ commutes with \hat{S} . Therefore, $\tau_1^*T_0$ belongs to the centralizer of \hat{S} and it must be of the form $(\xi, \eta) \rightarrow (\xi\Lambda_1(\xi\eta), \eta\Lambda_1^*(\xi\eta))$. Then $(\tau_1^*)^2 = I$ implies that

$$\Lambda_1(\xi\eta(\Lambda_1\Lambda_1^*)(\xi\eta))\Lambda_1^*(\xi\eta) = 1.$$

The latter implies, by induction on $d > 1$, that $\Lambda_1\Lambda_1^* = 1 + O(d)$ for all $d > 1$, i.e. $\Lambda_1\Lambda_1^* = 1$.

Let τ_i^* be given by (5.4). We want to achieve $\tilde{\Lambda}_{1j}\tilde{\Lambda}_{2j} = 1$ for $\tilde{\tau}_i = \psi_0^{-1}\tau_i^*\psi_0$ by applying a transformation ψ_0 in $\mathcal{C}^c(\hat{T}_1, \hat{T}_2)$ that commutes with \hat{S} . According to Definition 4.3, it has the form

$$\psi_0: \xi_j = \tilde{\xi}_j(1 + a_j(\tilde{\zeta})), \quad \eta_j = \tilde{\eta}_j(1 - a_j(\tilde{\zeta}))$$

with $a_j(0) = 0$. Here $\tilde{\zeta}_j := \tilde{\xi}_j\tilde{\eta}_j$ and $\tilde{\zeta} := (\tilde{\zeta}_1, \dots, \tilde{\zeta}_p)$. Computing the products ζ in $\tilde{\zeta}$ and solving $\tilde{\zeta}$ in ζ , we obtain

$$\psi_0^{-1}: \tilde{\xi}_j = \xi_j(1 + b_j(\zeta))^{-1}, \quad \tilde{\eta}_j = \eta_j(1 - b_j(\zeta))^{-1}.$$

Note that $(a_j^2)_P = \mathcal{A}_{j,P}(\{a\}_{|P|-1})$, and

$$\xi_j\eta_j = \tilde{\xi}_j\tilde{\eta}_j(1 - a_j^2(\tilde{\zeta})), \quad \tilde{\xi}_j\tilde{\eta}_j = \xi_j\eta_j(1 - b_j^2(\zeta))^{-1}.$$

From $\psi_0^{-1}\psi_0 = I$, we get

$$(5.31) \quad b_j(\zeta) = a_j(\tilde{\zeta}), \quad b_{j,P} = a_{j,P} + \mathcal{B}_{j,P}(\{a\}_{|P|-1}).$$

By a simple computation we see that $\tilde{\tau}_i = \psi_0^{-1}\tau_i^*\psi_0$ is given by

$$\tilde{\xi}'_j = \tilde{\eta}_j\tilde{\Lambda}_{ij}(\tilde{\zeta}), \quad \tilde{\eta}'_j = \tilde{\xi}_j\tilde{\Lambda}_{ij}^{-1}(\tilde{\zeta})$$

with

$$\tilde{\Lambda}_{1j}\tilde{\Lambda}_{2j}(\tilde{\zeta}) = (\Lambda_{1j}\Lambda_{2j})(\zeta)(1 + b_j(\zeta'))^{-2}(1 - a_j(\tilde{\zeta}))^2.$$

Here $\zeta'_j = \zeta_j(1 - a_j^2(\tilde{\zeta}))$. Using (5.32) and the implicit function theorem, we determine a_j uniquely to achieve $\tilde{\Lambda}_{1j}\tilde{\Lambda}_{2j} = 1$.

To identify the transformations that preserve the form of $\tilde{\tau}_1, \tilde{\tau}_2$, we first verify that each element $\psi_1 \in \mathcal{C}(\hat{T}_1, \hat{T}_2)$ preserves that form. According to (4.4), we have

$$\begin{aligned} \psi_1: \xi_j &= \tilde{\xi}_j\tilde{a}_j(\tilde{\zeta}), & \eta_j &= \tilde{\eta}_j\tilde{a}_j(\tilde{\zeta}), \\ \psi_1^{-1}: \tilde{\xi}_j &= \xi_j\tilde{b}_j(\zeta), & \tilde{\eta}_j &= \eta_j\tilde{b}_j(\zeta), \\ & & \tilde{b}_j(\zeta)\tilde{a}_j(\tilde{\zeta}) &= 1. \end{aligned}$$

This shows that $\psi_1^{-1}\tilde{\tau}_i$ is given by

$$\tilde{\xi}'_j = \tilde{\Lambda}_{ij}(\zeta)\tilde{b}_j(\zeta)\eta_j, \quad \tilde{\eta}'_j = \tilde{\Lambda}_{ij}^{-1}(\zeta)\tilde{b}_j(\zeta)\xi_j.$$

Then $\psi_1^{-1}\tilde{\tau}_i\psi_1$ is given by

$$\tilde{\xi}'_j = \tilde{\Lambda}_{ij}(\zeta)\tilde{\eta}_j, \quad \tilde{\eta}'_j = \tilde{\Lambda}_{ij}^{-1}(\zeta)\tilde{\xi}_j.$$

Since $\zeta_j = \tilde{\zeta}_j \tilde{a}_j^2(\tilde{\zeta})$, then $\psi_1^{-1} \tilde{\tau}_i \psi_1$ still satisfy (5.5). Conversely, suppose that ψ_1 preserves the forms of $\tilde{\tau}_1, \tilde{\tau}_2$. We apply Corollary 4.5 to decompose $\psi_1 = \phi_1 \phi_0^{-1}$ with $\phi_0 \in \mathcal{C}(\hat{T}_1, \hat{T}_2)$ and $\phi_1 \in \mathcal{C}^c(\hat{T}_1, \hat{T}_2)$. Since we just proved that each element in $\mathcal{C}(\hat{T}_1, \hat{T}_2)$ preserves the form of $\tilde{\tau}_i$, then $\phi_1 = \psi_1 \phi_0$ also preserves the forms of $\tilde{\tau}_1, \tilde{\tau}_2$. On the other hand, we have shown that there exists a unique mapping in $\mathcal{C}^c(\hat{T}_1, \hat{T}_2)$ which transforms $\{\tau_1^*, \tau_2^*\}$ into $\{\tilde{\tau}_1, \tilde{\tau}_2\}$. This shows that $\phi_0 = I$. We have verified all assertions in (i).

(ii). It is easy to see that $\mathcal{C}^c(\hat{S})$ and $\mathcal{C}^c(\hat{T}_1, \hat{T}_2)$ are invariant under conjugacy by ρ . We have $\Psi^{-1} \sigma \Psi = \sigma^*$ and $\Psi \in \mathcal{C}^c(\hat{S})$. Note that $\rho \sigma \rho = \sigma^{-1}$ and $\rho \sigma^* \rho$ have the same form as of $(\sigma^*)^{-1}$, i.e. they are in $\mathcal{C}(\hat{S})$ and have the same linear part. We have $\rho \Psi \rho \sigma \rho \Psi^{-1} \rho = \rho (\sigma^*)^{-1} \rho$. The uniqueness of Ψ implies that $\rho \Psi \rho = \Psi$ and $\tau_2^* = \rho \tau_1^* \rho$. Thus, we obtain relations (5.7)-(5.10). Analogously, $\rho \psi_0 \rho$ is still in $\mathcal{C}^c(\hat{T}_1, \hat{T}_2)$, and $\rho \phi_0 \rho$ preserves the form of $\tilde{\tau}_1, \tilde{\tau}_2$. Thus $\rho \psi_0 \rho = \psi_0$ and $\tilde{\tau}_2 = \rho \tilde{\tau}_1 \rho$, which gives us (5.11). q.e.d.

We will also need the following uniqueness result.

Corollary 5.3. *Suppose that σ has a non-resonant linear part \hat{S} . Let Ψ be the unique formal mapping in $\mathcal{C}^c(\hat{S})$ such that $\Psi^{-1} \sigma \Psi \in \mathcal{C}(\hat{S})$. If $\tilde{\Psi} \in \mathcal{C}^c(\hat{S})$ is a polynomial map of degree at most d such that $\tilde{\Psi}^{-1} \sigma \tilde{\Psi}(\xi, \eta) = \tilde{\sigma}(\xi, \eta) + O(|(\xi, \eta)|^{d+1})$ and $\tilde{\sigma} \in \mathcal{C}(\hat{S})$, then $\tilde{\Psi}$ is unique. In fact, $\Psi - \tilde{\Psi} = O(d+1)$.*

Proof. The proof is contained in the proof of Proposition 5.2. Let us recap it by using (5.27)-(5.28) and the proposition. We take a unique normalized mapping Φ such that $\Phi^{-1} \tilde{\Psi}^{-1} \sigma \tilde{\Psi} \Phi \in \mathcal{C}(\hat{S})$. By (5.27)-(5.28), $\Phi = I + O(d+1)$. From Proposition 5.2 it follows that $\psi_0 := \tilde{\Psi} \Phi \Psi^{-1} \in \mathcal{C}(\hat{S})$. We obtain $\tilde{\Psi} \Phi = \psi_0 \Psi$. Thus $\psi_0 \Psi = \tilde{\Psi} + O(d+1)$. Since $\psi_0 \in \mathcal{C}(\hat{S})$, and $\Psi, \tilde{\Psi}$ are in $\mathcal{C}^c(\hat{S})$, we conclude that $\Psi = \tilde{\Psi} + O(d+1)$. q.e.d.

For clarity, we state the following uniqueness results on normalization.

Corollary 5.4. *Let σ have a non-resonant linear part and let σ be given by*

$$\xi'_j = \mu_j^0(\xi, \eta) \xi_j + f_j^0(\xi, \eta), \quad \eta'_j = \nu_j^0(\xi, \eta) \eta_j + g_j^0(\xi, \eta).$$

Let $\Psi = I + (U, V) \in \mathcal{C}^c(\hat{S})$ and let $\sigma^ = \Psi^{-1} \sigma \Psi$ be given by*

$$\xi'_j = \mu_j(\xi, \eta) \xi_j + f_j(\xi, \eta), \quad \eta'_j = \nu_j(\xi, \eta) \eta_j + g_j(\xi, \eta).$$

Suppose that (f^0, g^0) and (f, g) are in $\mathcal{C}_2^c(\hat{S})$, and

$$\text{ord}(f^0, g^0) \geq d, \quad \text{ord}(f_j, g_j) \geq d, \quad d \geq 2.$$

Then $\text{ord}(U, V) \geq d$ and

$$(5.32) \quad \boldsymbol{\mu}_{j,P} = \boldsymbol{\mu}_{j,P}^0, \quad \boldsymbol{\nu}_{j,P} = \boldsymbol{\nu}_{j,P}^0, \quad 1 \leq 2|P| + 1 < 2d - 1.$$

Proof. By Corollary 5.3, we know that $\text{ord}(U, V) \geq d$. Expanding both sides of $\sigma\Psi = \Psi\sigma^*$ for terms of degree less than $2d - 1$, we obtain

$$\begin{aligned} & \boldsymbol{\mu}_j^0(\xi\eta)(\xi_j + U_j(\xi, \eta)) + D\boldsymbol{\mu}_j^0(\xi\eta)(\xi V + \eta U)\xi_j + f_j^0(\xi, \eta) \\ &= \boldsymbol{\mu}_j(\xi\eta)\xi_j + f_j(\xi, \eta) + U_j(\boldsymbol{\mu}(\xi\eta)\xi, \boldsymbol{\nu}(\xi\eta)\eta) + O(2d - 1). \end{aligned}$$

Note that $\xi_i V_i(\xi, \eta)\xi_j$ and $\eta_i U_i(\xi, \eta)\xi_j$ and $U_j(\boldsymbol{\mu}(\xi\eta)\xi, \boldsymbol{\nu}(\xi\eta)\eta)$ do not contain terms of the form $\xi^Q \eta^Q \xi_j$. Comparing the coefficients of $\xi^P \eta^P \xi_j$ for $2|P| + 1 < 2d - 1$, we obtain the first identity in (5.33). The second identity can be obtained similarly. q.e.d.

When $p = 1$, Proposition 5.2 is due to Moser and Webster [19]. In fact, they achieved

$$\tilde{\boldsymbol{\mu}}_1(\zeta_1) = \mu_1 e^{\delta \zeta_1^s}.$$

Here $\delta = 0, \pm 1$ for the elliptic case and $\delta = 0, \pm i$ for the hyperbolic case when μ_1 is not a root of unity, i.e. γ is *non-exceptional*. In particular the normal form is always convergent, although the normalizing transformations are generally divergent for the hyperbolic case.

Let us find out further normalization that can be performed to preserve the form of σ^* . In Proposition 5.2, we have proved that if σ is tangent to \hat{S} , there exists a unique $\Psi \in \mathcal{C}^c(\hat{S})$ such that $\Psi^{-1}\sigma\Psi$ is an element σ^* in the centralizer of \hat{S} . Suppose now that $\sigma = \tau_1\tau_2$ while τ_i is tangent to \hat{T}_i . Let $\tau_i^* = \Psi^{-1}\tau_i\Psi$. We have also proved that there is a unique $\psi_0 \in \mathcal{C}^c(\hat{T}_1, \hat{T}_2)$ such that $\tilde{\tau}_i = \psi_0^{-1}\tau_i^*\psi_0$, $i = 1, 2$, are of the form (5.5), i.e.

$$\begin{aligned} \tilde{\tau}_i: \xi'_j &= \tilde{\Lambda}_{ij}(\zeta)\eta_j, & \eta'_j &= \tilde{\Lambda}_{ij}^{-1}(\zeta)\xi_j; \\ \tilde{\sigma}: \xi'_j &= \tilde{\boldsymbol{\mu}}_j(\zeta)\xi_j, & \eta'_j &= \tilde{\boldsymbol{\mu}}_j^{-1}(\zeta)\eta_j. \end{aligned}$$

Here $\zeta = (\xi_1\eta_1, \dots, \xi_p\eta_p)$, $\tilde{\Lambda}_{2j} = \tilde{\Lambda}_{1j}^{-1}$ and $\tilde{\boldsymbol{\mu}}_j = \tilde{\Lambda}_{1j}^2$. We still have freedom to further normalize $\tilde{\tau}_1, \tilde{\tau}_2$ and to preserve their forms. However, any new coordinate transformation must be in $\mathcal{C}(\hat{T}_1, \hat{T}_2)$, i.e. it must have the form

$$\psi_1: \xi_j \rightarrow a_j(\xi\eta)\xi_j, \quad \eta_j \rightarrow a_j(\xi\eta)\eta_j.$$

When $\tau_{2j} = \rho\tau_{1j}\rho$, we require that ψ_1 commutes with ρ , i.e.

$$a_e = \bar{a}_e, \quad a_h = \bar{a}_h, \quad a_s = \bar{a}_{s+s*}.$$

In ζ coordinates, the transformation ψ_1 has the form

$$(5.33) \quad \varphi: \zeta_j \rightarrow b_j(\zeta)\zeta_j, \quad 1 \leq j \leq p$$

with $b_j = a_j^2$. Therefore, the mapping φ needs to satisfy

$$b_e > 0, \quad b_h > 0, \quad b_s = \bar{b}_{s+s*}.$$

Recall from (5.7)-(5.10) the reality conditions on $\tilde{\mu}_j$

$$\begin{aligned}\overline{\tilde{\mu}_e \circ \rho_z} &= \tilde{\mu}_e, & 1 \leq e \leq e_*; \\ \overline{\tilde{\mu}_h \circ \rho_z} &= \tilde{\mu}_h^{-1}, & e_* < h \leq h_* + e_*; \\ \tilde{\mu}_{s_*+s} &= \overline{\tilde{\mu}_s^{-1} \circ \rho_z}, & h_* + e_* < s \leq p - s_*.\end{aligned}$$

Here

$$\rho_z: \zeta_j \rightarrow \bar{\zeta}_j, \quad \zeta_s \rightarrow \bar{\zeta}_{s+s_*}, \quad \zeta_{s+s_*} \rightarrow \bar{\zeta}_s$$

for $1 \leq j \leq e_* + h_*$ and $e_* + h_* < s \leq p - s_*$.

Therefore, our normal form problem leads to another normal form problem which is interesting in its own right. To formulate a new normalization problem, let us define $F = (F_1, \dots, F_p)$ by

$$(5.34) \quad F_j(\zeta) := \begin{cases} \log(\tilde{\mu}_j(\zeta)/\tilde{\mu}_j(0)), & 1 \leq j \leq e_*, \\ -i \log(\tilde{\mu}_j(\zeta)/\tilde{\mu}_j(0)), & e_* < j \leq p. \end{cases}$$

Then the reality conditions on $\tilde{\mu}$ become

$$(5.35) \quad F = \rho_z F \rho_z.$$

The transformations (5.34) will then satisfy

$$\rho_z \varphi \rho_z = \varphi, \quad b_j(0) > 0, \quad 1 \leq j \leq e_* + h_*.$$

Under condition L on $\tilde{\mu}$, we have transformed the reality condition on μ into a linear condition (5.36). This will be useful to further normalize $\tilde{\mu}$. Therefore, when $F'(0)$ is additionally diagonal and invertible and its j th diagonal entry is positive for $j = e, h$, we apply a dilation φ satisfying the above condition so that F is tangent to the identity. Then any further change of coordinates must be tangent to the identity too. Thus, we need to normalize the formal holomorphic mapping F by composition $F \circ \varphi$, for which we study in next subsection.

5.2. A normal form for maps tangent to the identity. Let us consider a germ of holomorphic mapping $F(\zeta)$ in \mathbf{C}^p with an invertible linear part $\mathbf{A}\zeta$ at the origin. According to the inverse function theorem, there exists a holomorphic mapping Ψ with $\Psi(0) = 0$, $\Psi'(0) = I$ such that $F \circ \Psi(\zeta) = \mathbf{A}\zeta$. On the other hand, if we impose some restrictions on Ψ , we can no longer linearize F in general.

To focus on applications to CR singularity and to limit the scope of our investigation, we now deliberately restrict our analysis to the simplest case : F is tangent to the identity. We shall apply our result to F as defined by (5.35). In what follows, we shall devise a normal form of such an F under right composition by Ψ that *preserve all coordinate hyperplanes*, i.e. $\Psi_j(\zeta) = \zeta_j \tilde{\Psi}_j(\zeta)$, $j = 1, \dots, p$.

Lemma 5.5. *Let F be a formal holomorphic map of \mathbf{C}^p that is tangent to the identity at the origin.*

(i) There exists a unique formal biholomorphic map ψ which preserves all $\zeta_j = 0$ such that $\hat{F} := F \circ \psi$ has the form

$$(5.36) \quad \hat{F}(\zeta) = \zeta + \hat{f}(\zeta), \quad \hat{f}(\zeta) = O(|\zeta|^2); \quad \partial_{\zeta_j} \hat{f}_j = 0, \quad 1 \leq j \leq p.$$

(ii) If F is convergent, the ψ in (i) is convergent. If F commutes with ρ_z , so does the ψ .

(iii) The formal normal form in (i) has the form

$$(5.37) \quad \hat{f}_{j,Q} = f_{j,Q} - \{Df_j \cdot f\}_Q + \mathcal{F}_{j,Q}(\{f\}_{|Q|-2}), \quad q_j = 0, \quad |Q| > 1.$$

Here $\mathcal{F}_{j,Q}$ are universal polynomials and vanish at 0.

