CANCELLATION FOR SURFACES REVISITED. I

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Abstract. The celebrated Zariski Cancellation Problem asks as to when the existence of an isomorphism $X \times \mathbb{A}^n \cong X' \times \mathbb{A}^n$ for (affine) algebraic varieties X and X' implies that $X \cong X'$. In this and the subsequent papers we provide a criterion for cancellation by the affine line (that is, $n = 1$) in the case, where X is a normal affine surface admitting an $\mathbb{A}^1$-fibration $X \to B$ over a smooth affine curve B. If X does not admit such an $\mathbb{A}^1$-fibration, then the cancellation by the affine line is known to hold for X by a result of Bandman and Makar-Limanov. We show that, for a smooth $\mathbb{A}^1$-fibered affine surface X over B, the cancellation by an affine line holds if and only if $X \to B$ is a line bundle, and, for a normal such X, if and only if $X \to B$ is a cyclic quotient of a line bundle (an orbifold line bundle). When cancellation does not hold for X, we include X in a non-isotrivial deformation family $X_\lambda \to B$, $\lambda \in \Lambda$, of $\mathbb{A}^1$-fibered surfaces with cylinders $X_\lambda \times \mathbb{A}^1$ isomorphic over B. This gives large families of examples of non-cancellation for surfaces, which extend the known examples constructed by Danielewski, tom Dieck, Wilkens, Masuda and Miyanishi. Given two $\mathbb{A}^1$-fibered surfaces with reduced fibers and the same Danielewski-Fieseler quotient $\breve{B} \to B$, we provide a criterion as to when the corresponding cylinders are isomorphic over B. This criterion is expressed in terms of linear equivalence of certain 'type divisors' on $\breve{B}$.

Contents

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Introduction

The paper is divided into two parts addressed to as Part I and Part II. We introduce here into the results of the both parts.

Let X and Y be algebraic varieties over a field k. The celebrated Zariski Cancellation Problem, in its most general form, asks under which circumstances the existence of a biregular (resp., birational) isomorphism $X \times \mathbb{A}^n \cong Y \times \mathbb{A}^n$ implies that $X \cong Y$, where $\mathbb{A}^n$ stands for the affine n-space over k. In this and the subsequent papers, we are interested in the biregular cancellation problem, hence the symbol '$\cong$' stands for a biregular isomorphism. We say that X is a *Zariski factor* if $X \times \mathbb{A}^n \cong Y \times \mathbb{A}^n$ implies that $X \cong Y$, whatever is $n \in \mathbb{N}$, and a *strong Zariski factor* if any isomorphism $\Phi\colon X \times \mathbb{A}^n \to Y \times \mathbb{A}^n$, where Y is another algebraic variety, fits in a commutative diagram

$$\begin{array}{ccc} X \times \mathbb{A}^n & \xrightarrow{\Phi} & Y \times \mathbb{A}^n \\ \downarrow & & \downarrow \\ X & \xrightarrow[\varphi]{\cong} & Y \end{array}$$

where the vertical arrows are the canonical projections. This property is usually called a *strong cancellation*. We say that X is a *Zariski 1-factor* if $X \times \mathbb{A}^1 \cong Y \times \mathbb{A}^1$ implies that $X \cong Y$, and a *strong Zariski 1-factor* if the strong cancellation holds for X with $n = 1$. The latter means, in particular, that the cylinder structure on $X \times \mathbb{A}^1$ is unique, see [56, Thm. 2.18].

By the Iitaka-Fujita Theorem [49], any algebraic variety X of non-negative log-Kodaira dimension is a strong Zariski factor. Due to a theorem by Bandman and Makar-Limanov ([7, Lem. 2][1]), the following holds.

Theorem 0.1. *The affine varieties which do not admit any effective $\mathbb{G}_a$-action are strong Zariski 1-factors.*

[1] Cf. [18]; see [13, Thm. 3.1] for the positive characteristic case.

There are examples of smooth, rational affine surfaces of negative log-Kodaira dimension ($\mathbb{A}^1$-fibered over $\mathbb{P}^1$), which do not admit any effective $\mathbb{G}_a$-action, and so, are strong Zariski 1-factors (see [7, Ex. 3], [47, 3.7]). Some of these affine surfaces are not Zariski 2-factors, see [24, 25].

In this paper we concentrate on the Zariski Cancellation Problem for normal affine surfaces over an algebraically closed field k of characteristic zero. From Theorem 0.1 one can deduce the following criteria.

Corollary 0.2. *A normal affine surface X is a strong Zariski 1-factor if and only if it does not admit any effective $\mathbb{G}_a$-action, if and only if it is not fibered over a smooth affine curve C with general fibers isomorphic to the affine line $\mathbb{A}^1$.*

See, e.g., [56, Thm. 2.18] for the first part and [31, Lem. 1.6] for the second.

Recall (see e.g., [31]) that a *parabolic $\mathbb{G}_m$-surface* is a normal affine surface equipped with an $\mathbb{A}^1$-fibration $\pi\colon X \to C$ over a smooth affine curve C and with an effective $\mathbb{G}_m$-action along the fibers of π. All π-fibers of such a surface are irreducible curves isomorphic to the affine line $\mathbb{A}^1$. The singularities of X are cyclic quotients situated at the $\mathbb{G}_m$-fixed points on the multiple fibers of π. There is exactly one singular point of X in each multiple fiber. If a parabolic $\mathbb{G}_m$-surface $X \to C$ is smooth, then this is a line bundle over C. Any parabolic $\mathbb{G}_m$-surface admits an effective $\mathbb{G}_a$-action along the fibers of π ([32, Thm. 3.12]).

By the celebrated Miyanishi-Sugie-Fujita Theorem ([63, 39]; see also [62, Ch. 3, Thm. 2.3.1]) the affine plane $\mathbb{A}^2$ is a Zariski factor, hence also a Zariski 1-factor. An analogous result holds for the parabolic $\mathbb{G}_m$-surfaces. Moreover, we provide the following criterion.

Theorem 0.3. *Let X be a normal affine surface $\mathbb{A}^1$-fibered over a smooth affine curve. Then the following conditions are equivalent:*

(i) *X is a Zariski factor;*
(ii) *X is a Zariski 1-factor;*
(iii) *X is a parabolic $\mathbb{G}_m$-surface.*

The implication (i)⇒(ii) is immediate; see Theorem 7.16 for (ii)⇒(iii) and Theorem 6.8 for (iii)⇒(i).

From Theorems 0.1 and 0.3 one can deduce the following characterization.

Corollary 0.4. *A normal affine surface X is a Zariski 1-factor if and only if either X does not admit any effective $\mathbb{G}_a$-action, or X is a parabolic $\mathbb{G}_m$-surface.*

The Danielewski surfaces

$$X_m = \{z^m t - u^2 - 1 = 0\} \subset \mathbb{A}^3, \qquad m \in \mathbb{N},$$

are examples of non-Zariski 1-factors ([17, 29]). Indeed, these surfaces are pairwise non-homeomorphic ([29]), but have isomorphic cylinders: $X_m \times \mathbb{A}^1 \cong X_{m'} \times \mathbb{A}^1 \ \forall m, m' \in \mathbb{N}$. For non-Zariski 1-factors, one can consider the following problems.

Problems. *Given an affine algebraic variety X, describe the moduli space $\mathcal{C}_m(X)$ of isomorphism classes of the affine algebraic varieties Y such that $X \times \mathbb{A}^m \cong Y \times \mathbb{A}^m$. Study the behavior of $\mathcal{C}_m(X)$ upon deformation of X.*

Note that X is a Zariski 1-factor if and only if $\mathcal{C}_1(X) = \{X\}$. We don't have any example of an affine non-Zariski 1-factor X, where the moduli space $\mathcal{C}_1(X)$ were known.

For the first Danielewski surface X_1, the moduli space $\mathcal{C}_1(X_1)$ has infinite number of irreducible components. In [77] this sequence is extended to a family of surfaces in $\mathbb{A}^3$ with similar properties. The example in [77] shows that $\mathcal{C}_1(X_1)$ possesses an infinite number of components, which are infinite dimensional ind-varieties. Yet another family of examples including the Danielewski surfaces is considered in [60, Thm. 2.8]. The latter family has an infinite number of components of the same positive dimension, which can be chosen arbitrary.

In both Parts I and II, we concentrate on the normal affine surfaces X $\mathbb{A}^1$-fibered over affine curves. In particular, we show that, unless X is a parabolic $\mathbb{G}_m$-surface, X deforms in a large family of surfaces with isomorphic cylinders (see Theorems 5.7 and 5.9). Moreover, the deformation space contains infinitely many connected components of positive dimension.

In Part II we prove the following theorem. To an $\mathbb{A}^1$-fibered surface $\pi\colon X \to B$ over a smooth affine curve B with reduced fibers one associates a non-separated one-dimensional affine scheme $\breve{B}$ over B (the *Danielewski-Fieseler quotient*) and an effective divisor $\mathrm{tp}(\mathcal{D}(\pi))$ on $\breve{B}$, called a *type divisor*; see §2.3.

