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A note on the inapproximability of the Minimum Monotone Satisfying Assignment problem

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Abstract

The Minimum Monotone Satisfying Assignment problem (MMSA) consists, given a monotone boolean formula φ , in searching for a minimum number of true variables such that φ is satisfied. A polynomial inapproximability ratio was given by Dinur *et al.* However, this ratio depends on a parameter that is not the size of the MMSA instance. It is instead the size of the problem from which the reduction is done. Consequently, it is hard to reuse this result to prove other hardness of approximability. In this paper, we deepen the previous work and prove two inapproximability ratio for MMSA depending on the size of the formula and the number of variables and we prove that MMSA cannot be polylogarithmically approximated.

Keywords: Combinatorial optimization, Polynomial approximation, Minimum Monotone Satisfying Assignment problem

1. Introduction

Problem 1. Minimum Monotone Satisfying Assignment (MMSA). Given a set Y of boolean variables and a monotone boolean formula φ , minimize the number of variables assigned to true such that φ is satisfied.

We call MMSA_h the MMSA problem restricted to the instances in which the height of the formula is no more than h . In other words, this formula has h levels of alternating AND and OR gates, where the top level is an AND gate. MMSA_2 is equivalent to the Set Cover problem in which, given a set \mathcal{X} and a subset \mathcal{S} of $2^{\mathcal{X}}$, we search for a minimum cover of \mathcal{X} with sets of \mathcal{S} . Therefore, MMSA_2 is NP-Complete [6] and there exists a constant c such that MMSA_2 cannot be approximated to within a $c \log(|Y|)$ ratio unless $P = NP$ [4].

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The following theorem, from Dinur *et al.* and introduced in [3], is an harder inapproximability result for MMSA_3 .

Theorem 1 ([3], page 251). *For any $c < \frac{1}{2}$, it is NP-hard to approximate MMSA_3 to within $g_c(n) \stackrel{\text{def}}{=} 2^{\log(n)^{1 - \frac{1}{\log \log(n)^c}}}$.*

A major drawback with this result is that the parameter n given in this theorem is **not** the size of the MMSA_3 instance. This parameter is instead the size of the problem from which the reduction is done to prove this result. As a consequence, researchers using the Theorem of Dinur *et al.* to prove an inapproximability result, either cite the theorem without linking the parameter n with their problem (see for example [2]), or must carefully redefine the notations [1, 5].

Our ontributions. Firstly, as the Theorem of Dinur *et al.* is an important theoretical result we deepen the previous work in order to prove that MMSA_3 is hard to approximate to within a ratio depending only on the size of a formula φ : the number of AND gates Λ , of OR gates V and the number of boolean variables $|Y|$.

Secondly, we prove a theorem which has a pratical potential arising out of the Theorem of Dinur *et al.*: MMSA_3 is hard to approximate to within any polylogarithmic ratio depending on the number of AND gates, OR gates and the number of variables. This ratio is useful in case of a reduction from MMSA_h to a problem in which the size depends polynomially on Λ , V and $|Y|$.

The next section is dedicated to study the function $g_c(n)$ of the Theorem of Dinur *et al.* Section 3 describes the links between the parameter n and the size of MMSA_3 . Finally, Section 4 contains the proofs of our results.

2. The function $g_c(n)$

We prove three useful lemmas about the function $g_c(n)$ of the Theorem of Dinur *et al.* We first rewrite $g_c(n)$, for some $c \in]0, \frac{1}{2}[$.

$$\begin{aligned} g_c(n) &= 2^{\log(n)^{\left(1 - \frac{1}{(\log \log(n))^c}\right)}} \\ &= 2^{\log(n) \cdot \log(n)^{\left(-\frac{1}{(\log \log(n))^c}\right)}} \\ &= 2^{\log(n)} \cdot 2^{\left(-(\log \log(n))^{1-c}\right)} \end{aligned}$$

We firstly show that, on the first hand, it is asymptotically dominated by all the polynomials and, on the other hand, that it asymptotically dominates all the polylogarithms.