Proof. (i). Write $F(\zeta) = \zeta + f(\zeta)$ and

$$\psi: \zeta'_j = \zeta_j + \zeta_j g_j(\zeta), \quad g_j(0) = 0.$$

For $\hat{F} = F \circ \psi$, we need to solve for \hat{f}, g from

$$\hat{f}_j(\zeta) = \zeta_j g_j(\zeta) + f_j \circ \psi(\zeta).$$

Fix $Q = (q_1, \dots, q_p) \in \mathbf{N}^p$ with $|Q| > 1$. We obtain unique solutions

$$(5.38) \quad g_{j,Q-e_j} = -\{f_j(\psi(\zeta))\}_Q, \quad q_j > 0,$$

$$(5.39) \quad \hat{f}_{j,Q} := \{f_j(\psi(\zeta))\}_Q, \quad q_j = 0.$$

We first obtain $g_{j,Q-e_j} = -f_{j,Q} + \mathcal{G}_Q(\{f\}_{|Q|-1}, \{g\}_{|Q|-2})$. This determines

$$(5.40) \quad g_{j,Q-e_j} = -f_{j,Q} + \mathcal{G}_Q(\{f\}_{|Q|-1}).$$

Next, we expand $f_j(\psi(\zeta)) = f_j(\zeta) + Df_j(\zeta) \cdot (\zeta_1 g_1(\zeta), \dots, \zeta_p g_p(\zeta)) + \mathcal{R}_2 f_j(\zeta; \zeta g(\zeta))$. The last term, with $\text{ord } g \geq 1$, has the form

$$\{\mathcal{R}_2 f_j(\zeta; \zeta g(\zeta))\}_Q = \mathcal{F}_{j,Q}(\{f\}_{|Q|-2}, \{g\}_{|Q|-2}) = \tilde{\mathcal{F}}_{j,Q}(\{f\}_{|Q|-2}).$$

Combining (5.40), the expansion, and (5.41), we obtain (5.38).

(ii). Assume that F is convergent. Define $\bar{h}(\zeta) = \sum |h_Q| \zeta^Q$. We obtain for every multi-index $Q = (q_1, \dots, q_p)$ and for every j satisfying $q_j \geq 1$

$$\bar{g}_{j,Q-e_j} \leq \{\bar{f}_j(\zeta_1 + \zeta_1 \bar{g}_1(\zeta), \dots, \zeta_p + \zeta_p \bar{g}_p(\zeta))\}_Q.$$

Set $w(\zeta) = \sum \zeta_k \bar{g}_k(\zeta)$. We obtain

$$w(\zeta) \prec \sum \bar{f}_j(\zeta_1 + w(\zeta), \dots, \zeta_p + w(\zeta)).$$

Note that $f_j(\zeta) = O(|\zeta|^2)$ and $w(0) = 0$. By the Cauchy majorization and the implicit function theorem, w and hence g, ψ, \hat{f} are convergent.

(iii). Assume that $\rho_z F \rho_z = F$. Then $\rho_z \hat{F} \rho_z$ is normalized, $\rho_z \psi \rho_z$ is tangent to the identity, and the j th component of $\rho_z \hat{F} \rho_z(\zeta) - \zeta$ is independent of ζ_j . Thus $\rho_z \psi \rho_z$ normalizes F too. By the uniqueness of ψ , we obtain $\rho_z \psi \rho_z = \psi$.

By rewriting (5.40), we obtain

$$(5.41) \quad \hat{f}_{j,Q} = f_{j,Q} + \{f_j(\psi) - f_j\}_Q = f_{j,Q} + \mathcal{F}'_{j,Q}(\{f\}_{|Q|-1}, \{g\}_{|Q|-2}).$$

From (5.39), it follows that

$$g_{k,Q-e_k} = -f_{k,Q} + \mathcal{G}_{k,Q-e_k}(\{f\}_{|Q|-1}, \{g\}_{|Q|-2}), \quad |Q| > 1.$$

Note that $\{g\}_0 = 0$ and $\{f\}_1 = 0$. Using the identity repeatedly, we obtain $g_{k,Q-e_k} = -f_{k,Q} + \mathcal{G}_{k,Q-e_k}^*(\{f\}_{|Q|-1})$. Therefore, we can rewrite (5.42) as (5.38). q.e.d.

5.3. A unique formal normal form of a reversible map σ . We now state a normal form for $\{\tau_1, \tau_2, \rho\}$ under a condition on the third-order invariants of σ .

Theorem 5.6. *Let τ_1, τ_2 be a pair of holomorphic involutions with linear parts \hat{T}_i . Let $\sigma = \tau_1\tau_2$. Assume that the linear part of σ is*

$$\hat{S}: \xi'_j = \mu_j \xi_j, \quad \eta_j = \mu_j^{-1} \eta_j, \quad 1 \leq j \leq p$$

and μ_1, \dots, μ_p are non-resonant. Let $\Psi \in \mathcal{C}^c(\hat{S})$ be the unique formal mapping such that

$$\begin{aligned} \tau_i^* &= \Psi^{-1} \tau_i \Psi: \xi'_j = \Lambda_{ij}(\xi\eta) \eta_j, \quad \eta'_j = \Lambda_{ij}(\xi\eta)^{-1} \xi_j; \\ \sigma^* &= \Psi^{-1} \sigma \Psi: \xi'_j = \mu_j(\xi\eta) \xi_j, \quad \eta'_j = \mu_j(\xi\eta)^{-1} \eta_j \end{aligned}$$

with $\mu_j = \Lambda_{1j} \Lambda_{2j}^{-1}$. Suppose that μ satisfies condition L.

(i) Then there exists an invertible formal map $\psi_1 \in \mathcal{C}(\hat{S})$ such that

$$(5.42) \quad \begin{aligned} \hat{\tau}_i &= \psi_1^{-1} \tau_i^* \psi_1: \xi'_j = \hat{\Lambda}_{ij}(\xi\eta) \eta_j, \quad \eta'_j = \hat{\Lambda}_{ij}(\xi\eta)^{-1} \xi_j; \\ \hat{\sigma} &= \psi_1^{-1} \sigma^* \psi_1: \xi'_j = \hat{\mu}_j(\xi\eta) \xi_j, \quad \eta'_j = \hat{\mu}_j(\xi\eta)^{-1} \eta_j. \end{aligned}$$

Here $\hat{\Lambda}_{2j} = \hat{\Lambda}_{1j}^{-1}$, and \hat{T}_i is the linear part of $\hat{\tau}_i$. Moreover, $\hat{\mu}$ satisfies (1.7).

(ii) The centralizer of $\{\hat{\tau}_1, \hat{\tau}_2\}$ consists of 2^p dilations $(\xi, \eta) \rightarrow (a\xi, a\eta)$ with $a_j = \pm 1$. And $\hat{\Lambda}_{ij}$ are unique. If Λ_{ij} are convergent, then ψ_1 is convergent too.

(iii) Suppose that $\hat{\sigma}$ is divergent. If σ is formally equivalent to a mapping $\tilde{\sigma} \in \mathcal{C}(\hat{S})$ then $\tilde{\sigma}$ must be divergent too.

(iv) Let ρ be given by (3.8) and let $\tau_2 = \rho\tau_1\rho$. Then the above Ψ and ψ_1 commute with ρ . Moreover, $\hat{\tau}_i, \hat{\sigma}$ are unique.

Proof. Assertions in (i) are direct consequences of Proposition 5.2 and Lemma 5.5 in which F is the \tilde{M} in Proposition 5.2. The assertion in (ii) on the centralizer of $\{\hat{\tau}_1, \hat{\tau}_2\}$ is obtained from (5.6) of Proposition 5.2 in which $\tilde{\Lambda}_{ij} = \hat{\Lambda}_{ij}$. Indeed, by (5.6), if ψ preserves $\{\hat{\tau}_1, \hat{\tau}_2\}$, then $\psi(\xi, \eta) = (c(\xi\eta)\xi, c(\xi\eta)\eta)$ and $\hat{\mu}_j(c^2(\xi\eta)\xi\eta) = \hat{\mu}_j(\xi\eta)$. This shows that $\hat{\mu}_j \circ \tilde{\psi} = \hat{\mu}_j$ for $\tilde{\psi}(\zeta) = c^2(\zeta)\zeta$. Since $\hat{\mu}_j - \hat{\mu}_j(0) = \mu_j \zeta_j + o(|\zeta|^2)$ then

$\tilde{\psi}$ is the identity, i.e. $c_j = \pm 1$. Now (iii) follows from (ii) too. Indeed, suppose σ is formally equivalent to some convergent

$$\tilde{\sigma}: \xi_j = \tilde{\mu}_j(\xi\eta)\xi_j, \quad \eta'_j = \tilde{\mu}_j(\xi\eta)^{-1}\eta_j.$$

By Lemma 5.5, there exists a unique convergent mapping $\varphi: \zeta'_j = b_j(\zeta)\zeta_j$ ($1 \leq j \leq p$) with $b_j(0) = 1$ such that $\tilde{\mu} \circ \varphi$ is in the normal form (1.7). Then

$$(\xi'_j, \eta'_j) = (b_j^{1/2}(\xi\eta)\xi_j, b_j^{1/2}(\xi\eta)\eta_j), \quad 1 \leq j \leq p$$

transforms $\tilde{\sigma}$ into a convergent mapping σ_* . Since the normal form for μ is unique, then $\hat{\sigma} = \sigma_*$. In particular, $\hat{\sigma}$ is convergent.

(iv). Note that $\rho\sigma\rho = \sigma^{-1}$. Also $\rho(\sigma^*)^{-1}\rho$ has the same form as σ^* . By $(\rho\Psi^{-1}\rho)\sigma(\rho\Psi\rho) = (\rho\sigma^*\rho)^{-1}$, we conclude that $\rho\Psi\rho = \Psi$. The rest of assertions can be verified easily. q.e.d.

Note that $(\mu_1^{-1}, \dots, \mu_p^{-1})$ is also normalized in the form (1.7). Under condition L, the above theorem completely settles the formal classification of $\{\tau_1, \tau_2, \rho\}$. It also says that **the normal form $\hat{\tau}_1, \hat{\tau}_2$ can be achieved by a convergent transformation, if and only if σ^* can be achieved by some convergent transformation**, i.e. the Ψ in the theorem is convergent.

5.4. An algebraic manifold with linear σ . We conclude the section showing that when τ_1, τ_2 are normalized as in this section, $\{\tau_{ij}\}$ might still be very general; in particular $\{\tau_{1j}, \rho\}$ cannot always be simultaneously linearized even at the formal level. This is one of main differences between $p = 1$ and $p > 1$.

Example 5.7. Let $p = 2$. Let ϕ be a holomorphic mapping of the form

$$\phi: \xi'_i = \xi_i + q_i(\xi, \eta), \quad \eta'_i = \eta_i + \lambda_i^{-1}q_i(T_1(\xi, \eta)), \quad i = 1, 2.$$

Here q_i is a homogeneous quadratic polynomial map and

$$T_1(\xi, \eta) = (\lambda_1\eta_1, \lambda_2\eta_2, \lambda_1^{-1}\xi_1, \lambda_2^{-1}\xi_2).$$

Let $\tau_{1j} = \phi T_{1j} \phi^{-1}$ and $\tau_{2j} = \rho \tau_{1j} \rho$. Then ϕ commutes with T_1 and $\tau_1 = T_1$. In particular $\tau_2 = \rho T_1 \rho$ and $\sigma = \tau_1 \tau_2$ are in linear normal forms. However, τ_{11} is given by

$$\begin{aligned} \xi'_1 &= \lambda_1\eta_1 - q_1(\lambda\eta, \lambda^{-1}\xi) + q_1(\lambda_1\eta_1, \xi_2, \lambda_1^{-1}\xi_1, \eta_2) + O(3), \\ \xi'_2 &= \xi_2 - q_2(\xi, \eta) + q_2(\lambda_1\eta_1, \xi_2, \lambda_1^{-1}\xi_1, \eta_2) + O(3), \\ \eta'_1 &= \lambda_1^{-1}\xi_1 - \lambda_1^{-1}q_1(\xi, \eta) + \lambda_1^{-1}q_1(\xi_1, \lambda_2\eta_2, \eta_1, \lambda_2^{-1}\xi_2) + O(3), \\ \eta'_2 &= \eta_2 - \lambda_2^{-1}q_2(\lambda\eta, \lambda^{-1}\xi) + \lambda_2^{-1}q_2(\xi_1, \lambda_2\eta_2, \eta_1, \lambda_2^{-1}\xi_2) + O(3). \end{aligned}$$

Notice that the common zero set V of $\xi_1\eta_1$ and $\xi_2\eta_2$ is invariant under τ_1, τ_2, σ and ρ . In fact, they are linear on V . However, for $(\xi', \eta') = \tau_{11}(\xi, \eta)$, we have

$$\begin{aligned} \xi'_1\eta'_1 &= -\eta_1q_1(0, \xi_2, \eta) + \eta_1q_1(0, \lambda_2\eta_2, \eta_1, \lambda_2^{-1}\xi_2) - \lambda_1^{-1}\xi_1q_1(0, \lambda_2\eta_2, \lambda^{-1}\xi) \\ &\quad + \lambda_1^{-1}\xi_1q_1(0, \xi_2, \lambda_1^{-1}\xi_1, \eta_2) \pmod{(\xi_1\eta_1, \xi_2\eta_2, O(4))}. \end{aligned}$$

For a generic q , τ_{11} does not preserve V .

By Proposition 5.2, $\{\tau_{11}, \tau_{12}, \rho\}$ is not linearizable, when the above linear σ is non-resonant. By a simple computation, we can verify that $\sigma_j = \tau_{1j}\tau_{2j}$ for $j = 1, 2$ do not commute with each other. In fact, we proved in [12] that if the μ_1, \dots, μ_p are nonresonant, σ_j commute pairwise, and σ is linear as above, then τ_{1j} must be linear.

6. Divergence of all normal forms of a reversible map σ

Unlike the Birkhoff normal form for Hamiltonian systems, the normal form of Poincaré-Dulac is not unique for a general σ ; it just belongs to the centralizer of the linear part S of σ . One can obtain a divergent normal form easily from any non-linear Poincaré-Dulac normal form of $\sigma = \tau_1\tau_2$ by conjugating with a divergent transformation in the centralizer of S ; see (5.6). We have seen how the small divisors enter in the computation of the normalizing transformations via (5.27)-(5.28) and (5.22)-(5.23) in the computation of the normal forms. To see the effect of small divisors on normal forms, we first assume a condition, to be achieved later, on the third order invariants of σ and then we shall need to modify the normalization procedure. We will use two sequences of normalizing mappings to normalize σ . The composition of normalized mappings might not be normalized. Therefore, the new normal form $\tilde{\sigma}$ might not be the σ^* in Proposition 5.2. We will show that this $\tilde{\sigma}$, after it is transformed into the normal form $\hat{\sigma}$ in Theorem 5.6 (i), is divergent. Using the divergence of $\hat{\sigma}$, we will then show that any other normal forms of σ that are in the centralizer of S must be divergent too. This last step requires a convergent solution given by Lemma 5.5.

Our goal is to see a small divisor in a normal form $\tilde{\sigma}$; however they appear as a product. This is more complicated than the situation for the normalizing transformations, where a small divisor appears in a much simple way. In essence, a small divisor problem occurs naturally when one applies a Newton iteration scheme for a convergence proof. For a small divisor to show up in the normal form, we have to go beyond the Newton iteration scheme, measured in the degree or order of approximation in power series. Therefore, we first refine the formulae (5.22).

Lemma 6.1. *Let σ be a holomorphic mapping, given by*

$$\xi'_j = \mu_j^0(\xi\eta)\xi_j + f_j(\xi, \eta), \quad \eta'_j = \nu_j^0(\xi\eta)\eta_j + g_j(\xi, \eta), \quad 1 \leq j \leq p.$$

Here $\boldsymbol{\mu}_j^0(0) = \mu_j = \boldsymbol{\nu}_j^0(0)^{-1}$. Suppose that $\text{ord}(f, g) \geq d \geq 4$ and $I + (f, g) \in \mathcal{C}^c(S)$. There exist unique polynomials U, V of degree at most $2d - 1$ such that $\Psi = I + (U, V) \in \mathcal{C}^c(S)$ transforms σ into

$$\sigma^*: \xi' = \boldsymbol{\mu}(\xi\eta)\xi + \tilde{f}(\xi, \eta), \quad \eta' = \boldsymbol{\nu}(\xi\eta)\eta + \tilde{g}(\xi, \eta)$$

with $I + (\tilde{f}, \tilde{g}) \in \mathcal{C}^c(S)$ and $\text{ord}(\tilde{f}, \tilde{g}) \geq 2d$. Moreover,

$$(6.1) \quad U_{j,PQ} = (\mu^{P-Q} - \mu_j)^{-1}(f_{j,PQ} + \mathcal{U}_{j,PQ}^*(\delta_{\ell-2}, \{\boldsymbol{\mu}^0, \boldsymbol{\nu}^0\}_{\frac{\ell-d}{2}}; \{f, g\}_{\ell-2})),$$

$$(6.2) \quad V_{j,QP} = (\mu^{Q-P} - \mu_j^{-1})^{-1}(g_{j,QP} + \mathcal{V}_{j,QP}^*(\delta_{\ell-2}, \{\boldsymbol{\mu}^0, \boldsymbol{\nu}^0\}_d; \{f, g\}_{\ell-2})),$$

for $2 \leq |P| + |Q| = \ell \leq 2d - 1$ and $\mu^{P-Q} \neq \mu_j$. In particular, $\text{ord}(U, V) \geq d$. Also,

$$(6.3) \quad \boldsymbol{\mu}_{j,P} = \boldsymbol{\mu}_{j,P}^0, \quad 2|P| + 1 < 2d - 1,$$

$$(6.4) \quad \boldsymbol{\mu}_{j,P} = \boldsymbol{\mu}_{j,P}^0 + \{Df_j(U, V)\}_{(P+e_j)P}, \quad 2|P| + 1 = 2d - 1.$$

Assume further that $\boldsymbol{\mu}^0$ satisfies condition L. Then for $2|P| + 1 = 2d + 1$, we have

$$(6.5) \quad \boldsymbol{\mu}_{j,P} = \boldsymbol{\mu}_{j,P}^0 + \mu_j \left\{ 2(U_j V_j)_{PP} + (U_j^2)_{(P+e_j)(P-e_j)} \right\} + \{Df_j(U, V)\}_{(P+e_j)P}.$$

We recall that δ_l is defined by (5.29).

Remark 6.2. Note that (6.3) follows from (5.33). Formulae (6.3), (6.4), (6.5) give us an effective way to compute the Poincaré-Dulac normal form. Although (6.4) contains small divisors, it will be more convenient to associate small divisors to (6.5) when we have 3 elliptic components in σ .

Proof. Let D_i denote ∂_{ζ_i} . Let $Du(\xi, \eta)$ and $Dv(\zeta)$ denote the gradients of two functions. Let us expand both sides of the ξ_j components of $\Psi\sigma^* = \sigma\Psi$ for terms of degree $2d + 2$. For its left-hand side, Corollary 5.3 implies that $\text{ord}(U, V) \geq d$ and we can use $\text{ord} DU_j \cdot (\tilde{f}, \tilde{g}) \geq 2d - 1 + d \geq 2d + 2$ as $d \geq 3$. For its right-hand side, we use (5.16)-(5.19). We obtain

$$(6.6) \quad \boldsymbol{\mu}_j \xi_j + \tilde{f}_j(\xi, \eta) + U_j(\boldsymbol{\mu}\xi, \boldsymbol{\nu}\eta) = f_j(\xi, \eta) + Df_j(\xi, \eta)(U, V) + A_j(\xi, \eta) + (M_j^0 + D\boldsymbol{\mu}_j^0(\eta U + \xi V + UV))(\xi_j + U_j) + O(2d + 2),$$

where $\boldsymbol{\mu}, \boldsymbol{\nu}, \boldsymbol{\mu}^0, \boldsymbol{\nu}^0$ are evaluated at $\xi\eta$ and U, V are evaluated at (ξ, η) .

