



**HAL**  
open science

# Picard group of the forms of the affine line and of the additive group

Raphaël Achet

► **To cite this version:**

Raphaël Achet. Picard group of the forms of the affine line and of the additive group. 2016. hal-01377384v1

**HAL Id: hal-01377384**

**<https://hal.science/hal-01377384v1>**

Preprint submitted on 6 Oct 2016 (v1), last revised 21 Nov 2016 (v2)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Picard group of the forms of the affine line and of the additive group

Raphaël Achet \*

## Abstract

We obtain an explicit upper bound on the torsion of the Picard group of the forms of  $\mathbb{A}_k^1$  and  $\mathbb{G}_{a,k}$ , and a sufficient condition for this Picard group to be non trivial. We also give examples of non trivial forms of  $\mathbb{A}_k^1$  with trivial Picard groups.

## Contents

<b>1</b>	<b>Forms of the affine line and of the additive group</b>	<b>3</b>
1.1	Regular completion and invariants . . . . .	3
1.2	Structure of the forms of the additive group . . . . .	5
1.3	Examples . . . . .	6
1.4	Arithmetic genus of the regular completion . . . . .	8
1.5	Picard group of the forms of the affine line . . . . .	11
<b>2</b>	<b>Cocartesian diagram and Picard functor</b>	<b>12</b>
2.1	Unit group scheme . . . . .	13
2.2	Rigidified Picard functors . . . . .	14
2.3	An exact sequence of Picard schemes . . . . .	17
<b>3</b>	<b>Picard functor of the regular completion</b>	<b>17</b>
3.1	Torsion of the Picard functor . . . . .	17
3.2	Application to the Picard functor of the regular completion . . . . .	19
3.3	Rigidified Picard functor . . . . .	20
<b>4</b>	<b>Picard group</b>	<b>20</b>
4.1	Proof of the main theorem . . . . .	20
4.2	Examples of forms of the affine line with trivial Picard group . . . . .	21
<b>5</b>	<b>References</b>	<b>23</b>

---

\*Université Grenoble Alpes, Institut Fourier, raphael.achet@univ-grenoble-alpes.fr

# Introduction

With the recent progress in the structure of linear algebraic groups over an imperfect field [CGP15] [Tot13], it seems to be possible to study their Picard group if the Picard groups of unipotent algebraic groups are known well enough. As every unipotent smooth connected algebraic group is an iterated extension of forms of  $\mathbb{G}_{a,k}$  [SGAIII2, XVII 4.1.1], this motivates the study of the Picard group of forms of  $\mathbb{G}_{a,k}$ .

In this article, we consider more generally forms of the affine line, since our geometric approach applies to this setting without additional difficulties.

Let  $k$  be a field, let  $X, Y$  be schemes (resp. group schemes) over  $k$ . Recall that  $X$  is a form of  $Y$  if there is a field  $K \supset k$  such that the scheme (resp. group scheme)  $X_K$  is isomorphic to  $Y_K$ . We also recall that the affine line  $\mathbb{A}_k^1$  is the  $k$ -scheme  $\text{Spec}(k[t])$ ; and the additive group  $\mathbb{G}_{a,k}$  is the algebraic group of underlying scheme  $\mathbb{A}_k^1 = \text{Spec}(k[t])$ , which represent the group functor:

$$\begin{aligned} Sch/k^\circ &\rightarrow Groups \\ T &\mapsto (\mathcal{O}(T), +). \end{aligned}$$

If  $k$  is a perfect field, all forms of  $\mathbb{A}_k^1$  and  $\mathbb{G}_{a,k}$  are trivial. But non trivial forms of  $\mathbb{A}_k^1$  and  $\mathbb{G}_{a,k}$  exist over every imperfect field  $k$ ; there structure has been studied by P. Russell [Rus70], G. Greither [Gre86], T. Kambayashi, M. Miyanishi, and M. Takeuchi [KMT74]. In [Gre86] and [KMT74] the Picard group of some special forms of  $\mathbb{A}_k^1$  and  $\mathbb{G}_{a,k}$  is described [KMT74, Lem. 6.12.2] and [Gre86, Lem. 5.6].

More recently B. Totaro has obtained an explicit description of the class of extensions of a smooth connected unipotent group  $U$  by the multiplicative group as a subgroup of  $\text{Pic}(U)$  [Tot13, Lem. 9.2]. He has then applied this description to the structure of commutative pseudo-reductive groups [Tot13, Lem. 9.4] and [Tot13, Cor. 9.5]. Moreover he has constructed an example of a non trivial form of  $\mathbb{G}_{a,k}^2$ , such that  $\text{Pic}(U_{k_s})$  is trivial [Tot13, Exa. 9.7].

Given a form  $X$  of  $\mathbb{A}_k^1$ , it is known that there exists a finite purely inseparable extension  $K$  of  $k$  such that  $X_K \cong \mathbb{A}_K^1$ ; then  $\text{Pic}(X)$  is  $p^m$ -torsion, where  $p^m := [K : k]$  (see e.g. [Bri15, Lem. 2.4]). Our main theorem yields a sharper result:

**Theorem (4.1).** *Let  $X$  be a non trivial form of  $\mathbb{A}_k^1$ , and let  $n(X)$  be the smallest non-negative integer such that  $X_{k^{p^{-n}}} \cong \mathbb{A}_{k^{p^{-n}}}^1$ .*

- (i)  $\text{Pic}(X)$  is  $p^{n(X)}$ -torsion.
- (ii) If  $X$  has a rational point (e.g.  $X$  is a form of  $\mathbb{G}_{a,k}$  or  $k$  is separably closed), then  $\text{Pic}(X) \neq \{0\}$ .

The fact that a form of  $\mathbb{G}_{a,k}$  as a non trivial Picard group is a direct consequence of [KMT74, Th. 6.10.1 (ii)]. A form of  $\mathbb{A}_k^1$  does not necessary have a  $k$ -rational point. In Subsection 4.2 we present an explicit example of such a form with trivial Picard group (Lemma 4.2), and a more general construction (Proposition 4.4). We will finish by showing that the non trivial forms of  $\mathbb{G}_{a,k}$  are not special algebraic groups. This result has already been shown by D. T. Nguyễn [Ngu16], we are going to use a different method: we see it as a corollary of our main Theorem 4.1 and Proposition 4.4.

In order to show Theorem 4.1 we consider the regular completion  $C$  of the regular curve  $X$ , and we control the torsion of the Picard functor  $\text{Pic}_{C/k}^0$ . We obtain the following result:

**Theorem (3.4).** *Let  $X$  be a form of  $\mathbb{A}_k^1$  and  $C$  be the regular completion of  $X$ . Let  $n'(X)$  be the smallest non-negative integer  $n$  such that the function field of  $X_{k^{p^{-n}}}$  is isomorphic to  $k^{p^{-n}}(t)$ .*

*Then  $\text{Pic}_{C/k}^0$  is a smooth  $k$ -wound unipotent algebraic group of  $p^{n'(X)}$ -torsion.*

Theorem 3.4 is actually more precise but its complete formulation requires additional notations. The fact that  $\text{Pic}_{C/k}^0$  is smooth and  $k$ -wound is a direct consequence of results obtained in [BLR90, Chap.8 and Chap.9]. The fact that  $\text{Pic}_{C/k}^0$  is unipotent is obtained in [KMT74, Th. 6.6.10], so the original part of this theorem is the control on the torsion of  $\text{Pic}_{C/k}^0$ .

**Conventions:** Let  $k$  be a field, unless explicitly stated,  $k$  is of characteristic  $p > 0$ . We choose an algebraic closure of  $\bar{k}$ , and denote by  $k_s \subset \bar{k}$  the separable closure. For any non-negative integer  $n$  we denote  $k^{p^{-n}} = \{x \in \bar{k} \mid x^{p^n} \in k\}$ .

Let  $X$  be a scheme; we note  $\mathcal{O}_X$  the structural sheaf of  $X$ . We will denote  $\mathcal{O}(X)$  the ring of regular functions on  $X$ , and  $\mathcal{O}(X)^*$  the multiplicative group of invertible regular functions on  $X$ . Let  $x \in X$ , the stalk of  $\mathcal{O}_X$  at  $x$  is denoted  $\mathcal{O}_{X,x}$ , the residue field at  $x$  is denoted  $\kappa(x)$ .

The morphisms considered between two  $k$ -schemes are morphisms over  $k$ . An *algebraic variety* is a scheme of finite type on  $\text{Spec}(k)$ . Let  $K$  be a field extension of  $k$ , the base change  $X \times_{\text{Spec}(k)} \text{Spec}(K)$  is denoted  $X_K$ . Let  $X$  be an integral variety, the function field of  $X$  is denoted  $\kappa(X)$ . A group scheme of finite type over  $k$  will be called an *algebraic group*. A group scheme locally of finite type over  $k$  will be called a *locally algebraic group*.

A unipotent smooth connected algebraic group  $U$  over  $k$  is said to be  *$k$ -split* if  $U$  has a central composition series with successive quotients forms of  $\mathbb{G}_{a,k}$ . A unipotent smooth connected algebraic group  $U$  over  $k$  is said to be  *$k$ -wound* if every  $k$ -morphism  $\mathbb{A}_k^1 \rightarrow U$  is constant with image a point of  $U(k)$  (an equivalent definition is:  $U$  does not have a central subgroup isomorphic to  $\mathbb{G}_{a,k}$  [CGP15, Pro. B.3.2]).

**Acknowledgements:** I would like to thank Lara Thomas and Bruno Laurent for helping me to conjecture corollary 1.25, as well as Philippe Gille and Burt Totaro for their suggestions. I would very much like to thank Michel Brion for extremely useful ideas and remarks during the preparation of this article.

## 1 Forms of $\mathbb{A}_k^1$ and of $\mathbb{G}_{a,k}$

### 1.1 Regular completion and invariants

In this first subsection we introduce some notation and gather some results from Peter Russell's article [Rus70] that we will use in the rest of the article.

Let  $X$  be a form of  $\mathbb{A}_k^1$ , we will note  $C$  its regular completion, i.e., the unique projective regular curve such that there is an open dominant immersion  $j : X \rightarrow C$  satisfying the following universal property: for every morphism  $f : X \rightarrow Y$  to a proper scheme  $Y$  there exists a unique morphism  $\hat{f} : C \rightarrow Y$  such that  $\hat{f} \circ j = f$  [GW10, 15.21].

**Lemma 1.1.** [Rus70, 1.1]

*Let  $X$  be a form of  $\mathbb{A}_k^1$ , let  $C$  be the regular completion of  $X$ .*

- (i)  $C \setminus X$  is a point denoted  $P_\infty$  which is purely inseparable over  $k$ .
- (ii) There is a unique minimal field extension  $k'$  such that  $X_{k'} \cong \mathbb{A}_k^1$ , and  $k'$  is purely inseparable of finite degree over  $k$ .

Let  $\varphi_k$  be the Frobenius morphism of  $k$ , i.e. the morphism

$$\varphi_k : x \in k \mapsto x^p \in k.$$

In the following we will denote  $\varphi$  for  $\varphi_k$ .

Let  $X$  be a form of  $\mathbb{A}_k^1$ , by definition  $X = \text{Spec}(R)$  with  $R$  a  $k$ -algebra such that  $R \otimes_k k' \cong k'[t]$ . Let  $n$  be a non-negative integer, we consider

$$F_R^n : \begin{array}{ccc} R \otimes_k k & \rightarrow & R \\ r \otimes x & \mapsto & xr^{p^n} \end{array}$$

with  $k$  seen as a  $k$ -algebra via  $\varphi^n$ , the  $n$ th power of  $\varphi$ .

The morphism  $F_R^n$  corresponds at the scheme level to the  $n$ th relative Frobenius morphism  $F_X^n$ . Let  $X^{(p^n)}$  be the base change  $X \times_{\text{Spec}(k)} \text{Spec}(k)$  with  $k$  seen as a  $k$ -algebra via  $\varphi^n$ , in other words  $X^{(p^n)}$  is isomorphic to the base change of  $X$  by  $k^{p^{-n}}$ . We can then write

$$F_X^n : X \rightarrow X^{(p^n)}.$$

**Lemma 1.2.** [Rus70, 1.3]

There is an integer  $n \geq 0$  such that  $X^{(p^n)} \cong \mathbb{A}_{k^{p^{-n}}}^1$ .