Theorem 0.5. *For two $\mathbb{A}^1$-fibered surfaces $\pi\colon X \to B$ and $\pi'\colon X' \to B$ with reduced fibers over the same smooth affine curve B, the cylinders $X \times \mathbb{A}^1$ and $X' \times \mathbb{A}^1$ are isomorphic over B if and only if the corresponding type divisors $\mathrm{tp}(\mathcal{D}(\pi))$ and $\mathrm{tp}(\mathcal{D}(\pi'))$ on $\breve{B}$ are linearly equivalent.*

The proofs exploit the affine modifications ([54]), in particular, the Asanuma modification ([4]), and as well the flexibility techniques of [1], in particular, the interpolation by automorphisms. As an illustration, we analize from our viewpoint the examples of non-cancellation due to Danielewski [17], Fieseler [29], Wilkens [77], tom Dieck [76], and Miyanishi–Masuda [60].

Remark 0.6. The results of Part I and Part II were reported by the third author on the conference "Complex analyses and dynamical systems - VII" (Nahariya, Israel, May 10–15, 2015), in a seminar at the Bar Ilan University (Ramat Gan, Israel, May 24, 2015), and in the lecture course "Affine algebraic surfaces and the Zariski cancellation problem" at the University of Rome Tor Vergata (September–November, 2015; see the program in [79]). When this paper was written, the third author assisted in the lecture course by Adrien Dubouloz on the cancellation problem for affine surfaces in the 39th Autumn School in Algebraic Geometry (Lukecin, Poland, September 19–24, 2016). In this course, Adrien Dubouloz advertised a result on non-cancellation for smooth $\mathbb{A}^1$-fibered affine surfaces similar to our result (see, in particular, Theorem 1.2 below and Theorem 0.3 in the case of smooth surfaces), and indicated nice ideas of proofs done by completely different methods. He also posed the question whether the non-degenerate affine toric surfaces are Zariski 1-factors, which had been answered in affirmative by our Theorem 0.3.

1. Generalities

1.1. **Cancellation and the Makar-Limanov invariant.** The special automorphism group $\mathrm{SAut}\, X$ of an affine vaiety X is the subgroup of the group $\mathrm{Aut}\, X$ generated by all its $\mathbb{G}_a$-subgroups ([1]). The *Makar-Limanov invariant* $\mathrm{ML}(X)$ is the subring of invariants of the action of $\mathrm{SAut}\, X$ on $O(X)$. The $\mathrm{SAut}\, X$-orbits are locally closed

in X [1]. The complexity κ of the action of $\mathrm{SAut}\, X$ on X is the codimension of its general orbit, or, which is the same, the transcendence degree of the ring $\mathrm{ML}(X)$ ([1]). We design this integer κ as the *Makar-Limanov complexity* of X, and we say that X belongs to the class (ML_κ).

By the Miyanishi-Sugie Theorem ([63], [62, Ch. 2, Thm. 2.1.1, Ch. 3, Lem. 1.3.1 and Thm. 1.3.2]), a normal affine surface X with $\bar{k}(X) = -\infty$ contains a cylinder, i.e., a principal Zariski open subset U of the form $U \cong C \times \mathbb{A}^1$, where C is a smooth affine curve. It possesses as well an $\mathbb{A}^1$-fibration $\mu\colon X \to B$ over a smooth curve B, which extends the first projection $U \to C$ of the cylinder. If B is affine then X admits an effective action of the additive group $\mathbb{G}_a = \mathbb{G}_a(k)$ along the rulings of μ.

Conversely, suppose that there is an effective $\mathbb{G}_a$-action on X. Then the algebra of invariants $O(X)^{\mathbb{G}_a}$ is finitely generated and normal ([29, Lem. 1.1]). Hence $B = \mathrm{Spec}\, O(X)^{\mathbb{G}_a}$ is a smooth, affine curve, and the morphism $\mu\colon X \to B$ induced by the inclusion $O(X)^{\mathbb{G}_a} \hookrightarrow O(X)$ defines an $\mathbb{A}^1$-fibration (an affine ruling) on X. Such an $\mathbb{A}^1$-fibration is trivial over a Zariski open subset of B. It extends the first projection on a principal cylinder on X. If an $\mathbb{A}^1$-fibration on a surface X over an affine base is unique (non-unique, respectively), then X is of class (ML_1) (of class (ML_0), respectively). It is of class (ML_2) if X does not admit any $\mathbb{A}^1$-fibration over an affine curve. In the latter case, X could very well admit an $\mathbb{A}^1$-fibration over a projective curve, and this is so if and only if $\bar{k}(X) = -\infty$.

The cancellation problem is closely related to the problem on stability of the Makar-Limanov invariant under passing to a cylinder. The latter is discussed, e.g., in [5]–[7] and [12]-[14]. Suppose, for instance, that $\mathrm{ML}(X) = O(X)$. Then by [13, Thm. 3.1] (cf. also [18]), $\mathrm{ML}(X \times \mathbb{A}^1) = O(X)$. This means that the cylinder structure on $X \times \mathbb{A}^1$ is unique. Hence *an affine variety X, which does not admit any effective $\mathbb{G}_a$-action, is a Zariski 1-factor*. In particular, any smooth, affine surface of class (ML_2) is a Zariski 1-factor. Therefore, in the future we restrict to surfaces of classes (ML_0) and (ML_1).

In the Danielewski example, $Y_1 \in (\mathrm{ML}_0)$, whereas $Y_r \in (\mathrm{ML}_1)$ for $r \geq 2$. Thus, the Makar-Limanov complexity is not an invariant of cancellation. By contrast, the Euler characteristic, the Picard number (for a rational variety), the log-plurigenera, and the log-irregularity are cancellation invariants, see, e.g., Iitaka's Lemma in [62, Ch. 2, Lem. 1.15.1] and [39, (9.9)].

1.2. **Non-cancellation and Gizatullin surfaces.** Let X be a smooth, affine surface. Recall ([43]) that $\mathrm{SAut}\, X$ acts on X with an open orbit if and only if $X \in \mathrm{ML}_0$. In the latter case X is a *Gizatullin surface*, i.e., a normal affine surface completable by a chain of smooth rational curves, and different from $\mathbb{A}^1 \times (\mathbb{A}^1 \setminus \{0\})$. Furthermore, the group $\mathrm{SAut}(X \times \mathbb{A}^1)$ also acts with an open orbit in the cylinder $X \times \mathbb{A}^1$. Thus, the Makar-Limanov invariant $\mathrm{ML}(X \times \mathbb{A}^1)$ is trivial: $\mathrm{ML}(X \times \mathbb{A}^1) = \mathrm{ML}(X) = k$.

The following conjecture is inspired by [7, §4, Thm. 1] and the unpublished notes [8] kindly offered to one of us by the authors.

Conjecture 1.1. Let X be a normal affine surface such that the group $\mathrm{SAut}(X \times \mathbb{A}^1)$ acts with an open orbit in $X \times \mathbb{A}^1$. Then $\mathcal{C}_1(X)$ contains (the class of) a Gizatullin surface.

Due to [7, Thm. 1] (see also an alternative proof in Part II) this conjecture is true for the Danielewski-Fieseler surfaces, that is, for the $\mathbb{A}^1$-fibered surfaces $\pi\colon X \to \mathbb{A}^1$ with a unique degenerated fiber, provided this fiber is reduced.

1.3. **The Danielewski–Fieseler construction.** The Danielewski–Fieseler examples of non-cancellation exploit the properties of the *Danielewski–Fieseler quotient*. Assume that the $\mathbb{G}_a$-action on X is free. Then the geometric orbit space $X/\mathbb{G}_a$ is a non-separated pre-variety (an algebraic space), obtained by gluing together several copies of $B := \operatorname{Spec} \mathcal{O}(X)^{\mathbb{G}_a}$ along a common Zariski open subset. The morphism μ can be factorized into $X \to X/\mathbb{G}_a \to B$. An ingenious observation by Danielewski is as follows. Suppose that X and Y are non-isomorphic smooth affine $\mathbb{G}_a$-surfaces with free $\mathbb{G}_a$-actions and with the same Danielewski–Fieseler quotient $F = X/\mathbb{G}_a = Y/G_a$. Then the affine threefold $W = X \times_F Y$ carries two induced free $\mathbb{G}_a$-actions. Moreover, W carries two different structures of principal $\mathbb{G}_a$-bundles (torsors) over X and over Y, respectively. Since X and Y are affine varieties, by Serre's Theorem ([73]), both these bundles are trivial, and so, $X \times \mathbb{A}^1 \cong W \cong Y \times \mathbb{A}^1$. This is exactly what happens for two different Danielewski surfaces $X = X_r$ and $Y = X_s$.