Since $\tilde{f}(\boldsymbol{\mu}(\xi\eta)\xi, \boldsymbol{\nu}(\xi\eta)\eta) = O(|\xi, \eta|^{2d})$, then (6.1)-(6.2) follow from (5.27)-(5.28), where by Definition 5.1

$$\mathcal{U}_{j,PQ}^*(\cdot; 0) = \mathcal{V}_{j,QP}^*(\cdot; 0) = 0.$$

Next, we refine (5.30) to verify the remaining assertions. We recall from (5.18) the remainders

$$A_j(\xi, \eta) = R_2 \mathbf{\mu}_j^0(\xi\eta; \xi U + \eta V + UV)(\xi_j + U_j) + R_2 f_j(\xi, \eta; U, V).$$

Here by (5.12), we use the Taylor remainder formula

$$R_2 f(x; y) = 2 \int_0^1 (1-t) \sum_{|\alpha|=2} \frac{1}{\alpha!} \partial^\alpha f(x+ty) y^\alpha dt.$$

Since $\text{ord}(U, V) \geq d$, $\text{ord}(f, g) \geq d$, and $d \geq 4$, then A_j , defined by (5.18), satisfies

$$A_j(\xi, \eta) = O(|(\xi, \eta)|^{2d+2}).$$

Recall that $f_j(\xi, \eta)$ and $U_j(\xi, \eta)$ do not contain terms of the form $\xi_j \xi^P \eta^P$, while $g_j(\xi, \eta)$ and $V_j(\xi, \eta)$ do not contain terms of the form $\eta_j \xi^P \eta^P$. Comparing both sides of (6.6) for coefficients of $\xi_j \xi^P \eta^P$ with $|P| = d-1$, we get (6.4).

Assume now that condition L holds for $\mathbf{\mu}^0$. Assume that $i \neq j$. Then $D_i \mathbf{\mu}_j^0(\xi\eta) = O(|\xi\eta|)$. Thus, $D_i \mathbf{\mu}_j^0(\xi\eta) \eta_i U_i(\xi, \eta)$ and $D_i \mathbf{\mu}_j^0(\xi\eta) \xi_i V_i(\xi, \eta)$ do not contain terms of $\xi^P \eta^P$, and

$$D_i \mathbf{\mu}_j^0(\xi\eta) \xi_i U_i(\xi, \eta) V_i(\xi, \eta) = O(2d+3).$$

Since $(\tilde{f}, \tilde{g}) \in \mathcal{C}_2^s(S)$ and $(\tilde{f}, \tilde{g}) = O(2d)$, then $\tilde{f}_j(\mathbf{\mu}(\xi\eta)\xi, \mathbf{v}(\xi\eta)\eta)$ does not contain terms $\xi^P \eta^P \xi_j$ for $2|P|+1 = 2d+1$. Now (6.5) follows from a direction computation. q.e.d.

Set $|\delta_N(\mu)| := \max\{|\nu| : \nu \in \delta_N(\mu)\}$ for

$$\delta_N(\mu) = \bigcup_{j=1}^p \left\{ \mu_j, \mu_j^{-1}, \frac{1}{\mu^P - \mu_j} : P \in \mathbf{Z}^p, P \neq e_j, |P| \leq N \right\}.$$

Definition 6.3. We say that $\mu^{P_*-Q_*} - \mu_j$ and $\mu^{Q_*-P_*} - \mu_j^{-1}$ are small divisors of height N , if there exists a partition

$$\bigcup_{i=1}^p \left\{ |\mu^{P-Q} - \mu_i| : P, Q \in \mathbf{N}^p, |P| + |Q| \leq N, \mu^{P-Q} \neq \mu_i \right\} = S_N^0 \cup S_N^1$$

with $|\mu^{P_*-Q_*} - \mu_j| \in S_N^0$ and $S_N^1 \neq \emptyset$ such that

$$\max S_N^0 < C \min S_N^0, \quad \max S_N^0 < (\min S_N^1)^{L_N} < 1.$$

Here C depends only on an upper bound of $|\mu|$ and $|\mu|^{-1}$ and

$$L_N \geq N.$$

If $|\mu^{P_*-Q_*} - \mu_j|$ is in S_N^0 and if $P_*, Q_* \in \mathbf{N}^p$, we call $|P_* - Q_*|$ the degree of the small divisors $\mu^{P_*-Q_*} - \mu_j$ and $\mu^{Q_*-P_*} - \mu_j^{-1}$.

To avoid confusion, let us call $\mu^{P^*-Q^*} - \mu_j$ that appear in S_N^0 the *exceptional* small divisors. These small divisors have been used by Cremer [8] and Siegel [21]. The degree and height play different roles in computation. The height serves as the maximum degree of all small divisors that need to be considered in computation.

Roughly speaking, the quantities in S_N^0 are comparable but they are much smaller than the ones in S_N^1 . We will construct μ for any prescribed sequence of positive integers L_N so that

$$\max S_N^0 < (\min S_N^1)^{L_N} < 1$$

for a subsequence $N = N_k$ tending to ∞ . Furthermore, to use the small divisors we will identify all exceptional small divisors of height $2N_k + 1$ and all degrees of the exceptional small divisors with N_k being the smallest.

We start with the following lemma which gives us small divisors that decay as rapidly as we wish.

Lemma 6.4. *Let L_k be a strictly increasing sequence of positive integers. There exist a real number $\nu \in (0, 1/2)$ and a sequence $(p_k, q_k) \in \mathbf{N}^2$ such that $e, 1, \nu$ are linearly independent over \mathbf{Q} , and*

$$(6.7) \quad |q_k \nu - p_k - e| \leq \Delta(p_k, q_k)^{L_{p_k+q_k}},$$

$$(6.8) \quad \Delta(p_k, q_k) = \min \left\{ \frac{1}{2}, |q\nu - p - re| : 0 < |r| + |q| < 3(q_k + 1), \right. \\ \left. (p, q, r) \neq 0, \pm(p_k, q_k, 1), \pm 2(p_k, q_k, 1) \right\}.$$

Proof. We consider two increasing sequences $\{m_k\}_{k=1}^\infty, \{n_k\}_{k=1}^\infty$ of positive integers, which are to be chosen. For $k = 1, 2, \dots$, we set

$$\nu = \nu_k + \nu'_k, \quad \nu_k = \sum_{\ell=1}^k \frac{1}{m_\ell!} \sum_{j=0}^{n_\ell} \frac{1}{j!}, \quad \nu'_k = \sum_{\ell>k} \frac{1}{m_\ell!} \sum_{j=0}^{n_\ell} \frac{1}{j!}, \\ q_k = m_k!.$$

We choose $m_k > (m_\ell)!(n_\ell!)$ for $k > \ell$ and decompose

$$q_k \nu = p_k + e_k + e'_k, \\ p_k = m_k! \nu_{k-1} \in \mathbf{N}, \quad e_k = \sum_{\ell=0}^{n_k} \frac{1}{k!}, \quad e'_k = m_k! \nu'_k.$$

We have $e'_k < m_k! \sum_{\ell>k} \frac{e}{m_\ell!}$ and

$$q_k \nu = p_k + e + e'_k - \sum_{\ell=n_k+1}^{\infty} \frac{1}{\ell!}, \\ (6.9) \quad |q_k \nu - p_k - e| \leq m_k! \nu'_k + \sum_{\ell=n_k+1}^{\infty} \frac{1}{\ell!} < \{12(3(q_k + 1)^3)!\}^{-L_{p_k+q_k}}.$$

Here $(6.9)_k$ is achieved by choosing $(m_2, n_1), \dots, (m_{k+1}, n_k)$ successively. Clearly we can get $0 < \nu < 1/2$ if m_1 is sufficiently large.

Next, we want to show that $re + p + q\nu \neq 0$ for all integers p, q, r with $(p, q, r) \neq (0, 0, 0)$. Otherwise, we rewrite $-m_k!p = m_k!(q\nu + re)$ as

$$-m_k!p = qp_k + r \sum_{j=0}^{m_k} \frac{m_k!}{j!} + qe + q \left(e'_k - \sum_{\ell=n_k+1}^{\infty} \frac{1}{\ell!} \right) + r \sum_{j>m_k} \frac{m_k!}{j!}.$$

The left-hand side is an integer. On the right-hand side, the first two terms are integers, qe is a fixed irrational number, and the rest terms tend to 0 as $k \rightarrow \infty$. We get a contradiction.

To verify (6.7), we need to show that for each tuple (p, q, r) satisfying (6.8),

$$(6.10) \quad |q\nu - p - re| \geq |q_k\nu - p_k - e|^{\frac{1}{L_{p_k+q_k}}}.$$

We first note the following elementary inequality

$$(6.11) \quad |p + qe| \geq \frac{1}{(q-1)!} \min \left\{ 3 - e, \frac{1}{q+1} \right\}, \quad p, q \in \mathbf{Z}, \quad q \geq 1.$$

Indeed, the inequality holds for $q = 1$. For $q \geq 2$ we have $q!e = m + \epsilon$ with $m \in \mathbf{N}$ and

$$\epsilon := \sum_{k=q+1}^{\infty} \frac{q!}{k!} > \frac{1}{q+1}.$$

Furthermore, $1 - \epsilon > 1 - \frac{2}{q+1} = \frac{q-1}{q+1}$ as

$$\epsilon < \frac{1}{q+1} + \sum_{k \geq q+2} \frac{1}{k(k-1)} = \frac{2}{q+1}.$$

We may assume that $q \geq 0$. If $q = 0$, then $|r| < 3q_k + 3$ by condition in (6.8) and hence $|p + re| \geq \frac{1}{(3q_k+4)!}$ by (6.11). Now (6.10) follows from (6.9). Assume that $q > 0$. We have

$$(6.12) \quad \begin{aligned} | -q\nu + p + re | &\geq \left| -q \frac{p_k + e}{q_k} + p + re \right| - q \frac{|e + p_k - q_k\nu|}{q_k} \\ &= \left| \frac{q_k p - qp_k}{q_k} + \frac{rq_k - q}{q_k} e \right| - q \frac{|e + p_k - q_k\nu|}{q_k}. \end{aligned}$$

We first verify that $q_k p - qp_k$ and $q - rq_k$ do not vanish simultaneously. Assume that both are zero. Then $(p, q, r) = r(p_k, q_k, 1)$. Thus $|r| \neq 1, 2$, and $|r| \geq 3$ by conditions in (6.8); we obtain $|r| + |q| \geq 3(|q_k| + 1)$, a contradiction. Therefore, either $q_k p - qp_k$ or $rq_k - q$ is not zero. By

(6.11) and (6.12),

$$\begin{aligned} |-q\nu + p + re| &\geq \frac{1}{q_k} \cdot \frac{1}{3} \cdot \frac{1}{(|rq_k - q| + 1)!} - q \frac{|e + p_k - q_k\nu|}{q_k} \\ &\geq \frac{1}{(3q_k + 4)^{2!}} - 4|e + p_k - q_k\nu|. \end{aligned}$$

Using (6.9) twice, we obtain the next two inequalities:

$$|-q\nu + p + re| \geq \frac{1}{2} \left\{ (3q_k + 4)^{2!} \right\}^{-1} \geq |p_k + e - q_k\nu|^{\frac{1}{L_{p_k+q_k}}}.$$

The two ends give us (6.10).

q.e.d.

We now reformulate the above lemma as follows.

Lemma 6.5. *Let L_k be a strictly increasing sequence of positive integers. Let $\nu \in (0, 1/2)$, and let p_k and q_k be positive integers as in Lemma 6.4. Set $(\mu_1, \mu_2, \mu_3) := (e^{-1}, e^\nu, e^e)$. Then*

$$(6.13) \quad |\mu^{P_k} - \mu_3| \leq (C\Delta^*(P_k))^{L_{|P_k|}}, \quad P_k = (p_k, q_k, 0),$$

$$(6.14) \quad \Delta^*(P_k) = \min_j \left\{ |\mu^R - \mu_j| : R \in \mathbf{Z}^3, |R| \leq 2(q_k + p_k) + 1, \right. \\ \left. R - e_j \neq 0, \pm(p_k, q_k, -1), \pm 2(p_k, q_k, -1) \right\}.$$

Here C does not depend on k . Moreover, all exceptional small divisors of height $2|P_k| + 1$ have degree at least $|P_k|$. Moreover, $\mu^{P_k} - \mu_3$ is the only exceptional small divisor of degree $|P_k|$ and height $2|P_k| + 1$.

In the definition of $\Delta^*(P_k)$, equivalently we require that

$$R \neq P_k, R_k^1, R_k^2, R_k^3$$

with $R_k^1 := -P_k + 2e_3$, $R_k^2 := 2P_k - e_3$, and $R_k^3 := -2P_k + 3e_3$. Note that $|R_k^1| = |P_k| + 2$, $|R_k^2| = 2|P_k| + 1$, and $|R_k^3| = 2|P_k| + 3$ are bigger than $|P_k|$, i.e. the degree of the exceptional small divisor $\mu^{P_k} - \mu_3$. Each $\mu^{R_k^i} - \mu_3$ is a small divisor comparable with $\mu^{P_k} - \mu_3$. Finally, $\Delta^*(P_k)$ tends to zero as $|P_k| \rightarrow \infty$. Let us set $N := 2|P_k| + 1$, and

$$\begin{aligned} S_N^0 &:= \left\{ |\mu^{P_k} - \mu_3|, |\mu^{R_k^1} - \mu_3|, |\mu^{R_k^2} - \mu_3|, |\mu^{R_k^3} - \mu_3| \right\}, \\ S_N^1 &:= \bigcup_j \left\{ |\mu^R - \mu_j| : R \in \mathbf{Z}^3, |R| \leq 2(q_k + p_k) + 1, \right. \\ &\quad \left. R - e_j \neq 0, \pm(p_k, q_k, -1), \pm 2(p_k, q_k, -1) \right\}. \end{aligned}$$

This implies that the last paragraph of Lemma 6.5 holds when the L_N in Definition 6.3, denoted it by L'_N , takes the value $L'_N = L_{2N+1}$, while L_N is prescribed in Lemma 6.5.

Proof. By Lemma 6.4, we find a real number $\nu \in (0, 1/2)$ and positive integers p_k, q_k such that $e, 1, \nu$ are linearly independent over \mathbf{Q} and

$$(6.15) \quad |q_k \nu - e - p_k| \leq \Delta(p_k, q_k)^{L|P_k|},$$

$$\Delta(p_k, q_k) = \min \{ |q\nu - re - p| : 0 < |r| + |q| < 3(q_k + 1), \\ (p, q, r) \neq 0, \pm(p_k, q_k, 1), \pm 2(p_k, q_k, 1) \}.$$

Note that μ_1, μ_2, μ_3 are positive and non-resonant. We have

$$|\mu^{P_k} - \mu_3| = |\mu_3| \cdot |e^{q_k \nu - p_k - e} - 1|.$$

Let $\nu^* := (-1, \nu, e)$. If $|R \cdot \nu^* - \nu_j^*| < 2$, then by the intermediate value theorem

$$e^{-2} |\mu_j| |R \cdot \nu^* - \nu_j^*| \leq |\mu^R - \mu_j| \leq e^2 |\mu_j| |R \cdot \nu^* - \nu_j^*|.$$

If $R \cdot \nu^* - \nu_j^* > 2$ or $R \cdot \nu^* - \nu_j^* < -2$, we have

$$|\mu^R - \mu_j| \geq e^{-2} |\mu_j|.$$

Thus, we can restate the properties of ν^* as follows:

$$|\mu^{-(p_k, q_k, 0)} - \mu_3| \leq C'(C' \tilde{\Delta}(p_k, q_k))^{L|P_k|},$$

$$\tilde{\Delta}(p_k, q_k) = \min \{ |\mu^{(p, q, r)} - 1| : 0 < |r| + |q| < 3(q_k + 1), \\ (p, q, r) \neq 0, \pm(p_k, q_k, -1), \pm 2(p_k, q_k, -1) \}.$$

Recall that $0 < \nu < 1/2$. By (6.15), we have $|q_k \nu - e - p_k| < 1$. Since p_k, q_k are positive, then $p_k < \nu q_k < q_k/2$. Assume that $|\mu^R - \mu_j| = \Delta^*(P_k)$, $|R| \leq 2(p_k + q_k) + 1$, and

$$R - e_j \neq 0, \pm(p_k, q_k, -1), \pm 2(p_k, q_k, -1).$$

Set $R' := R - e_j$ and $(p, q, r) := R'$. Then $\Delta^*(P_k) = |\mu_j| |\mu^{R'} - 1|$. Also, $|r| + |q| \leq |R'| \leq |R| + 1 \leq 2(p_k + q_k) + 2 \leq q_k + 2q_k + 2 < 3(q_k + 1)$. This shows that $|\mu^{Q'} - 1| \geq \tilde{\Delta}(p_k, q_k)$. We obtain $\Delta^*(P_k) \geq \mu_j \Delta(p, q_k)$. We have verified (6.13). For the remaining assertions, see the remark following the lemma. q.e.d.

In the above we have retained $\mu_j > 0$ which are sufficient to realize $\mu_1, \mu_2, \mu_3, \mu_1^{-1}, \mu_2^{-1}, \mu_3^{-1}$ as eigenvalues of σ for an elliptic complex tangent. Indeed, with $0 < \mu_1 < 1$, interchanging ξ_1 and η_1 preserves ρ and changes the (ξ_1, η_1) components of σ into $(\mu_1^{-1} \xi_1, \mu_1 \eta_1)$.

We are ready to prove Theorem 1.4, which is restated here:

Theorem 6.6. *There exists a non-resonant elliptic real analytic 3-submanifold M in \mathbf{C}^6 such that M admits the maximum number of deck transformations and all Poincaré-Dulac normal forms of the σ associated to M are divergent.*

Proof. We will not construct the real analytic submanifold M directly. Instead, we will construct a family of involutions $\{\tau_{11}, \dots, \tau_{1p}, \rho\}$ so that all Poincaré-Dulac normal forms of σ are divergent. By the realization in Proposition 2.1, we get the desired submanifold.

We first give an outline of the proof. To prove the theorem, we first deal with the associated σ and its normal form $\tilde{\sigma}$, which belongs to the centralizer of S , the linear part of σ at the origin. Thus σ^* has the form

$$\sigma^*: \xi'_i = \mu_j(\xi\eta)\xi_j, \quad \eta'_j = \nu_j(\xi\eta)\eta_j, \quad j = 1, \dots, p.$$

We assume that μ satisfies condition L. We then normalize σ^* into the normal form $\hat{\sigma}$ stated in Theorem 5.6 (i). (In Lemma 6.1 we take F to be (5.35) and \hat{F} to be defined by (5.35) with \hat{M}_i replacing M_i .) We will show that $\hat{\sigma}$ is divergent if σ is well chosen. By Theorem 5.6 (iii), all normal forms of σ in the centralizer of S are divergent. To get σ^* , we use the normalization of Proposition 5.2 (i). To get $\hat{\sigma}$, we normalize further using Lemma 5.5. To find a divergent $\hat{\sigma}$, we need to tie the normalizations of two formal normal forms together, by keeping track of the small divisors in the two normalizations.