**Definition 1.3.** Let  $X$  be a form of  $\mathbb{A}_k^1$ .

- (i) The smallest non-negative integer  $n$  such that  $X^{(p^n)} \cong \mathbb{A}_{k^{p^{-n}}}^1$  is denoted  $n(X)$ .
- (ii) The smallest non-negative integer  $n$  such that  $\kappa(X^{(p^n)}) \cong k^{p^{-n}}(t)$  is denoted  $n'(X)$ .
- (iii) The point  $P_\infty$  is purely inseparable (Lemma 1.1), let  $r(X)$  be the integer such that  $\deg(P_\infty) = p^{r(X)}$ .

*Remark 1.4.* (i) We have  $n(X) \geq n'(X)$ , we will show in Example 1.19 that this inequality can be strict (but equality holds if  $X$  is a form of  $\mathbb{G}_{a,k}$  and if  $p \neq 2$  according to Lemma 1.18).

- (ii) Let  $n$  be  $n(X)$ , the morphism  $F_X^n$  extend to a finite dominant morphism  $\mathcal{F}_X^n : C \rightarrow \mathbb{P}_{k^{p^{-n}}}^1$  of degree  $p^n$  [Rus70, Lem 1.3]. Then  $p^{r(X)}$  is the residue class degree of the valuation associated to  $P_\infty$  in  $\kappa(C)$ , so

$$p^{r(X)} = [\kappa(P_\infty) : k] \leq \left[ \kappa(C) : \kappa\left(\mathbb{P}_{k^{p^{-n}}}^1\right) \right] = p^n.$$

Hence  $r(X) \leq n(X)$ .

**Definition 1.5.** Let  $m(X)$  be the positive integer such that the image of the group morphism  $\deg : \text{Pic}(C) \rightarrow \mathbb{Z}$  is  $m(X)\mathbb{Z}$ .

*Remark 1.6.*  $m(X)$  is the greatest common divisor of the degrees of the residue fields of the closed points of  $C$ , in particular  $m(X)$  divides  $[\kappa(P_\infty) : k] = p^{r(X)}$ . So  $m(X)$  is a power of  $p$  and  $m(X) \leq p^{r(X)}$ .

We have shown the following relations between the above invariants:

**Lemma 1.7.**

$$\begin{aligned} n(X) &\geq \max(n'(X), r(X)) \\ m(X) &| p^{r(X)}. \end{aligned}$$

## 1.2 Structures of the forms of $\mathbb{G}_{a,k}$

In this subsection we will gather some results mainly from Russell's article [Rus70] on the structure of the forms of  $\mathbb{G}_{a,k}$  and on the reasons why a form of  $\mathbb{A}_k^1$  can fail to have a group structure.

Let  $A = \text{End}_k(\mathbb{G}_{a,k})$  (endomorphisms of  $k$ -group scheme) and  $F = F_{\mathbb{G}_{a,k}}^1 \in A$  the relative Frobenius endomorphism. Then  $A = k\langle F \rangle$  is a non commutative ring of polynomials with the relations  $Fa = a^p F$  for all  $a \in k$ . Following [Rus70], we denote by  $A^*$  the subset of polynomials in  $A$  with non zero constant coefficients.

**Theorem 1.8.** [Rus70, 2.1]

Let  $G$  be a form of  $\mathbb{G}_{a,k}$ . Then  $G$  is isomorphic to the subgroup  $\text{Spec}(k[x, y]/I)$  of  $\mathbb{G}_{a,k}^2$ , where  $I$  is the ideal of  $k[x, y]$  generated by the separable polynomial  $y^{p^n} - (x + a_1 x^p + \cdots + a_m x^{p^m})$  for some  $a_1, \dots, a_m \in k$ . Equivalently,  $G$  is the kernel of the homomorphism

$$\begin{array}{ccc} \mathbb{G}_{a,k}^2 & \rightarrow & \mathbb{G}_{a,k} \\ (x, y) & \mapsto & y^{p^n} - (x + a_1 x^p + \cdots + a_m x^{p^m}). \end{array} \quad (1)$$

Thus, we can see  $G$  as a fibre product

$$\begin{array}{ccc} G & \longrightarrow & \mathbb{G}_{a,k} \\ \downarrow & & \downarrow \tau \\ \mathbb{G}_{a,k} & \xrightarrow{F^n} & \mathbb{G}_{a,k} \end{array}$$

where  $\tau = 1 + a_1 F + \cdots + a_m F^m \in A^*$ . Similarly, any  $G$  defined by such a product is a form of  $\mathbb{G}_{a,k}$ . We note  $G = (F^n, \tau)$ .

*Remark 1.9.* Let  $G$  be a form of  $\mathbb{G}_{a,k}$ , the proof of [Rus70, Th. 2.1] shows that in the equation (1) we can choose  $n$  to be  $n(G)$ .

Recall that any smooth connected unipotent algebraic group splits after base change by a finite purely inseparable extension [KMT74, 8.1]. In the particular case of the forms of  $\mathbb{G}_{a,k}$ , we have the following more precise result:

**Corollary 1.10.** [Rus70, 2.3.1]

Let  $G$  be the form of  $\mathbb{G}_{a,k}$  defined by the equation  $y^{p^n} = x + a_1 x^p + \cdots + a_m x^{p^m}$ . Then  $k' := k(a_1^{p^{-n}}, \dots, a_m^{p^{-n}})$  if the smallest extension of  $k$  such that  $G_{k'} \cong \mathbb{G}_{a,k'}$ .

Let  $X$  be a form of  $\mathbb{A}_k^1$ , P. Russell showed in his article [Rus70] that there are two reasons for  $X$  to fail to have a group structure. Firstly  $X$  may not have a  $k$ -rational point. Secondly  $X_{k_s}$  may have only finitely many automorphisms.

**Proposition 1.11.** [KMT74, 6.9.1]

Let  $X$  be a form of  $\mathbb{A}_k^1$ , such that  $X$  has a  $k$ -rational point  $P_0$ . Let  $C$  be the regular completion of  $X$ . Then the following are equivalent:

- (i)  $X$  has a group structure with neutral point  $P_0$ .
- (ii)  $X$  is isomorphic as a scheme to a form of  $\mathbb{G}_{a,k}$ .
- (iii)  $\text{Aut}(C_{k_s})$  is infinite.

**Proposition 1.12.** [Rus70, 4.1]

Let  $X$  be a form of  $\mathbb{A}_k^1$  and suppose that  $X_{k_s}$  admits a group structure. Then  $X$  is a principal homogeneous space for a form  $G$  of  $\mathbb{G}_{a,k}$  determined uniquely by  $X$ . Moreover  $X = \text{Spec}(k[x, y]/I)$ ,  $G = \text{Spec}(k[x, y]/J)$  where the ideals  $I$  and  $J$  are generated respectively by  $y^{p^n} - b - f(x)$  and  $y^{p^n} - f(x)$  with  $b \in k$  and  $f(x) := x + a_1x^p + \cdots + a_mx^{p^m}$ . Conversely, if  $X$  and  $G$  are defined as above, then  $X$  is a principal homogeneous space for  $G$ .

*Remark 1.13.* P. Russell in [Rus70] and T. Kambayashi, M. Miyanishi, and M. Takeuchi in [KMT74] have classified all forms of  $\mathbb{A}_k^1$  over a separably closed field such that the regular completion has arithmetic genus  $\leq 1$ .

M. Rosenlicht [KMT74, 6.9.3] has found an example of a form of  $\mathbb{A}_k^1$  with only finitely many automorphisms, of genus  $(p-1)/2$  for all  $p > 2$ .

More recently, T. Asanuma [Asa05, Th. 8.1] has found an explicit algebraic presentation of the forms of  $\mathbb{A}_k^1$ , for every field  $k$  of characteristics  $p > 2$ .

### 1.3 Examples

Let  $X$  be a form of  $\mathbb{A}_k^1$ , first we will compare the minimal field  $k'$  such that  $X_{k'} \cong \mathbb{A}_{k'}^1$  and the residue field  $\kappa(P_\infty)$  of  $P_\infty$ . There is an inclusion  $\kappa(P_\infty) \subset k'$ , which may be strict, as shown by the example below.

*Example 1.14.* Let  $k = \mathbb{F}_p(t_1, t_2)$  and  $G$  be the form of  $\mathbb{G}_{a,k}$  defined by the equation

$$y^{p^2} = x + t_1x^p + t_2x^{p^2},$$

then  $C$  is defined as a curve of  $\mathbb{P}_k^2$  by the equation

$$y^{p^2} = xz^{p^2-1} + t_1x^p z^{p^2-p} + t_2x^{p^2}.$$

In this case  $\kappa(P_\infty) = k(t_2^{p-2}) \subsetneq k' = k(t_1^{p-2}, t_2^{p-2})$ .

The inequality  $p^{r(X)} = [\kappa(P_\infty) : k] \leq p^{n(X)}$  (Lemma 1.7) may also be strict, as shown by:

*Example 1.15.* Let  $k = \mathbb{F}_p(t)$  and  $G$  be the form of  $\mathbb{G}_{a,k}$  defined by the equation

$$y^{p^3} = x + tx^p + t^{p^2}x^{p^2},$$

then  $n(G) = 3$  and after the change of variable  $w = tx - y^p$  we remark that  $G$  is also defined by the equation

$$-t^{1-p}y^{p^2} - t^{-1}y^p = t^{-1}w + t^{1-p}w^p + w^{p^2}.$$

So  $C$  is defined in  $\mathbb{P}_k^2$  by

$$-t^{1-p}y^{p^2} - t^{-1}y^p z^{p^2-p} = t^{-1}wz^{p^2-1} + t^{1-p}w^p z^{p^2-p} + w^{p^2},$$

the residue field of the point at infinity is  $\kappa(P_\infty) = k(t^{p-2})$ .

We will now present some results on the forms of  $\mathbb{A}_k^1$  with regular completion  $\mathbb{P}_k^1$ .

**Lemma 1.16.** Let  $X$  be a form of  $\mathbb{A}_k^1$  such that  $X(k) \neq \emptyset$ . The following are equivalent:

(i)  $C \cong \mathbb{P}_k^1$ .

(ii)  $X$  is the complement of a purely inseparable point of  $\mathbb{P}_k^1$ .

(iii)  $C$  is smooth.

*Proof.* We begin with (i)  $\Leftrightarrow$  (ii), the implication (i)  $\Rightarrow$  (ii) is a consequence of [Rus70, Lem. 1.1]. The converse is clear.

Now we show (i)  $\Leftrightarrow$  (iii), the implication (i)  $\Rightarrow$  (iii) is clear. Suppose  $C$  is smooth, let  $k'$  the smallest field such that  $X_{k'} \cong \mathbb{A}_{k'}^1$ . Then  $C_{k'}$  is smooth; so  $C_{k'} \cong \mathbb{P}_{k'}^1$  and  $C(k) \neq \emptyset$ . According to [Liu06, Pro. 7.4.1 (b)] it follows that  $C \cong \mathbb{P}_k^1$ .  $\square$

*Remark 1.17.* If  $X$  is a non trivial form of  $\mathbb{A}_k^1$ , then  $P_\infty$  is not  $k$ -rational. Indeed if  $P_\infty$  is  $k$ -rational then  $C$  is smooth at  $P_\infty$  [Liu06, Pro. 4.3.30] so it is smooth everywhere. According to Lemma 1.16  $C$  is isomorphic to  $\mathbb{P}_k^1$  and  $X$  is the complement of a  $k$ -rational point of  $\mathbb{P}_k^1$ , thus  $X \cong \mathbb{A}_k^1$ .

