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# Entropies and fractal dimensions

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Abstract: In this paper, we discuss the relation between entropy and the fractal dimension, a statistical index which is measuring the complexity of a given pattern, embedded in given spatial dimensions. We will consider the Shannon entropy and the generalized entropies of Tsallis and Kaniadakis.

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In 1948 [1], Claude Shannon defined the entropy  $H$  for a discrete random variable  $X$ , as given by  $H(X) = -\sum_i p(x_i) \log_b p(x_i)$ . In this expression, the probability of  $i$ -event is  $p_i$  and  $b$  is the base of the used logarithm (common value is Euler's number  $e$ ). In information theory, this entropy became the measure of information. Several other entropic formalisms are available for having different approaches to the measure of information, which is present in a given distribution. Here, we will discuss in a simple approach, the link between entropy and the fractal dimension.

The fractal dimension is a statistical index, measuring the complexity of a given pattern, which is embedded in given spatial dimensions. It has also been characterized as a measure of the space-filling capacity of a pattern that tells how a fractal scales differently from the space it is embedded in.[2-4] The idea of a fractional approach to calculus has a long history in mathematics (see references in [5]), but the term became popular with the works of Benoit Mandelbrot, in particular from his 1967 paper where he discussed the fractional dimensions [6]. In [6], Mandelbrot cited a previous work by Lewis Fry Richardson, who was discussing how a coastline's measured length can change with the length of the rigid stick used for measurements. In this manner, the fractal dimension of a coastline is provided by the number of rigid sticks, required to measure the coastline, and by the scale of the used stick. [7]

Several formal mathematical definitions of fractal dimension exist: in this framework, we can give the following formulas, where  $N$  stands for the number of sticks used to cover the coastline,  $\epsilon$  is the scaling factor, and  $D$  the fractal dimension:

$$N \propto \epsilon^{-D}$$
$$\log_{\epsilon} N = -D = \frac{\log N}{\log \epsilon}$$

Let us show an example. We can use Google Earth satellite images and GIMP (the GNU Image Manipulation Program) to have a map and rigid sticks to repeat what Richardson considered. Here, in the Figure 1, it is shown the same approach for a part of Grand Canyon. The ruler tool of Google Earth is used to establish the reference length. In the left-upper panel, we have the rulers for 6 km, 3 km and 1 km. To determine the fractal dimension, we choose as reference length that of 6 km. In the left-

lower panel, we can see that we need about 13 rigid sticks, one-half the reference length long, to follow the rim of this part of the canyon. In the case that we used a stick, which is 1/6 long, we need 44 sticks. We can go on reducing the length of sticks.

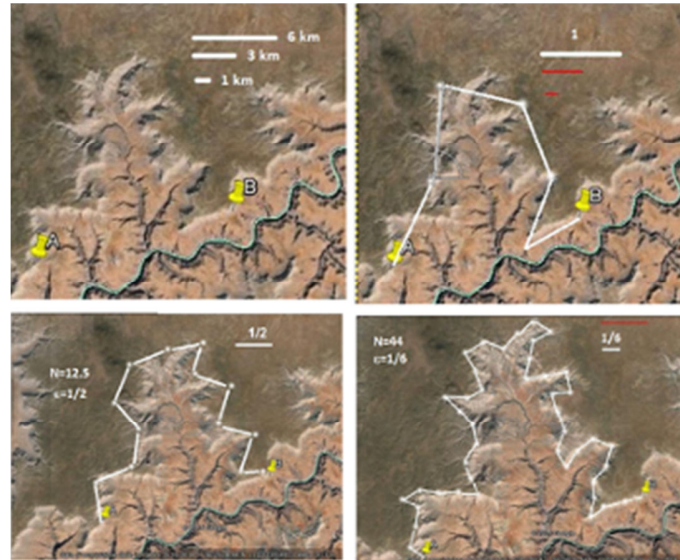


Figure 1: This example is based on Google Earth satellite images of a part of Grand Canyon, Arizona. Rigid sticks are created by GIMP. The ruler tool of Google Earth is used to establish the reference length. For evaluating the fractal dimension of the rim of the canyon, we choose as reference length that of 6 km. In the left-lower panel, we can see that we need about 13 rigid sticks, one-half the reference length long, to follow its rim. With a stick 1/6 long, we need 44 sticks.