The question arises, how universal is the Danielewski-Fieseler construction. More precisely,

Question. *Let X and Y be non-isomorphic smooth affine surfaces with isomorphic cylinders $X \times \mathbb{A}^1 \cong Y \times \mathbb{A}^1$. Assume that both X and Y possess free $\mathbb{G}_a$-actions. Do there exist $\mathbb{A}^1$-fibrations on X and on Y over the same affine base and with the same Danielewski–Fieseler quotient?*

Recall ([20, Def. 0.1]) that a *Danielewski-Fieseler surface* is a smooth affine surface X equipped with an $\mathbb{A}^1$-fibration $\mu\colon X \to \mathbb{A}^1$, which represents a (trivial) line bundle over $\mathbb{A}^1 \setminus \{0\}$, and such that the divisor $\mu^*(0)$ is reduced. Such a surface admits a free $\mathbb{G}_a$-action if and only if it is isomorphic to a surface in $\mathbb{A}^3$ with equation $xy - p(z) = 0$, where $p \in k[z]$ has simple roots ([20, Cor. 4.13]). Theorem 5.7 below deals, more generally, with normal affine surfaces $\mathbb{A}^1$-fibered over affine curves, such that any fiber of the given $\mathbb{A}^1$-fibration is reduced. Abusing the language, we abbreviate these as *GDF-surfaces*, see Definition 2.1. The Danielewski trick does not work for them, in general, because such a surface does not need to admit a free $\mathbb{G}_a$-action. However, we show (see Theorems 5.7 and 7.9)

Theorem 1.2. *A GDF-surface is a Zariski 1-factor if and only if it is the total space of a line bundle.*

The proof involves affine modifications, in particular, the *Asanuma modification*.

1.4. **Affine modifications.** Most of the known examples of non-cancellable affine surfaces exploit the Danielewski–Fieseler quotient, see, e.g., [60, 77]. By contrast, in this paper we use an alternative construction of non-cancellation due to T. Asanuma [4]. Recall first the notion of an affine modification (see [54]).

Definition 1.3 (*Affine modification*)**.** Let $X = \operatorname{Spec} A$ be a normal affine variety, where $A = \mathcal{O}_X(X)$ is the structure ring of X. Let further $I \subset A$ be an ideal, and let $f \in I$, $f \neq 0$. Consider the Rees algebra $A[tI] = \bigoplus_{n \geq 0} t^n I^n$ with $I^0 = A$, where t is an independent variable. Consider further the quotient $A' = A[tI]/(1 - tf)$ by the principal ideal of $A[tI]$ generated by $1 - tf$. The affine variety $X' = \operatorname{Spec} A'$ is called the

affine modification of X along divisor $D = f^*(0)$ with center I. The inclusion $A \hookrightarrow A'$ induces a birational morphism $\varrho\colon X' \to X$, which contracts the exceptional divisor $E = (f \circ \varrho)^{-1}(0)$ on X' to the center $\mathbb{V}(I) \subset X$. Actually, any birational morphism of affine varieties $X' \to X$ is an affine modification ([54, Thm. 1.1]).

Remarks 1.4. 1. If $I = (a_1, \dots, a_l)$, where $a_i \in A$, $i = 1, \dots, l$, then $A' = A[I/f] = A[a_1/f, \dots, a_l/f]$.

2. Assume that $f \in I_1 \subset I$, where I_1 is an ideal of A. Letting $A_1 = A[I_1/f]$ we obtain the equality $A' = A_1[I_2/f]$, where I_2 is the ideal generated by I in A_1. The inclusion $A \hookrightarrow A_1 \hookrightarrow A'$ leads to a factorization of the morphism $X' \to X$ into a composition of affine modifications (i.e., birational morphisms of affine varieties) $X' \to X_1 \to X$, where $X_1 = \operatorname{spec} A_1$ (cf. also [54, Proposition 1.2] for a different kind of factorization).

3. Geometrically speaking, the variety $X' = \operatorname{Spec} A'$ is obtained by blowing up $X = \operatorname{Spec} A$ at the ideal $I \subset A$ and deleting a certain transform of the divisor D on X', see [54] for details. However, in general $\mathbb{V}(I)$ can have components of codimension 1, which are then also components of the divisor $f^*(0)$. These components survive the modification. Thus, it is worth to distinguish between a *geometric* affine modification and an *algebraic* one.

Indeed, given a birational morphism of affine varieties $\sigma\colon X' \to X$ with exceptional divisor $E \subset X'$ and center $C = \sigma_*(E)$ of codimension at least 2, the divisor D of the associated modification can be defined as the closure of $X \setminus \sigma(X')$ in X. However, this D is not necessarily a principal divisor. So, in order to represent $\sigma\colon X' \to X$ via an affine modification, we have to find a principal divisor on X with support containing D. Thus, although the data (D, C) is uniquely defined for σ, there are many different affine modifications which induce the same birational morphism $\sigma\colon X' \to X$ (cf. [19] and also Remark 2.27 for the case of $\mathbb{A}^1$-fibered affine surfaces).

The following lemma will be used on several occasions. It generalizes [54, Cor. 2.2], with a similar proof.

Lemma 1.5. *Let $X' \to X$ and $Y' \to Y$ be affine modifications along principal divisors $D_X = \operatorname{div} f_X$ and $D_Y = \operatorname{div} f_Y$ with centers I_X and I_Y, respectively, where $f_X \in I_X \setminus \{0\}$ and $f_Y \in I_Y \setminus \{0\}$. If an isomorphism $\varphi\colon X \xrightarrow{\cong} Y$ sends f_Y to f_X (hence, D_X to D_Y) and I_Y onto I_X, then it lifts to an isomorphism $\varphi'\colon X' \xrightarrow{\cong} Y'$.*

We need also the following version of this lemma.

Lemma 1.6. *Let $\mathcal{M}$ and $\mathcal{N}$ be affine varieties, and let $\sigma\colon \mathcal{M} \to \mathcal{N}$ be an affine modification along a principal divisor $\mathcal{D} = f^*(0)$ in $\mathcal{N}$ with center an ideal $I \subset \mathcal{O}_{\mathcal{N}}(\mathcal{N})$, where $z \in I \setminus \{0\}$. If $\alpha \in \operatorname{Aut}\mathcal{N}$ with $\alpha(f) = f$ satisfies the congruences* [2]

$$\alpha \equiv \mathrm{id} \mod f^s \quad \text{and} \quad \alpha^{-1} \equiv \mathrm{id} \mod f^s \quad \text{for some} \quad s \in \mathbb{N},$$

then α can be lifted to an automorphism $\widetilde{\alpha} \in \operatorname{Aut}\mathcal{M}$ such that

$$\widetilde{\alpha} \equiv \mathrm{id} \mod f^{s-1} \quad \text{and} \quad \widetilde{\alpha}^{-1} \equiv \mathrm{id} \mod f^{s-1}. \tag{1}$$

Proof. Let $A = \mathcal{O}_N(N)$ and $A' = \mathcal{O}_M(M) = A[a_1/f, \dots, a_l/f]$, where $a_1, \dots, a_l$ are generators of I. We have $\alpha^*(a_i) - a_i \in (f^s)$, that is, $\alpha^*(a_i) = a_i + f^s b_i$ for some $b_i \in A$, $i = 1, \dots, l$. Extending α^* to an automorphism of the fraction field $\operatorname{Frac} A$ denoted by

[2] That is, α and α^{-1} induce both the identity on the sth infinitesimal neighborhood of $\mathcal{D}$.

the same symbol, we obtain $\alpha^*(a_i/f) = a_i/f + f^{s-1}b_i$, $i = 1, \dots, l$. Thus, $\alpha^*(A') \subset A'$ and, similarly, $(\alpha^{-1})^*(A') \subset A'$. So, α^* extends to an automorphism $\widetilde{\alpha}^* \in \operatorname{Aut} A'$ such that (1) holds. □

In the next remark we discuss a converse to Lemma 1.5.

Remark 1.7. Let $\sigma\colon X' \to X$ be an affine modification along divisor $f^*(0)$. Let $A = \mathcal{O}_X(X)$ and $A' = \mathcal{O}_{X'}(X')$, where $A \hookrightarrow A'$. Among all ideals, which give the same modification σ, the largest one $I \subset A$ is given by $I = (f)_{A'} \cap A$.

Suppose we have a commutative diagram of birational morphisms

$$\begin{array}{ccc} X_1 & \xrightarrow{\ \varphi\ } & X_2 \\ & {\scriptstyle\sigma_1}\searrow \quad \swarrow{\scriptstyle\sigma_2} & \\ & X & \end{array} \tag{2}$$

where σ_1 and σ_2 are affine modifications along the same divisor $f^*(0)$ on X. Letting $I_1, I_2 \subset A$ be the largest ideals of σ_1 and σ_2, respectively, we have $\varphi^*(I_2) \subset I_1$. Furthermore, $\varphi^*(I_2) = I_1$ provided φ is an isomorphism. The next example shows that the inclusion $\varphi^*(I_2) \subset I_1$ does not hold any longer, in general, if σ_1 and σ_2 are affine modifications along two different divisors.