**Lemma 1.18.** [Ros55] [Rus70] [KMT74, 6.9.2]

Let  $G$  be a form of  $\mathbb{G}_{a,k}$ . If  $C \cong \mathbb{P}_k^1$  then either  $G \cong \mathbb{G}_{a,k}$  or  $p = 2$  and  $n(G) = 1$ .

*Example 1.19.* Let  $p = 2$ , and  $G$  be the form of  $\mathbb{G}_{a,k}$  defined by the equation

$$y^2 = x + ax^2$$

with  $a \in k \setminus k^2$  where  $k^2 = \{x^2 \mid x \in k\}$ . Then  $G$  is a non trivial form of  $\mathbb{G}_{a,k}$ , the regular completion  $C$  is defined as a curve of  $\mathbb{P}_k^2$  by the equation

$$y^2 = xz + ax^2.$$

We remark that  $C$  is smooth (because it is smooth at  $P_\infty$ ), so according to Lemma 1.16  $C \cong \mathbb{P}_k^1$  (this follows more directly from the fact that  $C$  is a conic with a  $k$ -rational point).

*Remark 1.20.* We can combine examples 1.15 and 1.19: let  $p = 2$  and  $G$  be the form of  $\mathbb{G}_{a,k}$  defined by

$$y^{p^3} = x + tx^p + t^{p^2}x^{p^2},$$

then  $r(G) = 2$  and  $n(G) = 3$ . Moreover  $G_{k^{p-2}}$  is isomorphic to the form of  $\mathbb{G}_{a,k}$  defined by the equation  $y^2 = x + t^{p-2}x^2$ , so  $n'(G) = 2$ . So we have constructed an example of a form of  $\mathbb{G}_{a,k}$  where the inequality  $n(X) \geq \max(n'(X), r(X))$  (Lemma 1.7) is strict.

*Example 1.21.* Let  $Q$  be an inseparable point of  $\mathbb{P}^1$ , then  $X = \mathbb{P}_k^1 \setminus \{Q\}$  is a form of  $\mathbb{A}_k^1$  with regular completion  $\mathbb{P}_k^1$ . If  $Q$  is not  $k$ -rational then  $X$  is a non trivial form of  $\mathbb{A}_k^1$  and if  $\deg(Q) > 2$  then according to Lemma 1.18,  $X_{k_s}$  does not have a group structure. In this case  $n'(X) = 0$  and  $n(X) = r(X)$  can be arbitrary big.

*Example 1.22.* Let  $X$  be a form of  $\mathbb{A}_k^1$ , if  $C \cong \mathbb{P}_k^1$  then  $k' = \kappa(P_\infty)$ . The converse is false: let  $G$  be the form of  $\mathbb{G}_{a,k}$  defined by the equation  $y^p = x + ax^p$  where  $a \in k \setminus k^p$ . Then  $C$  is defined by the equation  $y^p = xz^{p-1} + ax^p$  in  $\mathbb{P}_k^2$ , so  $\kappa(P_\infty) = k[a^{p-1}] = k'$ . If  $p \geq 3$ , then  $C$  isn't smooth (because  $C_{k'}$  is not regular at  $P_\infty$ ) so  $C$  is not isomorphic to  $\mathbb{P}_k^1$ .

## 1.4 Arithmetic genus of the regular completion

First let us consider a field  $k$  of arbitrary characteristic. Let  $a$ ,  $b$  and  $c$  be three positive integers, recall that the weighted projective space  $\mathbb{P}_k(a, b, c)$  is defined as  $\text{Proj}(k[x, y, z])$  where  $k[x, y, z]$  is the graded polynomial  $k$ -algebra with weight  $a$  for  $x$ ,  $b$  for  $y$  and  $c$  for  $z$ . If  $w$  is a homogeneous element of  $k[x, y, z]$ , we will denote  $D_+(w)$  the open subset of  $\mathbb{P}_k(a, b, c)$  consisting of the homogeneous ideals of  $\text{Proj}(k[x, y, z])$  not containing the ideal  $(w)$ . Then  $(D_+(w), \mathcal{O}_{\mathbb{P}_k(a, b, c)}|_{D_+(w)})$  is an affine scheme.

Let  $C$  be a geometrically integral curve of degree  $d$  in  $\mathbb{P}_k^2$ , we denote  $p_a(C)$  the arithmetic genus of the curve  $C$ . It is well known that  $p_a(C) = (d-1)(d-2)/2$ . In this subsection we will generalize this result for some curves in some weighted projective planes (Proposition 1.24). I. Dolgachev has computed the geometric genus of a smooth curve in a weighted projective plane [Dol82, 3.5.2] under the assumption that the characteristic of the field does not divide the weights of the projective plane. But we need to compute the arithmetic genus of a curve in a weighted projective plane where one of the weights is a power of the characteristic (Corollary 1.25). Even though the result of Proposition 1.24 is certainly already known, we did not find a reference with an appropriate setting so we include the proof here for the sake of completeness.

**Lemma 1.23.** *Let  $k$  be a field of arbitrary characteristic, let  $a$  be a positive integer and let  $n$  be an integer. Let  $S = \bigoplus_{d \in \mathbb{N}} S_d$  be the graded polynomial  $k$ -algebra  $k[x, y, z]$  with weight 1 for  $x$ , 1 for  $y$  and  $a$  for  $z$ . We denote  $\mathbb{P}$  the weighted projective space  $\mathbb{P}_k(1, 1, a) = \text{Proj}(S)$ .*

*Then  $\mathcal{O}_{\mathbb{P}}(na)$  is an invertible sheaf on  $\mathbb{P}$  and  $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(na)) = S_{na}$ .*

*Proof.* First we will show that  $S_{na} = H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(na))$  (in the case where  $a = 1$  and  $\mathbb{P} = \mathbb{P}_k^2$  it is a well known fact). Let  $g \in \mathcal{O}_{\mathbb{P}}(na)(\mathbb{P})$ , then by definition of  $\mathcal{O}_{\mathbb{P}}(na)$ :

$$\begin{aligned} g|_{D_+(x)} &= P/x^{m_x} \text{ with } P \in S_{m_x+na}, \\ g|_{D_+(y)} &= Q/y^{m_y} \text{ with } Q \in S_{m_y+na}. \end{aligned}$$

We can suppose that  $m_x = m_y = m$  (if this not the case, for example  $m_x > m_y$ , consider  $Q' = Qy^{m_x-m_y}$ , then  $g|_{D_+(y)} = Q'/y^{m_x}$ ). The two local sections  $g|_{D_+(x)}$  and  $g|_{D_+(y)}$  coincide on  $D_+(xy)$ , so

$$g|_{D_+(xy)} = P/x^m = Q/y^m \in S \left[ \frac{1}{xy} \right].$$

Then, in  $S$  we have the equality  $x^m Q = y^m P$ . So  $y^m$  divide  $Q$ , and  $g = Q/y^m$  is a homogeneous polynomial of degree  $m + na - m = na$ . Thus  $g \in S_{na}$ , so  $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(na)) \subset S_{na}$ . Conversely, it is clear that  $S_{na} \subset H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(na))$ .

Next, to show that  $\mathcal{O}_{\mathbb{P}}(na)$  is an invertible sheaf on  $\mathbb{P}$ , it is enough to show that for  $U = D_+(x)$ ,  $D_+(y)$  and  $D_+(z)$ , the  $\mathcal{O}_{\mathbb{P}}(U)$ -module  $\mathcal{O}_{\mathbb{P}}(na)(U)$  is isomorphic to  $\mathcal{O}_{\mathbb{P}}(U)$ .

Let  $w$  be  $x$  or  $y$ , the multiplication by  $w^{na}$ :

$$\begin{aligned} \text{mult}_{w^{na}} : \mathcal{O}_{\mathbb{P}}(D_+(w)) &\rightarrow \mathcal{O}_{\mathbb{P}}(na)(D_+(w)) \\ P/w^{\deg(P)} &\mapsto w^{na} P/w^{\deg(P)}, \end{aligned}$$

has for inverse the multiplication by  $1/w^{na}$ . So  $\text{mult}_{w^{na}}$  is an isomorphism.

For  $D_+(z)$ , the isomorphism is the multiplication by  $z^n$ :

$$\begin{aligned} \text{mult}_{z^n} : \mathcal{O}_{\mathbb{P}}(D_+(z)) &\rightarrow \mathcal{O}_{\mathbb{P}}(na)(D_+(z)) \\ P/z^m &\mapsto z^n P/z^m, \end{aligned}$$

where  $P \in S_{ma}$ . □

**Proposition 1.24.** *Let  $k$  be a field of arbitrary characteristic, let  $a$  be a positive integer. We denote  $\mathbb{P}$  the weighted projective space  $\mathbb{P}_k(1, 1, a)$ .*

*Let  $C$  be a geometrically integral curve of degree  $d$  in  $\mathbb{P}$ , such that  $d$  is a multiple of  $a$ . Let  $h$  be the integer  $d/a$ . Then the arithmetic genus of  $C$  is:*

$$p_a(C) = \frac{(h-1)(d-2)}{2}. \quad (2)$$

*Proof.* Let  $n$  be a positive integer, according to Riemann-Roch Theorem [Liu06, Th. 7.3.17],

$$\dim_k H^0(C, \mathcal{O}_C(na)) - \dim_k H^1(C, \mathcal{O}_C(na)) = \deg(\mathcal{O}_C(na)) + 1 - p_a(C). \quad (3)$$

According to [Liu06, Th. 5.3.2], if  $n$  is large enough, then  $H^1(C, \mathcal{O}_C(na)) = 0$ . We make this assumption throughout this proof.

We denote  $f : C \rightarrow \mathbb{P}$  the inclusion, and  $\mathcal{I}_C$  the sheaf of ideal of  $\mathcal{O}_{\mathbb{P}}$  that defines the closed subvariety  $C$ ; then

$$0 \rightarrow \mathcal{I}_C \rightarrow \mathcal{O}_{\mathbb{P}} \rightarrow f_* \mathcal{O}_C \rightarrow 0$$

is an exact sequence of sheaves on  $\mathbb{P}$ . Moreover  $\mathcal{I}_C \cong \mathcal{O}_{\mathbb{P}}(-ha) = \mathcal{O}_{\mathbb{P}}(-ha)$ , and the sheaf  $\mathcal{O}_{\mathbb{P}}(na)$  is invertible (Lemma 1.23), so in particular flat. Then

$$0 \rightarrow \mathcal{O}_{\mathbb{P}}(na - ha) \rightarrow \mathcal{O}_{\mathbb{P}}(na) \rightarrow f_* \mathcal{O}_C(na) \rightarrow 0 \quad (4)$$

is an exact sequence of sheaves on  $\mathbb{P}$ .

As above, we can take  $n$  large enough, so that  $H^1(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(na - ha)) = 0$ . Then the cohomological exact sequence induced by the sequence (4) is

$$0 \rightarrow H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(na - ha)) \rightarrow H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(na)) \rightarrow H^0(C, \mathcal{O}_C(na)) \rightarrow 0.$$

Thus

$$\dim_k H^0(C, \mathcal{O}_C(na)) = \dim_k H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(na)) - \dim_k H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(na - ah)). \quad (5)$$

Next, we compute  $\dim_k H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(\delta a))$ . As in Lemma 1.23, let  $S = \bigoplus_{d \in \mathbb{N}} S_d$  be the graded  $k$ -algebra  $k[x, y, z]$  with weight 1 for  $x$ , 1 for  $y$  and  $a$  for  $z$ . According to [Bou07, Chap. 5 §5.1 Pro. 1],  $\dim_k S_{\delta a}$  is the  $\delta a$ -th coefficient of the formal series

$$\frac{1}{(1-t)^2(1-t^a)} = \left( \sum_{l \geq 0} (l+1)t^l \right) \left( \sum_{i \geq 0} t^{ia} \right).$$

Then

$$\begin{aligned}
\dim_k H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(\delta a)) &= \dim_k S_{\delta a} = \sum_{l+ai=\delta a} l+1 \\
&= \sum_{i=0}^{\delta} \delta a - ia + 1 \\
&= (\delta a + 1)(\delta + 1) - a \frac{\delta(\delta + 1)}{2} \\
&= \frac{(\delta + 1)(\delta a + 2)}{2}.
\end{aligned}$$

By combining the equations (3), (5) and the equation above we obtain:

$$\begin{aligned}
\deg(\mathcal{O}_C(na)) + 1 - p_a(C) &= \frac{(n+1)(na+2)}{2} - \frac{(n-h+1)(na-ha+2)}{2} \\
&= nah + \frac{2h+ah-ah^2}{2}.
\end{aligned}$$

Finally:

$$p_a(C) = 1 + \frac{ah^2 - 2h - ah}{2} = \frac{(h-1)(ah-2)}{2}.$$

□

We are going to apply Proposition 1.24 to the study of the arithmetic genus of the regular completion of the forms of  $\mathbb{G}_{a,k}$ . This genus has been studied by C. Greither for a form  $X$  of  $\mathbb{A}_k^1$  in the particular case when the minimal field  $k'$  such that  $X_{k'} \cong \mathbb{A}_{k'}^1$  is of degree  $p$  [Gre86, Th. 3.4] and [Gre86, Th. 4.6].