Table 1

| N   | $\epsilon$ | D    |
|-----|------------|------|
| 13  | 1/2        | 3.70 |
| 44  | 1/6        | 2.11 |
| 119 | 1/12       | 1.92 |
| 405 | 1/30       | 1.76 |
| 871 | 1/60       | 1.65 |

In the Table I, considering the case of the Figure 1, we give the fractal dimension of the rim. Of course, when the scaled sticks are smaller, we need more images, here not shown. The process should be further iterated, to reach the limit of smaller scales. Therefore, the fractal dimension of the rim of the canyon, defined as the boundary between flat soil and steep terrain, is a number between 1 and 2. The approach here proposed is just an example to show the method to evaluate experimentally a fractal dimension (a more rigorous approach will be the subject of a future paper).

In the given framework, let us consider the role of probability. Each rigid stick has the same probability to be used and then sticks have a uniform distribution. In probability, the discrete uniform distribution is a probability distribution of a finite number N of values, which are equally likely to be observed; every one of N values has then the equal probability 1/N. An example of discrete uniform distribution is that we obtain by throwing a die. If the die has 6 faces, the possible values are 1, 2, 3, 4, 5, 6; each

time the die is thrown the probability of a given score is 1/6.

Let us consider the Shannon entropy in the case of a discrete uniform distribution. It is immediate to see that its value is equal to  $H = -\sum_i p(x_i) \ln(x_i) = -\sum_i (1/N) \ln(1/N) = -\ln(1/N) = \ln N$ . Then, an entropic definition of the fractional dimension is given by:

$$(1) \quad D = \lim_{\varepsilon \rightarrow 0} \frac{\ln N}{\ln \frac{1}{\varepsilon}} = \lim_{\varepsilon \rightarrow 0} \frac{H}{\ln \frac{1}{\varepsilon}}$$

This is an immediate relation, when H is the Shannon entropy. Besides Shannon entropy, several other entropies exist; among them the Rényi entropy R [8]. The “generalized dimension” or the “Rényi dimension” of order  $\alpha$ , is defined in the following manner [9,10]:

$$(2) \quad D_\alpha = \lim_{\varepsilon \rightarrow 0} \frac{R}{\ln \frac{1}{\varepsilon}} = \lim_{\varepsilon \rightarrow 0} \frac{\frac{1}{1-\alpha} (\ln \sum_{i=1}^N p_i^\alpha)}{\ln \frac{1}{\varepsilon}}$$

In the case of the equiprobable distribution:

$$(3) \quad D_\alpha = \lim_{\varepsilon \rightarrow 0} \frac{\frac{1}{1-\alpha} (\ln \sum_{i=1}^N \frac{1}{N^\alpha})}{\ln \frac{1}{\varepsilon}} = \lim_{\varepsilon \rightarrow 0} \frac{\frac{1}{1-\alpha} (\ln N^{1-\alpha})}{\ln \frac{1}{\varepsilon}} = \lim_{\varepsilon \rightarrow 0} \frac{\ln N}{\ln \frac{1}{\varepsilon}}$$

Other generalized entropies exist, connected to the Rényi entropy. Let us consider two of them, the Tsallis [11] and the Kaniadakis entropies [12] (for formulas and relations between entropies, see please [13,14]). In a naive approach, in analogy with Equations (1) and (3), we can define the same for these entropies. Let us start from the Tsallis entropy T (with q as entropic index), and define a Tsallis generalized dimension, substituting the Rényi entropy R:

$$(4) \quad D_q = \lim_{\varepsilon \rightarrow 0} \frac{T}{\ln \frac{1}{\varepsilon}} = \lim_{\varepsilon \rightarrow 0} \frac{\frac{1}{q-1} (1 - \sum_{i=1}^N p_i^q)}{\ln \frac{1}{\varepsilon}}$$

In the case of the equiprobable distribution:

$$(5) \quad D_q = \lim_{\varepsilon \rightarrow 0} \frac{\frac{1}{q-1} (1 - \sum_{i=1}^N \frac{1}{N^q})}{\ln \frac{1}{\varepsilon}} = \lim_{\varepsilon \rightarrow 0} \frac{\frac{1}{q-1} (1 - N^{1-q})}{\ln \frac{1}{\varepsilon}}$$

In (5), when  $0 < q < 1$ , D is greater than 0.