Example 1.8. Letting

$$A = k[u, v], \quad A_1 = k[x_1, y_1] = k[u, v/u^2], \quad \text{and} \quad A_2 = k[x_2, y_2] = k[u, v/u]\,,$$

we have $A \hookrightarrow A_2 \hookrightarrow A_1$. The corresponding morphisms

$$\sigma_1\colon (x_1, y_1) \mapsto (x_1, x_1^2 y_1), \quad \sigma_2\colon (x_2, y_2) \mapsto (x_2, x_2 y_2), \quad \text{and} \quad \varphi\colon (x_1, y_1) \mapsto (x_1, x_1 y_1)$$

fit in (2), that is, $\sigma_2 \circ \varphi = \sigma_1$. We have $I_1 = (u^2, v) \subset A$, $I_2 = (u, v) \subset A$, and $(\varphi^*)(I_2) = (u, uv)$. However, there is no inclusion between the ideals $\varphi^*(I_2)$ and I_1.

It is easily seen that the affine modification of the linear space $\mathbb{A}^n$ with center in a linear subspace of codimension ≥ 2 and with divisor a hyperplane is isomorphic to $\mathbb{A}^n$. Similarly, certain affine *Asanuma modifications* of a cylinder give again a cylinder. This simple and elegant fact is due to Asanuma ([4]); we follow here [52, Lem. 7.9].

Lemma 1.9. *Let X be an affine variety, D a principal effective divisor on X, and I an ideal of $\mathcal{O}_X(X)$ with support contained in D. Let $X' \to X$ be the affine modification of X along D with center I. Consider the cylinder $\mathcal{X} = X \times \mathbb{A}^1 = \operatorname{Spec} \mathcal{O}_X(X)[v]$, the divisor $\mathcal{D} = D \times \mathbb{A}^1$ on $\mathcal{X}$, the ideal $\tilde{I} \subset \mathcal{O}_{\mathcal{X}}(\mathcal{X})$ generated by I, and the ideal $J = (\tilde{I}, v) \subset A[v]$ supported on $D \times \{0\} \subset \mathcal{D}$. Then the affine modifications of $\mathcal{X}$ along $\mathcal{D}$ with center $\tilde{I}$ and with center J are both isomorphic to the cylinder $X' \times \mathbb{A}^1$.*

Proof. The affine modification of the cylinder $\mathcal{X}$ along $\mathcal{D}$ with center $\tilde{I}$ yields the cylinder $\mathcal{X}' = X' \times \mathbb{A}^1$. Let $A = \mathcal{O}_X(X)$, f and $a_1, \dots, a_l$ be as in 1.3 and 1.4. Then

$$\mathcal{O}_{\mathcal{X}'}(\mathcal{X}') = A[a_1/f, \dots, a_l/f, v] \cong A[a_1/f, \dots, a_l/f, v'/f] = \mathcal{O}_{\mathcal{X}''}(\mathcal{X}'')\,,$$

where $v' = vf$ is a new variable, and $\mathcal{X}'' \to \mathcal{X}$ is the affine modification of $\mathcal{X}$ along $\mathcal{D}$ with center J. This gives the desired isomorphism. □

2. $\mathbb{A}^1$-fibered surfaces via affine modifications

2.1. **Covering trick and GDF surfaces.** Throughout the paper, we deal with the following class of $\mathbb{A}^1$-fibered surfaces.

Definition 2.1 (*a GDF surface*)**.** Let X be a normal affine surface over k. A morphism $\pi\colon X \to B$ onto a smooth affine curve B is called an $\mathbb{A}^1$-*fibration* if the fiber $\pi^*(b)$ over a general point $b \in B$ is isomorphic to the affine line $\mathbb{A}^1$ over k. An $\mathbb{A}^1$-fibered surface $\pi\colon X \to B$ is called a *generalized Danielewski-Fieseler surface*, or a *GDF surface* for short, if all the fibers $\pi^*(b)$, $b \in B$, are reduced. According to Lemma 2.24(b) below, any GDF surface is smooth.

We say that a GDF surface $\pi\colon X \to B$ is *marked* if a regular function $z \in \mathcal{O}_B(B)$ is given such that $z \circ \pi \in \mathcal{O}_X(X)$ vanishes along any degenerate fiber of π, and has only first order zeros along fiber components of π. Abusing notation, we often consider z as a function on X, identifying it with $z \circ \pi$.

A GDF surface $\pi\colon X \to B$ equipped with actions of a finite group G on X and on B making the morphism π G-equivariant is called a *GDF G-surface*. Assume that $G = \mu_d$ is the group of $d$ths roots of unity, and let z be a μ_d-quasi-invariant marking on X of weight 1. Then we say that $\pi\colon X \to B$ is a *marked GDF μ_d-surface*.

When $B = \mathbb{A}^1$ and $\pi^{-1}(0)$ is the only reducible fiber of π, such surfaces were studied in [20] under the name *Danielewski-Fieseler surfaces*.

Lemma 2.3 below is well known; for the sake of completeness, we indicate a proof. This lemma says that, starting with a normal affine $\mathbb{A}^1$-fibered surface and applying a suitable cyclic Galois base change, one obtains a marked GDF μ_d-surface. The proof uses the following branched covering construction.

Definition 2.2 (*Branched covering construction*)**.** Consider a normal affine $\mathbb{A}^1$-fibered surface $\pi'\colon Y \to C$ over a smooth affine curve C. Fix a finite set of points $p_1, \dots, p_t \in C$ such that for any $p \in C \setminus \{p_1, \dots, p_t\}$ the fiber $\pi'^*(p)$ is reduced and irreducible. Let d be the least common multiple of the multiplicities of the components of the divisor $\sum_{i=1}^t \pi'^*(p_i)$ on Y. Choose a regular function $h \in \mathcal{O}_C(C)$ with only simple zeros, which vanishes in the points $p_1, \dots, p_t$.[3] Letting $\mathbb{A}^1 = \operatorname{spec} k[z]$, consider the smooth curve $B \subset C \times \mathbb{A}^1$ given by equation $z^d - h(p) = 0$, where $(p, z) \in C \times \mathbb{A}^1$, along with the morphism $\mathrm{pr}_1\colon B \to C$ and the function $z|_B \in \mathcal{O}_B(B)$; by abuse of notation, we denote it still by z. Let X be the normalization of the cross-product $Y \times_C B$, and let $\pi\colon X \to B$ and $\varphi\colon X \to Y$ be the induced morphisms. By abuse of notation, the pullback $\pi^*(z) \in \mathcal{O}_X(X)$ will be also denoted by z.

Lemma 2.3. *In the notation of 2.2 the following holds.*

- *The cyclic group μ_d of order d acts naturally on B so that $C = B/\mu_d$;*
- *the morphism $\mathrm{pr}_1\colon B \to C$ is ramified with order d over the zeros of h, $z \in \mathcal{O}_B(B)$ is a μ_d-quasi-invariant of weight 1, and $\operatorname{div}_0 z = \mathrm{pr}_1^*(\operatorname{div}_0 h)$ is a reduced effective μ_d-invariant divisor on B;*
- *the morphism $\varphi\colon X \to Y$ of $\mathbb{A}^1$-fibrations is a Galois covering with Galois group μ_d and the reduced branching divisor $(h \circ \pi')^*(0)$ on Y and ramification divisor $z^*(0)$ on X;*

[3] These points do not necessarily exhaust the set of zeros of h.