**Corollary 1.25.** *Let  $k$  be a field of characteristic  $p > 0$ , and  $G$  be a form of  $\mathbb{G}_{a,k}$ . We note  $n = n(G)$  and  $m$  the smallest integer such that  $G$  is defined by  $y^{p^n} = x + a_1x^p + \dots + a_mx^{p^m}$ . Let  $C$  be the regular completion of  $G$ , then*

$$p_a(C) \leq \frac{(p^{\min(n,m)} - 1)(p^{\max(n,m)} - 2)}{2}. \quad (6)$$

Moreover,  $a_m \notin k^p$  if and only if (6) is an equality.

In order to show this Corollary we are going to introduce a "naive completion"  $\widehat{C}$  of  $G$ . The "naive completion" will give us a geometrical interpretation of the condition  $a_m \notin k^p$ : this is equivalent to  $\widehat{C}$  being regular.

First we suppose that  $n \leq m$ . Let  $\widehat{C}$  be the closure of  $G$  in  $\mathbb{P}_k(1, p^{m-n}, 1)$ , then  $\widehat{C}$  is defined by the homogeneous polynomial

$$y^{p^n} - (xz^{p^n-1} + a_1x^p z^{p^n-p} + \dots + a_mx^{p^m}), \quad (7)$$

where  $x$  has weight 1,  $y$  has weight  $p^{m-n}$ , and  $z$  has weight 1.

Let  $A$  be the graded  $k$ -algebra defined as the quotient of the graded algebra  $k[x, y, z]$  (with weights as above) by the ideal generated by the homogeneous polynomial (7), then  $\widehat{C} = \text{Proj}(A)$ .

Let us consider the affine open  $D_+(x)$  of  $\mathbb{P}_k(1, p^{m-n}, 1)$ , the affine variety  $\widehat{C} \cap D_+(x)$  is the spectrum of  $A_{(x)}$ , the sub-algebra of  $A[\frac{1}{x}]$  of elements of degree 0. Then  $A_{(x)}$  is generated by  $y/x^{p^{m-n}}$  and  $z/x$ .

Let  $Y = y/x^{p^{m-n}}$  and  $Z = z/x$ , then

$$A_{(x)} = k[Y, Z] / (Y^{p^n} - (Z^{p^{n-1}} + a_1 Z^{p^{n-p}} + \cdots + a_m)).$$

Also  $\widehat{C} \setminus G$  is a unique point that we will note  $\infty$ . A straightforward computation shows that

$$\frac{\mathcal{O}_{\widehat{C}, \infty}}{(z)} \cong \frac{k[y]}{(y^{p^n} - a_m)}.$$

If  $a_m \notin k^p$  then  $\frac{k[y]}{(y^{p^n} - a_m)}$  is a field, so  $\widehat{C}$  is regular, thus  $\widehat{C}$  is the regular completion  $C$ .

Let us consider the morphism  $\widehat{C} \rightarrow \mathbb{P}_k^1$  induced by the projection  $p_x : G \rightarrow \mathbb{G}_{a,k}$ . The scheme theoretic fibre of this morphism at  $[1 : 0]$  is  $\text{Spec}(\mathcal{O}_{\widehat{C}, \infty}/(z))$  so  $z$  is a uniformizing parameter of  $\widehat{C}$  at  $\infty$  if and only if  $\widehat{C}$  is regular if and only if  $a_m \notin k^p$ .

If  $n > m$ , the construction of the naive completion  $\widehat{C}$  is almost the same, except that  $\widehat{C}$  is the closure of  $G$  in  $\mathbb{P}_k(p^{n-m}, 1, 1)$ . The curve  $\widehat{C}$  is defined by the homogeneous equation

$$y^{p^n} = xz^{p^n-1} + a_1 x^p z^{p^m-p} + \cdots + a_m x^{p^m},$$

where  $x$  has weight  $p^{n-m}$ ,  $y$  has weight 1, and  $z$  has weight 1.

And  $\widehat{C} \setminus G$  is a unique point still denoted  $\infty$ , then

$$\frac{\mathcal{O}_{\widehat{C}, \infty}}{(z)} \cong \frac{k[x]}{(x^{p^m} - a_m^{-1})}.$$

By the same argument as above  $a_m \notin k^p$  if and only if  $\widehat{C}$  is regular.

*Proof.* Assume  $a_m \notin k^p$ . Then we have shown that  $\widehat{C}$  is regular, so (by unicity of the regular completion)  $\widehat{C}$  is the regular completion  $C$ . And according to Proposition 1.24, we have

$$p_a(C) = \frac{1}{2}(p^{\min(n,m)} - 1)(p^{\max(n,m)} - 2).$$

On the other hand, if  $a_m \in k^p$ , then  $\widehat{C}$  is not normal. Let  $\pi : C \rightarrow \widehat{C}$  be the normalisation. There is an exact sequence of sheaves on  $\widehat{C}$ :

$$0 \rightarrow \mathcal{O}_{\widehat{C}} \rightarrow \pi_* \mathcal{O}_C \rightarrow \mathcal{F} \rightarrow 0$$

where  $\mathcal{F}$  is a non trivial sheaf with support  $\infty$ . So  $p_a(C) = p_a(\widehat{C}) - \dim_k H^0(\widehat{C}, \mathcal{F})$ , then

$$p_a(C) < p_a(\widehat{C}) = \frac{1}{2}(p^{\min(n,m)} - 1)(p^{\max(n,m)} - 2).$$

□

## 1.5 Picard group of the forms of $\mathbb{A}_k^1$ and of their regular completion

Let  $X$  be a form of  $\mathbb{A}_k^1$ , in this subsection we will reduce the study of the Picard group of  $X$  to that of the Picard group of  $C$  by adapting the argument of [KMT74, Th. 6.10.1]. The curves  $C$  and  $X$  are regular, so we can identify the Picard group

with the divisor class group, thus we note  $[P_\infty]$  the class of the point  $P_\infty$  in the Picard group of  $C$ . The following sequences are exact [Har13, Pro. II.6.5]:

$$0 \rightarrow \mathbb{Z}[P_\infty] \rightarrow \text{Pic}(C) \rightarrow \text{Pic}(X) \rightarrow 0, \quad (8)$$

and

$$0 \rightarrow \text{Pic}^0(C) \rightarrow \text{Pic}(C) \rightarrow m(X)\mathbb{Z} \rightarrow 0 \quad (9)$$

where  $m(X)$  is the invariant of  $X$  defined in 1.5.

By combining the two exact sequences (8) and (9) we obtain the following exact sequence:

$$0 \rightarrow \text{Pic}^0(C) \rightarrow \text{Pic}(X) \rightarrow m(X)\mathbb{Z}/p^{r(X)}\mathbb{Z} \rightarrow 0. \quad (10)$$

*Example 1.26.* As in Example 1.19, let  $p = 2$  and let  $G$  be the form of  $\mathbb{G}_{a,k}$  defined by the equation  $y^2 = x + ax^2$  where  $a \notin k^2$ . Since  $C \cong \mathbb{P}_k^1$ , we obtain  $\text{Pic}(G) \cong \mathbb{Z}/2\mathbb{Z}$ .

More generally, let  $Q$  be a purely inseparable point of  $\mathbb{P}_k^1$ , then  $X = \mathbb{P}_k^1 \setminus \{Q\}$  is a non trivial form of  $\mathbb{A}_k^1$  and  $\text{Pic}(X) \cong \mathbb{Z}/\deg(Q)\mathbb{Z}$ .

*Example 1.27.* Let  $k$  be a field of characteristic  $p \neq 2$ , let  $G$  be the form of  $\mathbb{G}_{a,k}$  defined by the equation  $y^2 = x + ax^2$  where  $a \notin k^2$ .

Let  $P_0$  be the neutral element of  $G$ , the morphism

$$P \in G(k) \mapsto [P] - [P_0] \in \text{Pic}^0(C)$$

is injective [KMT74, Th. 6.7.9], so if  $G(k)$  is infinite (e.g. if  $k$  is separably closed), then  $\text{Pic}(G)$  is an infinite group (Recall that over a perfect field, the Picard group of a affine connected smooth algebraic group is finite [San81, Lem. 6.9]).

*Remark 1.28.* For every extension  $K$  of  $k$ , there is a normal completion  $C^K$  of  $X_K$  which is not necessary the base change  $C_K$  (if  $K$  is not a separable extension of  $k$ , then  $C_K$  can be no longer regular). So there is an exact sequence

$$0 \rightarrow \text{Pic}^0(C^K) \rightarrow \text{Pic}(X_K) \rightarrow m(X_K)\mathbb{Z}/p^{r(X_K)}\mathbb{Z} \rightarrow 0.$$

This motivates the study of  $\text{Pic}_{C/k}^0$  which is going to be done in Section 3. Before that we are going to gather some results on the Picard functor.

## 2 Cocartesian diagram and Picard functor

The main result of this section is Theorem 2.8, it is stated and proved in Subsection 2.3. In Subsection 2.1 we gather some auxiliary results on the unit group scheme, in Subsection 2.2 we show Proposition 2.7 that is the main tool for the proof of Theorem 2.8.

Throughout this section  $S$  is a base scheme, we consider schemes and morphisms over  $S$ . And if  $X$  and  $T$  are two  $S$ -schemes we will note  $X_T$  for the product  $X \times_S T$ .

We will use this level of generality, in a future work, to study the  $G$ -torsors for  $G$  a form of  $\mathbb{G}_{a,k}$ .

## 2.1 Unit group scheme

Let  $f : X \rightarrow S$  be a proper morphism, flat and of finite presentation. The functor

$$\begin{aligned} Sch/S^\circ &\rightarrow Rings \\ T &\mapsto \mathcal{O}(X_T) \end{aligned}$$

is represented by a  $S$ -scheme  $V_X$  which is smooth if and only if  $f$  is cohomologically flat in dimension 0 [BLR90, Cor. 8.1.8] (i.e. the formation of  $f_*(\mathcal{O}_X)$  commutes with base change). Moreover the functor

$$\begin{aligned} Sch/S^\circ &\rightarrow Groups \\ T &\mapsto \mathcal{O}(X_T)^* \end{aligned}$$

is represented by an open sub-scheme  $\mu^X$  of  $V_X$ , so it is an  $S$ -group scheme [BLR90, Lem. 8.1.10].

An important particular case is the following, let  $k$  be a field of arbitrary characteristic, let  $A$  be a  $k$ -algebra of finite dimension (as a  $k$ -vector space), then the group functor

$$\begin{aligned} k\text{-algebras} &\rightarrow Groups \\ R &\mapsto (A \otimes_k R)^* \end{aligned}$$

is represented by an affine smooth commutative connected algebraic group denoted  $\mu^A$  whose Lie algebra is  $A$  with the trivial bracket [DG70, II §1 2.3].