Now, we can define the Kaniadakis generalized dimension for Kaniadakis entropy  $K$  ( $\kappa$  is the entropic index):

$$(6) \quad D_{\kappa} = \lim_{\varepsilon \rightarrow 0} \frac{K}{\ln \frac{1}{\varepsilon}} = \lim_{\varepsilon \rightarrow 0} \frac{\frac{1}{2\kappa} (\sum_{i=1}^N p_i^{1-\kappa} - \sum_{i=1}^N p_i^{1+\kappa})}{\ln \frac{1}{\varepsilon}}$$

In the case of the equiprobable distribution:

$$(7) \quad D_{\kappa} = \lim_{\varepsilon \rightarrow 0} \frac{K}{\ln \frac{1}{\varepsilon}} = \lim_{\varepsilon \rightarrow 0} \frac{\frac{1}{2\kappa} \left( \sum_{i=1}^N \frac{1}{N^{1-\kappa}} - \sum_{i=1}^N \frac{1}{N^{1+\kappa}} \right)}{\ln \frac{1}{\varepsilon}}$$

$$(8) \quad D_{\kappa} = \lim_{\varepsilon \rightarrow 0} \frac{\frac{1}{2\kappa} (N^{\kappa} - N^{-\kappa})}{\ln \frac{1}{\varepsilon}}$$

To see the behavior of these generalized dimension when their indices are varied, we can apply them to the parameters of the Koch snowflake ( $N=4$  and  $\varepsilon=1/3$ ). In the Figure 2, we propose the generalized fractal dimensions in the case of Shannon  $S$ , Rényi  $R$ , Kaniadakis  $K$ -entropy, as a function of  $\alpha$ , using Equations (1),(3), (5) and (8). Note that  $\kappa=\alpha$ . In the case of the Tsallis entropy, we show  $T'$ , as a function of  $(1-\kappa)$ , and like  $T''$ , given as a function of  $(1+\kappa)$ .

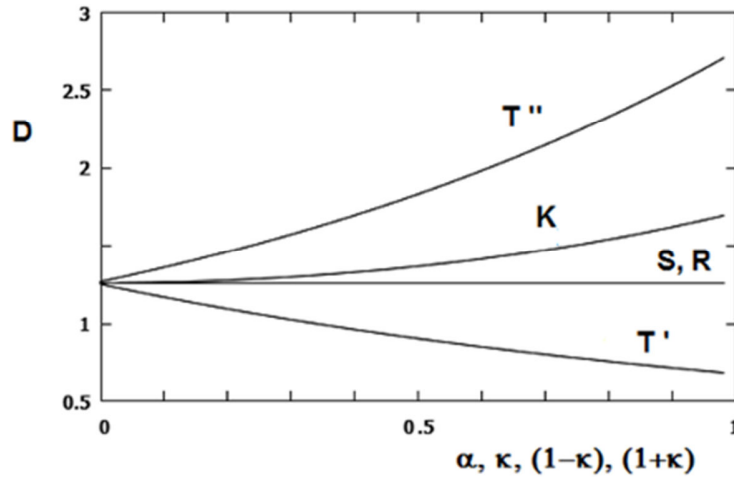


Figure 2: Behavior of the generalized fractal dimensions in the case of Shannon  $S$ , Rényi  $R$ , Kaniadakis  $K$ , as a function of  $\alpha$ . Note that  $\kappa=\alpha$ . In the case of the Tsallis entropy, we show  $T'$ , as a function of  $(1-\kappa)$ ;  $T''$  is given as a function of  $(1+\kappa)$ .

Of course, the proposed discussion was made in the framework of a equiprobable distribution and a

simple vision of fractal dimension as given for Shannon entropy. A more refined discussion for Tsallis and Kaniadakis entropies are necessary of course. Since a connection between fractals and Tsallis entropy exists [15], and the Tsallis generalized statistics seems the natural frame for studying fractal systems, we can aim considering the same also for other entropies, such as the K-entropy.

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