Lemma 2. *Let $\delta > 0$ and $0 < c < \frac{1}{2}$, then $g_c(n) = o(n^\delta)$.*

PROOF. Let $0 < c < \frac{1}{2}$ and $\delta > 0$, we are interested in the limit of $\frac{g_c(n)}{n^\delta}$ as n approaches infinity.

$$\begin{aligned}\frac{g_c(n)}{n^\delta} &= \left(2^{\log(n)} \cdot 2^{-(\log \log(n))^{1-c}}\right) / \left(2^{\log(n)\delta}\right) \\ &= 2^{\log(n) \cdot \left(2^{-(\log \log(n))^{1-c}} - \delta\right)}\end{aligned}\quad (1)$$

As $1 - c > 0$

$$\begin{aligned}\lim_{n \rightarrow +\infty} -(\log \log(n))^{1-c} &= -\infty \\ \lim_{n \rightarrow +\infty} 2^{-(\log \log(n))^{1-c}} - \delta &= -\delta \\ \lim_{n \rightarrow +\infty} \log(n) \cdot \left(2^{-(\log \log(n))^{1-c}} - \delta\right) &= -\infty\end{aligned}$$

By Equation (1)

$$\lim_{n \rightarrow +\infty} \frac{g_c(n)}{n^\delta} = 0$$

□

Lemma 3. Let $\delta > 0$ and $0 < c < \frac{1}{2}$, then $\log(n)^\delta = o(g_c(n))$.

PROOF. Let $0 < c < \frac{1}{2}$ and $\delta > 0$.

We now give the limit of $\frac{\log^\delta(n)}{g_c(n)}$ as n approaches infinity.

$$\begin{aligned}\frac{\log^\delta(n)}{g_c(n)} &= \left(2^{\log \log(n) \cdot \delta}\right) / \left(2^{\log(n)} \cdot 2^{-(\log \log(n))^{1-c}}\right) \\ &= 2^{\log \log(n) \cdot \left(\delta - \left(\frac{\log(n) \cdot 2^{-(\log \log(n))^{1-c}}}{\log \log(n)}\right)\right)} \\ \frac{\log(n) \cdot 2^{-(\log \log(n))^{1-c}}}{\log \log(n)} &= 2^{\log \log(n) - (\log \log(n))^{1-c} - \log \log \log(n)} \\ &= 2^{\log \log(n) \cdot \left(1 - \frac{1}{(\log \log(n))^c} - \frac{\log \log \log(n)}{\log \log(n)}\right)}\end{aligned}\quad (2)$$

$$\quad (3)$$

However, as $c > 0$

$$\lim_{n \rightarrow +\infty} \frac{1}{(\log \log(n))^c} = 0$$

Moreover, as $\log \log \log$ is asymptotically dominated by $\log \log$

$$\lim_{n \rightarrow +\infty} \frac{\log \log \log(n)}{\log \log(n)} = 0$$

As a consequence,

$$\lim_{n \rightarrow +\infty} \log \log(n) \cdot \left(1 - \frac{1}{(\log \log(n))^c} - \frac{\log \log \log(n)}{\log \log(n)} \right) = +\infty$$

Consequently, by Equation (3)

$$\begin{aligned} \lim_{n \rightarrow +\infty} \frac{\log(n) \cdot 2^{-(\log \log(n))^{1-c}}}{\log \log(n)} &= +\infty \\ \lim_{n \rightarrow +\infty} \log \log(n) \cdot \left(\delta - \frac{\log(n) \cdot 2^{-(\log \log(n))^{1-c}}}{\log \log(n)} \right) &= -\infty \end{aligned}$$

Finally, by Equation (2)

$$\lim_{n \rightarrow +\infty} \frac{\log^\delta(n)}{g_c(n)} = 0$$

□

The following lemma shows a useful property of $g_c(n)$ that is used later in this paper.

Lemma 4. *Let $c < \frac{1}{2}$, $|F| = \Theta(g_c(n))$ and $D = O((\log \log(n))^c)$, then $|F|^D = o(n)$.*

PROOF. There are four constants c_F , n_F , c_D and n_D such that, for all $n \geq n_D$ and $n \geq n_F$

$$\begin{aligned} |F| &\leq c_F \cdot g_c(n) \\ D &\leq c_D (\log \log(n))^c \end{aligned}$$

Consequently

$$\begin{aligned} \frac{|F|^D}{n} &\leq \left(c_F^{c_D \cdot (\log \log(n))^c} \right) \cdot \left(2^{\log(n)} \cdot 2^{-(\log \log(n))^{1-c}} \cdot (c_D \cdot (\log \log(n))^c) \right) \cdot \frac{1}{n} \\ \frac{|F|^D}{n} &\leq \left(2^{c_D \cdot \log(c_F) \cdot (\log \log(n))^c} \right) \cdot \left(2^{c_D \cdot \log(n)} \cdot 2^{-(\log \log(n))^{1-c} + c \cdot \log \log \log(n)} \right) \cdot 2^{-\log(n)} \\ \frac{|F|^D}{n} &\leq 2^{\left(\log(n) \cdot \left((c_D \cdot \log(c_F) \cdot \frac{(\log \log(n))^c}{\log(n)} \right) + \left(c_D \cdot 2^{-(\log \log(n))^{1-c} + c \cdot \log \log \log(n)} \right) - 1 \right)} \end{aligned}$$