*Remark 2.1.* Let  $k$  be a field of arbitrary characteristic, then  $\mu^k$  is the multiplicative group  $\mathbb{G}_{m,k}$ .

More generally if  $A$  is a  $k$ -algebra of finite dimension, then  $\mu^A = R_{A/k}(\mathbb{G}_{m,A})$ , where  $R_{A/k}$  is the Weil restriction.

*Remark 2.2.* Let  $A \subset A'$  be two  $k$ -algebras of finite dimension. The inclusion  $f : A \hookrightarrow A'$  induces a morphism of algebraic groups  $f^* : \mu^A \rightarrow \mu^{A'}$  which is injective on  $\bar{k}$ -rational points and induces an injection on the Lie algebras. So the scheme theoretic kernel of  $f^*$  is trivial, and  $f^*$  is a closed immersion [DG70, II §5 5.1].

The co-kernel of  $f^*$  is a smooth commutative connected affine algebraic group denoted  $\mu^{A'/A}$ .

**Lemma 2.3.** *Let  $k$  be a field of arbitrary characteristic.*

(i) *Let  $A$  be a local  $k$ -algebra, of finite dimension. Let  $M$  be the maximal ideal of  $A$  and  $K$  the residue field of  $A$ . We have an exact sequence of algebraic groups*

$$0 \rightarrow 1 + M \rightarrow \mu^A \rightarrow \mu^K \rightarrow 0$$

*where  $1 + M$  is a  $k$ -split smooth connected unipotent algebraic group.*

*Moreover if the residue field  $K$  is  $k$ , this sequence has a unique splitting and we have a canonical isomorphism  $\mu^A \cong (1 + M) \times_k \mu^k$ .*

(ii) *Let  $A \subset A'$  be two local  $k$ -algebras, of finite dimension and having the same residue field  $K$ , then  $\mu^{A'/A}$  is a  $k$ -split smooth connected unipotent group.*

*Proof.* (i) First we look at the composition series associated to the  $k$ -sub-algebras  $k \oplus M^n$ , the successive quotients are vector groups associated with the  $k$ -vector spaces  $M^n/M^{n+1}$ . So  $1 + M$  is a  $k$ -split unipotent algebraic group.

The quotient map  $p : A \twoheadrightarrow A/M \cong K$  induces a morphism of algebraic groups  $\mu^A \rightarrow \mu^K$ , then  $\mu^A(\bar{k}) \twoheadrightarrow \mu^K(\bar{k})$ , so  $\mu^A \rightarrow \mu^K \rightarrow 0$  is exact and the kernel of  $\mu^A \rightarrow \mu^K$  is  $1 + M$ .

Moreover if  $K = k$ , then  $A = k \oplus M$  and the inclusion  $k \subset A$  is the unique morphism  $k \rightarrow A$ . So there is a unique section of the morphism  $\mu^A \rightarrow \mu^k$ , and  $\mu^A \cong (1 + M) \times_k \mu^K$  canonically.

(ii) According to (i), the rows of the commutative diagram below are exact.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & 1 + M & \longrightarrow & \mu^A & \longrightarrow & \mu^K & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & 1 + M' & \longrightarrow & \mu^{A'} & \longrightarrow & \mu^K & \longrightarrow & 0. \end{array}$$

So there is an isomorphism  $\mu^{A'/A} = \mu^{A'}/\mu^A \cong (1 + M')/(1 + M)$ . In particular  $\mu^{A'/A}$  is a  $k$ -split unipotent group (as a quotient of a split unipotent group [Bor12, Th. V.15.4]).  $\square$

*Remark 2.4.* Let  $k$  be a field of positive characteristic.

Let  $A \subset A'$  be two local  $k$ -algebras of finite dimension, with residue fields  $K$  and  $K'$ , then  $\mu^{A'/A}$  is not necessary  $k$ -split. For example, if  $A = K = k$  and  $A' = K'$  is a purely inseparable extension of finite degree of  $k$ , then according to [Oes84, Lem. VI.5.1]  $\mu^{K/k}$  is  $k$ -wound.

## 2.2 Rigidified Picard functors

The main result of this section is the exact sequence (12) which relates the Picard functor and the rigidified Picard functor.

Let  $X \rightarrow S$  be a proper, flat morphism of finite presentation.

**Definition 2.5.** Following [BLR90] we will define the rigidified Picard functor. First we define a sub-scheme  $Y \subset X$  which is finite, flat, and of finite presentation over  $S$ , to be a rigidificator (also called rigidifier) of  $\text{Pic}_{X/S}$  if for all  $S$ -schemes  $T$  the map  $\mathcal{O}(X_T) \rightarrow \mathcal{O}(Y_T)$  induced by the inclusion of schemes  $Y_T \rightarrow X_T$  is injective.

Let  $Y$  be a rigidificator of  $\text{Pic}_{X/S}$ , a rigidified line bundle on  $X$  along  $Y$  is by definition a pair  $(\mathcal{L}, \alpha)$  where  $\mathcal{L}$  is a line bundle on  $X$  and  $\alpha$  is an isomorphism  $\mathcal{O}_Y \xrightarrow{\sim} \mathcal{L}|_Y$ .

Let  $(\mathcal{L}, \alpha)$  and  $(\mathcal{L}', \alpha')$  be two rigidified line bundle on  $X$  along  $Y$ . A morphism of rigidified line bundle  $f : (\mathcal{L}, \alpha) \rightarrow (\mathcal{L}', \alpha')$  is a morphism of line bundle  $f : \mathcal{L} \rightarrow \mathcal{L}'$  such that  $f|_Y \circ \alpha = \alpha'$ .

We can now define the rigidified Picard functor as the functor

$$(\text{Pic}_{X/S}, Y) : (\text{Sch}/S)^0 \rightarrow (\text{Set})$$

which associates to the  $S$ -scheme  $T$  the set of isomorphisms of rigidified line bundles on  $X_T$  along  $Y_T$ .

There is a map

$$\delta : \begin{array}{ccc} \mu^Y & \rightarrow & (\text{Pic}_{X/S}, Y) \\ a \in \mathcal{O}(Y \times_S T)^* & \mapsto & (\mathcal{O}_{X \times_S T}, \text{mult}_a) \end{array}$$

where the map  $\text{mult}_a : \mathcal{O}_{X \times_S T} \xrightarrow{\sim} \mathcal{O}_{X \times_S T}$  is the multiplication by  $a \in \mathcal{O}(Y \times_S T)^*$ . There is also a map  $(\text{Pic}_{X/S}, Y) \rightarrow \text{Pic}_{X/S}$  which forgets the rigidification and whose kernel is the image of  $\delta$ .

According to [Ray70, Pro. 2.1.2] and [Ray70, Pro. 2.4.1], the sequence

$$0 \rightarrow \mu^X \rightarrow \mu^Y \rightarrow (\text{Pic}_{X/S}, Y) \rightarrow \text{Pic}_{X/S} \rightarrow 0 \quad (11)$$

is an exact sequence of sheaves for the étale topology.

Under the above hypotheses we can apply [Ray70, Th. 2.3.1], so the rigidified Picard functor  $(\text{Pic}_{X/S}, Y)$  is represented by an algebraic space of finite presentation on  $S$ .

In Remark 2.6 and in Proposition 2.7 we will present particular cases where  $(\text{Pic}_{X/S}, Y)$  is represented by an  $S$ -group scheme.

*Remark 2.6.* Let  $X \rightarrow S$  be cohomologically flat in dimension 0, then  $\text{Pic}_{X/S}$  is represented by an  $S$  group scheme locally of finite type. Moreover if  $S$  is a field, then  $(\text{Pic}_{X/S}, Y)$  is represented by a  $S$ -group scheme locally of finite type [Art69, Lem. 4.2].

**Proposition 2.7.** *Let  $X \rightarrow S$  be a projective flat morphism of finite presentation, with geometrically integral fibres and let  $Y \subset X$  be a rigidificator. Then,*

- (i) *The quotient  $\mu^Y/\mu^X$  is represented by an affine, flat  $S$ -group scheme of finite presentation.*
- (ii) *The functor  $(\text{Pic}_{X/S}, Y)$  is represented by an  $S$ -group scheme, locally of finite presentation.*
- (iii) *The sequence*

$$0 \rightarrow \mu^Y/\mu^X \rightarrow (\text{Pic}_{X/S}, Y) \rightarrow \text{Pic}_{X/S} \rightarrow 0 \quad (12)$$

*is an exact sequence of  $S$ -group schemes, locally of finite presentation.*

*Proof.* The Picard functor  $\text{Pic}_{X/S}$  is represented by a separated  $S$ -scheme locally of finite presentation [BLR90, Th. 8.2.1]. Moreover  $\mu^X = \mathbb{G}_{m,S}$  and  $\mu^Y = R_{Y/S}(\mathbb{G}_{m,Y})$ , so the  $S$ -group scheme  $\mu^Y$  is affine [DG70, Pro. I §16.6]. According to [SGAIII2, Th. VIII.5.1], the quotient  $\mu^Y/\mu^X$  is an affine  $S$ -scheme ( $\mu^X \rightarrow \mu^Y$  is an immersion [BLR90, Pro. 8.1.9], so  $\mathbb{G}_{m,S}$  acts freely on  $\mu^Y$ ). In addition  $\mu^Y$  is smooth and of finite presentation over  $S$  [BLR90, Pro. 7.6.5], so  $\mu^Y/\mu^X$  is according to [SGAIII2, Pro. 8.5.8] of finite presentation on  $S$  and according to [EGAIV2, Cor. 2.2.11 (ii)]  $\mu^Y/\mu^X \rightarrow S$  is faithfully flat.

We are going to show that  $\mu^Y/\mu^X$  is an  $S$ -group scheme. Let  $m' : \mu^Y \times \mu^Y \rightarrow \mu^Y$  be the multiplication and  $p : \mu^Y \rightarrow \mu^Y/\mu^X$  be the quotient. Then we have a morphism  $p \circ m' : \mu^Y \times \mu^Y \rightarrow \mu^Y/\mu^X$  which is  $\mu^X \times \mu^X$ -invariant. So according to [SGAIII2, Th. VIII.5.1], the quotient  $(\mu^Y \times \mu^Y)/(\mu^X \times \mu^X)$  exists. By the universal property of the categorical quotient (the torsors are categorical quotients [MFK94, Pro. 0.1]), there is a unique morphism  $m$  such that the diagram

$$\begin{array}{ccc} \mu^Y \times \mu^Y & \xrightarrow{p \circ m'} & (\mu^Y \times \mu^Y)/(\mu^X \times \mu^X) \\ & \searrow p \circ m' & \downarrow m \\ & & \mu^Y/\mu^X \end{array}$$

is commutative. Moreover  $(\mu^Y \times \mu^Y)/(\mu^X \times \mu^X) = \mu^Y/\mu^X \times \mu^Y/\mu^X$ , so we have show that there is a morphism  $m : \mu^Y/\mu^X \times \mu^Y/\mu^X \rightarrow \mu^Y/\mu^X$ . Likewise, there are two morphisms  $e : S \rightarrow \mu^Y/\mu^X$  and  $i : \mu^Y/\mu^X \rightarrow \mu^Y/\mu^X$ . We only have to remark that by the universal property of quotients, the diagrams

$$\begin{array}{ccc} \mu^Y/\mu^X \times \mu^Y/\mu^X \times \mu^Y/\mu^X & \xrightarrow{m \times id} & \mu^Y/\mu^X \times \mu^Y/\mu^X \\ \downarrow id \times m & & \downarrow m \\ \mu^Y/\mu^X \times \mu^Y/\mu^X & \xrightarrow{m} & \mu^Y/\mu^X, \end{array}$$

$$\begin{array}{ccccc} \mu^Y/\mu^X & \xrightarrow{e \times id} & \mu^Y/\mu^X \times \mu^Y/\mu^X & \xleftarrow{id \times e} & \mu^Y/\mu^X \\ & \searrow id & \downarrow m & \swarrow id & \\ & & \mu^Y/\mu^X & & \end{array}$$

and

$$\begin{array}{ccccc} \mu^Y/\mu^X & \xrightarrow{id \times i} & \mu^Y/\mu^X \times \mu^Y/\mu^X & \xleftarrow{i \times id} & \mu^Y/\mu^X \\ & \searrow e \circ f & \downarrow m & \swarrow e \circ f & \\ & & \mu^Y/\mu^X & & \end{array}$$

(where  $f$  is the structural morphism of  $\mu^Y/\mu^X$ ) are commutative. Thus we have shown (i).

