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VARIATIONAL INEQUALITIES

by Guido STAMPACCHIA

In these last few years the theory of variational inequalities, is being developed very fast, having as model the variational theory of boundary value problems for partial differential equations. The theory of variational inequalities represents, in fact, a very natural generalization of the theory of boundary value problems and allows us to consider new problems arising from many fields of applied Mathematics, such as Mechanics, Physics, the Theory of convex programming and the Theory of control.

While the variational theory of boundary value problems has its starting point in the method of orthogonal projection, the theory of variational inequalities has its starting point in the projection on a convex set.

The first existence theorem for variational inequalities [1] was proved in connection with the theory of second order equations with discontinuous coefficients [2] in order to bring together again, as it was at the beginning, potential theory and the theory of elliptic partial differential equations.

It turned out that many other problems could be fitted in this theory and that many other theories were closely related.

Let X be a reflexive Banach space over the reals with norm $\| \cdot \|$ and denote by X' its dual and by $\langle \cdot, \cdot \rangle$ the pairing between X and X' . Let A be an operator from X into X' ; fix a closed convex subset K of X and consider the following problem :

PROBLEM 1. — *To find $u \in K$ such that*

$$(1) \quad \langle Au, v - u \rangle \geq 0 \quad \text{for all } v \in K$$

Problem 1 is what is called a variational inequality and any element $u \in K$ which satisfies (1) is called a solution.

Note that in the case when $K = X$ (or u is an interior point of K), (1) reduces to the equation $Au = 0$, since then the $(v - u)$ ranges over a neighborhood of the origin in X .

When A is a coercive linear operator from an Hilbert space H to its dual H' , the existence and uniqueness of the solution was proved in [1] and [3]. In [3] also the case when A is assumed to be only positive or semicoercive was considered. This last case includes the problem of Signorini [4].

When A is a non linear operator the existence theorem of the solution of Problem 1 was proved by Hartman-Stampacchia [5] and Browder [6], assuming that A is a monotone hemicontinuous operator.

We recall that A is a *monotone* operator from X into X' if the condition

$$(2) \quad \langle Au - Av, u - v \rangle \geq 0 \quad \text{for all } u, v \in X$$

holds. If in (2) equality holds only for $u = v$ the operator is called *strictly monotone*. The operator A is *hemicontinuous* if the map $t \rightarrow \langle A((1-t)u + tv), w \rangle$ is continuous from $[0, 1]$ into R^1 for u, v, w in X .

The theorem mentioned above is the following

THEOREM 1. — *Let A be a monotone, hemicontinuous operator from X into X' , and K a bounded, closed convex set of X . Then there exists at least one solution of Problem 1.*

Moreover, the set of all solutions of Problem 1 is a closed convex subset of K , which reduces to a single point of K if A is strictly monotone.

When K is unbounded, consider the closed ball Σ_R in X with center in the origin and radius R and the closed convex set

$$K_R = K \cap \Sigma_R$$

Then

THEOREM 2. — *Problem 1 has solution if and only if there exists an $R > 0$ such that at least one solution of the problem*

$$u_R \in K_R : \langle Au_R, v - u_R \rangle \geq 0 \quad \text{for all } v \in K_R$$

(which exists because of Theorem 1) satisfies the inequality

$$\|u_R\| < R.$$

A sufficient condition in order that the condition of Theorem 2 hold is the *coerciveness* of the operator A ; i.e. we assume that there exists $v_0 \in K$ such that

$$(3) \quad \langle Av, v - v_0 \rangle / \|v\| \rightarrow +\infty \quad \text{as } \|v\| \rightarrow +\infty, v \in K.$$

The variational inequalities generalize the theory of equations ; on the other hand a variational inequality can be reduced to an equation with the following device.

Define the multivalued map χ from K into X' in the following way : set for $u \in K$ $\chi(u) = 0$ and for $u \in \partial K$ let $\chi(u)$ be the set of elements of X' such that

$$(4) \quad \langle \chi(u), v - u \rangle \geq 0 \quad \text{for all } v \in K ;$$

(4) defines the set of all supporting planes to K in u .

Then (1) can be written as

$$A(u) \in \chi(u)$$

or

$$A(u) - \chi(u) \ni 0.$$

In fact (1) means that $\langle Au, v - u \rangle = 0$ is a supporting plane of K at u , and the convex K is in the half space $X^+(u)$ where $\langle Au, v - u \rangle \geq 0$.

In general the variational inequality is satisfied, not only by the ν 's in K , but by all the ν 's in $X^+(u)$.

Assume that A is a strictly monotone hemicontinuous operator from X into X' and coercive, i.e. (3) is satisfied. Let u_0 be the solution of the equation

$$A(u_0) = 0.$$

Let K be a closed convex set of X and u the solution of the related variational inequality.