- *the μ_d-equivariant morphism $\pi\colon X \to B$ and the function $z \in \mathcal{O}_X(X)$ define a structure of a marked GDF μ_d-surface on X.*

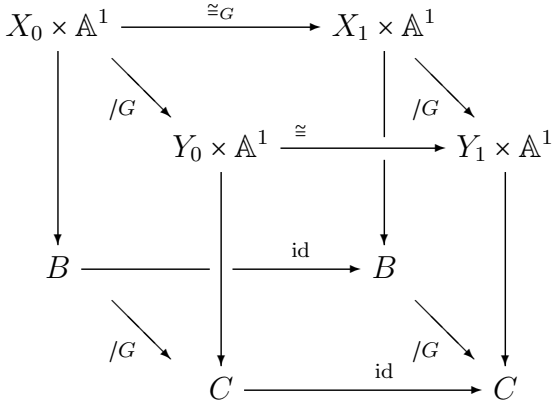
Proof. The map $\nu_d\colon \mathbb{A}^1 \to \mathbb{A}^1$, $z \mapsto z^d$, is the quotient morphism of the natural μ_d-action on $\mathbb{A}^1$. The first three statements follow from the fact that the curve B along with the morphism $z\colon B \to \mathbb{A}^1$ is obtained using the morphism $h\colon C \to \mathbb{A}^1$ via the base change $\nu_d\colon \mathbb{A}^1 \to \mathbb{A}^1$ that fits in the commutative diagram

(3)
$$\begin{array}{ccc} X & \xrightarrow{/\mu_d} & Y \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi'} \\ B & \xrightarrow{/\mu_d} & C \\ {\scriptstyle z}\downarrow & & \downarrow{\scriptstyle h} \\ \mathbb{A}^1 & \xrightarrow[\nu_d]{} & \mathbb{A}^1 \end{array}$$

The remaining assertions can be reduced to a simple computation in local charts. Indeed, let (t, u) be coordinates in a local analytic chart U in Y centered at a smooth point $y \in Y$, which is a general point of a fiber component F over p_i of multiplicity n in the divisor $(\pi')^*(p_i)$. We may choose t so that $h \circ \pi'|_U = t^n$ and $F \cap U = t^*(0)$. Then $Y \times_C B$ is given locally in $\mathbb{A}^3$ with coordinates (z, t, u) by equation $z^d - t^n = 0$, where $n|d$ by our choice of d. This is a union of n smooth surface germs $z^{d/n} - \zeta t = 0$, where $\zeta^n = 1$, meeting transversely along the line $z = t = 0$ that projects in Y onto $F \cap U$. After the normalization we get n smooth disjoint surface germs, say, $V_1, \dots, V_n$ in X over U. The function $z \in \mathcal{O}_X(X)$ gives in each chart V_j a local coordinate such that $\varphi^*(F) = z^*(0)$ has multiplicity one in V_j. We leave the further details to the reader. □

2.4 (*Cancellation Problem for surfaces: a reduction*)**.** The following reasoning is borrowed in [60, 61, 76]. It occurs that, in order to construct (families of) $\mathbb{A}^1$-fibered surfaces with isomorphic cylinders, it suffices to construct (families of) $\mathbb{A}^1$-fibered GDF G-surfaces with G-equivariantly isomorphic cylinders.

Suppose that a Galois base change $B \to C$ with a Galois group G applied to two distinct $\mathbb{A}^1$-fibered surfaces $\pi'_j\colon Y_j \to C$, $j = 0, 1$, yields two $\mathbb{A}^1$-fibered GDF G-surfaces $\pi_j\colon X_j \to B$, $j = 0, 1$, with G-isomorphic cylinders $X_0 \times \mathbb{A}^1 \cong_G X_1 \times \mathbb{A}^1$ over B, where in both cases G acts identically on the second factor $\mathbb{A}^1$. Clearly, for the quotients we have $(X_j \times \mathbb{A}^1)/G \cong Y_j \times \mathbb{A}^1$, $j = 0, 1$. Hence passing to the quotients yields an isomorphism of the cylinders $Y_0 \times \mathbb{A}^1 \cong Y_1 \times \mathbb{A}^1$ over C that fits in the commutative diagram



Thus, in the sequel we will concentrate on the following problem. Consider the cylinders $X \times \mathbb{A}^1$ and $X' \times \mathbb{A}^1$ over two $\mathbb{A}^1$-fibered GDF surfaces $\pi\colon X \to B$ and $\pi'\colon X' \to B$

with the same smooth affine base B. Suppose that π and π' are equivariant with respect to actions of a finite group G on X, X', and B. We extend these actions to G-actions on the cylinders $X \times \mathbb{A}^1$ and $X' \times \mathbb{A}^1$ identically on the second factor.

Problem 2.5. *Find conditions on a pair of GDF G-surfaces X and X' which guarantee that the cylinders $X \times \mathbb{A}^1$ and $X' \times \mathbb{A}^1$ are G-equivariantly isomorphic.*

In Theorems 5.7 and 5.9 we provide such sufficient conditions in the case where G is a cyclic group. Moreover, these theorems guarantee the existence of an equivariant B-isomorphism $X \times \mathbb{A}^1 \xrightarrow{\cong} X' \times \mathbb{A}^1$, that is, an isomorphism which respects the natural projections $X \times \mathbb{A}^1 \to B$ and $X' \times \mathbb{A}^1 \to B$ and induces the identity on B.

2.2. Pseudominimal completion and extended divisor.

Definition 2.6 (*Pseudominimal resolved completion*)**.** Any $\mathbb{A}^1$-fibration $\pi\colon X \to B$ on a normal affine surface X over a smooth affine curve B extends to a $\mathbb{P}^1$-fibration $\tilde{\pi} : \tilde{X} \to \bar{B}$ on a complete surface $\tilde{X}$ over a smooth completion $\bar{B}$ of B such that $D = \tilde{X} \smallsetminus X$ is a simple normal crossing divisor carrying no singular point of $\tilde{X}$. Let $\varrho : \bar{X} \to \tilde{X}$ be the minimal resolution of singularities (all of these singularities are located in X). Abusing notation, we consider D as a divisor in $\bar{X}$. We call $(\bar{X}, D)$ a *resolved completion* of X.

Consider the induced $\mathbb{P}^1$-fibration $\bar{\pi} := \tilde{\pi} \circ \varrho : \bar{X} \to B$. There is a unique (*horizontal*) component S of D which is a section of $\bar{\pi}$, while all the other (*vertical*) components of D are fiber components. Let $\bar{B} \smallsetminus B = \{c_1, \dots, c_s\}$. Contracting subsequently the (-1)-components of D different from S we may assume in addition that D does not have any (-1)-component. Such a resolved completion $(\bar{X}, D)$ is called *pseudominimal*.[4]

Definition 2.7 (*Extended divisor*)**.** Let $(\bar{X}, D)$ be a resolved completion of X along with the associate $\mathbb{P}^1$-fibration $\bar{\pi}\colon \bar{X} \to \bar{B}$, and let $b_1, \dots, b_n$ be the points of B such that the fibers $\tilde{\pi}^*(b_i)$ over b_i in $\bar{X}$ are degenerate, i.e., are either non-reduced, or reducible. The reduced divisor

$$D_{\mathrm{ext}} = D \cup \Lambda, \quad \text{where} \quad \Lambda = \bigcup_{j=1}^{n} \bar{\pi}^{-1}(b_j)\,, \tag{4}$$

is called the *extended divisor* of $(\bar{X}, D)$, and the weighted dual graph Γ_{ext} of D_{ext} the *extended graph* of $(\bar{X}, D)$. The graph Γ_{ext} is a rooted tree with the horizonthal section $S \subset D$ as a root. The dual graph $\Gamma(D)$ of the boundary divisor D is a rooted subtree of Γ_{ext}. The connected components of $D_{\mathrm{ext}} \ominus D$ are called *feathers* of D_{ext}. Under the pseudominimality assumption all the (-1)-components of Λ are among the feather components. In this case we say that Γ_{ext} is *pseudominimal*.

Definition 2.8 (*Standard completion*)**.** The fibers $\bar{\pi}^{-1}(c_i)$, $i = 1, \dots, s$ in a pseudominimal resolved completion $\bar{\pi}\colon \bar{X} \to \bar{B}$ are reduced, irreducible 0-curves. Performing, if necessary, elementary transformations in one of them we may assume that also the section S is a 0-curve. Such a completion will be called *standard*, cf., e.g., [34, 5.11]. By [34, Lem. 5.12], if two $\mathbb{A}^1$-fibrations $\pi\colon X \to B$ and $\pi'\colon X' \to B$ are B-isomorphic, then the corresponding standard extended divisors D_{ext} and D'_{ext} and the corresponding extended graphs Γ_{ext} and Γ'_{ext} are.

[4] We do not carry always this assumption; see, e.g., Proposition 2.29 and Corollary 2.30.

Remark 2.9 (*Fiber structure*)**.** Recall (see [62, Ch. 3, Lem. 1.4.1 and 1.4.4]) that any degenerate fiber of $\pi\colon X \to B$ is a disjoint union of components isomorphic to $\mathbb{A}^1$, any singular point of X is a cyclic quotient singularity, and two such singular points cannot belong to the same component. The minimal resolution of a singular point has as exceptional divisor in $\bar{X}$ a chain of rational curves without (-1)-component and with a negative definite intersection form. This chain meets just one other fiber component at a terminal component of the chain.