We will start with our initial pair of involutions $\{\tau_1^0, \tau_2^0\}$ satisfying $\tau_2^0 = \rho\tau_1^0\rho$ such that σ^0 is a third order perturbation of S . We require that τ_1^0 be the composition of $\tau_{11}^0, \dots, \tau_{1p}^0$. The latter can be realized by a real analytic submanifold by using Proposition 2.1. We will then perform a sequence of holomorphic changes of coordinates φ_k such that $\tau_1^k = \varphi_k\tau_1^{k-1}\varphi_k^{-1}$, $\tau_2^k = \rho\tau_1^k\rho$, and $\sigma^k = \tau_1^k\tau_2^k$. By abuse of notation, τ_i^k, σ^k , etc. do not stand for iterating the maps k times. Each φ_k is tangent to the identity to order d_k . For a suitable choice of φ_k , we want to show that the coefficients of order d_k of the normal form of σ^k increase rapidly to the effect that the coefficients of the normal form of the limit mapping σ^∞ increase rapidly too. Here we will use the exceptional small divisors to achieve the rapid growth of the coefficients of the normal forms. Roughly speaking, the latter requires us to keep track the rapid growth for a sequence of coefficients in the normal form in a sequence of two-step normalizations. Recall from Lemma 6.1 that if we have

$$\sigma: \xi'_j = \mu_j(\xi\eta)\xi_j + f_j(\xi, \eta), \quad \eta'_j = \nu_j(\xi\eta)\eta_j + g_j(\xi, \eta), \quad 1 \leq j \leq p$$

with $\text{ord}(f, g) \geq d \geq 4$ and $(f, g) \in \mathcal{C}_2^d(S)$, then there is a polynomial mapping $\Psi: \xi' = \xi + \hat{U}(\xi, \eta), \eta' = \eta + \hat{V}(\xi, \eta)$ in $\mathcal{C}^c(S)$ that has order d and degree at most $2d - 1$ such that for the new mapping

$$\hat{\sigma} := \Psi^{-1}\sigma\Psi: \xi'_j = \hat{\mu}_j(\xi\eta)\xi_j + \hat{f}_j(\xi, \eta), \quad \eta'_j = \hat{\nu}_j(\xi\eta)\eta_j + \hat{g}_j(\xi, \eta),$$

the coefficients of $\boldsymbol{\mu}_j(\xi\eta)\xi_j$ of degree $2d + 1$ have the form

$$(6.16) \quad \hat{\boldsymbol{\mu}}_{j,P} = \boldsymbol{\mu}_{j,P} + \mu_j \left\{ 2(\hat{U}_j \hat{V}_j)_{PP} + (\hat{U}_j^2)_{(P+e_j)(P-e_j)} \right\} \\ + \{Df_j(\hat{U}, \hat{V})\}_{(P+e_j)P}.$$

It is crucial that for suitable multi-indices, *both* \hat{U}_j, \hat{V}_j contain the exceptional small divisors of degree d as formulated in Lemma 6.5 (see also Definition 6.3). Although $Df_j(\hat{U}, \hat{V})$ contains (exceptional) small divisors, they can only appear at most once in each term, provided f_j contains no small divisor of degree d . The formula (6.16) appears as simple as it is, it requires that σ has been normalized to degree d . To achieve such a σ , we need to use a preliminary change of coordinates $\Phi: \xi' = \xi + U(\xi, \eta), \eta' = \eta + V(\xi, \eta)$ via polynomials of degree less than d . The Φ depends only on small divisors of degree $< d$, but none of them are exceptionally small. Therefore, by composing $\Psi\Phi$, we obtain (6.16) where small divisors of degree $< d$ are absorbed into terms $\boldsymbol{\mu}_{j,P}$ and the products of two exceptional small divisors in (6.16), if they exist, dominate the other terms in $\hat{\boldsymbol{\mu}}_{j,P}$. Of course, we need to apply a sequence of transformations Φ, Ψ and we should leave the coefficients of a certain degree unchanged in the process once they become large, which are possible by Corollary 5.4.

We now present the proof. Let $\sigma^0 = \tau_1^0 \tau_2^0$, $\tau_2^0 = \rho \tau_1^0 \rho$, and

$$\tau_1^0: \xi'_j = \Lambda_{1j}^0(\xi\eta)\eta_j, \quad \eta'_j = (\Lambda_{1j}^0(\xi\eta))^{-1}\xi_j, \\ \sigma^0: \xi'_j = (\Lambda_{1j}^0(\xi\eta))^2 \xi_j, \quad \eta'_j = (\Lambda_{1j}^0(\xi\eta))^{-2}(\xi\eta)\eta_j.$$

Since we consider the elliptic case, we require that $(\Lambda_{1j}^0(\xi\eta))^2 = \mu_j e^{\xi_j \eta_j}$. So $\zeta \rightarrow (\Lambda_{1j}^0)^2(\zeta)$ is biholomorphic. Recall that σ^0 can be realized by $\{\tau_{11}^0, \dots, \tau_{1p}^0, \rho\}$. We will take

$$(6.17) \quad \varphi_k: \xi'_j = (\xi - h^{(k)}(\xi), \eta), \quad \text{ord } h^{(k)} = d_k > 3,$$

$$(6.18) \quad d_k \geq 2d_{k-1}, \quad |h_P^{(k)}| \leq 1.$$

We will also choose each $h_j^{(k)}(\xi)$ to have one monomial only. Let $\Delta_r := \Delta_r^3$ denote the polydisc of radius r . Let $\|\cdot\|$ be the sup norm on \mathbf{C}^3 . Let $H^{(k)}(\xi) = \xi - h^{(k)}(\xi)$ and we first verify that $H_k = H^{(k)} \circ \dots \circ H^{(1)}$ converges to a holomorphic function on the polydisc Δ_{r_1} for $r_1 > 0$ sufficiently small; consequently, $\varphi_k \circ \dots \circ \varphi_1$ converges to a germ of holomorphic map φ^∞ at the origin. Note that $H^{(k)}$ sends Δ_{r_k} into $\Delta_{r_{k+1}}$ for $r_{k+1} = r_k + r_k^{d_k}$. We want to show that when r_1 is sufficiently small,

$$(6.19) \quad r_k \leq s_k := \left(2 - \frac{1}{k}\right)r_1.$$

It holds for $k = 1$. Let us show that $r_{k+1}/r_k - 1 \leq \theta_k := s_{k+1}/s_k - 1$, i.e.

$$r_k^{d_k-1} \leq \theta_k = \frac{1}{(k+1)(2k-1)}.$$

We have $(2r_1)^{d_k-1} \leq (2r_1)^k$ when $0 < r_1 < 1/2$. Fix r_1 sufficiently small such that $(2r_1)^k < \frac{1}{(k+1)(2k-1)}$ for all k . By induction, we obtain (6.19) for all k . In particular, we have $\|h^{(k)}(\xi)\| \leq \|\xi\| + \|H^{(k)}(\xi)\| \leq 2r_{k+1}$ for $\|\xi\| < r_k$. To show the convergence of H_k , we write $H_k - H_{k-1}(\xi) = -h^{(k)} \circ H_{k-1}$. By the Schwarz lemma, we obtain

$$\|h^{(k)} \circ H_{k-1}(\xi)\| \leq \frac{2r_{k+1}}{r_1^{d_k}} \|\xi\|^{d_k}, \quad \|\xi\| < r_1.$$

Note that the above estimate is uniform under conditions (6.17)-(6.18). Therefore, H_k converges to a holomorphic function on $\|\xi\| < r_1$.

Throughout the proof, we make initial assumptions that d_k and $h^{(k)}$ satisfy (6.17)-(6.18), $e^{-1} \leq \mu_j \leq e^e$, and $\mu^Q \neq 1$ for $Q \in \mathbf{Z}^3$ with $Q \neq 0$. Set $\sigma^k = \tau_1^k \tau_2^k$, $\tau_2^k = \rho \tau_1^k \rho$, and

$$\tau_1^k = \varphi_k \tau_1^{k-1} \varphi_k^{-1}.$$

We want σ^k not to be holomorphically equivalent to σ^{k-1} . Thus we have chosen a φ_k that does not commute with ρ in general. Note that σ^k is still generated by a real analytic submanifold; indeed, when $\tau_i^{k-1} = \tau_{i1}^{k-1} \cdots \tau_{ip}^{k-1}$ and $\tau_{2j}^{k-1} = \rho \tau_{1j}^{k-1} \rho$, we still have the same identities if the superscript $k-1$ is replaced by k and τ_{1j}^k equals $\varphi_k \tau_{1j}^{k-1} \varphi_k^{-1}$. It is clear that $\sigma^k = \sigma^{k-1} + O(d_k)$. As power series, we have

$$(6.20) \quad \sigma^\ell = \sigma^{k-1} + O(d_k), \quad k \leq \ell \leq \infty.$$

Note that as limits in convergence, $\tau_{ij}^\infty = \lim_{k \rightarrow \infty} \tau_{ij}^k$, $\tau_i^\infty = \lim_{k \rightarrow \infty} \tau_i^k$ and $\sigma^\infty = \lim_{k \rightarrow \infty} \sigma^k$ satisfy

$$\tau_{2j}^\infty = \rho \tau_{1j}^\infty \rho, \quad \tau_i^\infty = \tau_{i1}^\infty \cdots \tau_{ip}^\infty, \quad \tau_2^\infty = \rho \tau_1^\infty \rho, \quad \sigma^\infty = \tau_1^\infty \tau_2^\infty.$$

Of course, $\{\tau_{11}, \dots, \tau_{1p}, \rho\}$ satisfies all the conditions that ensure it can be realized by a real analytic submanifold.

We know that σ^∞ does not have a unique normal form in the centralizer S . Therefore, we will choose a procedure that arrives at a unique formal normal form in S . We show that this unique normal form is divergent; and hence by Theorem 5.6 (iii) any normal form of σ that is in the centralizer of S must diverge.

We now describe the procedure. For a formal mapping F , we have a unique decomposition

$$F = NF + N^c F, \quad NF \in \mathcal{C}(S), \quad N^c F \in \mathcal{C}^c(S).$$

Set $\hat{\sigma}_0^\infty = \sigma^\infty$. For $k = 0, 1, \dots$, we take a normalized polynomial map $\Phi_k \in \mathcal{C}_2^c(S)$ of degree less than d_k such that $\sigma_k^\infty := \Phi_k^{-1} \hat{\sigma}_k^\infty \Phi_k$ is normalized up to degree $d_k - 1$. Specifically, we require that

$$\deg \Phi_k \leq d_k - 1, \quad \Phi_k \in \mathcal{C}^c(S); \quad N^c \sigma_k^\infty(\xi, \eta) = O(d_k).$$

Take a normalized polynomial map Ψ_{k+1} such that Ψ_{k+1} and $\hat{\sigma}_{k+1}^\infty := \Psi_{k+1}^{-1} \sigma_k^\infty \Psi_{k+1}$ satisfy

$$\deg \Psi_{k+1} \leq 2d_k - 1; \quad \Psi_{k+1} \in \mathcal{C}_2^c(S), \quad N^c \hat{\sigma}_{k+1}^\infty = O(2d_k).$$

We can repeat this for $k = 0, 1, \dots$. Thus we apply two sequences of normalization as follows

$$\hat{\sigma}_{k+1}^\infty = \Psi_{k+1}^{-1} \circ \Phi_k^{-1} \cdots \Psi_1^{-1} \circ \Phi_0^{-1} \circ \sigma^\infty \circ \Phi_0 \circ \Psi_1 \cdots \Phi_k \circ \Psi_{k+1}.$$

We will show that $\Psi_{k+1} = I + O(d_k)$ and $\Phi_k = I + O(2d_{k-1})$. This shows that the sequence $\Phi_0 \Psi_1 \cdots \Phi_k \Psi_{k+1}$ defines a formal biholomorphic mapping Φ so that

$$(6.21) \quad \hat{\sigma}^\infty := \Phi^{-1} \sigma^\infty \Phi$$

is in a normal form. Finally, we need to combine the above normalization with the normalization for the unique normal form in Lemma 5.5. We will show that the unique normal form diverges.

Let us recall previous results to show that Φ_k, Ψ_{k+1} are uniquely determined. Set

$$(6.22) \quad \hat{\sigma}_k^\infty: \begin{cases} \xi'_j = \hat{\mu}_j^{(k)}(\xi\eta)\xi_j + \hat{f}_j^{(k)}(\xi, \eta), \\ \eta'_j = \hat{\nu}_j^{(k)}(\xi\eta)\eta_j + \hat{g}_j^{(k)}(\xi, \eta), \end{cases}$$

$$(6.23) \quad (\hat{f}^{(k)}, \hat{g}^{(k)}) \in \mathcal{C}_2^c(S).$$

Recall that $\hat{\sigma}_0 = \sigma^\infty$. Assume that we have achieved

$$(6.24) \quad (\hat{f}^{(k)}, \hat{g}^{(k)}) = O(2d_{k-1}).$$

Here we take $d_{-1} = 2$ so that (6.22)-(6.24) hold for $k = 0$. By Proposition 5.2, there is a unique normalized mapping $\tilde{\Phi}_k$ that transforms $\hat{\sigma}_k^\infty$ into a normal form. We denote by Φ_k the truncated polynomial mapping of $\tilde{\Phi}_k$ of degree $d_k - 1$. We write

$$\begin{aligned} \Phi_k: \xi' &= \xi + U^{(k)}(\xi, \eta), \quad \eta' = \eta + V^{(k)}(\xi, \eta), \\ (U^{(k)}, V^{(k)}) &= O(2), \quad \deg(U^{(k)}, V^{(k)}) \leq d_k - 1. \end{aligned}$$

By Corollary 5.3, Φ_k satisfies

$$(6.25) \quad \sigma_k^\infty = \Phi_k^{-1} \hat{\sigma}_k^\infty \Phi_k: \begin{cases} \xi'_j = \mu_j^{(k)}(\xi\eta)\xi_j + f_j^{(k)}(\xi, \eta), \\ \eta'_j = \nu_j^{(k)}(\xi\eta)\eta_j + g_j^{(k)}(\xi, \eta), \end{cases}$$

$$(f^{(k)}, g^{(k)}) \in \mathcal{C}_2^c(S), \quad \text{ord}(f^{(k)}, g^{(k)}) \geq d_k.$$

In fact, by (5.27)-(5.28) (or (5.25)-(5.26)), we have

$$(6.26) \quad U_{j,PQ}^{(k)} = (\mu^{P-Q} - \mu_j)^{-1} \left\{ f_{j,PQ}^{(k)} + \mathcal{U}_{j,PQ}(\delta_{d-1}, \{\hat{\boldsymbol{\mu}}^{(k)}, \hat{\boldsymbol{\nu}}^{(k)}\}_{\lfloor \frac{d-1}{2} \rfloor}; \{\hat{f}^{(k)}, \hat{g}^{(k)}\}_{d-1}) \right\},$$

$$(6.27) \quad V_{j,QP}^{(k)} = (\mu^{Q-P} - \mu_j^{-1})^{-1} \left\{ g_{j,QP}^{(k)} + \mathcal{V}_{j,QP}(\delta_{d-1}, \{\hat{\boldsymbol{\mu}}^{(k)}, \hat{\boldsymbol{\nu}}^{(k)}\}_{\lfloor \frac{d-1}{2} \rfloor}; \{\hat{f}^{(k)}, \hat{g}^{(k)}\}_{d-1}) \right\},$$

for $|P| + |Q| = d < d_k$ and $\mu^{P-Q} \neq \mu_j$. By (5.30)-(5.31) (or (5.22)-(5.23)), we have

$$(6.28) \quad \boldsymbol{\mu}_P^{(k)} = \hat{\boldsymbol{\mu}}_P^{(k)} + \mathcal{M}_P(\delta_{2|P|-1}, \{\hat{\boldsymbol{\mu}}^{(k)}, \hat{\boldsymbol{\nu}}^{(k)}\}_{|P|-1}; \{\hat{f}^{(k)}, \hat{g}^{(k)}\}_{2|P|-1}),$$

$$(6.29) \quad \boldsymbol{\nu}_P^{(k)} = \hat{\boldsymbol{\nu}}_P^{(k)} + \mathcal{N}_P(\delta_{2|P|-1}, \{\hat{\boldsymbol{\mu}}^{(k)}, \hat{\boldsymbol{\nu}}^{(k)}\}_{|P|-1}; \{\hat{f}^{(k)}, \hat{g}^{(k)}\}_{2|P|-1})$$

for $2|P|-1 < d_k$. Recall that $\mathcal{U}_{j,PQ}$, $\mathcal{V}_{j,QP}$, $\mathcal{M}_{j,P}$, and $\mathcal{N}_{j,P}$ are universal polynomials in their variables. In notation defined by Definition 5.1,

$$\mathcal{U}_{j,PQ}(\boldsymbol{\cdot}; 0) = \mathcal{V}_{j,QP}(\boldsymbol{\cdot}; 0) = 0, \quad \mathcal{M}_P(\boldsymbol{\cdot}; 0) = \mathcal{N}_P(\boldsymbol{\cdot}; 0) = 0.$$

We apply (6.26)-(6.27) for $d < 2d_{k-1} \leq d_k$ and (6.28)-(6.29) for $2|P|-1 < 2d_{k-1} \leq d_k$ to obtain

$$(6.30) \quad \Phi_k - I = (U^{(k)}, V^{(k)}) = O(2d_{k-1}),$$

$$(6.31) \quad \boldsymbol{\mu}_P^{(k)} = \hat{\boldsymbol{\mu}}_P^{(k)}, \quad \boldsymbol{\nu}_P^{(k)} = \hat{\boldsymbol{\nu}}_P^{(k)}, \quad |P| \leq d_{k-1}.$$

In fact, by Corollary 5.4, the above holds for $|P| < 2d_{k-1} - 1$.

By Lemma 6.1, there is a unique normalized polynomial mapping

$$\begin{aligned} \Psi_{k+1}(\xi, \eta) &= (\xi + \hat{U}^{(k+1)}(\xi, \eta), \eta + \hat{V}^{(k+1)}(\xi, \eta)), \\ (\hat{U}^{(k+1)}, \hat{V}^{(k+1)}) &\in \mathcal{C}_2^c(S), \\ (\hat{U}^{(k+1)}, \hat{V}^{(k+1)}) &= O(2), \quad \deg(\hat{U}^{(k+1)}, \hat{V}^{(k+1)}) \leq 2d_k - 1 \end{aligned}$$

such that $\hat{\sigma}_{k+1}^\infty = \Psi_{k+1}^{-1} \Phi_k^{-1} \sigma_k^\infty \Phi_k \Psi_{k+1}$ satisfies the following:

$$\begin{aligned} \hat{\sigma}_{k+1}^\infty : \xi'_j &= \hat{\boldsymbol{\mu}}_j^{(k+1)}(\xi\eta)\xi_j + \hat{f}_j^{(k+1)}, \quad \eta'_j = \hat{\boldsymbol{\nu}}_j^{(k+1)}(\xi\eta)\eta_j + \hat{g}_j^{(k+1)}, \\ (\hat{f}^{(k+1)}, \hat{g}^{(k+1)}) &\in \mathcal{C}_2^c(S), \quad \text{ord}(\hat{f}^{(k+1)}, \hat{g}^{(k+1)}) \geq 2d_k. \end{aligned}$$

By (6.1)-(6.2), we know that

$$(6.32) \quad \hat{U}_{j,PQ}^{(k+1)} = (\mu^{P-Q} - \mu_j)^{-1} \left\{ f_{j,PQ}^{(k)} + \mathcal{U}_{j,PQ}^*(\delta_{\ell-1}, \{\boldsymbol{\mu}^{(k)}, \boldsymbol{\nu}^{(k)}\}_{\lfloor \frac{\ell-1}{2} \rfloor}; \{f^{(k)}, g^{(k)}\}_{\ell-1}) \right\},$$

$$(6.33) \quad \hat{V}_{j,QP}^{(k+1)} = (\mu^{Q-P} - \mu_j^{-1})^{-1} \left\{ g_{j,QP}^{(k)} + \mathcal{V}_{j,QP}^*(\delta_{\ell-1}, \{\boldsymbol{\mu}^{(k)}, \boldsymbol{\nu}^{(k)}\}_{\lfloor \frac{\ell-1}{2} \rfloor}; \{f^{(k)}, g^{(k)}\}_{\ell-1}) \right\},$$

for $d_k \leq |P| + |Q| = \ell \leq 2d_k - 1$ and $\mu^{P-Q} \neq \mu_j$. Recall that $\mathcal{U}_{j,PQ}^*$ and $\mathcal{V}_{j,QP}^*$ are universal polynomials in their variables. In notation defined by Definition 5.1, $\mathcal{U}_{j,PQ}^*(\cdot; 0) = \mathcal{V}_{j,QP}^*(\cdot; 0) = 0$. Thus

$$(6.34) \quad \Psi_{k+1} - I = (\hat{U}^{(k+1)}, \hat{V}^{(k+1)}) = O(d_k),$$

$$(6.35) \quad \hat{U}_{j,PQ}^{(k+1)} = \frac{f_{j,PQ}^{(k)}}{\mu^{P-Q} - \mu_j}, \quad \hat{V}_{j,QP}^{(k+1)} = \frac{g_{j,QP}^{(k)}}{\mu^{Q-P} - \mu_j^{-1}},$$

for $|P| + |Q| = d_k$. Here $\mu^{P-Q} \neq \mu_j$. By (6.3)-(6.5), we have

$$(6.36) \quad \hat{\mu}_{j,P'}^{(k+1)} = \mu_{j,P'}^{(k)}, \quad |P'| < d_k - 1;$$

$$(6.37) \quad \hat{\mu}_{j,P'}^{(k+1)} = \mu_{j,P'}^{(k)} + \left\{ Df_j^{(k)}(\xi, \eta)(\hat{U}^{(k+1)}, \hat{V}^{(k+1)}) \right\}_{(P+e_j)P}, \\ |P_k| = d_k - 1;$$

$$(6.38) \quad \hat{\mu}_{j,P}^{(k+1)} = \mu_{j,P}^{(k)} + \mu_j \left\{ 2(\hat{U}_j^{(k+1)} \hat{V}_j^{(k+1)})_{PP} \right. \\ \left. + ((\hat{U}_j^{(k+1)})^2)_{(P+e_j)(P-e_j)} \right\} \\ + \left\{ Df_j^{(k)}(\xi, \eta)(\hat{U}^{(k+1)}, \hat{V}^{(k+1)}) \right\}_{(P+e_j)P}, \quad |P| = d_k.$$

As stated in Corollary 5.4, the coefficients of $\hat{\mu}_j^{(k+1)}(\xi\eta)\xi_j$ of degree $2d_k + 1$ do not depend on the coefficients of $f^{(k)}, g^{(k)}$ of degree $\geq 2d_k$, provided $(f^{(k)}, g^{(k)}) = O(d_k)$ is in $\mathcal{C}_2^2(S)$ as it is assumed.