Let us show (ii). The morphism  $(\text{Pic}_{X/S}, Y) \rightarrow \text{Pic}_{X/S}$  is a  $\mu^Y/\mu^X$ -torsor [DG70, Cor. III §4 1.8]. The  $S$ -group  $\mu^Y/\mu^X$  is affine, so by [DG70, Pro. III §4 1.9 a)]  $(\text{Pic}_{X/S}, Y)$  is represented by a  $S$ -scheme. Moreover recall that  $(\text{Pic}_{X/S}, Y)$  is an algebraic space [Ray70, Th. 2.3.1]. A consequence of [BLR90, Pro. 8.3.5] is that the morphisms of algebraic spaces between two schemes are exactly the morphisms of schemes.

There is only (iii) left, according to [DG70, Cor. III §4 1.7] and [DG70, Cor. III §1 2.11] if the morphism  $(\text{Pic}_{X/S}, Y) \times_{\text{Pic}_{X/S}} (\text{Pic}_{X/S}, Y) \rightarrow (\text{Pic}_{X/S}, Y)$  is faithfully flat of finite presentation, then  $(\text{Pic}_{X/S}, Y) \rightarrow \text{Pic}_{X/S}$  has the same property. According to [DG70, III §1 2.4],

$$(\text{Pic}_{X/S}, Y) \times_{\text{Pic}_{X/S}} (\text{Pic}_{X/S}, Y) \cong (\text{Pic}_{X/S}, Y) \times_S \mu^Y/\mu^X.$$

Moreover we have already shown that  $\mu^Y/\mu^X \rightarrow S$  is faithfully flat of finite presentation, so

$$(\text{Pic}_{X/S}, Y) \rightarrow \text{Pic}_{X/S}$$

is also faithfully flat of finite presentation.

To conclude we remark that

$$(\text{Pic}_{X/S}, Y) \times_{\text{Pic}_{X/S}} (\text{Pic}_{X/S}, Y) \cong (\text{Pic}_{X/S}, Y) \times_S \mu^Y/\mu^X$$

so  $\mu^Y/\mu^X$  is the kernel of  $(\text{Pic}_{X/S}, Y) \rightarrow \text{Pic}_{X/S}$ , hence (iii).  $\square$

## 2.3 An exact sequence of Picard schemes

**Theorem 2.8.** *Let*

$$\begin{array}{ccc} Y' & \xrightarrow{v} & X' \\ g \downarrow & & \downarrow f \\ Y & \xrightarrow{u} & X \end{array}$$

*be a commutative square of  $S$ -schemes, cocartesian in the category of ringed spaces. We make the following hypotheses:*

- (i) *The morphisms  $u$  and  $v$  are closed immersions, the morphisms  $g$  and  $f$  are affine.*
- (ii) *The structural morphisms  $X \rightarrow S$  and  $X' \rightarrow S$  are projective, flat of finite presentation with geometrically integral fibres .*
- (iii)  *$Y$  is a rigidificator of  $\mathrm{Pic}_{X/S}$ , and likewise  $Y'$  is a rigidificator of  $\mathrm{Pic}_{X'/S}$ .*

*Then the sequence*

$$0 \rightarrow \mu^Y \rightarrow \mu^{Y'} \rightarrow \mathrm{Pic}_{X/S} \rightarrow \mathrm{Pic}_{X'/S} \rightarrow 0 \quad (13)$$

*is an exact sequence of  $S$ -group schemes locally of finite presentation.*

*Proof.* According to Proposition 2.7, the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mu^X & \longrightarrow & \mu^Y & \longrightarrow & (\mathrm{Pic}_{X/S}, Y) \longrightarrow \mathrm{Pic}_{X/S} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow f^* \\ 0 & \longrightarrow & \mu^{X'} & \longrightarrow & \mu^{Y'} & \longrightarrow & (\mathrm{Pic}_{X'/S}, Y') \longrightarrow \mathrm{Pic}_{X'/S} \longrightarrow 0 \end{array}$$

is a diagram of  $S$ -group schemes with exact lines. By [Bri14, Lem. 2.2]  $f^*$  is an isomorphism, and  $\mu^X \cong \mu^{X'} \cong \mathbb{G}_{m,S}$ . So by diagram chasing the sequence (13) is exact.  $\square$

## 3 Picard functor of the regular completion

### 3.1 Torsion of the Picard functor

Let  $X$  be a form of  $\mathbb{A}_k^1$ , let  $C$  be the regular completion of  $X$ . Let  $K$  be a field such that the regular completion of  $X_K$  is  $\mathbb{P}_K^1$  (e.g.  $K = k'$  or  $K = k^{p^{-n'(X)}}$ ,  $n'(X)$  being the integer defined in 1.3). The base change  $C_K$  is not necessary normal, but the normalisation of  $C_K$  is  $\mathbb{P}_K^1$  because it is the regular completion of  $X_K$ , and the regular completion is unique up to unique isomorphism. Let  $\pi : \mathbb{P}_K^1 \rightarrow C_K$  be the normalisation. Following [Fer03] we show how  $C_K$  is obtained from  $\mathbb{P}_K^1$  via "pinching".

Let  $\mathcal{C}$  be the conductor of  $\mathcal{O}_{C_K}$  in  $\mathcal{O}_{\mathbb{P}_K^1}$ , i.e. the sheaf of ideals of  $\pi_*\mathcal{O}_{\mathbb{P}_K^1}$  given by:

$$\mathcal{C}(U) = \left\{ a \in \mathcal{O}_{\mathbb{P}_K^1}(\pi^{-1}(U)) \mid a \cdot \mathcal{O}_{\mathbb{P}_K^1}(\pi^{-1}(U)) \subset \mathcal{O}_{C_K}(U) \right\}$$

for any open sub-scheme  $U$  of  $C_K$ .

Then  $\mathcal{C}$  is also a sheaf of ideals of  $\mathcal{O}_{C_K}$ . Let  $Y^K$  be the closed sub-scheme of  $C_K$  associated to the sheaf of ideals  $\mathcal{C}$ . Then  $C_K$  is regular outside of  $P_\infty$ , so  $\pi$  induces

an isomorphism between  $C_K \setminus P_\infty$  and  $\mathbb{P}_K^1 \setminus \infty$  (where  $P_\infty$  is the unique point of  $C_K \setminus X_K$ , and  $\infty$  is the unique point of  $\mathbb{P}_K^1$  above  $P_\infty$ ). So as a set,  $Y^K$  is the point  $P_\infty$  and by construction there is a closed immersion  $Y^K \rightarrow C_K$ . Finally let  $Z^K$  be the fibre product  $Y^K \times_{C_K} \mathbb{P}_K^1$ .

We have obtained a commutative diagram of  $K$ -varieties:

$$\begin{array}{ccc} Z^K & \longrightarrow & \mathbb{P}_K^1 \\ \downarrow & & \downarrow \pi \\ Y^K & \longrightarrow & C_K. \end{array} \quad (14)$$

By construction the diagram (14) is cartesian, in fact according to the scholium [Fer03, 4.3] the diagram is also cocartesian.

First we will explicit  $Y^K$  and  $Z^K$ . The morphism  $\pi$  induces a morphism of local rings  $\pi_{P_\infty}^\# : \mathcal{O}_{C_K, P_\infty} \rightarrow \mathcal{O}_{\mathbb{P}_K^1, \infty}$  which is the normalisation. Let  $\mathfrak{C}$  be the conductor of  $\mathcal{O}_{C_K, P_\infty}$  in  $\mathcal{O}_{\mathbb{P}_K^1, \infty}$ , i.e.

$$\mathfrak{C} = \left\{ x \in \mathcal{O}_{\mathbb{P}_K^1, \infty} \mid x \cdot \mathcal{O}_{\mathbb{P}_K^1, \infty} \subset \mathcal{O}_{C_K, P_\infty} \right\} = \mathcal{C}_{P_\infty}.$$

we then have explicitly  $Z^K = \text{Spec}(\mathcal{O}_{\mathbb{P}_K^1, \infty} / \mathfrak{C})$  and  $Y^K = \text{Spec}(\mathcal{O}_{C_K, P_\infty} / \mathfrak{C})$ .

By construction the cocartesian diagram (14) satisfies the hypotheses of Theorem 2.8. Thus we have an exact sequence of locally algebraic groups over  $K$ :

$$0 \rightarrow \mu^{Z^K/Y^K} \rightarrow \text{Pic}_{C_K/K} \rightarrow \text{Pic}_{\mathbb{P}_K^1/K} \rightarrow 0.$$

The neutral component of  $\text{Pic}_{\mathbb{P}_K^1/K}$  is trivial and  $\mu^{Z^K/Y^K}$  is connected. So we have an isomorphism of algebraic groups over  $K$ :

$$\text{Pic}_{C_K/K}^0 \cong \mu^{Z^K/Y^K}.$$

In particular  $\text{Pic}_{C_K/K}^0$  is smooth.

*Remark 3.1.* If  $K = k'$ , then according to Lemma 2.3, the algebraic group  $\mu^{Z^K/Y^K}$  is  $k'$ -split unipotent, so  $\text{Pic}_{C_{k'}/k'}^0$  is  $k'$ -split unipotent.

And if we look at points over  $\bar{k}$  we have the following isomorphisms:

$$\text{Pic}_{C_K/K}^0(\bar{k}) \cong \mu^{Z^K/Y^K}(\bar{k}) = \frac{\mu^{Z^K}(\bar{k})}{\mu^{Y^K}(\bar{k})} = \frac{\left( \bar{k} \otimes_k \frac{\mathcal{O}_{\mathbb{P}_K^1, \infty}}{\mathfrak{C}} \right)^*}{\left( \bar{k} \otimes_k \frac{\mathcal{O}_{C_K, P_\infty}}{\mathfrak{C}} \right)^*}. \quad (15)$$

**Lemma 3.2.** *If  $K = k^{p^{-n'(X)}}$ , then the algebraic group  $\mu^{Z^K/Y^K}$  is of  $p^{n'(X)}$ -torsion.*

*Proof.*  $\mu^{Z^K/Y^K}$  is a smooth algebraic group, so it is enough to show that the group of  $\bar{k}$ -points  $\mu^{Z^K/Y^K}(\bar{k})$  is of  $p^{n'(X)}$ -torsion.

Let  $n$  be a non-negative integer, then  $\kappa(X^{(p^n)}) = k \otimes_k \kappa(X)$  (where  $k$  is seen as a  $k$ -algebra via  $\varphi_k^n$ ). By definition of  $C$  we have  $\kappa(X) = \kappa(C)$ , take  $n = n'(X)$ ; then  $\kappa(X^{(p^n)}) = \kappa(\mathbb{P}_K^1)$ . Thus,  $\kappa(\mathbb{P}_K^1) = k \otimes_k \kappa(C)$ . With this identification, the image of  $\varphi_{\kappa(\mathbb{P}_K^1)}^n : x \in \kappa(\mathbb{P}_K^1) \mapsto x^{p^n} \in \kappa(\mathbb{P}_K^1)$  is contained in  $\kappa(C)$ .

The discrete valuation ring  $\mathcal{O}_{\mathbb{P}_K^1, \infty}$  is defined by the valuation  $\text{mult}_\infty$  on  $\kappa(\mathbb{P}_K^1)$ , and  $\text{mult}_\infty$  is an extension of the valuation  $\text{mult}_{P_\infty}$  on  $\kappa(C)$ . If  $x \in \mathcal{O}_{\mathbb{P}_K^1, \infty}$ , then

of course  $x^{p^n} \in \mathcal{O}_{\mathbb{P}_K^1, \infty}$ ; and we have shown that  $x^{p^n} \in \kappa(C)$ , so  $x^{p^n} \in \mathcal{O}_{C, P_\infty} \subset \mathcal{O}_{C_K, P_\infty}$ .