If $u_0 \in K$, then $u = u_0$; if $u_0 \notin K$ then $u \in \partial K$ and u belongs to that part of ∂K where at least one of the supporting planes separates u_0 from K . In other words this means that u can be seen from u_0 in $X - K$ or that the segment joining u_0 to u is completely outside of K .

Let $u_0 \notin K$; if the fact just mentioned were not true, we would have

$$\langle A(u), u_0 - u \rangle \leq 0$$

and thus

$$\langle A(u_0) - A(u), u_0 - u \rangle \leq 0$$

which implies $u_0 = u$, contradicting $u \in K$ and $u_0 \notin K$.

An analogous relation holds between the solutions related to two convex sets K_1, K_2 such that $K_1 \supset K_2$.

These facts have been used in order to compute solutions of variational inequalities in finite dimensional spaces [7].

Another approach to variational inequalities is to consider Problem 1 as limit for $\epsilon \rightarrow 0$ of a sequence u_ϵ of solutions of monotone equations. This approach has been used by H. Lewy and G. Stampacchia [8] in a special case. J.L. Lions [9] has shown that this can be done in a very general situation; it is enough that the norm of X and that of X' are strictly convex.

In this case it is possible to reduce Problem 1 to the sequences of problems

$$A(u_\epsilon) + \frac{1}{\epsilon} \beta(u_\epsilon) = 0$$

where β is called a "penalization", namely a bounded, hemicontinuous monotone operator such that

$$\{\nu \mid \nu \in X, \beta(\nu) = 0\} \equiv K.$$

This method of penalization can also lead to theorems of regularization of which we shall speak later on.

A very important tool for variational inequalities is a lemma due essentially to Minty [5], [6].

LEMMA 1. — *Let A be a monotone hemicontinuous operator; u is a solution of the variational inequality (1) if and only if*

$$u \in K, \quad \langle A\nu, \nu - u \rangle \geq 0 \quad \text{for all } \nu \in K.$$

Many of the results I have mentioned above hold under a more general assumption about A , namely that A is pseudo-monotone. This notion is due to Brezis [10]. It requires that (i) A is bounded and (ii) if $u_i \rightarrow u$ weakly in X and if

$$\limsup \langle A(u_i), u_i - u \rangle \leq 0 \quad \text{then}$$

$$\liminf \langle A(u_i), u_i - v \rangle \geq \langle A(u), u - v \rangle \quad \text{for all } v \in X.$$

2. Following the thought that the theory of variational inequalities generalizes the theory of boundary value problems, the next problem which appears to be natural is the problem of the regularity of the solutions of variational inequalities.

This problem has been studied from an abstract point of view by H. Brezis and Stampacchia [11]. Let us write our variational inequality (1) in the form

$$(5) \quad u \in K \quad \langle Au - f, v - u \rangle \geq 0 \quad \text{for all } v \in K$$

where f is an element of X' . Assuming that f belongs to a subspace W of X' , when can we assure that $A(u) \in W$?

Assume that W is a reflexive Banach space such that

$$(6) \quad (i) W \subset X', \quad (ii) \| \cdot \|_{X'} \leq \text{const} \| \cdot \|_W, \quad (iii) W \text{ is dense in } X'.$$

Let J be a duality map between W and W' (dual of W) and assume that for any $u \in K$ and $\epsilon > 0$ there exists $u_\epsilon \in K$ such that $Au_\epsilon \in W$ and

$$u_\epsilon + \epsilon J(Au_\epsilon) = u.$$

Then the existence theorem 1 and its generalization for the variational inequality (1) give

$$Au \in W.$$

Theorems on the regularity of the solutions of variational inequalities have been considered for many special problems. In order to describe some of these results, I would like to confine myself to special operators.

Let $a(p)$ be a continuous vector field defined on \mathbb{R}^N . The field is called *monotone* if for any vectors p, q in \mathbb{R}^N , the following condition holds

$$(6) \quad (a(p) - a(q)) \cdot (p - q) \geq 0.$$

If in (6) equality holds only if $p = q$, the field $a(p)$ is called *strictly monotone*. We shall say that the vector field $a(p)$ is *locally coercive* if, for any compact set C of \mathbb{R}^N , there exists a positive constant $\nu(C)$, such that

$$(7) \quad (a(p) - a(q)) \cdot (p - q) \geq \nu(C) |p - q|^2 \quad \text{for all } p, q \in C.$$