Next, we need to estimate the size of coefficients of $\mu^{(k)}$ that appear in (6.36)-(6.38). Recall that we apply two sequences of normalization. We have

$$\hat{\sigma}_{k+1}^\infty = \Psi_{k+1}^{-1} \circ \Phi_k^{-1} \cdots \Psi_1^{-1} \circ \Phi_0^{-1} \circ \sigma^\infty \circ \Phi_0 \circ \Psi_1 \cdots \Phi_k \circ \Psi_{k+1}.$$

Thus, $M^{(k)}, N^{(k)}$ depend only on $\sigma^\infty, \Phi_0, \Psi_1, \Phi_1, \dots, \Psi_{k-1}, \Phi_k$.

Recall that if u_1, \dots, u_m are power series, then $\{u_1, \dots, u_m\}_d$ denotes the set of their coefficients of degree at most d , and $|\{u_1, \dots, u_m\}_d|$ denotes the sup norm. We need some crude estimates on the growth of Taylor coefficients. If $F = I + f$ and $f = O(2)$ is a map in formal power series, then (5.1)-(5.3) imply

$$(6.39) \quad |\{F^{-1}\}_m| \leq (2 + |\{f\}_m|)^{\ell_m},$$

$$|\{G \circ F\}_m| \leq (2 + |\{f, G\}_m|)^{\ell_m},$$

$$(6.40) \quad |\{F^{-1} \circ G \circ F\}_m| \leq (2 + |\{f, G\}_m|)^{\ell_m},$$

In general, if F_j are formal mappings of \mathbf{C}^n that are tangent to the identity, then

$$|\{F_k^{-1} \cdots F_1^{-1} G F_1 \cdots F_k\}_m| \leq (2 + |\{F_1, \dots, F_k, G\}_m|)^{\ell_{m,k}}, \quad 1 \leq k < \infty$$

In particular, if $F_j = I + O(j)$ for $j = 1, \dots, m$, then for any $k \geq |P| := m$ we have

$$(F_k^{-1} \cdots F_1^{-1} G F_1 \cdots F_k)_P = (F_m^{-1} \cdots F_1^{-1} G F_1 \cdots F_m)_P,$$

$$|\{F_k^{-1} \cdots F_1^{-1} G F_1 \cdots F_k\}_m| \leq (2 + |\{F_1, \dots, F_m, G\}_m|)^{\ell'_m}, \quad 1 \leq k \leq \infty.$$

We may take ℓ'_m by ℓ_m , while ℓ_m depends only on m . We have similar estimates for $F_k \cdots F_1 G F_1^{-1} \cdots F_k^{-1}$. Recall that $1/\sqrt{2} < \lambda_j < e^{e/2} < 4$. Using $\tau_{1j}^k = \varphi_k \tau_{1j}^{k-1} \varphi_k^{-1} = \varphi_k \cdots \varphi_1 \tau_{1j}^0 \varphi_1^{-1} \cdots \varphi_k^{-1}$ and hence $\tau_1^k = \varphi_k \cdots \varphi_1 \tau_1^0 \varphi_1^{-1} \cdots \varphi_k^{-1}$ and $\sigma_k = \tau_1^k(\rho \tau_2^k \rho)$, we obtain $|\{\tau_1^k\}_m| \leq (2 + |\{\tau_1^0, \varphi_1, \dots, \varphi_m\}_m|)^{\ell'_m} \leq 6^{\ell'_m}$ and $|\{\sigma^k\}_m| \leq (8^{\ell'_m})^{\ell'_m}$. Thus we obtain

$$(6.41) \quad |\{\sigma^k\}_m| \leq 8^{\ell_m \ell'_m}, \quad |\{\sigma^\infty\}_m| \leq 8^{\ell_m \ell'_m}.$$

Here we have used (6.17)-(6.18).

For simplicity, let δ_i denote $\delta_i(\mu)$. Inductively, let us show that for $k = 0, 1, \dots$,

$$(6.42) \quad |\{\hat{\boldsymbol{\mu}}^{(k)}, \hat{\boldsymbol{v}}^{(k)}\}_P| \leq |\delta_{d_{k-1}-1}|^{L_m}, \quad m = 2|P| + 1 < 2d_{k-1} - 1,$$

$$(6.43) \quad |\{\hat{\sigma}_k^\infty\}_{PQ}| \leq |\delta_{2d_{k-1}-1}|^{L_m}, \quad m = |P| + |Q| \geq 2d_{k-1} - 1.$$

$$(6.44) \quad |\boldsymbol{\mu}_{j,P}^{(k)}| + |\boldsymbol{v}_{j,P}^{(k)}| \leq |\delta_{d_k-1}|^{L_m}, \quad m = 2|P| + 1,$$

$$(6.45) \quad |f_{j,PQ}^{(k)}| + |g_{j,PQ}^{(k)}| \leq |\delta_{d_k-1}|^{L_m}, \quad m = |P| + |Q| \geq d_k.$$

Note that the last inequalities are equivalent to $|\{\sigma_k^\infty\}_m| \leq |\delta_{d_k-1}|^{L_m}$. Here and in what follows L_m does not depend on the choices of $\mu_j, d_k, h^{(k)}$ which satisfy the initial conditions, i.e. $1/e \leq \mu_j \leq e^e$ and (6.17)-(6.18) but are arbitrary otherwise. However, it suffices to find constants $L_{m,k}$ replacing L_m and depending on k such that (6.42)-(6.45) hold. Indeed, by (6.30) and (6.34) we have $\Psi_{k+1} = I + O(d_k)$ and $\Phi_k = I + O(2d_{k-1})$. Since $\hat{\sigma}_{k+1}^\infty = \Psi_{k+1}^{-1} \sigma_k^\infty \Psi_{k+1}$ and $\sigma_k^\infty = \Phi_k^{-1} \hat{\sigma}_k^\infty \Phi_k$, then

$$\hat{\sigma}_{k+1}^\infty = \hat{\sigma}_k^\infty + O(2d_{k-1}), \quad \sigma_{k+1}^\infty = \sigma_k^\infty + O(d_k)$$

as $d_k \geq 2d_{k-1}$. Since d_k increases to ∞ with k , then (6.42)-(6.45) with $L_{m,k}$ in place of L_m imply that they also hold for

$$L_m = \min_k \max\{L_{m,1}, \dots, L_{m,k} : (\hat{\sigma}_\ell^\infty - \hat{\sigma}_k^\infty, \sigma_\ell^\infty - \sigma_k^\infty) = O(m), \ell > k\}.$$

Therefore, in the following the dependence of L_m on k will not be indicated. The estimates (6.42)-(6.43) hold trivially for $\hat{\sigma}_0^\infty = \sigma^\infty$, $k = 0$ and $d_{-1} = 2$ by (6.20) and (6.41). Assuming (6.42)-(6.43), we want to verify (6.44)-(6.45). We also want to verify (6.42)-(6.43) when k is replaced by $k + 1$.

The $\Phi_k = I + (U^{(k)}, V^{(k)})$ is a polynomial mapping. Its degree is at most $d_k - 1$ and its coefficients are polynomials in $\{\hat{\sigma}_k\}_{d_k-1}$ and δ_{d_k-1} ;

see (6.26)-(6.27). Hence

$$(6.46) \quad |U_{j,PQ}^{(k)}| + |V_{j,QP}^{(k)}| \leq |\delta_{d_k-1}|^{L_m}, \quad m = |P| + |Q|.$$

Applying (6.40) to $\sigma_k^\infty = \Phi_k^{-1} \hat{\sigma}_k^\infty \Phi_k$, we obtain (6.44)-(6.45) from (6.42)-(6.43). Here we use that fact that since $d_k \geq 2d_{k-1}$, the small divisors in $\delta_{2d_{k-1}-1}$ appear in δ_{d_k-1} too. To obtain (6.42)-(6.43) when k is replaced by $k+1$, we note that Ψ_{k+1} is a polynomial map that has degree at most $2d_k - 1$ and the coefficients of degree m bounded by $\delta_{2d_k-1}^{L_m}$; see (6.32)-(6.33). This shows that

$$(6.47) \quad |\hat{U}_{j,PQ}^{(k+1)}| + |\hat{V}_{j,QP}^{(k+1)}| \leq |\delta_{2d_k-1}|^{L_m}, \quad |P| + |Q| = m.$$

We then obtain (6.42)-(6.43) when k is replaced by $k+1$ for $\hat{\sigma}_{k+1}^\infty$ by applying (6.40) to $\hat{\sigma}_{k+1}^\infty = \Psi_{k+1}^{-1} \sigma_k^\infty \Psi_{k+1}$ and by using (6.44)-(6.45) for σ_k^∞ and (6.47) for Ψ_{k+1} .

Let us summarize the above computation for $\hat{\sigma}^\infty$ defined by (6.21). We know that $\hat{\sigma}^\infty$ is the unique power series such that $\hat{\sigma}^\infty - \hat{\sigma}_k^\infty = O(d_k)$ for all k , and $\hat{\sigma}^\infty$ is a formal normal form of σ^∞ . Let us write

$$\hat{\sigma}^\infty: \begin{cases} \xi'_j = \hat{\mu}_j^\infty(\xi\eta)\xi_j, \\ \eta'_j = \hat{\nu}_j^\infty(\xi\eta)\eta_j. \end{cases}$$

Let $|P| \leq d_k$. By (6.31), we get $\hat{\mu}_P^{(k+1)} = \mu_P^{(k+1)}$; by (6.36) in which k is replaced by $k+1$, we get $\hat{\mu}_P^{(k+2)} = \mu_P^{(k+1)}$ as $|P| \leq d_k < d_{k+1} - 1$. Therefore,

$$(6.48) \quad \hat{\mu}_P^\infty = \hat{\mu}_P^{(k+1)}, \quad |P| \leq d_k.$$

For $|P| < d_k - 1$, (6.36) says that $\hat{\mu}_{j,P}^{(k+1)} = \mu_{j,P}^{(k)}$; by (6.44) that holds for any P , we obtain

$$(6.49) \quad |\hat{\mu}_P^\infty| = |\hat{\mu}_{j,P}^{(k+1)}| \leq |\delta_{d_k-1}|^{L_m}, \quad m = 2|P| + 1, \quad |P| < d_k - 1,$$

$$(6.50) \quad |\hat{\mu}_{j,P}^\infty| \leq |\delta_{d_k-1}|^{L_m} (1 + |\delta_{d_k}|), \quad m = 2|P| + 1 = 2d_k - 1.$$

We have verified (6.42)-(6.45). The sequence L_m depend only on

$$(6.51) \quad m = d_k + 1, \quad d_k, d_{k-1}, \dots, d_0, \quad d_j \geq 2d_{j-1}, \quad d_j > 3.$$

To obtain rapid increase of coefficients of $\hat{\mu}_{j,P}^{(k+1)}$, we want to use both small divisors hidden in $\hat{U}_{j,PQ}^{(k)}$ and $\hat{V}_{j,QP}^{(k)}$ in (6.38). Therefore, if $\mu_{j,P}^{(k)}$ is already sufficiently large for $|P| = d_k$ that will be specified later, we take φ_k to be the identity, i.e. $\tau_1^k = \tau_1^{k-1}$. Otherwise, we need to achieve it by choosing

$$\tau_1^k = \varphi_k \tau_1^{k-1} \varphi_k^{-1}.$$

Therefore, we examine the effect of a coordinate change by φ_k on these coefficients.

Recall that we are in the elliptic case. We have $\rho(\xi, \eta) = (\bar{\eta}, \bar{\xi})$ and $\tau_2^k = \rho\tau_1^k\rho$. Recall that

$$\varphi_k : \xi'_j = (\xi - h^{(k)}(\xi), \eta), \quad \text{ord } h^{(k)} = d_k > 3.$$

By a simple computation, we obtain

$$\begin{aligned} \tau_1^k(\xi, \eta) &= \tau_1^{k-1}(\xi, \eta) + (-h^{(k)}(\lambda\eta), \lambda^{-1}h^{(k)}(\xi)) + O(|(\xi, \eta)|^{d_k+1}), \\ \tau_2^k(\xi, \eta) &= \tau_2^{k-1}(\xi, \eta) + (\lambda^{-1}\overline{h^{(k)}}(\eta), -\overline{h^{(k)}}(\lambda\xi)) + O(|(\xi, \eta)|^{d_k+1}). \end{aligned}$$

Then we have

$$(6.52) \quad \begin{aligned} \sigma^k &= \sigma^{k-1} + (r^{(k)}, s^{(k)}) + O(d_k + 1); \\ r^{(k)}(\xi, \eta) &= -\lambda\overline{h^{(k)}}(\lambda\xi) - h^{(k)}(\lambda^2\xi), \\ s^{(k)}(\xi, \eta) &= \lambda^{-2}\overline{h^{(k)}}(\eta) + \lambda^{-1}h^{(k)}(\lambda^{-1}\eta). \end{aligned}$$

Since σ^k converges to σ^∞ , from (6.52) it follows that

$$(6.53) \quad \sigma^\infty = \sigma^{k-1} + (r^{(k)}, s^{(k)}) + O(d_k + 1).$$

For $|P| + |Q| = d_k$, we have

$$\begin{aligned} r_{j,PQ}^{(k)} &= \left\{ -\lambda_j\overline{h_j^{(k)}}(\lambda\xi) - h_j^{(k)}(\lambda^2\xi) \right\}_{PQ}, \\ s_{j,QP}^{(k)} &= \left\{ \lambda_j^{-2}\overline{h_j^{(k)}}(\eta) + \lambda_j^{-1}h_j^{(k)}(\lambda^{-1}\eta) \right\}_{QP}. \end{aligned}$$

We obtain

$$(6.54) \quad r_{j,P0}^{(k)} = -\lambda^{P+e_j}\overline{h_{j,P}^{(k)}} - \lambda^{2P}h_{j,P}^{(k)},$$

$$(6.55) \quad s_{j,0P}^{(k)} = \lambda_j^{-2}\overline{h_{j,P}^{(k)}} + \lambda^{-P-e_j}h_{j,P}^{(k)}, \quad |P| = d_k,$$

$$(6.56) \quad r_{j,PQ}^{(k)} = s_{j,QP}^{(k)} = 0, \quad |P| + |Q| = d_k, \quad Q \neq 0.$$

The above computation is actually sufficient to construct a divergent normal form $\tilde{\sigma} \in \mathcal{C}(S)$. To show that all normal forms of σ in $\mathcal{C}(S)$ are divergent, We need to related it to the normal form $\hat{\sigma}$ in Theorem 5.6, which is unique. This requires us to keep track of the small divisors in the normalization procedure in the proof of Lemma 5.5.

Recall that $F^{(k+1)}$ is defined by (5.35) in which $\tilde{\mu}_j$ are replaced by $\hat{\mu}_j^{(k+1)}$. Thus

$$(6.57) \quad F_j^{(k+1)}(\zeta) = \zeta_j + a_j^{(k+1)}(\zeta), \quad 1 \leq j \leq 3.$$

We also define F_j^∞ by (5.35) in which $\tilde{\mu}_j$ are replaced by $\hat{\mu}_j^\infty$. Then $F_j^\infty(\zeta) = \zeta_j + a_j^\infty(\zeta)$. Then by (6.48),

$$(6.58) \quad a_{j,P}^\infty = a_{j,P}^{(k+1)}, \quad |P| \leq d_k.$$

By (6.57) and $\log(1+x) = x + \frac{x^2}{2} + O(3)$, we have for $|P| > 1$

$$(6.59) \quad a_{j,P}^{(k+1)}(\zeta) = \mu_j^{-1} \hat{\mu}_{j,P}^{(k+1)} + \mu_j^{-1} \hat{\mu}_{j,P-e_j}^{(k+1)} + \mathcal{A}_{j,P}(\{\hat{\mu}_j^{(k+1)}\}_{|P|-2}).$$

By (6.49)-(6.50), we estimate the last two terms as follows

$$(6.60) \quad |\mathcal{A}_{j,P}(\{\hat{\mu}_j^{(k+1)}\}_{|P|-2})| \leq |\delta_{d_k-1}|^{L_m^* L_m}, \quad |P| = d_k, \quad m = 2|P| + 1,$$

$$(6.61) \quad |\hat{\mu}_{j,Q}^{(k+1)}| \leq |\delta_{d_k-1}|^{L_m} (1 + |\delta_{d_k}|), \quad m = 2|Q| + 1 = 2d_k - 1.$$

Here $L_m^* \geq 1$ is independent of k and depends only on the degrees of the polynomials $\mathcal{A}_{j,P}$. Recall from the formula (5.38) that $F^{(k+1)}$, F^∞ have the normal forms $\hat{F}^{(k+1)} = I + \hat{a}^{(k+1)}$ and $\hat{F}^\infty = I + \hat{a}^\infty$, respectively. The coefficients of $\hat{a}_{j,Q}^{(k+1)}$ and $\hat{a}_{j,Q}^\infty$ are zero, except the ones given by

$$\begin{aligned} \hat{a}_{j,Q}^{(k+1)} &= a_{j,Q}^{(k+1)} - \{Da_j^{(k+1)} \cdot a^{(k+1)}\}_Q + \mathcal{B}_{j,Q}(\{a^{(k+1)}\}_{|Q|-2}), \\ \hat{a}_{j,Q}^\infty &= a_{j,Q}^\infty - \{Da_j^\infty \cdot a^\infty\}_Q + \mathcal{B}_{j,Q}(\{a^\infty\}_{|Q|-2}), \end{aligned}$$

for $Q = (q_1, \dots, q_p)$, $q_j = 0$, and $|Q| > 1$. Derived from the same normalization, the $\mathcal{B}_{j,Q}$ in both formulae stands for the same polynomial and independent of k . Hence $\hat{a}_{j,P}^{(\infty)} = \hat{a}_P^{(k+1)}$ for $|P| \leq d_k$, by (6.58).