$$\begin{aligned}
& \text{As } \lim_{n \rightarrow +\infty} \frac{(\log \log(n))^c}{\log(n)} = 0 \text{ and } \lim_{n \rightarrow +\infty} -(\log \log(n))^{1-c} + c \cdot \log \log \log(n) = -\infty \\
& \lim_{n \rightarrow +\infty} \left(c_D \cdot \log(c_F) \cdot \frac{(\log \log(n))^c}{\log(n)} + c_D \cdot 2^{-(\log \log(n))^{1-c} + c \cdot \log \log \log(n)} - 1 \right) = -1 \\
& \lim_{n \rightarrow +\infty} 2^{\left(\log(n) \cdot \left((c_D \cdot \log(c_F) \cdot \frac{(\log \log(n))^c}{\log(n)})^c + (c_D \cdot 2^{-(\log \log(n))^{1-c} + c \cdot \log \log \log(n)}) - 1 \right) \right)} = 0
\end{aligned}$$

Consequently, $\lim_{n \rightarrow +\infty} \frac{|F|^D}{n} = 0$. □

3. The reduction

In this section, we describe the parameters of the formula φ with the parameter n given in the Theorem of Dinur *et al.*

We use in this section the same notations as the ones describes in [3]. The given reduction is from the PCP theorem. The parameters of the PCP instance are $n \in \mathbb{N}$, $c < \frac{1}{2}$, $c' < c$, $D = O((\log \log(n))^{c'})$, $|F| = \Theta(g_{c'}(n))$.

The following MMSA₃ instance is built where $T(x, \psi', r|_x)$ is a variable.

$$\varphi = \bigwedge_{\psi \in \Psi} \bigvee_{r \in R_\psi} \bigwedge_{x \in \psi} \bigwedge_{\psi' \in \Psi_x} T(x, \psi', r|_x)$$

The parameters satisfy: $|\Psi| = n$, $|R_\psi| = |F|^D$, $|\psi| = D$, $|\Psi_x| = n$ and the set T contains $n|F|^D$ variables. By Lemma 4, $|R_\psi|$ is polynomial.

Remark 1. Note that there are 4 levels in this formula, however the levels 3 and 4 are two AND Gates levels and can be merged into one level.

Any $g_c(n)$ -approximation for MMSA₃ could be used to solve the PCP instance in polynomial time and this proves the Theorem of Dinur *et al.*

Lemma 5. *Let $\varepsilon > 0$, then $\Lambda = O(n^{2+\varepsilon})$, $V = n$ and $|Y| = O(n^{1+\varepsilon})$.*

PROOF. The AND root gate of φ has $|\Psi| = n$ children. Any OR gate has $|R_\psi| = |F|^D$ children. The AND gates of level 3 and 4 have respectively $|\psi| = D$ and $|\Psi_x| = n$ children. The number of variables of φ is $|T| = n|F|^D$.

Consequently, $\Lambda = 1 + n|F|^D(1 + D)$, $V = n$ and $|Y| = n|F|^D$. The result for V is proven. We now have to detail the values of $n|F|^D D$ and $n|F|^D$.

Let $0 < \delta < \varepsilon$, by Lemma 2

$$\begin{aligned}
|F| &= \Theta(g_{c'}(n)) = O(n^\delta) \\
n|F|^D &= O(n^{1+\delta}(\log \log(n))^c) \\
n|F|^D &= O(n^{1+\delta}(\log \log(n))^c) \\
n|F|^D &= O(n^{1+\varepsilon})
\end{aligned}$$

This gives the result for $|Y|$.

By Lemma 4

$$\begin{aligned} |F|^D &= o(n) \\ n|F|^D D &= O(n^2(\log \log(n))^c) \\ n|F|^D D &= O(n^{2+\varepsilon}) \end{aligned}$$

And this gives the result for Λ and concludes the proof. \square

4. Main result

This section is dedicated to proving the main results of this paper. The first theorem is a rewriting of the theorem of [3] with the parameters Λ , V and $|Y|$ using the results of Lemma 5.