So according to the equation (15)  $\mu^{Z^K/Y^K}(\bar{k})$  is of  $p^{n'(X)}$ -torsion.  $\square$

To conclude we have shown the following result:

**Proposition 3.3.** *The algebraic group  $\text{Pic}_{C/k}^0$  is unipotent of  $p^{n'(X)}$ -torsion, and  $\text{Pic}_{C_{k'}/k'}^0$  is  $k'$ -split.*

## 3.2 Application to the Picard functor of the regular completion

**Theorem 3.4.** *Let  $X$  be a form of  $\mathbb{A}_k^1$  and  $C$  be the regular completion of  $X$ . Then  $\text{Pic}_{C/k}^0$  is a smooth connected unipotent algebraic group of  $p^{n'(X)}$ -torsion which is  $k$ -wound and splits over  $k'$  (the smallest field such that  $X_{k'} \cong \mathbb{A}_{k'}^1$ ).*

Moreover if  $X$  is a principal homogeneous space for a form  $G$  of  $\mathbb{G}_{a,k}$ , then

$$\dim \text{Pic}_{C/k}^0 \leq \frac{(p^{\min(n,m)} - 1)(p^{\max(n,m)} - 2)}{2}$$

where  $n = n(G)$  and  $m$  is the smallest integer such that  $G$  is defined by an equation of the form  $y^{p^n} = x + a_1x^p + \dots + a_mx^{p^m}$ .

In addition if  $X$  is a non trivial form of  $\mathbb{G}_{a,k}$  and  $p \neq 2$ , then  $k'$  is a smallest field such that  $\text{Pic}_{C/k}^0$  splits over  $k'$ .

*Proof.* The assertion on the torsion and the fact that  $\text{Pic}_{C/k}^0$  is unipotent and splits over  $k'$  are direct consequences of Proposition 3.3. According to [BLR90, Pro. 8.4.2]  $\text{Pic}_{C/k}^0$  is smooth and by [BLR90, Th. 8.4.1],  $\dim \text{Pic}_{C/k}^0 = \dim_k H^1(C, \mathcal{O}_C) = p_a(C)$ . The variety  $C$  is normal and geometrically integral, so according to [BLR90, Pro. 9.2.4] and [CGP15, Pro. B.3.2] the unipotent algebraic group  $\text{Pic}_{C/k}^0$  is  $k$ -wound.

In the case where  $X$  is a principal homogeneous space for a form  $G$  of  $\mathbb{G}_{a,k}$ , the assertion on the dimension of  $\text{Pic}_{C/k}^0$  is a direct consequence of Corollary 1.25, in view of the fact that  $C_{k_s}$  is still regular [EGAIV2, Cor. 6.14.2] and that the arithmetic genus is invariant by field extensions.

We will now show the last assertion. Let  $K$  be a field such that  $k \subset K \subsetneq k'$ , we will show that the unipotent group  $\text{Pic}_{C/k}^0$  does not split on  $K$ , or equivalently that  $\text{Pic}_{C_K/K}^0$  is not split. First of all if  $C_K$  is normal, then the unipotent group  $\text{Pic}_{C_K/K}^0$  is wound, so in particular it is not split. Else let  $g: C^K \rightarrow C_K$  be the normalisation of  $C_K$ . We are going to make the same conductor base construction as in Subsection 3.1. Let  $\mathcal{C}$  be the conductor of  $\mathcal{O}_{C_K}$  in  $\mathcal{O}_{C^K}$  i.e. the sheaf defined by:

$$\mathcal{C}(U) = \{a \in \mathcal{O}_{C^K}(g^{-1}(U)) \mid a \cdot \mathcal{O}_{C^K}(g^{-1}(U)) \subset \mathcal{O}_{C_K}(U)\}.$$

Then  $\mathcal{C}$  is a sheaf of ideals of  $\mathcal{O}_{C^K}$ , and of  $\mathcal{O}_{C_K}$ . Let  $Y$  be the closed sub scheme of  $C_K$  defined by the sheaf  $\mathcal{C}$ , let  $Z$  be the fibre product  $Y \times_{C_K} C^K$ . Then we have a cocartesian square of  $K$ -varieties:

$$\begin{array}{ccc} Z & \longrightarrow & C^K \\ \downarrow & & \downarrow g \\ Y & \longrightarrow & C_K \end{array}$$

which satisfies the hypotheses of Theorem 2.8. So we have an exact sequence of algebraic groups over  $K$

$$0 \rightarrow \mu^{Z/Y} \rightarrow \mathrm{Pic}_{C^K/K}^0 \rightarrow \mathrm{Pic}_{C^K/K}^0 \rightarrow 0.$$

By hypothesis  $K \not\subseteq k'$  and  $p > 2$ , so  $C^K$  isn't isomorphic to  $\mathbb{P}_K^1$  (else according to Lemma 1.18, we would have  $X_K \cong \mathbb{G}_{a,K}$ ). Thus  $\mathrm{Pic}_{C^K/K}^0$  is a non trivial  $K$ -wound algebraic group. Every morphism from a connected smooth unipotent split algebraic group to a connected smooth unipotent wound algebraic group is trivial [CGP15, B.3.4], thus  $\mathrm{Pic}_{C^K/K}^0$  isn't  $K$ -split.  $\square$

### 3.3 Rigidified Picard functor

Let  $X$  be a form of  $\mathbb{A}_k^1$ , let  $C$  be the regular completion of  $X$  and let  $P_\infty$  be the unique point of  $X \setminus C$  (Lemma 1.1).

A geometric invariant of  $X$  is the rigidified Picard functor  $(\mathrm{Pic}_{C/k}, Y)$  where  $Y \subset C$  is a rigidificator of  $\mathrm{Pic}_{C/k}$ . In fact the rigidified Picard functor has the remarkable property of being "invariant by cocartesian square", i.e. if

$$\begin{array}{ccc} Y' & \xrightarrow{v} & X' \\ g \downarrow & & \downarrow f \\ Y & \xrightarrow{u} & C \end{array}$$

is a commutative diagram of rigidificators, cocartesian in the category of ringed spaces, then according to [Bri14, Lem. 2.2],  $f^* : (\mathrm{Pic}_{C/k}, Y) \rightarrow (\mathrm{Pic}_{X'/k}, Y')$  is an isomorphism.

According to Proposition 2.7 the sequence

$$0 \rightarrow \mu^Y / \mu^C \rightarrow (\mathrm{Pic}_{C/k}, Y)^0 \rightarrow \mathrm{Pic}_{C/k}^0 \rightarrow 0 \quad (16)$$

is an exact sequence of algebraic groups.

**Proposition 3.5.** *If  $Y = \mathrm{Spec}(\kappa(P_\infty))$ , then  $Y$  is a rigidificator of  $C$  and  $(\mathrm{Pic}_{C/k}, Y)^0$  is a unipotent  $k$ -wound algebraic group which splits over  $k'$ .*

*Proof.* The algebraic group  $\mu^C$  is isomorphic to  $\mathbb{G}_m$ , so  $\mu^Y / \mu^C \cong \mu^{\kappa(P_\infty)/k}$  is a unipotent algebraic group which is  $k$ -wound according to Remark 2.4, and which splits over  $\kappa(P_\infty) \subset k'$ . The group  $\mathrm{Pic}_{C/k}^0$  is  $k$ -wound unipotent and splits over  $k'$  according to Theorem 3.4. So the algebraic group  $(\mathrm{Pic}_{C/k}, Y)^0$  is an extension of two  $k$ -wound algebraic groups, so  $(\mathrm{Pic}_{C/k}, Y)^0$  is a  $k$ -wound unipotent group [Oes84, V.3.5].

Moreover  $\mu^{\kappa(P_\infty)/k}$  and  $\mathrm{Pic}_{C/k}^0$  split over  $k'$ , so  $(\mathrm{Pic}_{C/k}, Y)^0$  also splits over  $k'$ .  $\square$

## 4 Picard group

### 4.1 Proof of the main theorem

**Theorem 4.1.** *Let  $X$  be a non trivial form of  $\mathbb{A}_k^1$ .*

(i)  $\text{Pic}(X)$  is  $p^{n(X)}$ -torsion.

(ii) If  $X$  has a rational point (e.g.  $X$  is a form of  $\mathbb{G}_{a,k}$  or  $k = k_s$ ), then  $\text{Pic}(X) \neq \{0\}$ .

*Proof.* (i) Consider again the exact sequences (8) and (9):

$$0 \rightarrow \mathbb{Z}[P_\infty] \rightarrow \text{Pic}(C) \rightarrow \text{Pic}(X) \rightarrow 0,$$

and

$$0 \rightarrow \text{Pic}^0(C) \rightarrow \text{Pic}(C) \rightarrow m(X)\mathbb{Z} \rightarrow 0.$$

According to Theorem 3.4,  $\text{Pic}_{C/k}^0(k)$  is of  $p^{n'(X)}$ -torsion, so its subgroup  $\text{Pic}^0(C)$  [BLR90, Th. 9.3.1] is of  $p^{n'(X)}$ -torsion. Also  $n'(X) \leq n(X)$ , thus the image of

$$p^{n(X)} : \text{Pic}(C) \rightarrow \text{Pic}(C)$$

is isomorphic via the degree map to  $p^{n(X)}\mathbb{Z}$ . But  $\deg([P_\infty]) \leq p^{n(X)}$  (Lemma 1.7), so the image of  $p^{n(X)}$  is a subgroup of  $\mathbb{Z}[P_\infty]$ . By exactness of the sequence (8), the group  $\text{Pic}(X)$  is of  $p^{n(X)}$ -torsion.

(ii) If there is a  $k$ -rational point on  $X$  then  $m(X) = 1$ . By hypothesis  $X$  is a non trivial form of  $\mathbb{A}_k^1$ , so  $P_\infty$  is a non  $k$ -rational purely inseparable point (Remark 1.17), and  $\mathbb{Z}[P_\infty]$  is a strict subgroup of  $\text{Pic}(C)$ . So  $\text{Pic}(X)$  is non trivial.  $\square$

## 4.2 Examples of forms of the affine line with trivial Picard group

First, we give an explicit example of a form of  $\mathbb{A}_k^1$  with trivial Picard group.

**Lemma 4.2.** *Let  $k = \mathbb{F}_2(t, u)$  and let  $X$  be the form of  $\mathbb{A}_k^1$  defined by the equation  $y^2 = u + x + tx^2$ , then  $X(k) = \emptyset$  and  $\text{Pic}(X)$  is trivial.*

*Proof.* In order to show that  $X(k) = \emptyset$ , it is enough to show that the only solution of  $P^2 = uQ^2 + QR + tR^2$ , where  $P, Q, R \in \mathbb{F}_2[t, u]$  is trivial. We denote  $\deg$  the total degree of a polynomial. If  $\deg(Q) \neq \deg(R)$ , for example  $\deg(Q) < \deg(R)$ , then

$$\deg(P^2) = \deg(uQ^2 + QR + tR^2) = 1 + 2\deg(R).$$

So  $\deg(P^2)$  is odd, contradiction. So  $\deg(Q) = \deg(R)$  and the monomials of highest degree of  $uQ^2$  and  $tR^2$  must cancel (if they don't cancel we have the same contradiction). But this is impossible because the monomial of highest degree of  $uQ^2$  has an odd partial degree in  $u$  whereas the monomial of highest degree of  $tR^2$  has an even partial degree in  $u$ .