Combining (6.38) and (6.48) yields $\hat{a}_{3,Q}^\infty = \hat{a}_{3,Q}^{(k+1)}$ and

$$(6.62) \quad \begin{aligned} \hat{a}_{3,Q}^\infty &= 2(\hat{U}_3^{(k+1)} \hat{V}_3^{(k+1)})_{QQ} + ((\hat{U}_3^{(k+1)})^2)_{(Q+e_3)(Q-e_3)} \\ &+ \mu_3^{-1} \{Df_3^{(k)}(\xi, \eta)(\hat{U}^{(k+1)}, \hat{V}^{(k+1)})\}_{(Q+e_3)P_k} + \mathcal{A}_Q(\{\hat{\mu}^{(k+1)}\}_{|Q|-2}) \\ &+ \mu_3^{-1} \mu_{3,Q}^{(k)} + \mu_3^{-1} \hat{\mu}_{Q-e_3}^{(k+1)} - \{Da_j^{(k+1)} \cdot a^{(k+1)}\}_Q. \end{aligned}$$

The above formula holds for any Q with $|Q| = d_k$. To examine the effect of small divisors, we assume that

$$P_k = (p_k, q_k, 0), \quad |P_k| = d_k$$

are given by Lemma 6.5, so are μ_1, μ_2 , and μ_3 . **However, P_k and μ depend on a sequence L_m (to be renamed as L'_m) in Lemma 6.5. We will determine the sequence L'_m and hence P_k and μ_j later.**

Note that the second term in (6.62) is 0 as the third component of $P_k - e_3$ is negative. We apply the above computation to the P_k . Taking a subsequence of P_k if necessary, we may assume that $d_k \geq 2d_{k-1}$ and $d_{k-1} > 3$ for all $k \geq 1$. The 4 exceptional small divisors of height $2|P_k| + 1$ in (6.14) are

$$\mu^{P_k} - \mu_3, \quad \mu^{-P_k} - \mu_3^{-1}, \quad \mu^{2P_k - e_3} - \mu_3, \quad \mu^{-2P_k + e_3} - \mu_3^{-1}.$$

The last two cannot show up in \hat{a}_{3,P_k}^∞ , since their degree, $2d_k + 1$, is larger than the degrees of Taylor coefficients in \hat{a}_{3,P_k} . We have 3 products of the two exceptional small divisors of height $2|P_k| + 1$ and degree $|P_k|$, which are

$$(\mu^{P_k} - \mu_3)(\mu^{-P_k} - \mu_3^{-1}), \quad (\mu^{P_k} - \mu_3)(\mu^{P_k} - \mu_3), \quad (\mu^{-P_k} - \mu_3^{-1})(\mu^{-P_k} - \mu_3^{-1}).$$

The $(\hat{U}_3^{(k+1)}\hat{V}_3^{(k+1)})_{P_k P_k}$ contains the first product, but none of the other two. The third term and $f_3^{(k)}$ in \hat{a}_{3,P_k}^∞ do not contain exceptional small divisors of degree $|P_k| = d_k > 2d_{k-1} - 1$. Since $f_3^{(k)} = O(d_k)$ by (6.25), the exceptional small divisors of height $2|P_k| + 1$ can show up at most once in the fourth term of \hat{a}_{3,P_k}^∞ . Therefore, we arrive at

$$\begin{aligned} \hat{a}_{3,P_k}^\infty &= 2\hat{U}_{3,P_k 0}^{(k+1)}\hat{V}_{3,0P_k}^{(k+1)} + \hat{\mathcal{A}}_{P_k}^1(\delta_{d_k-1}, \frac{1}{\mu_{P_k} - \mu_3}; \{f^{(k)}, g^{(k)}\}_{d_k}) \\ &\quad + \hat{\mathcal{A}}_{P_k}^2(\delta_{d_k-1}; \{f^{(k)}, g^{(k)}\}_{d_k}) + \mu_3^{-1}\boldsymbol{\mu}_{3,P_k}^{(k)} + \mathcal{A}_{P_k}(\{\hat{\boldsymbol{\mu}}^{(k+1)}\}_{|P_k|-2}) \\ &\quad + \mu_3^{-1}\boldsymbol{\mu}_{3,P_k-1}^{(k+1)} - \{Da_3^{(k+1)} \cdot a^{(k+1)}\}_{P_k}, \\ \hat{\mathcal{A}}_{P_k}^1(\delta_{d_k-1}, \frac{1}{\mu_{P_k} - \mu_3}; \{f^{(k)}, g^{(k)}\}_{d_k}) &= (\hat{U}_{3,P_k 0}^{(k+1)}, \hat{V}_{3,0P_k}^{(k+1)}) \\ &\quad \cdot \hat{\mathcal{A}}_{P_k}^3(\delta_{d_k-1}; \{f^{(k)}, g^{(k)}\}_{d_k}). \end{aligned}$$

Note that $\hat{\mathcal{A}}_{P_k}^i$ and \mathcal{A}_{P_k} are polynomials independent of k . Set

$$m = 2d_k + 1.$$

In the following we can increase the value of L_m^* in (6.60) or when it reappears for a finite number of times such that the estimates involving L_m^* are valid for all k . By (6.44) and (6.60), we obtain $|\boldsymbol{\mu}_{3,P_k}^{(k)}| + |\mathcal{A}_{P_k}^3(\{\hat{\boldsymbol{\mu}}^{(k+1)}\}_{|P_k|-2})| \leq \delta_{d_k-1}^{L_m^* L_m}$. By (6.59), the smallest $|Q|$ for which $a_{i,Q}$ contains an exceptional small divisor in δ_{d_k} is $2|Q| + 1 = 2d_k - 1$. Now, $\{Da_3^{(k+1)} \cdot a^{(k+1)}\}_{P_k}$ is a linear combination of products of two terms and at most one of the two terms contains an exceptional small divisor; if the both terms contain an exceptional small divisor, one term is $a_{3,Q'}^{(k+1)}$ with $2|Q'| + 1 \geq 2d_k - 1$, while another is $a_{i,Q''}^{(k+1)}$ with $2|Q''| + 1 \geq 2d_k - 1$. (Here $Q' - e_i + Q'' = P_k$ and the i th component of Q' is positive.) Then $d_k = |P_k| = |Q'| + |Q''| - 1 \geq 2d_k - 2$, a contradiction. Therefore, by (6.59)-(6.61), we have

$$|\{a_3^{(k+1)} \cdot a^{(k+1)}\}_{P_k}| \leq |\delta_{d_k-1}|^{L_m^* L_m} (1 + |\delta_{d_k}|).$$

By (6.61), we also have $|\boldsymbol{\mu}_{j,P_k-1}^{(k+1)}| \leq |\delta_{d_k-1}|^{L_m} (1 + |\delta_{d_k}|)$. Omitting the arguments in the polynomial functions, we obtain from (6.46)-(6.47), and (6.48) that

$$\begin{aligned} &|\hat{\mathcal{A}}_{P_k}^1| + |\hat{\mathcal{A}}_{P_k}^2| + |\boldsymbol{\mu}_{3,P_k-1}^{(k+1)}| + |\boldsymbol{\mu}_{3,P_k}^{(k)}| + |\mathcal{A}_{P_k}| + |\{Da_3^{(k+1)} \cdot a^{(k+1)}\}_{P_k}| \\ &\leq \frac{|\delta_{d_k-1}(\mu)|^{L_m^* L_m}}{|\mu_{P_k} - \mu_3|}, \end{aligned}$$

for $m = 2|P_k| + 1$ and a possibly larger L_m . We remark that although each term in the inequality depends on the choices of the sequences $\mu_i, d_j, h^{(\ell)}$, the L_m does not depend on the choices, provided

that $\mu_j, d_k, h^{(i)}$ satisfy our initial conditions. Therefore, we have

$$|\hat{a}_{3,P_k}^\infty| \geq 2|\hat{U}_{3,P_k 0}^{(k+1)} \hat{V}_{3,0P_k}^{(k+1)}| - |\delta_{d_k-1}(\mu)|^{L_{2|P_k|+1}^* L_{2|P_k|+1}} |\mu^{P_k} - \mu_3|^{-1}.$$

Recall that $\sigma_k^\infty = \Phi_k^{-1} \Psi_{k-1}^{-1} \cdots \Phi_0^{-1} \sigma^\infty \Phi_0 \Psi_1 \cdots \Phi_k$. Set

$$\tilde{\sigma}_k^\infty := \Phi_k^{-1} \Psi_{k-1}^{-1} \cdots \Phi_0^{-1} \sigma^{k-1} \Phi_0 \Psi_1 \cdots \Phi_k.$$

By (6.53), we get

$$(6.63) \quad \sigma_k^\infty = \tilde{\sigma}_k^\infty + (r^{(k)}, s^{(k)}) + O(d_k + 1).$$

Recall that Φ_k depends only on coefficients of $\hat{\sigma}_{k-1}^\infty = \Psi_{k-1}^{-1} \sigma_{k-2}^\infty \Psi_{k-1}$ of degree less than d_k , while Ψ_{k-1} depends only on coefficients of $\sigma_{k-1}^\infty = \Phi_{k-1}^{-1} \hat{\sigma}_{k-1} \Phi_{k-1}$ of degree at most $2d_{k-1} - 1$ which is less than d_k too. Therefore, $\Phi_k, \Psi_{k-1}, \dots, \Phi_0$ depend only on coefficients of σ^∞ of degree less than d_k . On the other hand, $\sigma^\infty = \sigma^{k-1} + O(d_k)$. Therefore, $\tilde{\sigma}_k^\infty$ depends only on σ^{k-1} , and hence it depends only on $h^{(\ell)}$ for $\ell < k$. By (6.63), we can express

$$(6.64) \quad f_{j,PQ}^{(k)} = \tilde{f}_{j,PQ}^{(k)} + r_{j,PQ}^{(k)}, \quad g_{j,QP}^{(k)} = \tilde{g}_{j,QP}^{(k)} + s_{j,QP}^{(k)},$$

where $|P| + |Q| = d_k$ and $\tilde{f}_{j,PQ}^{(k)}, \tilde{g}_{j,QP}^{(k)}$ depend only on $h^{(\ell)}$ for $\ell < k$. Collecting (6.35), (6.64), and (6.54)-(6.56), we obtain

$$|\hat{a}_{3,P_k}^\infty| \geq 2 \frac{|T_k|}{|\mu^{P_k} - \mu_3| |\mu^{-P_k} - \mu_3^{-1}|} - \frac{|\delta_{d_k-1}(\mu)|^{L_{2d_k+1}^* L_{2d_k+1}}}{|\mu^{P_k} - \mu_3|}$$

with

$$\begin{aligned} T_k &= \left(-\lambda^{P_k+e_3} \overline{h_{3,P_k}^{(k)}} - \lambda^{2P_k} h_{3,P_k}^{(k)} + \tilde{f}_{3,P_k 0}^{(k-1)} \right) \\ &\quad \times \left(\lambda_3^{-2} \overline{h_{3,P_k}^{(k)}} + \lambda^{-P_k-e_3} h_{3,P_k}^{(k)} + \tilde{g}_{3,0P_k}^{(k-1)} \right) \\ &= -\lambda^{2P_k-2e_3} \left(\lambda^{e_3-P_k} \overline{h_{3,P_k}^{(k)}} + h_{3,P_k}^{(k)} - \lambda^{-2P_k} \tilde{f}_{3,P_k 0}^{(k-1)} \right) \\ &\quad \times \left(\lambda^{e_3-P_k} h_{3,P_k}^{(k)} + \overline{h_{3,P_k}^{(k)}} + \lambda_3^2 \tilde{g}_{3,0P_k}^{(k-1)} \right). \end{aligned}$$

Set $\tilde{T}_k(h_{3,P_k}^{(k)}) := -\lambda^{2e_3-2P_k} T_k$. We are ready to choose $h_{3,P_k}^{(k)}$ to get a divergent normal form. We have $|\lambda^{P_k-e_3} + 1| \geq 1$. Then one of $|\tilde{T}_k(0)|, |\tilde{T}_k(1)|, |\tilde{T}_k(-1)|$ is at least $1/4$; otherwise, we would have

$$2|\lambda^{P_k-e_3} + 1|^2 = |\tilde{T}_k(1) + \tilde{T}_k(-1) - 2\tilde{T}_k(0)| < 1,$$

which is a contradiction. This shows that by taking $h_{3,P_k}^{(k)}$ to be one of $0, 1, -1$, we have achieved

$$|T_k| \geq \frac{1}{4} \mu^{P_k-e_3}.$$

Therefore,

$$(6.65) \quad |\hat{a}_{3,P_k}^\infty| \geq \frac{\mu^{P_k - e_3}}{2|\mu^{P_k} - \mu_3||\mu^{-P_k} - \mu_3^{-1}|} - \frac{|\delta_{d_k-1}(\mu)|^{L_{2d_k+1}^* L_{2d_k+1}}}{|\mu^{P_k} - \mu_3|}.$$

Recall that $\mu_3 = e^e$. If $|\mu^{P_k} - \mu_3| < 1$ then $1/2 < \mu^{P_k - e_3} < 2$. The above inequality implies

$$(6.66) \quad |\hat{a}_{3,P_k}^\infty| \geq \frac{\mu^{2P_k}}{4|\mu^{P_k} - \mu_3|^2},$$

provided

$$|\mu^{P_k} - \mu_3| \leq \frac{1}{4} |\delta_{d_k-1}(\mu)|^{-L_{2d_k+1}^* L_{2d_k+1}}, \quad |P_k| = d_k.$$

For the last inequality to hold, it suffices have

$$(6.67) \quad |\mu^{P_k} - \mu_3| \leq |\delta_{d_k-1}(\mu)|^{-L_{2d_k+1}^* L_{2d_k+1} - 1}, \quad |\delta_{d_k-1}(\mu)|^{-1} < 1/4.$$

When (6.66), we still have (6.65). Thus we have derived universal constants $L_{2d_k+1}, L_{2d_k+1}^*$ for any $P_k = (p_k, q_k, 0)$ as long as $|P_k| = d_k > 3$. The sequence L_m^*, L_m do not depend on the choice of λ and they are independent of k ; however it depends on d_0, d_1, \dots, d_k as described in (6.51). Let us denote the constants $L_{2d_k+1}, L_{2d_k+1}^*$ in (6.67) respectively by $(L_{2d_k+1}(d_0, \dots, d_k), L_{2d_k+1}^*(d_0, \dots, d_k))$. We now remove the dependence of L_m on the partition d_0, \dots, d_k and define L_m for $m > 7$ as follows. For each $m > 7$, define

$$\mathcal{D}_m = \{(d_0, \dots, d_k) : 3 < d_0 \leq d_1/2 \leq \dots \leq d_k/2^k, \\ 2d_k + 1 \leq m, k = 0, 1, \dots\},$$

$$L'_N = N + 2 \max\{(L_{2d_k+1} L_{2d_k+1}^*)(d_0, \dots, d_k) : (d_0, \dots, d_k) \in \mathcal{D}_{2N+1}\}.$$

Let us apply Lemma 6.5 to the sequence L'_N . Therefore, there exist μ and a sequence of $P_k = (p_k, q_k, 0)$ satisfying $|\mu^{P_k} - \mu_3| \leq (C\Delta^*(P_k))^{L'_{|P_k|}}$. Taking a subsequence if necessary, we may assume that $d_k = |P_k| \geq 2d_{k-1}$ and $d_k > 3$. Thus

$$\begin{aligned} |\mu^{P_k} - \mu_3| &\leq (C\Delta^*(P_k))^{L'_{|P_k|}} \leq (\Delta^*(P_k)^{1/2})^{L'_{|P_k|}} \\ &\leq (\delta_{d_k-1}(\mu))^{-L'_{|P_k|}/2} \leq |\delta_{d_k-1}(\mu)|^{-L'_{2d_k+1}} \\ &\leq |\delta_{d_k-1}(\mu)|^{-L_{2d_k+1}^*(d_0, \dots, d_k) L_{2d_k+1}(d_0, \dots, d_k) - 1}, \end{aligned}$$

which gives us (6.67). Here the second inequality follows from

$$C(\Delta^*(P_k))^{1/2} < 1$$

when k is sufficiently large. The third inequality is obtained as follows. The definition of $\Delta^*(P_k)$ and $|P_k| = d_k$ imply that any small divisor in $\delta_{d_k-1}(\mu)$ is contained in $\Delta^*(P_k)$. Also, $\Delta^*(P_k) < \mu_i^{-1}$ for $i = 1, 2, 3$ and

k sufficiently large. Hence, $\Delta^*(P_k) \leq \delta_{d_k-1}^{-1}(\mu)$, which gives us the third inequality. We have that $L_k \geq k$. From (6.66) and (6.67) it follows that

$$|\hat{a}_{3,P_k}^\infty| > \delta_{d_k-1}^{d_k+1}(\mu) = \delta_{d_k-1}^{|P_k|+1}(\mu),$$

for k sufficiently large. As $\delta_{d_k}(\mu) \rightarrow +\infty$, this shows that the divergence of \hat{F}_3 and the divergence of the normal form $\hat{\sigma}$.

As mentioned earlier, Theorem 5.6 (iii) implies that any normal form of σ that is in the centralizer of \hat{S} must diverge. q.e.d.

7. A unique formal normal form of a real submanifold

Recall that we consider submanifolds of which the complexifications admit the maximum number of deck transformations. The deck transformations of π_1 are generated by $\{\tau_{i1}, \dots, \tau_{1p}\}$. We also set $\tau_{2j} = \rho\tau_{1j}\rho$. Each of $\tau_{i1}, \dots, \tau_{ip}$ fixes a hypersurface and $\tau_i = \tau_{i1} \cdots \tau_{ip}$ is the unique deck transformation of π_i whose set of fixed points has the smallest dimension. We first normalize the composition $\sigma = \tau_1\tau_2$. This normalization is reduced to two normal form problems. In Proposition 5.2 we obtain a transformation Ψ to transform τ_1, τ_2 , and σ into

$$\begin{aligned} \tau_i^* : \xi_j' &= \Lambda_{ij}(\xi\eta)\eta_j, & \eta_j' &= \Lambda_{ij}^{-1}(\xi\eta)\xi_j, \\ \sigma^* : \xi_j' &= \mu_j(\xi\eta)\xi_j, & \eta_j' &= \mu_j^{-1}(\xi\eta)\eta_j, \quad 1 \leq j \leq p. \end{aligned}$$

Here $\Lambda_{2j} = \Lambda_{1j}^{-1}$ and $\mu_j = \Lambda_{1j}^2$ are power series in the product $\zeta = (\xi_1\eta_1, \dots, \xi_p\eta_p)$. We also normalize the map μ by a transformation φ which preserves all coordinate hyperplanes. This is the second normal form problem, which is solved formally in Theorem 5.6 under condition L on μ . This gives us a map Ψ_1 which transforms τ_1, τ_2 , and σ into $\hat{\tau}_1, \hat{\tau}_2, \hat{\sigma}$ of the above form where Λ_{ij} and μ_j become $\hat{\Lambda}_{ij}, \hat{\mu}_j$.

In this section, we derive a **unique formal normal form** for

$$\{\tau_{11}, \dots, \tau_{1p}, \rho\}$$

under the above condition that $\hat{\mu}$ is in the norm form (1.7). In this case, we know from Theorem 5.6 that $\mathcal{C}(\hat{\sigma})$ consists of only 2^p dilatations

$$(7.1) \quad R_\epsilon : (\xi_j, \eta_j) \rightarrow (\epsilon_j \xi_j, \epsilon_j \eta_j), \quad \epsilon_j = \pm 1, \quad 1 \leq j \leq p.$$

We will consider two cases. In the first case, we impose no restriction on the linear parts of $\{\tau_{ij}\}$ but the coordinate changes are restricted to mappings that are tangent to the identity. The second is for the family $\{\tau_{ij}\}$ that arises from a higher order perturbation of a product quadric, while no restriction is imposed on the changes of coordinates. We will show that in both cases, if the normal form of σ can be achieved by a convergent transformation, the normal form of $\{\tau_{11}, \dots, \tau_{1p}, \rho\}$ can be achieved by a convergent transformation too.