Theorem 6. *For all $\varepsilon > 0$ and $c > \frac{1}{2}$, it is NP-hard to approximate $MMSA_3$ to within $\max(g_c(\sqrt[2+\varepsilon]{\Lambda}), g_c(V), g_c(\sqrt[1+\varepsilon]{|Y|}))$.*

PROOF. The proof for $g_c(V)$ is immediate by Lemma 5. We now prove that there is no $g_c(\sqrt[1+\varepsilon]{|Y|})$ -approximation. The last result is similar. Lemma 5 proves that $|Y| = O(n^{1+\varepsilon})$. As g_c is a non-decreasing function, $g_c(\sqrt[1+\varepsilon]{|Y|}) \leq g_c(O(n))$. The conclusion follows from the property that $\lim_{n \rightarrow +\infty} \frac{g_c(O(n))}{g_{c'}(n)} = 0$ for every $c' \in]c; \frac{1}{2}[$: a $g_c(\sqrt[1+\varepsilon]{|Y|})$ -approximation would lead to a $g_{c'}(n)$ -approximation and this is a contradiction by the Theorem of Dinur *et al.* \square

The second theorem is a more easily to use theorem arising out of the Theorem of Dinur *et al.* It proves $MMSA$ cannot be approximated to within a polylogarithmic ratio.

Theorem 7. *Let Q be a polynomial with three variables. Unless $P = NP$, there is no polynomial approximation with ratio $Q(\log(|Y|), \log(\Lambda), \log(V))$ for $MMSA_3$.*

PROOF. The proof is given for the following polynomial $Q(x, y, z) = x^\delta$, for some $\delta > 0$. The proof for other polynomials are similar.

We assume that there is a $Q(\log(|Y|))$ -approximation for $MMSA_3$.

Let now be a PCP instance of size n and $0 < c' < c < \frac{1}{2}$. We build an instance of $MMSA_3$ such that $|Y(n)| = O(n^{1+\varepsilon})$ by Lemma 5. Thus

(A) for all $\varepsilon > 0$, there are two constants $k_\varepsilon > 0$ and $n_\varepsilon \in \mathbb{N}$ such that, for every integer $n > n_\varepsilon$, $|Y(n)| \leq k_\varepsilon \cdot n^{1+\varepsilon}$.

For all $\delta' > \delta$, \log^δ is asymptotically dominated by $\log^{\delta'}$, consequently:

(B) for all $\varepsilon > 0$, $k_\varepsilon > 0$, $\delta' > \delta$, there is an integer $n' = n(\varepsilon, k_\varepsilon, \delta') \in \mathbb{N}$ such that for every integer $n > n'$, $(\log(k_\varepsilon) + (1 + \varepsilon) \log(n))^\delta < \log^{\delta'}(n)$.

Finally, by Lemma 3, for all $0 < c < \frac{1}{2}$ and $\delta' > 0$, $g_c(n)$ asymptotically dominates $\log^{\delta'}(n)$, thus

(C) for all $\delta' > 0$, there is an integer $n_{\delta'} \in \mathbb{N}$ such that for every integer $n > n_{\delta'}$, $\log^{\delta'}(n) < g_c(n)$.

We are going to use those three properties to prove the inapproximability result. Let $\varepsilon > 0$ and $\delta' > \delta$ be two real numbers. The constants k_ε , n_ε , $n' = n(\varepsilon, k_\varepsilon, \delta')$ and $n_{\delta'}$ are given by the previous properties. We focus on the PCP instances such that $n > \max(n_\varepsilon, n', n_{\delta'})$.

$$Q(\log(|Y(n)|)) = \log^\delta(|Y(n)|)$$

By Property (A)

$$\begin{aligned} Q(\log(|Y(n)|)) &\leq \log^\delta(k_\varepsilon \cdot n^{1+\varepsilon}) \\ Q(\log(|Y(n)|)) &\leq (\log(k_\varepsilon) + (1 + \varepsilon) \log(n))^\delta \end{aligned}$$

By Property (B)

$$Q(\log(|Y(n)|)) \leq \log^{\delta'}(n)$$

By Property (C)

$$Q(\log(|Y(n)|)) \leq g_c(n)$$

As a consequence, there would be $g_c(n)$ -approximation for the MMSA₃ problem, and this would be a contradiction by the Theorem of Dinur *et al.* \square

A consequence of such a result is that, if we now consider a reduction from MMSA_h to a problem Π such that

- the size s of the instances of Π is a polynomial in $|Y|$, Λ and V ;
- the reduction proves Π cannot be approximated to within $Q(\log(|Y|), \log(\Lambda), \log(V))$ for any polynomial Q with Theorem 7,

then we can prove Π cannot be approximated to within $Q'(\log(s))$ for any polynomial Q' .

5. Conclusion

In this paper we deepened the inapproximability result of MMSA_3 presented in [3] and gave a new inapproximability ratio only depending on the size of a formula. This allows researchers to have a practical use of this important theoretical result.

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