After a field extension to  $k_s$ , by Proposition 1.12  $X_{k_s}$  is isomorphic, as a scheme, to the non trivial form of  $\mathbb{G}_{a,k_s}$  of equation  $y^2 = x + tx^2$ . We have seen in Example 1.19 that the regular completion of this form of  $\mathbb{G}_{a,k_s}$  is  $\mathbb{P}_{k_s}^1$ , so by uniqueness of the regular completion  $C_{k_s} \cong \mathbb{P}_{k_s}^1$ . Then  $\text{Pic}_{C_{k_s}/k_s}^0$  is trivial, and  $\text{Pic}_{C/k}^0$  too. By [BLR90, Th. 9.3.1] again,  $\text{Pic}^0(C)$  is a subgroup of  $\text{Pic}_{C/k}^0(k)$ , hence trivial.

Moreover  $X$  has no  $k$ -rational point so according to a theorem by T. A. Springer [EKM08, Cor. 18.5]  $X$  has no rational point on any extension of odd degree. Thus  $\deg : \mathbb{Z}[P_\infty] \xrightarrow{\sim} \text{Pic}(C)$ , and by exactness of the sequence (8),  $\text{Pic}(X)$  is trivial.  $\square$

*Remark 4.3.* The regular completion of the form consider in Lemma 4.2 is a non trivial form of  $\mathbb{P}_k^1$ .

We will now construct a family of forms of the affine line with trivial Picard group. Let  $G$  be a non trivial form of  $\mathbb{G}_{a,k}$ , by Theorem 1.8 there is an exact sequence:

$$0 \rightarrow G \rightarrow \mathbb{G}_{a,k}^2 \rightarrow \mathbb{G}_{a,k} \rightarrow 0.$$

Let  $\eta$  be the generic point of  $\mathbb{G}_{a,k}$ , then  $\eta = \text{Spec}(k(t))$  and comes with a map  $\eta \rightarrow \mathbb{G}_{a,k}$ . We denote  $X$  the fibre product  $\mathbb{G}_{a,k}^2 \times_{\mathbb{G}_{a,k}} \eta$ .

**Proposition 4.4.** *With the above notation  $X$  is a non trivial form of  $\mathbb{A}_{k(t)}^1$  and  $\text{Pic}(X)$  is trivial.*

*Proof.* The morphism  $\mathbb{G}_{a,k}^2 \rightarrow \mathbb{G}_{a,k}$  is a  $G$ -torsor. So  $X \rightarrow \eta$  is a  $G_{k(t)}$ -torsor, and in particular a form of  $\mathbb{A}_{k(t)}^1$ . Let  $K$  be a separable closure of  $k(t)$ , then  $G_K$  is still  $K$ -wound ( $K/k(t)$  and  $k(t)/k$  are separable extensions, and being wound is not changed by separable extension [CGP15, B.3.2]). By definition of  $k$ -wound,  $G_K$  is a non trivial form of  $\mathbb{A}_K^1$ . Moreover  $X_K$  is an homogeneous space under  $G_K$ , so  $X_K$  is a non trivial form of  $\mathbb{A}_K^1$  and in particular  $X$  is a non trivial form of  $\mathbb{A}_{k(t)}^1$ .

Finally, at the algebraic level the morphism  $X \rightarrow \mathbb{G}_{a,k}^2$  is the localisation morphism

$$k[x, y] \rightarrow k[x, y] \otimes_{k[T]} k(T),$$

where  $T = y^{p^n} - (x + a_1x^p + \dots + a_mx^{p^m})$  is a polynomial that defines  $G$ . Then  $\text{Pic}(\mathbb{G}_{a,k}^2) \rightarrow \text{Pic}(X)$  [Bou06, Chap. 7 §2 n°10 Pro. 17], thus  $\text{Pic}(X)$  is trivial.  $\square$

*Remark 4.5.* Let  $k$  be an imperfect field, with the construction of Proposition 4.4 we have an example of a non trivial form of  $\mathbb{A}_{k(t)}^1$  with trivial Picard group.

Let  $G$  be a smooth affine algebraic group, we recall that  $G$  is said to be *special* if for any field extension  $K$  of  $k$ , any  $G_K$ -torsor  $X \rightarrow \text{Spec}(K)$  is trivial.

J.-P. Serre initiated the study of special groups over an algebraically closed field in [Ser58] and A. Grothendieck classified these groups [Gro58]. More recently J.-L. Colliot-Thélène and J.-J. Sansuc characterised special tori over an arbitrary field [CS87, Pro. 7.4], and M. Hurugen characterised the special reductive groups over an arbitrary field [Hur16, Th. 4.1]. It is known that  $\mathbb{G}_{a,k}$  is special, by the arguments of [Ser58, 4.4.a], and more generally that every smooth connected  $k$ -split unipotent algebraic group is special. D. T. Nguyễn had shown, under a mild assumption on the base field, that a smooth unipotent algebraic group is special if and only if it is  $k$ -split [Ngu13, Cor. 6.10]. In an unpublished note he generalised the result to an arbitrary base field [Ngu16]. We are going to show this result, in the particular case of the forms of  $\mathbb{G}_{a,k}$ , by using a different method; we see it as a corollary of our main Theorem 4.1 and Proposition 4.4:

**Corollary 4.6.** *Let  $G$  be a non trivial form of  $\mathbb{G}_{a,k}$ , then the  $G_{k(t)}$ -torsor  $X \rightarrow \text{Spec}(k(t))$  of Proposition 4.4 is non trivial, thus  $G$  is not special.*

*Proof.* Assume that  $X \rightarrow \eta$  is a trivial  $G_{k(t)}$ -torsor, in particular  $\text{Pic}(X) \cong \text{Pic}(G)$ . According to Theorem 4.1,  $\text{Pic}(G)$  is not trivial, but  $\text{Pic}(X)$  is trivial (Proposition 4.4).  $\square$

## 5 References

- [Art69] M. Artin. *Algebraization of formal moduli I. in Global Analysis, Papers in Honor of K. Kodaira.*, volume 29 of *Princeton Mathematical Series*. University of Tokyo Press, Princeton University Press, 1969.
- [Asa05] T. Asanuma. Purely inseparable  $k$ -forms of affine algebraic curves. In *Affine algebraic geometry. Contributions of the special session on affine algebraic geometry at the 1st joint AMS-RSME meeting, Seville, Spain, June 18–21, 2003*, pages 31–46. Providence, RI: American Mathematical Society (AMS), 2005.
- [BLR90] S. Bosch, W. Lütkebohmert, and M. Raynaud. *Néron models*. Springer-Verlag, 1990. *Ergebnisse der Mathematik und ihrer Grenzgebiete 3. Folge Band 21*.
- [Bor12] A. Borel. *Linear algebraic groups*, volume 126 of *Graduate Texts in Mathematics*. Springer, 2012.
- [Bou06] N. Bourbaki. *Algèbre commutative. Chapitres 5 à 7*. *Éléments de mathématique*. Springer-Verlag Berlin Heidelberg, reprint of the 1985 original edition, 2006.
- [Bou07] N. Bourbaki. *Groupes et algèbres de Lie. Chapitres 4 à 6*. *Éléments de mathématique*. Berlin: Springer, reprint of the 1968 original edition, 2007.
- [Bri14] M. Brion. Which algebraic groups are Picard varieties? *Science China Mathematics*, 58:461–478, 2014.
- [Bri15] M. Brion. On linearization of line bundles. *J. Math. Sci., Tokyo*, 22(1):113–147, 2015.
- [CGP15] B. Conrad, O. Gabber, and G. Prasad. *Pseudo-reductive groups*, volume 26 of *New Mathematical Monographs*. Cambridge University Press, second edition, 2015.
- [CS87] J.-L. Colliot-Thélène and J.-J. Sansuc. Principal homogeneous spaces under flasque tori; applications. *J. Algebra*, 106:148–205, 1987.
- [DG70] M. Demazure and P. Gabriel. *Groupes algébriques Tome 1*. Masson&cie éditeur-Paris north-Holland publishing company-Amsterdam, 1970.
- [Dol82] I. Dolgachev. *"Weighted projective varieties", Group actions and vector fields (Vancouver, B.C., 1981)*, volume 956 of *Lecture Notes in Mathematics*. Springer, 1982.
- [EGAIII2] A. Grothendieck and J. Dieudonné. *Éléments de géométrie algébrique III, Étude cohomologique des faisceaux cohérents seconde partie*, volume 17 of *Publications Mathématiques*. Institut des hautes études scientifiques, 1963.
- [EGAIV2] A. Grothendieck and J. Dieudonné. *Éléments de géométrie algébrique: IV. Étude locale des schémas et des morphismes de schémas, Seconde partie*, volume 24 of *Publications Mathématiques*. Institut des hautes études scientifiques, 1965.

- [EKM08] R. S. Elman, N. Karpenko, and A. Merkurjev. *The algebraic and geometric theory of quadratic forms*, volume 56. American Mathematical Soc., 2008.
- [Fer03] D. Ferrand. Conducteur, descente et pincement. *Bulletin de la société mathématique de France*, 131 (4):553–585, 2003.
- [Gre86] C. Greither. Forms of the affine line and their genus. *Journal of Pure and Applied Algebra*, 39:105–118, 1986.
- [Gro58] A. Grothendieck. Torsion homologique et sections rationnelles. *Séminaire Claude Chevalley 3*, 1958. exposé n°5.
- [GW10] U. Görtz and T. Wedhorn. *Algebraic geometry I. Schemes. With examples and exercises*. Advanced Lectures in Mathematics. Vieweg+Teubner Verlag, 2010.
- [Har13] R. Hartshorne. *Algebraic geometry*, volume 52 of *Lecture notes in mathematics*. Springer, 2013.
- [Hur16] M. Huruguen. Special reductive groups over an arbitrary field. *Transformation Groups*, pages 1–26, 2016.
- [KMT74] T. Kambayashi, M. Miyanishi, and M. Takeuchi. *Unipotent Algebraic groups*, volume 414 of *Lecture Notes in Mathematics*. Springer, 1974.
- [Liu06] Q. Liu. *Algebraic Geometry and Arithmetic Curves*, volume 6 of *Oxford graduate texts in mathematics*. Oxford University Press, second edition, 2006.
- [MFK94] D. Mumford, J. Fogarty, and F. Kirwan. *Geometric invariant theory*, volume 34 of *Ergeb. Math. Grenzgeb.* Springer, third edition, 1994.
- [Ngu13] D. T. Nguyễn. On the essential dimension of unipotent algebraic groups. *Journal of Pure and Applied Algebra*, 217(3):432 – 448, 2013.
- [Ngu16] D. T. Nguyễn. A note on special unipotent groups. Unpublished note, 2016.
- [Oes84] J. Oesterlé. Nombres de tamagawa et groupes unipotents en caractéristique p. *Invent. math.*, 78:13–88, 1984.
- [Ray70] M. Raynaud. Spécialisation du foncteur de Picard. *Publications Mathématiques de l’I.H.E.S.*, 38:27–76, 1970.
- [Ros55] M. Rosenlicht. Automorphisms of function fields. *Transactions of the American Mathematical Society*, 79(1):1–11, 1955.
- [Rus70] P. Russell. Forms of the affine line and its additive group. *Pacific journal of mathematics*, 32:527–539, 1970.
- [San81] J.-J. Sansuc. Groupe de brauer et arithmétique des groupes algébriques linéaires sur un corps de nombres. *Journal für die reine und angewandte Mathematik*, 327:12–80, 1981.
- [SGAIII2] A. Grothendieck and M. Demazure. *Schémas en groupes. Tome II, Groupes de type multiplicatif et structure des schémas en groupes généraux*, volume 152 of *Lecture notes in mathematics*. Springer, 1970. Actes du 3e séminaire de géométrie algébrique tenu au Domaine Du Bois Marie, Bures-sur-Yvette, de 1962 à 1964.

- [Ser58] J.-P. Serre. Espaces fibrés algébriques. *Séminaire Claude Chevalley 3*, 1958. exposé n°1.
- [Tit67] J. Tits. *Lectures on algebraic groups*. Yale University, Department of Mathematics, 1966-1967. Mimeographed Notes.
- [Tot13] B. Totaro. Pseudo-abelian varieties. *Ann. Sci. Éc. Norm. Supér. (4)*, 46(5):693–721, 2013.