We now restrict our real submanifolds to some classes. First, we assume that σ and τ_1, τ_2 are already in the normal form $\hat{\sigma}$ and $\hat{\tau}_1, \hat{\tau}_2$ such that

$$(7.2) \quad \hat{\tau}_i: \xi' = \hat{\Lambda}_i(\xi\eta)\eta, \quad \eta' = \hat{\Lambda}_i(\xi\eta)^{-1}\xi, \quad \hat{\Lambda}_2 = \hat{\Lambda}_1^{-1},$$

$$(7.3) \quad \hat{\sigma}: \xi' = \hat{\mu}(\xi\eta)\xi, \quad \eta' = \hat{\mu}(\xi\eta)^{-1}\eta, \quad \hat{\mu} = \hat{\Lambda}_1^2.$$

Let us start with the general situation without imposing condition L on $\hat{\mu}$. Assume that $\hat{\sigma}$ and $\hat{\tau}_i$ are in the above forms. We want to describe $\{\tau_{1j}, \rho\}$. Let us start with the linear normal forms described in Lemma 3.5 or Proposition 3.10. Recall that $\mathbf{Z}_j = \text{diag}(1, \dots, -1, \dots, 1)$ with -1 at the $(p+j)$ -th place, and $\mathbf{Z} := \mathbf{Z}_1 \cdots \mathbf{Z}_p$. Let Z_j (resp. Z) be the linear transformation with the matrix \mathbf{Z}_j (resp. \mathbf{Z}). We also use notation

$$(7.4) \quad \mathbf{B}_* = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{B} \end{pmatrix}, \quad \mathbf{E}_{\hat{\Lambda}_i} = \begin{pmatrix} \mathbf{I} & \hat{\Lambda}_i \\ -\hat{\Lambda}_i^{-1} & \mathbf{I} \end{pmatrix}.$$

Here \mathbf{B} , as well as $\hat{\Lambda}_i$ given by (7.2), is a non-singular complex $(p \times p)$ matrix. Assume that \mathbf{B}_1 and \mathbf{B}_2 are invertible $p \times p$ matrices. Define

$$(7.5) \quad \begin{aligned} (B_i)_* &: \begin{pmatrix} \xi \\ \eta \end{pmatrix} \rightarrow (B_i)_* \begin{pmatrix} \xi \\ \eta \end{pmatrix}, \\ E_{\hat{\Lambda}_i} &: \begin{pmatrix} \xi \\ \eta \end{pmatrix} \rightarrow \begin{pmatrix} \mathbf{I} & \hat{\Lambda}_i(\xi\eta) \\ -\hat{\Lambda}_i^{-1}(\xi\eta) & \mathbf{I} \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}. \end{aligned}$$

Let us assume that in suitable linear coordinates, the linear parts $L\tau_{ij} = T_{ij}$ of two families of involutions $\{\tau_{i1}, \dots, \tau_{ip}\}$ for $i = 1, 2$ are given by

$$(7.6) \quad T_{ij} := E_{\Lambda_i, \mathbf{B}_i} \circ Z_j \circ E_{\Lambda_i, \mathbf{B}_i}^{-1},$$

$$(7.7) \quad E_{\Lambda_i, \mathbf{B}_i} := E_{\Lambda_i} \circ (B_i)_*, \quad \Lambda_i := \hat{\Lambda}_i(0).$$

Note that $(B_i)_*$ commutes with Z . Also, $E_{\hat{\Lambda}_i} \circ \hat{\tau}_i = Z \circ E_{\hat{\Lambda}_i}$. We have the decomposition

$$(7.8) \quad \hat{\tau}_i = \hat{\tau}_{i1} \cdots \hat{\tau}_{ip},$$

$$(7.9) \quad E_{\hat{\Lambda}_i, \mathbf{B}_i} := E_{\hat{\Lambda}_i} \circ (B_i)_*, \quad \hat{\tau}_{ij} := E_{\hat{\Lambda}_i, \mathbf{B}_i} \circ Z_j \circ E_{\hat{\Lambda}_i, \mathbf{B}_i}^{-1}.$$

As before, we assume that S is non resonant. For real submanifolds, we still impose the reality condition $\tau_{2j} = \rho\tau_{1j}\rho$ where ρ is given by (1.3). Then we take $T_{2j} = \rho T_{1j}\rho$, $\mathbf{B}_1 = \mathbf{B}$ in (7.4)-(7.9). The following lemma describes a way to classify all involutions $\{\tau_{11}, \dots, \tau_{1p}, \rho\}$ provided that σ is in a normal form.

Lemma 7.1. *Let $\{\tau_{1j}\}$ and $\{\tau_{2j}\}$ be two families of formal holomorphic commuting involutions. Let $\tau_i = \tau_{i1} \cdots \tau_{ip}$ and $\sigma = \tau_1\tau_2$. Suppose*

that

$$\begin{aligned}\tau_i &= \hat{\tau}_i: \xi'_j = \hat{\Lambda}_{ij}(\xi\eta)\eta_j, & \eta'_j &= \hat{\Lambda}_{ij}(\xi\eta)^{-1}\xi_j; \\ \sigma &= \hat{\sigma}: \xi'_j = \hat{\mu}_j(\xi\eta)\xi_j, & \eta'_j &= \hat{\mu}_j(\xi\eta)^{-1}\eta_j\end{aligned}$$

with $\hat{\mu}_j = \hat{\Lambda}_{1j}^2$ and $\hat{\mu}_j(0) = \mu_j$. Suppose that $\mu_1, \dots, \mu_p, \mu_1^{-1}, \dots, \mu_p^{-1}$ satisfy the non-resonant condition (1.4). Assume further that the linear parts T_{ij} of τ_{ij} are given by (7.6). Then we have the following :

- (i) For $i = 1, 2$ there exists $\Phi_i \in \mathcal{C}(\hat{\tau}_i)$, tangent to the identity, such that $\Phi_i^{-1}\tau_{ij}\Phi_i = \hat{\tau}_{ij}$ for $1 \leq j \leq p$.
- (ii) Let $\{\tilde{\tau}_{1j}\}$ and $\{\tilde{\tau}_{2j}\}$ be two families of formal holomorphic commuting involutions. Let $\tilde{\tau}_i = \tilde{\tau}_{i1} \cdots \tilde{\tau}_{ip}$ and $\tilde{\sigma} = \tilde{\tau}_1\tilde{\tau}_2$. Suppose that $\tilde{\tau}_i = \hat{\tau}_i$ and $\tilde{\sigma} = \hat{\sigma}$ and $\tilde{\Phi}_i^{-1}\tilde{\tau}_{ij}\tilde{\Phi}_i = \hat{\tau}_{ij}$ with $\tilde{\Phi}_i \in \mathcal{C}(\hat{\tau}_i)$ being tangent to the identity and

$$\hat{\tau}_{ij} = E_{\hat{\Lambda}_i, \tilde{\mathbf{B}}_i} \circ Z_j \circ E_{\hat{\Lambda}_i, \tilde{\mathbf{B}}_i}^{-1}.$$

Here for $i = 1, 2$, the matrix $\tilde{\mathbf{B}}_i$ is non-singular. Then

$$\Upsilon^{-1}\tau_{ij}\Upsilon = \tilde{\tau}_{i\nu_i(j)}, \quad i = 1, 2, \quad j = 1, \dots, p$$

if and only if there exist $\Upsilon \in \mathcal{C}(\hat{\tau}_1, \hat{\tau}_2)$ and $\Upsilon_i \in \mathcal{C}(\hat{\tau}_i)$ such that

$$(7.10) \quad \begin{aligned}\tilde{\Phi}_i &= \Upsilon^{-1} \circ \Phi_i \circ \Upsilon_i, \quad i = 1, 2, \\ \Upsilon_i^{-1}\hat{\tau}_{ij}\Upsilon_i &= \hat{\tau}_{i\nu_i(j)}, \quad 1 \leq j \leq p.\end{aligned}$$

Here each ν_i is a permutation of $\{1, \dots, p\}$.

- (iii) Assume further that $\tau_{2j} = \rho\tau_{1j}\rho$ with ρ being defined by (1.3). Define $\hat{\tau}_{1j}$ by (7.8) and let $\hat{\tau}_{2j} := \rho\hat{\tau}_{1j}\rho$. Then we can choose $\Phi_2 = \rho\Phi_1\rho$ for (i). Suppose that $\tilde{\Phi}_2 = \rho\tilde{\Phi}_1\rho$ where $\tilde{\Phi}_1$ is as in (ii). Then $\{\tilde{\tau}_{1j}, \rho\}$ is equivalent to $\{\tau_{1j}, \rho\}$ if and only if there exist Υ_i, ν_i with $\nu_2 = \nu_1$, and Υ satisfying the conditions in (ii) and $\Upsilon_2 = \rho\Upsilon_1\rho$. The latter implies that $\Upsilon\rho = \rho\Upsilon$.

Proof. (i). Note that $\hat{\tau}_{ij}$ is conjugate to Z_j via the map $E_{\hat{\Lambda}_i, \tilde{\mathbf{B}}_i}$. Fix i . Each $\hat{\tau}_{ij}$ is an involution and its set of fixed-point is a hypersurface. Furthermore, $\text{Fix}(\tau_{11}), \dots, \text{Fix}(\tau_{1p})$ intersect transversally at the origin. By [12, Lemma 2.4], there exists a formal mapping ψ_i such that $\psi_i^{-1}\tau_{ij}\psi_i = L\tau_{ij}$. Now $L\psi_i$ commutes with $L\tau_{ij}$, Replacing ψ_i by $\psi_i(L\psi_i)^{-1}$, we may assume that ψ_i is tangent to the identity. We also find a formal mapping $\hat{\psi}_i$, which is tangent to the identity, such that $\hat{\psi}_i^{-1}\hat{\tau}_{ij}\hat{\psi}_i = L\hat{\tau}_{ij} = L\tau_{ij}$. Then $\Phi_i = \psi_i\hat{\psi}_i^{-1}$ fulfills the requirements.

(ii). Suppose that

$$\tau_{ij} = \Phi_i\hat{\tau}_{ij}\Phi_i^{-1}, \quad \tilde{\tau}_{ij} = \tilde{\Phi}_i\hat{\tau}_{ij}\tilde{\Phi}_i^{-1}.$$

Assume that there is a formal biholomorphic mapping Υ that transforms $\{\tau_{ij}\}$ into $\{\tau_{ij}\}$ for $i = 1, 2$. Then

$$(7.11) \quad \Upsilon^{-1}\tau_{ij}\Upsilon = \tilde{\tau}_{i\nu_i(j)}, \quad j = 1, \dots, p, \quad i = 1, 2.$$

Here ν_i is a permutation of $\{1, \dots, p\}$. Then

$$(7.12) \quad \hat{\tau}_i\Upsilon = \Upsilon\hat{\tau}_i, \quad \hat{\sigma}\Upsilon = \Upsilon\hat{\sigma}.$$

Set $\Upsilon_i := \Phi_i^{-1}\Upsilon\tilde{\Phi}_i$. We obtain

$$(7.13) \quad \Upsilon_i^{-1}\hat{\tau}_{ij}\Upsilon_i = \hat{\tau}_{i\nu_i(j)}, \quad 1 \leq j \leq p,$$

$$(7.14) \quad \tilde{\Phi}_i = \Upsilon^{-1}\Phi_i\Upsilon_i, \quad i = 1, 2.$$

Conversely, assume that (7.12)-(7.14) are valid. Then (7.11) holds as

$$\Upsilon^{-1}\tau_{ij}\Upsilon = \Upsilon^{-1}\Phi_i\hat{\tau}_{ij}\Phi_i^{-1}\Upsilon = \tilde{\Phi}_i\Upsilon_i^{-1}\hat{\tau}_{ij}\Upsilon_i\tilde{\Phi}_i^{-1} = \tilde{\tau}_{i\nu_i(j)}.$$

(iii). Assume that we have the reality assumption $\tau_{2j} = \rho\tau_{1j}\rho$ and $\tilde{\tau}_{2j} = \rho\tilde{\tau}_{1j}\rho$. As before, we take Φ_1 , tangent to the identity, such that $\tau_{1j} = \Phi_1\hat{\tau}_{1j}\Phi_1^{-1}$. Let $\Phi_2 = \rho\Phi_1\rho$. By $\hat{\tau}_{2j} = \rho\hat{\tau}_{1j}\rho$, we get $\tau_{2j} = \rho\tau_{1j}\rho = \Phi_2\hat{\tau}_{2j}\Phi_2^{-1}$ for $\nu_2 = \nu_1$. Suppose that $\tilde{\Phi}_i$ associated with $\tilde{\tau}_{1j}$ and ρ satisfy the analogous properties. Suppose that $\Upsilon^{-1}\tau_{ij}\Upsilon = \tilde{\tau}_{i\nu_i(j)}$ with $\nu_2 = \nu_1$, and $\Upsilon\rho = \rho\Upsilon$. Letting $\Upsilon_1 = \Phi_1^{-1}\Upsilon\tilde{\Phi}_1$ we get $\Upsilon_2 = \rho\Upsilon_1\rho$. Conversely, if Υ_1 and Υ_2 satisfy $\Upsilon_2 = \rho\Upsilon_1\rho$, then

$$\rho\Upsilon\rho = \rho\Phi_1\Upsilon_1\tilde{\Phi}_1^{-1}\rho = \Phi_2\Upsilon_2\tilde{\Phi}_2^{-1} = \Upsilon.$$

This shows that Υ satisfies the reality condition.

q.e.d.

Now we assume that $\hat{\mu}$ is in the normal form (1.7). We assume that the linear part T_{ij} of τ_{ij} are given by (7.6), where the non-singular matrix \mathbf{B} is arbitrary. As mentioned earlier in this section, the group of formal biholomorphisms that preserve the form of $\hat{\sigma}$ consists of only linear involutions R_ϵ defined by (7.1). This restricts the holomorphic equivalence classes of the quadratic parts of M . By Proposition 3.10, such quadrics are classified by a more restricted equivalence relation, namely, $(\tilde{\mathbf{B}}_1, \tilde{\mathbf{B}}_2) \sim (\mathbf{B}_1, \mathbf{B}_2)$, if and only if

$$\tilde{\mathbf{B}}_i = (\text{diag } \mathbf{a})^{-1}\mathbf{B}_i \text{diag}_{\nu_i} \mathbf{d}, \quad i = 1, 2.$$

To deal with a general situation, let us assume for the moment that $\mathbf{B}_1, \mathbf{B}_2$ are arbitrary invertible matrices.

Using the normal form $\{\hat{\tau}_1, \hat{\tau}_2\}$ and the matrices $\mathbf{B}_1, \mathbf{B}_2$, we first decompose $\hat{\tau}_i = \hat{\tau}_{i1} \cdots \hat{\tau}_{ip}$. By Lemma 7.1 (i), we then find Φ_i such that

$$\tau_{ij} = \Phi_i\hat{\tau}_{ij}\Phi_i^{-1}, \quad 1 \leq j \leq p.$$

For each i , Φ_i commutes with $\hat{\tau}_i$. It is within this family of $\{\mathbf{B}_i, \Phi_i; i = 1, 2\}$ with $\Phi_i \in \mathcal{C}(\hat{\tau}_i)$ for $i = 1, 2$ that we will find a normal form for $\{\tau_{ij}\}$. When restricted to $\tau_{2j} = \rho\tau_{1j}\rho$, the classification of the real

submanifolds is within the family of $\{\tau_{1j}, \rho\}$ as described in Lemma 7.1 (iii).

From Lemma 7.1 (ii), the equivalence relation on $\mathcal{C}(\hat{\tau}_i)$ is given by

$$\tilde{\Phi}_i = \Upsilon^{-1} \Phi_i \Upsilon_i, \quad i = 1, 2.$$

Here Υ_i and Υ satisfy

$$\Upsilon_i^{-1} \hat{\tau}_{ij} \Upsilon_i = \hat{\tau}_{i\nu_i(j)}, \quad 1 \leq j \leq p; \quad \Upsilon^{-1} \hat{\tau}_i \Upsilon = \hat{\tau}_i, \quad i = 1, 2.$$

We now construct a normal form for $\{\tau_{ij}\}$ within the above family. Let us first use the centralizer of $\mathcal{C}^c(Z_1, \dots, Z_p)$, described in Lemma 4.6, to define the complement of the centralizer of the family of non-linear commuting involutions $\{\hat{\tau}_{11}, \dots, \hat{\tau}_{1p}\}$. Recall that the mappings $E_{\hat{\Lambda}_i}$ and $(B_i)_*$ are defined by (7.5). According to Lemma 4.6, we have the following.

Lemma 7.2. *Let $i = 1, 2$. Let $\{\hat{\tau}_{i1}, \dots, \hat{\tau}_{ip}\}$ be given by (7.9). Then*

$$\begin{aligned} \mathcal{C}(\hat{\tau}_{i1}, \dots, \hat{\tau}_{ip}) &= \left\{ E_{\hat{\Lambda}_i, \mathbf{B}_i} \circ \phi_0 \circ E_{\hat{\Lambda}_i, \mathbf{B}_i}^{-1} : \phi_0 \in \mathcal{C}(Z_1, \dots, Z_p) \right\}, \\ \mathcal{C}(\hat{\tau}_i) &= \left\{ E_{\hat{\Lambda}_i, \mathbf{B}_i} \circ \phi_0 \circ E_{\hat{\Lambda}_i, \mathbf{B}_i}^{-1} : \phi_0 \in \mathcal{C}(Z) \right\}. \end{aligned}$$

Set

$$\mathcal{C}^c(\hat{\tau}_{i1}, \dots, \hat{\tau}_{ip}) := \left\{ E_{\hat{\Lambda}_i, \mathbf{B}_i} \circ \phi_1 \circ E_{\hat{\Lambda}_i, \mathbf{B}_i}^{-1} : \phi_1 \in \mathcal{C}^c(Z_1, \dots, Z_p) \right\}.$$

For each i , every formal biholomorphic mapping ψ admits a unique decomposition $\psi_1 \psi_0^{-1}$ with

$$\psi_1 \in \mathcal{C}^c(\hat{\tau}_{i1}, \dots, \hat{\tau}_{ip}), \quad \psi_0 \in \mathcal{C}(\hat{\tau}_{i1}, \dots, \hat{\tau}_{ip}).$$

If $\hat{\tau}_{ij}$ and ψ are convergent, then ψ_0, ψ_1 are convergent.

When $\tau_{2j} = \rho \tau_{1j} \rho$, only $\mathcal{C}(\hat{\tau}_{11}, \dots, \hat{\tau}_{1p})$ and its complement will be used in the normal forms constructed in Propositions 7.4 and 7.6 and Theorem 7.7.