We consider, formally, the operator

$$(8) \quad Au = - \frac{\partial}{\partial x_i} a_i(u_x)$$

where u_x denotes the gradient of a function u defined in a bounded open set Ω . The operator (8) is defined on the Sobolev spaces $H^{1,\alpha}(\Omega)$ of functions u such that $u \in L^\alpha(\Omega)$ and $u_x \in (L^\alpha)^N$, $1 < \alpha \leq +\infty$, only if some conditions on

the growth of the $a_i(p)$ are fulfilled. For instance, A is defined on the space $H^1 = H^{1,2}$ if the $a_i(p)$ are linear functions or on the space $H^{1,\alpha}(\Omega)$ if the $a_i(p)$ are bounded by $\text{const. } |p|^{\alpha-1}$ ($\alpha > 1$). Only in the case of Lipschitz functions (that we will denote by $H^{1,\infty}(\Omega)$) no conditions on the growth of the $a_i(p)$ are needed. In any case the operator is defined in the sense of distributions and in its domain it is monotone and hemicontinuous. Denote by $H^{1,\alpha}(\Omega)$ the domain of the operator (8) and consider the following example of variational inequality.

Let K_α be the closed convex subset of $H_0^{1,\alpha}(\Omega)$ (functions of $H^{1,\alpha}(\Omega)$ vanishing on the boundary of Ω)

$$(9) \quad K_\alpha \equiv \{v(x) \in H_0^{1,\alpha}(\Omega) \ ; \ v(x) \geq \psi(x) \text{ in } \Omega \}$$

where $\psi(x)$ is a given function of the domain of A subject to the condition of being negative on $\partial\Omega$. For $\alpha > 1$, it follows from theorem 1 and theorem 2 that there exists a solution u such that

$$(10) \quad u \in K_\alpha \ ; \ \int_{\Omega} a_i(u_x) (v - u)_{x_i} dx \geq 0 \quad \text{for all } v \in K_\alpha$$

provided that A satisfies a coerciveness condition ; for instance,

$$(11) \quad a_i(p) p_i \geq c |p|^\alpha \quad c > 0.$$

Recently H. Lewy and G. Stampacchia [12] have proved that in the case $\alpha = +\infty$ condition (11) can be dropped if Ω is supposed to be convex. We have proved

THEOREM 3. – *There exists, for any monotone field $a(p)$ a function $u \in H_0^{1,\infty}(\Omega)$ such that*

$$(12) \quad u \in K_\infty \ ; \ \int_{\Omega} a_i(u_x) (v - u)_{x_i} dx \geq 0 \quad \text{for all } v \in K_\infty$$

The Lipschitz coefficient of u is no greater than that of the obstacle ψ . Such a solution u is unique provided that the field $a(p)$ is strictly monotone.

This result contains, as special case, the problem of minimizing the integral

$$\int_{\Omega} f(\text{grad } v) dx$$

where f is a convex function in \mathbf{R}^N and v ranges in K_∞ . The special case

$$f(p) = (1 + p^2)^{1/2}$$

deals with the surface of minimal area among all surfaces $u(x)$ with given boundary values which stay above the obstacle represented by ψ . In the same paper [12] we have proved that the solution of theorem 3 has Hölder continuous first derivatives provided that $a(p)$, Ω and ψ are suitably smooth.

A similar problem, for linear $a(p)$, was treated in a previous paper by the same Authors [8]. Other results were obtained in [11] in the case of a non linear field $a(p)$. All these papers deal with coercive problems. Problems of the type considered in theorem 3 for the case of minimal surfaces have been the subject of investigation by several Authors. Minimal surfaces with obstacles have been studied by

J.C.C. Nitsche [13] for $N = 2$. Also for $N = 2$, Kinderlehrer [14] has considered the interesting question of the nature of the set where the solution and the obstacle coincide. M. Miranda [15] solves for all N the *parametric* minimal surface problem with obstacles.

In the special case of minimal surface the condition of convexity on Ω could be weakened with a condition on the mean curvature on $\partial\Omega$ as was done by Serrin [16] in the case of Dirichlet problem.

Other interesting problems of the same type arise when the obstacle does not belong to the domain of A . The case of linear $a(p)$ has been treated in a paper just appeared [17] and the case of continuous ψ in the situation of theorem 3 is treated in [12]. It turns out that the solution is the greatest lower bound of all smooth supersolutions which are $\geq \psi$ in Ω and non negative on $\partial\Omega$.

For lacking of space I cannot treat many other interesting examples of variational inequalities. I refer to the book of Lions [9] and to the expository papers [18] [19] [22]. I would like to add only that recently some achievements have been obtained on the problem of Signorini of unilateral constraints on the boundary by H. Beirao da Veiga [20] and H. Brézis [21].

For more general results and for the problems of evolution, I refer to the lecture of J.L. Lions [23].

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