Definition 2.10 (*Bridges*)**.** Any feather $\mathfrak{F}$ of D_{ext} (see 2.7) is a linear chain of smooth rational curves on $\tilde{X}$ with dual graph

$$\Gamma(\mathfrak{F})\colon \quad \overset{F_0}{\circ}\!\!-\!\!\!-\overset{F_1}{\circ}\!\!-\!\!\!-\cdots-\!\!\!-\overset{F_k}{\circ}\,,$$

where the subchain $\mathfrak{R} = \mathfrak{F} \ominus F_0 = F_1 + \ldots + F_k$ (if non-empty) contracts to a cyclic quotient singularity of X, and the component F_0, called the *bridge* of $\mathfrak{F}$, is attached to a unique component C of D. The bridge F_0 is the closure in $\bar{X}$ of a fiber component $F_0 \setminus C \cong \mathbb{A}^1$ of π. Vice versa, for each fiber component F of π, the closure $\bar{F}$ in $\bar{X}$ of the proper transform of F is a bridge of a unique feather. In the case of a smooth surface X one has $k = 0$, i.e., any feather $\mathfrak{F}$ consists in a bridge: $\mathfrak{F} = F_0$.

2.3. **Graph divisors and type divisors.**

Definition 2.11 (*Fiber trees, levels, and types*)**.** Given a completion $\bar{\pi}\colon \bar{X} \to \bar{B}$ and a point $b \in B$, the dual graph $\Gamma_b = \Gamma_b(\bar{X})$ of the fiber $\bar{\pi}^{-1}(b)$ will be called a *fiber tree*. It depends on the completion chosen. This is a rooted tree with a root $v_0 \in \Gamma_b$ being the neighbor of S in Γ_{ext}. We say that a vertex v of Γ_b has *level* l if the tree distance between v and v_0 equals l. Thus, the root v_0 is a unique vertex of Γ_b on level 0. By a *hight* $\mathrm{ht}(\Gamma_b)$ we mean the highest level of the vertices in Γ_b. The *leaves* of a rooted tree are its extremal vertices different from the root. By the *type* $\mathrm{tp}(\Gamma_b)$ we mean the sequence of nonnegative integers $(n_1, n_2, \ldots, n_h)$, where $h = \mathrm{ht}(\Gamma_b)$ and n_i is the number of leaves of Γ_b on level i.

Definition 2.12 (*Graph divisor*)**.** Let $\mathfrak{G}$ be the set of all finite weighted rooted trees contractible to the root with weight zero. By a *graph divisor* on a smooth affine curve B we mean a formal sum

$$\mathcal{D} = \sum_{b \in B}^{n} \Gamma_b b, \quad \text{where} \quad \Gamma_b \in \mathfrak{G}\,,$$

and Γ_b consists just of the root $[[0]]$ for all but finite set of points $b \in B$. If all the Γ_b are chains, then we call $\mathcal{D}$ a *chain divisor*. The *height* of a graph divisor $\mathcal{D}$ is the maximal hight of the trees Γ_b, $b \in B$.

To any $\mathbb{A}^1$-fibered surface $\pi\colon X \to B$ with a marking $z \in \mathcal{O}_B(B)$, a resolved completion $\bar{\pi}\colon \hat{X} \to \bar{B}$, and the corresponding extended graph Γ_{ext}, we associate the graph divisor $\mathcal{D}(\pi) = \sum_{b \in B}^{n} \Gamma_b b$, where for any zero b_j of z, $j = 1, \ldots, n$, Γ_{b_j} is the fiber tree of the fiber $\pi^{-1}(b_j)$, and $\Gamma_b = [[0]]$ otherwise. If $\pi\colon X \to B$ is a μ_d-surface and the marking z is μ_d-quasi-invariant, then there is a natural μ_d-action on the graph divisor $\mathcal{D}(\pi)$.

Definitions 2.13 (*Type divisor*)**.** 1. Let $\pi\colon X \to B$ be a GDF surface, and let $(\bar{X}, D)$ be its resolved completion with the associated extended graph divisor $\mathcal{D}(\pi)$. Given a fiber $\pi^{-1}(b) = F_1 \cup \ldots \cup F_N$, its *type* is the one of the fiber tree $\Gamma_b = \Gamma_b(\bar{X})$, see

Definition 2.11. This is a sequence $\bar{n}_b = (n_i)_{i\geq 0} \in \mathbb{Z}^\infty$ of nonnegative integers, where n_i is the number of components F_j on level i, so that $n_i = 0\ \forall i > \mathrm{ht}(\Gamma_b)$. Thus, $\bar{n}_b = \bar{0} \in \mathbb{Z}^\infty$ if and only if $\pi^{-1}(b) = F_1$, where $l(F_1) = 0$, that is, Γ_b is the zero chain $[[0]]$.

2. By the *type divisor* of D_{ext} we mean the $\mathbb{Z}^\infty$-divisor

$$\mathrm{tp}\,(D_{\mathrm{ext}}) = \sum_{b\in B} \bar{n}_b b$$

on B, where the sum has a finite number of nonzero terms. We let $\mathrm{Div}(B, \mathbb{Z}^\infty)$ be the $\mathbb{Z}$-module of all $\mathbb{Z}^\infty$-divisors on B with finite support.

3. Given an effective divisor $A = \sum_{i=1}^n a_i b_i \in \mathrm{Div}(B)$, where $a_i \in \mathbb{N}\ \forall i = 1, \dots, n$, we define its action on $\mathrm{Div}(B, \mathbb{Z}^\infty)$ via a shift as follows. For $T \in \mathrm{Div}(B, \mathbb{Z}^\infty)$, we let

- $(A.T)(b) = T(b)$, if $b \notin \{b_1, \dots, b_n\}$;
- $(A.T)(b_i) = (0, \dots, 0, 1_{a_i}, 0, 0, \dots.)$, if $T(b_i) = \bar{0}$;
- $(A.T)(b_i)$ is the sequence $T(b_i)$ shifted by a_i positions to the right, otherwise.

4. We say that two $\mathbb{Z}^\infty$-divisors T_1 and T_2 are *linearly equivalent*, and we write $T_1 \sim T_2$, if $A_1.T_1 = A_2.T_2$ for some linearly equivalent effective divisors $A_1, A_2 \in \mathrm{Div}(B)$.

This terminology serves to formulate our main results. Namely, Theorem 5.7 says that

Given two GDF surfaces X, X' over B with isomorphic graph divisors, the cylinders over X and X' are isomorphic over B.

In Part II we strengthen this by showing that

The B-isomorphism classes of cylinders over GDF surfaces with base B are in one-to-one correspondence with the linear equivalence classes of their type divisors.

Remark 2.14. The type divisor can be expressed via a usual divisor on a non-separated one-dimensional affine scheme. Indeed, given a GDF surface $\pi\colon X \to B$, we define a *DF quotient* $\mathrm{DF}(X)$ to be the quotient of X by the equivalence relation determined by the fiber components of π. Thus, π factorizes into the quotient morphism $X \to \mathrm{DF}(X)$ with reduced, irreducible fibers followed by the induced morphism $\mathrm{DF}(X) \to B$. The latter morphism is an isomorphism over $B \setminus \{b_1, \dots, b_n\}$, while the preimage of b_j in $\mathrm{DF}(X)$ consists of N_j points $(b_{i,j})_{i=1,\dots,N_j}$, where N_j is the number of the fiber components $F_{i,j}$ in $\pi^{-1}(b_j)$. It is an isomorphism if and only if $\pi\colon X \to B$ admits a structure of a line bundle.

Letting $l_{i,j} = l(F_{i,j})$ be the level of $F_{i,j}$, the type divisor of X can be defined as the effective divisor $\sum_{i,j} l_{i,j} b_{i,j}$ on $\mathrm{DF}(X)$. There is an easy way to reconstruct the type divisor $\mathrm{tp}(\mathcal{D}(\pi))$ as in Definition 2.13.2 starting with the latter one, and vice versa. With the new definition, the linear equivalence of type divisors is the usual linear equivalence, that is, equivalence modulo the principal divisors on $\mathrm{DF}(X)$, that are just principal divisors on B lifted to $\mathrm{DF}(X)$. The Picard group $\mathrm{Pic}\,\mathrm{DF}(X)$ is defined in a usual way. To a GDF surface $\pi\colon X \to B$ we associate its Picard class $[\pi] = [-\mathrm{tp}(\mathcal{D}(\pi))] \in \mathrm{Pic}\,\mathrm{DF}(X)$. If $\pi\colon X \to B$ is a line bundle L, then $[\pi] = [L] \in \mathrm{Pic}\,B$. One can reformulate Theorem 0.5 as follows.

Theorem 2.15. *The B-isomorphism classes of cylinders over GDF surfaces with base B are in one-to-one correspondence with the elements in the Picard group $\mathrm{Pic}\,\mathrm{DF}(X)$, which can be represented in the anti-effective cone.*

Next we define an action of an effective divisor on the set of graph divisors on B via a *stretching*.