Proposition 7.3. *Let $\hat{\tau}_i, \hat{\sigma}$ be given by (7.2)-(7.3) in which $\hat{\mu}$ is in the formal normal form (1.7). Let $\{\hat{\tau}_{ij}\}$ be given by (7.9). Suppose that*

$$(7.15) \quad \tau_{ij} = \Phi_i \hat{\tau}_{ij} \Phi_i^{-1}, \quad \tilde{\tau}_{ij} = \tilde{\Phi}_i \hat{\tau}_{ij} \tilde{\Phi}_i^{-1} \quad 1 \leq j \leq p,$$

$$(7.16) \quad \Phi_i \in \mathcal{C}(\hat{\tau}_i), \quad \tilde{\Phi}_i \in \mathcal{C}(\hat{\tau}_i), \quad \tilde{\Phi}'_i(0) = \Phi'_i(0) = \mathbf{I}, \quad i = 1, 2.$$

Then $\{\Upsilon^{-1} \tau_{ij} \Upsilon\} = \{\tilde{\tau}_{ij}\}$ for $i = 1, 2$ and for some invertible $\Upsilon \in \mathcal{C}(\hat{\tau}_1, \hat{\tau}_2)$, if and only if there exist formal biholomorphisms $\Upsilon, \Upsilon_1^*, \Upsilon_2^*$ such that

$$(7.17) \quad \Upsilon^{-1} \circ (B_i)_* \circ Z_j \circ (B_i)_*^{-1} \circ \Upsilon = (B_i)_* \circ Z_{\nu_i(j)} \circ (B_i)_*^{-1},$$

$$(7.18) \quad \tilde{\Phi}_i = \Upsilon^{-1} \Phi_i \Upsilon_i^* \Upsilon, \quad \Upsilon_i^* \in \mathcal{C}(\hat{\tau}_{i1}, \dots, \hat{\tau}_{ip}), \quad i = 1, 2,$$

$$(7.19) \quad \Upsilon \hat{\sigma} \Upsilon^{-1} = \hat{\sigma},$$

where each ν_i is a permutation of $\{1, \dots, p\}$. Assume further that $\hat{\tau}_{2j} = \rho \hat{\tau}_{1j} \rho$ and $\tilde{\Phi}_2 = \rho \tilde{\Phi}_1 \rho$ and $\tilde{\Phi}_2 = \rho \tilde{\Phi}_1 \rho$. We can take $\Upsilon_2^* = \rho \Upsilon_1^* \rho$ and $\nu_2 = \nu_1$, if additionally

$$\Upsilon \rho = \rho \Upsilon.$$

Proof. Recall that

$$\tau_{ij} = \Phi_i \hat{\tau}_{ij} \Phi_i^{-1}, \quad \Phi_i \in \mathcal{C}(\hat{\tau}_i); \quad \tilde{\tau}_{ij} = \tilde{\Phi}_i \hat{\tau}_{ij} \tilde{\Phi}_i^{-1}, \quad \tilde{\Phi}_i \in \mathcal{C}(\hat{\tau}_i).$$

Suppose that

$$(7.20) \quad \Upsilon^{-1} \tau_{ij} \Upsilon = \tilde{\tau}_{i\nu_i(j)}, \quad j = 1, \dots, p, \quad i = 1, 2.$$

By Lemma 7.1, there are invertible Υ_i such that

$$(7.21) \quad \Upsilon_i^{-1} \hat{\tau}_{ij} \Upsilon_i = \hat{\tau}_{i\nu_i(j)}, \quad 1 \leq j \leq p,$$

$$(7.22) \quad \tilde{\Phi}_i = \Upsilon^{-1} \circ \Phi_i \circ \Upsilon_i, \quad i = 1, 2.$$

Let us simplify the equivalence relation. By Theorem 5.6, $\mathcal{C}(\hat{\tau}_1, \hat{\tau}_2)$ consists of 2^p dilations Υ of the form $(\xi, \eta) \rightarrow (a\xi, a\eta)$ with $a_j = \pm 1$. Since $\Phi_i, \tilde{\Phi}_i$ are tangent to the identity, then $D\Upsilon_i(0)$ is diagonal because

$$L\Upsilon_i = \Upsilon.$$

Clearly, Υ commutes with each non-linear transformation $E_{\hat{\Lambda}_i}$. Simplifying the linear parts of both sides of (7.21), we get

$$(7.23) \quad \Upsilon^{-1} \circ ((B_i)_* \circ Z_j \circ (B_i)_*^{-1}) \circ \Upsilon = (B_i)_* \circ Z_{\nu_i(j)} \circ (B_i)_*^{-1}.$$

From the commutativity of Υ and $E_{\hat{\Lambda}_i}$ again and the above identity, it follows that

$$(7.24) \quad \Upsilon^{-1} \circ \hat{\tau}_{ij} \circ \Upsilon = \hat{\tau}_{i\nu_i(j)}, \quad j = 1, \dots, p, \quad i = 1, 2.$$

Using (7.15) and (7.24), we can rewrite (7.20) as

$$\Upsilon^{-1} \Phi_i \hat{\tau}_{ij} \Phi_i^{-1} \Upsilon = \tilde{\tau}_{i\nu_i(j)} = \tilde{\Phi}_i \Upsilon^{-1} \hat{\tau}_{ij} \Upsilon \tilde{\Phi}_i^{-1}.$$

It is equivalent to $\Upsilon_i^* \hat{\tau}_{ij} = \hat{\tau}_{ij} \Upsilon_i^*$, where we define

$$\Upsilon_i^* := \Phi_i^{-1} \Upsilon \tilde{\Phi}_i \Upsilon^{-1}.$$

Therefore, by (7.10), in $\mathcal{C}(\hat{\tau}_i)$, $\tilde{\Phi}_i$ and Φ_i are equivalent, if and only if

$$\tilde{\Phi}_i = \Upsilon^{-1} \Phi_i \Upsilon_i^* \Upsilon, \quad \Upsilon_i^* \in \mathcal{C}(\hat{\tau}_{i1}, \dots, \hat{\tau}_{ip}), \quad i = 1, 2.$$

Conversely, if Υ_i^*, Υ satisfy (7.17)-(7.19), we take $\Upsilon_i = \Upsilon_i^* \Upsilon$ to get (7.22) by (7.18). Note that (7.19) ensures that Υ commutes with $\hat{\tau}_i$ and $E_{\hat{\Lambda}_i}$. Then (7.24), or equivalently (7.23) (i.e. (7.17)) as Υ commutes with $E_{\hat{\Lambda}_i}$, gives us (7.21). By Lemma 7.1, (7.21)-(7.22) are equivalent to (7.20). q.e.d.

Proposition 7.4. *Let $\{\tau_{ij}\}$, $\{\tilde{\tau}_{ij}\}$, Φ_i , and $\tilde{\Phi}_i$ be as in Proposition 7.3. Decompose $\Phi_i = \Phi_{i1} \circ \Phi_{i0}^{-1}$ with $\Phi_{i1} \in \mathcal{C}^c(\hat{\tau}_{i1}, \dots, \hat{\tau}_{1p})$ and $\Phi_{i0} \in \mathcal{C}(\hat{\tau}_{i1}, \dots, \hat{\tau}_{1p})$, and decompose $\tilde{\Phi}_i$ analogously. Then $\{\{\tau_{1j}\}, \{\tau_{2j}\}\}$ and $\{\{\tilde{\tau}_{1j}\}, \{\tilde{\tau}_{2j}\}\}$ are equivalent under a mapping that is tangent to the identity if and only if $\Phi_{i1} = \tilde{\Phi}_{i1}$ for $i = 1, 2$. Assume further that $\tau_{2j} = \rho \tau_{1j} \rho$ and $\tilde{\tau}_{2j} = \rho \tilde{\tau}_{1j} \rho$. Then two families are equivalent under a mapping that is tangent to the identity and commutes with ρ if and only if $\Phi_{11} = \tilde{\Phi}_{11}$.*

Proof. When restricting to changes of coordinates that are tangent to the identity, we have $\Upsilon = I$ in (7.20). Also (7.17) holds trivially as ν_i is the identity. By the uniqueness of the decomposition $\Phi_i = \Phi_{i1} \Phi_{i0}^{-1}$, (7.18) becomes $\Phi_{i1} = \tilde{\Phi}_{i1}$. q.e.d.

We consider a general case without restriction on coordinate changes.

Lemma 7.5. *Let $\Upsilon = \text{diag}(\mathbf{a}, \mathbf{a})$ with $\mathbf{a} \in \{-1, 1\}^p$. Let \mathbf{B} be a nonsingular $p \times p$ matrix and let ν be a permutation of $\{1, \dots, p\}$. Then*

$$(7.25) \quad \Upsilon^{-1} \circ B_* \circ Z_j \circ B_*^{-1} \circ \Upsilon = B_* \circ Z_{\nu(j)} \circ B_*^{-1}, \quad 1 \leq j \leq p$$

if and only if

$$(7.26) \quad \mathbf{B} = (\text{diag } \mathbf{a})^{-1} \mathbf{B} (\text{diag } \mathbf{d}).$$

In particular, if \mathbf{B} is an upper or lower triangular matrix, then $\nu = I$ and $\mathbf{d} = \mathbf{a}$.

Proof. Let $\tilde{\mathbf{Z}}_j = \text{diag}(1, \dots, -1, \dots, 1)$ be the matrix where -1 at the j -th place. Set $\mathbf{C} := \mathbf{B}^{-1} \text{diag } \mathbf{a} \mathbf{B}$ and $\mathbf{C} = (c_{ij})$. In 2×2 block matrices, we see that (7.25) is equivalent to $\mathbf{C} \tilde{\mathbf{Z}}_{\nu(j)} = \tilde{\mathbf{Z}}_j \mathbf{C}$, i.e.

$$-c_{i\nu(j)} = c_{i\nu(j)}, \quad i \neq j.$$

Therefore, $\mathbf{C} = \text{diag}_{\nu} \mathbf{d}$ with $d_j = c_{j\nu(j)}$, by (3.20). q.e.d.

We will assume that M is a higher order perturbation of non-resonant product quadric. Let us recall $\hat{\sigma}$ be given by (7.3) and define $\hat{\tau}_{ij}$ as follows:

$$\hat{\sigma} : \begin{cases} \xi'_j = \hat{\mu}_j(\xi\eta)\xi_j \\ \eta'_j = \hat{\mu}_j^{-1}(\xi\eta)\eta_j, \end{cases} \quad \hat{\tau}_{ij} : \begin{cases} \xi'_j = \hat{\Lambda}_{ij}(\xi\eta)\eta_j \\ \eta'_j = \hat{\Lambda}_{ij}^{-1}(\xi\eta)\xi_j \\ \xi'_k = \xi_k \\ \eta'_k = \eta_k, \quad k \neq j \end{cases}$$

with $\hat{\Lambda}_{2j} = \hat{\Lambda}_{1j}^{-1}$ and $\hat{\mu}_j = \hat{\Lambda}_{1j}^2$. Let $\hat{\tau}_i = \hat{\tau}_{i1} \cdots \hat{\tau}_{1p}$. Recall that $E_{\hat{\Lambda}_i}$ in (7.5).

Proposition 7.6. *Let $\{\tau_{11}, \dots, \tau_{1p}, \rho\}$ be the family of involutions with ρ be given by (1.3). Suppose that the linear parts of τ_{1j} are given by (7.6) and associated σ is non-resonant, while the associated matrix \mathbf{B}*

for $\{T_{1j}\}$ satisfies the non-degeneracy condition that (7.26) holds only for $\nu = I$. Let $\hat{\sigma}$ be the formal normal form of the σ associated to M that is given by (7.3) in which $\hat{\mu}$ is in the formal normal form (1.7). Let $\hat{\tau}_{1j}$ be given by (7.9) and $\hat{\tau}_{2j} = \rho\hat{\tau}_{1j}\rho$. Then in suitable formal coordinates the involutions τ_{ij} have the form

$$(7.27) \quad \begin{aligned} \tau_{1j} &= \Psi\hat{\tau}_{1j}\Psi^{-1}, & \tau_{2j} &= \rho\tau_{1j}\rho, \\ \Psi &\in \mathcal{C}(\hat{\tau}_1) \cap C^c(\hat{\tau}_{11}, \dots, \hat{\tau}_{1p}), & \Psi'(0) &= \mathbf{I}. \end{aligned}$$

Assume further that $\tilde{\tau}_{11}, \dots, \tilde{\tau}_{1p}$ have the form (7.27) in which Ψ is replaced by $\tilde{\Psi}$. Then there exists a formal mapping R commuting with ρ and transforming the family $\{\tilde{\tau}_{11}, \dots, \tilde{\tau}_{1p}\}$ into $\{\tau_{11}, \dots, \tau_{1p}\}$ if and only if R is an R_ϵ defined by (7.1) and

$$(7.28) \quad \tilde{\Psi} = R_\epsilon^{-1}\Psi R_\epsilon, \quad R_\epsilon\rho = \rho R_\epsilon.$$

In particular, $\{\tau_{11}, \dots, \tau_{1p}, \rho\}$ is formally equivalent to $\{\hat{\tau}_{11}, \dots, \hat{\tau}_{1p}, \rho\}$ if and only if Ψ in (7.27) is the identity map.

Proof. We apply Proposition 7.3. We need to refine the equivalence relation (7.17)-(7.19). First we know that (7.19) means that $\Upsilon = R_\epsilon$ and it commutes with ρ . It remains to refine (7.18). We have $\Phi_2 = \rho\Phi_1\rho$. By assumption, we know that ν_1 in (7.17) must be the identity. Then $\Phi_1 \in \mathcal{C}^c(\hat{\tau}_{11}, \dots, \hat{\tau}_{1p})$ implies that $\Upsilon^{-1}\Phi_1\Upsilon \in \mathcal{C}^c(\hat{\tau}_{11}, \dots, \hat{\tau}_{1p})$; indeed by (7.26) we have

$$\Upsilon E_{\Lambda_i} \circ (B_i)_* = E_{\Lambda_i} \circ \Upsilon \circ (B_i)_* = E_{\Lambda_i} \circ (B_i)_* \circ D, \quad \tilde{\mathbf{D}} = \text{diag}(\text{diag } \mathbf{a}, \text{diag } \mathbf{d}).$$

Note that $\psi_0 = (U, V)$ is in $\mathcal{C}_2(Z_1, \dots, Z_p)$ if and only if

$$U(\xi, \eta) = \tilde{U}(\xi, \eta_1^2, \dots, \eta_p^2), \quad V_j(\xi, \eta) = \eta_j \tilde{V}_j(\xi, \eta_1^2, \dots, \eta_p^2).$$

Let $\psi_1 = (U, V)$ be in $\mathcal{C}_2^c(Z_1, \dots, Z_p)$, i.e.

$$\begin{aligned} U(\xi, \eta) &= \sum_i \eta_i \tilde{U}_i(\xi, \eta_1^2, \dots, \eta_i^2), \\ V_j(\xi, \eta) &= V_j^*(\xi, \eta) + \eta_j \sum_i \eta_i \tilde{V}_i(\xi, \eta_1^2, \dots, \eta_i^2), \end{aligned}$$

where $V_j^*(\xi, \eta)$ is independent of η_j . Since \mathbf{D} is diagonal, then $D\psi_1 D^{-1}$ is in $\mathcal{C}_2^c(Z_1, \dots, Z_p)$. This shows that conjugation by Υ preserves the set $\mathcal{C}^c(\hat{\tau}_{11}, \dots, \hat{\tau}_{1p})$. Also Υ commutes with each $\hat{\tau}_{1j}$. Hence, it preserves $\mathcal{C}(\hat{\tau}_{11}, \dots, \hat{\tau}_{1p})$. By the uniqueness of decomposition, (7.18) becomes

$$\tilde{\Phi}_{11} = \Upsilon^{-1}\Phi_{11}\Upsilon, \quad \tilde{\Phi}_{10}^{-1} = \Upsilon^{-1}\Phi_{10}^{-1}\Upsilon_1^*\Upsilon.$$

The second equation defines Υ_1^* that is in $\mathcal{C}(\hat{\tau}_{11}, \dots, \hat{\tau}_{1p})$ as $\Upsilon, \Phi_{10}, \tilde{\Phi}_{10}$ are in the centralizer. Rename $\Phi_{11}, \tilde{\Phi}_{11}$ by $\Psi, \tilde{\Psi}$. This shows that the equivalence relation is reduced to (7.28). q.e.d.

We now derive the following formal normal form.

Theorem 7.7. *Let M be a real analytic submanifold that is a third order perturbation of a non-resonant product quadric. Assume that the formal normal form $\hat{\sigma}$ of the map σ associated to μ satisfying condition L and the $\hat{\mu}$ associated with $\hat{\sigma}$ is in the formal normal form (1.7). Let $E_{\hat{\Lambda}_1}$ be defined by (7.4). Then M is formally equivalent to a formal submanifold in the (z_1, \dots, z_{2p}) -space defined by*

$$\tilde{M}: z_{p+j} = (\lambda_j^{-1}U_j(\xi, \eta) - V_j(\xi, \eta))^2, \quad 1 \leq j \leq p,$$

where $(U, V) = E_{\hat{\Lambda}_1(0)}^{-1}\Psi^{-1}$, Ψ is tangent to the identity and is in $\mathcal{C}(\hat{\tau}_1) \cap \mathcal{C}^c(\hat{\tau}_{11}, \dots, \hat{\tau}_{1p})$, defined in Lemma 7.2, and ξ, η are solutions to $z_j = U_j(\xi, \eta) + \lambda_j V_j(\xi, \eta)$, $\bar{z}_j = \overline{U_j \circ \rho(\xi, \eta) + \lambda_j V_j \circ \rho(\xi, \eta)}$, $1 \leq j \leq p$. Furthermore, the Ψ is uniquely determined up to conjugacy $R_\epsilon \Psi R_\epsilon^{-1}$ by an involution $R_\epsilon: \xi_j \rightarrow \epsilon_j \xi_j, \eta_j \rightarrow \epsilon_j \eta_j$ for $1 \leq j \leq p$ that commutes with ρ , i.e. $\epsilon_{s+s_*} = \epsilon_s$ and $\epsilon_j^2 = 1$. The formal holomorphic automorphism group of \tilde{M} consists of involutions of the form

$$L_\epsilon: z_j \rightarrow \epsilon_j z_j, \quad z_{p+j} \rightarrow z_{p+j}, \quad 1 \leq j \leq p$$

with ϵ satisfying $R_\epsilon \Psi = \Psi R_\epsilon$ and $\epsilon_{s+s_*} = \epsilon_s$. If the σ associated to M is holomorphically equivalent to a Poincaré-Dulac normal form, then \tilde{M} can be achieved by a holomorphic transformation too.

Proof. We first choose linear coordinates so that the linear parts of $\{\tau_{11}, \dots, \tau_{1p}, \rho\}$ are in the normal form in Lemma 3.2. We apply Proposition 7.6 and assume that τ_{ij} are already in the normal form. The rest of the proof is essentially in Proposition 2.1 and we will be brief. Write $T_{1j} = E_{\hat{\Lambda}_1(0)} \circ Z_j \circ E_{\hat{\Lambda}_1(0)}^{-1}$. Let $\psi = (U, V)$ with U, V being given in the theorem. We obtain

$$\tau_{1j} = \Psi \hat{\tau}_{1j} \Psi^{-1} = \psi^{-1} T_{1j} \psi, \quad 1 \leq j \leq p.$$

Let $f_j = \xi_j + \lambda_j \eta_j$ and $h_j = (\lambda_j \xi_j - \eta_j)^2$. The invariant functions of $\{T_{11}, \dots, T_{1p}\}$ are generated by $f_1, \dots, f_p, h_1, \dots, h_p$. This shows that the invariant functions of $\{\tau_{11}, \dots, \tau_{1p}\}$ are generated by $f_1 \circ \psi, \dots, f_p \circ \psi, h_1 \circ \psi, \dots, h_p \circ \psi$. Set $g := f \circ \psi \circ \rho$. We can verify that $\phi = (f \circ \psi, g)$ is biholomorphic. Now $\phi \rho \phi^{-1} = \rho_0$. Let M be defined by

$$z_{p+j} = E_j(z', \bar{z}'), \quad 1 \leq j \leq p,$$

where $E_j = h_j \circ \phi^{-1}$. Then $E_j \circ \phi$ and $z_j \circ \phi = f_j$ are invariant by $\{\tau_{1k}\}$. This shows that $\{\phi \tau_{ij} \phi^{-1}\}$ has the same invariant functions as deck transformations of π_1 of the complexification \mathcal{M} of M . By Lemma 2.5 in [12], $\{\phi \tau_{1j} \phi^{-1}\}$ agrees with the unique set of generators for the deck transformations of π_1 . Then M is a realization of $\{\tau_{11}, \dots, \tau_{1p}, \rho\}$.

Finally, we identify the formal automorphisms of M , which fix the origin. For such an automorphism F on \mathbf{C}^n , define

$$\tilde{F}(z', w') = (F(z', E(z', w')), \bar{F}(w', \bar{E}'(w', z')))$$

on \mathcal{M} . Then $\phi^{-1}\tilde{F}\phi$ preserves $\{\tau_{11}, \dots, \tau_{1p}, \rho\}$. By Proposition 7.6, $\phi^{-1}\tilde{F}\phi = R_\epsilon$, $R_\epsilon\rho = \rho R_\epsilon$, and $R_\epsilon\Psi = \Psi R_\epsilon$. Given (7.1), we write $R_\epsilon = (L'_\epsilon, L'_\epsilon)$. In view of $(U, V) = E_{\hat{\Lambda}_1(0)}E_{\hat{\Lambda}_1}^{-1}\Psi^{-1}$, we obtain that $L'_\epsilon U = UR_\epsilon$ and $L'_\epsilon V = VR_\epsilon$. Since $z_j = U_j(\xi, \eta) + \lambda_j V_j(\xi, \eta)$ and $z_{p+j} = (\lambda_j^{-1}U_j(\xi, \eta) - V_j(\xi, \eta))^2$, then $z' \circ \tilde{F} = L'_\epsilon z'$ and $z'' \circ \tilde{F} = z''$ as functions in (z', w') . This shows that $z' \circ F = L'_\epsilon z'$ and $z'' \circ F = z''$ as functions in (z', z'') . Therefore, $F = L_\epsilon$. q.e.d.

Remark 7.8. Let b be on the unit circle with $0 \leq \arg b < \pi$. Let

$$\mathbf{B} = \begin{pmatrix} 1 & b \\ \tilde{b} & 1 \end{pmatrix}, \quad |\tilde{b}| \leq 1, \quad b\tilde{b} \neq 1.$$

One can check that (7.26) admits a solution $\nu \neq I$ if and only if $\tilde{b} = -\bar{b}$.

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