Definition 2.16. With an effective divisor $A = \sum_{i=1}^{n} a_i b_i \in \mathrm{Div}(B)$, where $a_i \in \mathbb{Z}_{\geq 0}$ and $b_i \in B$, we associate a chain divisor $\mathcal{D}(A) = \sum_{i=1}^{n} L(a_i) b_i$, where $L(a_i)$ is a chain with weights $[[-2, -2, \ldots, -2, -1]]$ of length a_i if $a_i > 0$, and $L(0) = \varnothing$. Let $\mathcal{D} = \sum_{b \in B} \Gamma_b b$ be a graph divisor such that $\mathrm{ht}\,(\Gamma_{b_i}) \geq m_i \geq -1$ for each $i = 1, \ldots, n$. We let $(A.\mathcal{D})_{\bar{m}} = \mathcal{D}' = \sum_{b \in B} \Gamma'_b b$, where $\bar{m} = (m_1, \ldots, m_n)$ and

- $\Gamma'_b = \Gamma_b$ if $b \notin \{b_1, \ldots, b_n\}$;
- otherwise, Γ'_{b_i} is obtained from Γ_{b_i} by inserting the chain L_i between each vertex v of Γ_{b_i} on level m_i and its neighbors on level $m_i + 1$, so that the left end l_i of L_i becomes a vertex on level $m_i + 1$ of Γ'_{b_i}, and its right end r_i is joint with the vertices of Γ_{b_i} on level $m_i + 1$ over v. The weights change accordingly: the weight of v decreases by 1, and the weight of r_i becomes $-1 - s(v)$, where $s(v)$ is the number of vertices on level $m_i + 1$ in Γ_{b_i} joint with v.

Clearly, $\mathrm{tp}(\mathcal{D}') = A.\mathrm{tp}(\mathcal{D})$. This operation of A on $\mathcal{D}$ will be called a (combinatorial) $(A, \bar{m})$-*stretching*.

2.4. **Blowup construction.**

Definition 2.17 (*Blowup construction*)**.** Let as before $\pi\colon X \to B$ be an $\mathbb{A}^1$-fibration on a normal affine surface X over a smooth affine curve B, and let $(\bar{X}, D)$ be a resolved completion of X along with the associate $\mathbb{P}^1$-fibration $\bar{\pi}\colon \bar{X} \to \bar{B}$ and with a section 'at infinity' S. In any degenerate fiber $\bar{\pi}^*(b_i)$ on $\bar{X}$, $i = 1, \ldots, n$, there is a unique component, say, D_i meeting S. The rest of the fiber $\bar{\pi}^{-1}(b_i) \ominus D_i$ can be blown down to a smooth point. This fact is well known; for the reader's convenience we provide a brief argument.

Lemma 2.18. *Let C_0 be the component of a fiber $\bar{\pi}^{-1}(b)$, $b \in B$, which corresponds to the root v_0 of Γ_b. Then the rest of the fiber $\bar{\pi}^{-1}(b) \ominus C_0$ can be blown down to a smooth point.*

Proof. Since $S \cdot \bar{\pi}^*(b) = S \cdot C_0 = 1$, C_0 has multiplicity 1 in the fiber. We proceed by induction on the number n of components in the fiber $\bar{\pi}^{-1}(b_i)$. The statement is clearly true for $n = 2$. Suppose now that $n > 1$. Then there exists a (-1)-component E in the fiber. If $E \neq C_0$, then contracting E we can use the induction conjecture. Assume now that $C_0^2 = -1$, and C_0 is the only (-1)-component of $\bar{\pi}^{-1}(b_i)$. Since C_0 has multiplicity 1, it appears as a result of an outer blowup on a component, say, C_1 of multiplicity 1. Hence after blowing down C_0, by the induction hypothesis, the rest of the fiber but C_1 can be blown down. Thus there is a (-1)-component of the fiber disjoint from C_0. However, the latter contradicts our assumption that C_0 is a unique (-1)-component of the fiber $\bar{\pi}^{-1}(b_i)$. □

Performing such a contraction for every $i = 1, \ldots, n$ we arrive at a geometrically minimal ruling (that is, a locally trivial $\mathbb{P}^1$-fibration) $\bar{\pi}_0\colon \bar{X}_0 \to \bar{B}$. The image of S on $\bar{X}_0$ is a section of $\bar{\pi}_0$. Thus $\bar{X}$ can be obtained starting with a geometrically ruled surface $\bar{X}_0$ via a sequence of blowups of points

$$\bar{X} = \bar{X}_m \xrightarrow{\sigma_m} \bar{X}_{m-1} \longrightarrow \ldots \longrightarrow \bar{X}_1 \xrightarrow{\sigma_1} \bar{X}_0 \tag{5}$$

with centers in the images of $D_i \smallsetminus S$ in $\bar{X}_0$ and at infinitely near points, $i = 1, \ldots, n$.

Definition 2.19 (*Well ordered blowup construction*)**.** In the rooted tree Γ_{ext} with a root S, the (-1)-vertices on a maximal distance from S are disjoint with S and mutually disjoint, due to Lemma 2.18. Hence the corresponding fiber components can be simultaneously contracted. Repeating this procedure, we arrive finally at a smooth geometrically ruled surface $\bar{\pi}_0\colon \bar{X}_0 \to \bar{B}$ along with a specific sequence (5) of blowups, where every σ_i, $i = 1, \ldots, n$, is a blowup with center in a reduced zero dimensional subscheme of $\bar{X}_{i-1} \smallsetminus (F \cup S)$. We call such a sequence (5) a *well ordered blowup construction*.

The following lemma is a generalization of Theorem 2.1 in [29].

Lemma 2.20. *Let $\pi\colon X \to B$ be an $\mathbb{A}^1$-fibered GDF G-surface., where G is a finite group. Then there is a G-equivariant resolved completion $(\bar{X}, D)$ of X obtained via a G-equivariant well ordered blowup construction (5).*

Proof. By Sumihiro Theorem ([75, Thm. 3]), there exists a G-equivariant projective completion $(\tilde{X}, \tilde{D})$ of X. The singularities of the pair $(\tilde{X}, \tilde{D})$ can be resolved via a minimal G-equivariant resolution. In this way we arrive at a G-equivariant smooth projective completion $(\bar{X}, D)$ of X by a G-stable simple normal crossing divisor D. The closures in $\bar{X}$ of the fibers of $\pi\colon X \to B$ form a (nonlinear) G-invariant pencil. Its base points also admit a G-equivariant resolution. Hence we may assume that $\bar{X}$ comes equipped with a G-equivariant $\mathbb{P}^1$-fibration $\bar{\pi}\colon \bar{X} \to \bar{B}$, along with a G-stable section S of $\bar{\pi}$.

In particular, the root S of the extended graph Γ_{ext} of $(\bar{X}, D)$ is fixed by the induced G-action on Γ_{ext}. This action stabilizes as well the set of all (-1)-vertices on a maximal distance from S. Therefore, the simultaneous contraction of the corresponding fiber components is G-equivariant. Continuing by recursion leads to a G-equivariant well ordered blowup construction. □

Remarks 2.21. 1. Under a (well ordered) blowup construction, no blowup in (5) is done near the section S of $\bar{\pi}_0$. We may assume also that no blowup is done with center over the points $c_i \in \bar{B} \smallsetminus B$, $i = 1, \ldots, s$, and so, the fibers in $\bar{X}$ over these points remain reduced and irreducible.

2. Let a component F of D_{ext} different from S be created by one of the blowups $\sigma_\nu\colon \tilde{X}_\nu \to \tilde{X}_{\nu-1}$ in (5). We claim that then the center of the blowup σ_ν belongs to the image of D in $\bar{X}_{\nu-1}$. Indeed, otherwise the last (-1)-curve, say, E over p_ν would neither be a bridge of a feather, nor a component of D. Hence E should be a component of a feather, say, $\mathfrak{F}$, different from the bridge component F_0. However, the latter contradicts the minimality of $\mathfrak{F} \ominus F_0$, that is, the minimality of the resolution of singularities of X.

Recall the following notions.

2.22. Let D be a simple normal crossing divisor on a smooth surface Y. A blowup of Y at a point $p \in D$ is called *outer* if p is a smooth point of D, and *inner* if p is a node.

We use the following notation.

Notation 2.23. Given a blowup construction (5) we let

$$D_{0,\mathrm{ext}} = S_0 \cup \Delta_0 \cup \Lambda_0 \subset \bar{X}_0, \quad \text{where} \quad \Delta_0 = \bigcup_{i=1}^{k} \bar{\pi}_0^{-1}(c_i) \ \text{ and } \ \Lambda_0 = \bigcup_{j=1}^{n} \bar{\pi}_0^{-1}(b_j)\,. \tag{6}$$

The following lemma should be well known; see, e.g., [20, (2.2)] for (b).

Lemma 2.24. [5] *Let $\pi\colon X \to B$ be a normal affine $\mathbb{A}^1$-fibered surface over a smooth affine curve B. Consider a resolved completion $(\bar{X} = \bar{X}_m, D)$ of X obtained via a well ordered blowup construction (5) starting with a ruled surface $\bar{\pi}_0\colon \bar{X}_0 \to \bar{B}$. Then the following hold.*

(a) *X is a GDF surface if and only if all the blowups σ_ν in (5), $\nu = 1, \dots, m$, are outer*[6].

(b) *If X is a GDF surface, then X is smooth, and every feather $\mathfrak{F}$ of $D_{\mathrm{ext}} = D_{m,\mathrm{ext}}$ consists in a single (-1)-component F_0, which is a bridge.*

(c) *For a fiber component F of a GDF surface $\pi\colon X \to B$ with a pseudominimal resolved completion*[7] *$(\bar{X}, D)$ the following are equivalent:*
- *$\bar{F}$ is a leave (an extremal vertex) of the rooted tree Γ_{ext};*
- *$\bar{F}$ is a feather;*
- *$\bar{F}$ is a (-1)-vertex of Γ_{ext}.*

Proof. Suppose that for some $\nu \in \{1, \dots, m\}$, the blowup σ_ν is inner. Assume also that the center $P_\nu \in \bar{X}_{\nu-1}$ of σ_ν lies on the fiber over $b_i \in B$ and on the image $D_{\nu-1,\mathrm{ext}}$ of D_{ext}. Then all the components of the fiber $\bar{\pi}^*(b_i)$ which appear over P_ν, including the last (-1)-component, say, $\bar{F}$, have multiplicities > 1. However, $\bar{F} = \bar{F}_0$ is a bridge component of a feather, say, $\mathfrak{F}$. Hence $\bar{F}$ is the closure in $\bar{X}$ of a component F of the fiber $\pi^*(b_i) \subset X$. Thus, the fiber $\pi^*(b_i)$ is not reduced. It follows that for a GDF surface $\pi\colon X \to B$ all the blowups σ_ν, $\nu = 1, \dots, m$, are outer.

To show the converse, suppose that all the blowups σ_ν in (5), $\nu = 1, \dots, m$, are outer. Then starting with the reduced divisor $D_{0,\mathrm{ext}}$, all the resulting degenerate fibers $\bar{\pi}^*(b_j)$, $j = 1, \dots, n$ are reduced as well. Hence $\pi\colon X \to B$ is a GDF surface. This proves (a).

Assume further that a feather $\mathfrak{F}$ of D_{ext} has more than one component. The component of $\mathfrak{F}$ which appears the last in the blowup construction (5) is the bridge component F_0 of $\mathfrak{F}$. Hence F_0 appears in a blowup σ_ν with center P_ν, which belongs to the image in $\bar{X}_{\nu-1}$ of a component C of D and the component $\bar{F}_1$ of $\mathfrak{F}$; see Remark 2.21. Thus P_ν is a nodal point of the divisor $D_{\nu-1,\mathrm{ext}}$ on $\bar{X}_{\nu-1}$. It follows that the blowup σ_ν is inner. So, the bridge component $\bar{F}_0$ of $\mathfrak{F}$ has multiplicity > 1 in its fiber.

This shows that for a GDF surface $\pi\colon X \to B$, every feather $\mathfrak{F}$ of D_{ext} consists in a single bridge component $\bar{F}_0$. Consequently, the surface X is smooth. Furthermore, assuming that $\bar{F}_0^2 < -1$, an outer blowup was done in (5) with center on F_0 creating a new component, say, E of D. The graph distance $\mathrm{dist}(E, S)$ in Γ_{ext} is bigger than $\mathrm{dist}(\bar{F}_0, E)$. Hence $\bar{F}_0$ disconnects S and E in D. The latter contradicts the facts that the affine surface X is connected at infinity, i.e., its boundary divisor D is connected. Therefore, $\bar{F}_0^2 = -1$. This shows (b).

The same argument proves that $\bar{F}_0$ is an extremal vertex (a *tip*) of Γ_{ext}. Conversely, if $\bar{F}$ is a tip of Γ_{ext} different from S, then $\bar{F}^2 = -1$. Indeed, since all the blowups in (5) are outer, then after creating $\bar{F}$ no further blowup was done near $\bar{F}$. Due to the pseudominimality assumption, $\bar{F}$ is a feather of D_{ext}. Now (c) follows. □

Remark 2.25. Let Γ_b be the fiber tree of a special fiber $\pi^{-1}(b)$ on a GDF surface $\pi\colon X \to B$ in its pseudominimal completion. It is viewed as an unweighted tree. However, one can easily reconstruct the weights due to the fact that Γ_b can be grown up starting

[5] Cf. the proof of Proposition 6.3.23 in [35].

[6] With respect to the divisor $D_{0,\mathrm{ext}}$ on $\bar{X}_0$ and its subsequent total transforms $D_{\nu,\mathrm{ext}}$ on $\bar{X}_\nu$.

[7] See Definition 2.6.

with the root v_0 of weight 0 via outer blowups, by Lemma 2.24. Namely, for a vertex v of weight $w(v)$ and of degree $\deg(v)$ in Γ_b one has $w(v) = -\deg(v)$. Thus, the (-1)-vertices are the leaves, and all linear vertices are (-2)-vertices.

2.5. **GDF surfaces via affine modifications.** Let $X \to B$ be an affine $\mathbb{A}^1$-fibered GDF surface over a smooth affine curve B. In Corollary 2.30 below we describe a recursive procedure, which allows to recover X starting with the product $B \times \mathbb{A}^1$ via a sequence of fibered modifications. This special type of affine modifications (see Definition 1.3) is defined as follows (cf. [20, Def. 4.2]).

Definition 2.26 (*Fibered modification*)**.** A *fibered modification* between two $\mathbb{A}^1$-fibered GDF surfaces $\pi\colon X \to B$ and $\pi'\colon X' \to B$ is an affine modification $\varrho\colon X' \to X$, which consists in blowing up a reduced zero-dimensional subscheme of X and deleting the proper transform of the union of those fiber components of π which carry centers of blowups. Such a modification is a *B-morphism*, that is, it fits in the commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{\ \varrho\ } & X \\ & {\scriptstyle\pi'}\searrow \quad \swarrow{\scriptstyle\pi} & \\ & B & \end{array}$$

Remark 2.27. Let F be a reduced curve on a smooth affine surface X, let $\Sigma \subset F$ be a reduced zero dimensional subscheme, and let $\sigma\colon X' \to X$ be composed of a blowing up X with center Σ and deleting the proper transform F' of F. We claim that X' is again affine, and so, by [54, Thm. 1.1], the birational morphism $X' \to X$ is an affine modification.

Indeed, there exists a completion $\bar{X}$ of X and an ample divisor A on $\bar{X}$ with support $\operatorname{supp} A = \bar{X} \setminus X$. Let $\bar{X}'$ be the surface obtained from $\bar{X}$ by blowing up with center Σ. By Kleiman ampleness criterion, the divisor $nA' + \bar{F}'$ on $\bar{X}'$, where A' and $\bar{F}'$ are the proper transforms on $\bar{X}'$ of A and of the closure $\bar{F}$ of F in $\bar{X}$, respectively, is again ample provided that n is sufficiently large. Hence, the surface $X' = \bar{X}' \setminus \operatorname{supp}(nA' + \bar{F}')$ is affine, as claimed.

In general, F is not a principal divisor on X. To represent $\sigma\colon X' \to X$ via an affine modification, let us choose functions $f, g \in \mathcal{O}_X(X)$ such that f vanishing on F to order 1, and the restriction $g|_F$ vanishes with order 1 on Σ. Let $I \subset \mathcal{O}_X(X)$ be the ideal generated by f, g, and by all regular functions on X vanishing on $\Sigma \cup (\mathbb{V}(f) \setminus F)$. Then $\sigma\colon X' \to X$ is the affine modification along divisor $f^*(0)$ with center I.

Let $\pi\colon X \to B$ be a GDF surface, F be a fiber component of π, and $f = \pi^* z$, where $z \in \mathcal{O}_B(B)$ has a simple zero at the point $\pi(F) \in B$. Then $\pi' = \pi \circ \sigma\colon X' \to B$ is again a GDF surface, and $\sigma\colon X' \to X$ is a fibered modification. This justifies Definition 2.26.

For a general GDF surface $\pi\colon X \to B$ we have the following decomposition.

Proposition 2.28. *(a) Any GDF surface $\pi\colon X \to B$ can be obtained starting with a line bundle $\pi_0\colon X_0 \to B$ over B via a sequence of fibered modifications*

$$X = X_m \xrightarrow{\varrho_m} X_{m-1} \longrightarrow \ldots \longrightarrow X_1 \xrightarrow{\varrho_1} X_0\,, \tag{7}$$

which can be extended to corresponding completions yielding a well ordered blowup construction (5)*.*

(b) Suppose, furthermore, that $\pi\colon X \to B$ is a GDF G-surface, where G is a finite group. Then (7) can be chosen so that the intermediate surfaces X_ν come equipped with G-actions making the morphisms $\varrho_{\nu+1}\colon X_{\nu+1} \to X_\nu$ and $\pi_\nu\colon X_\nu \to B$ G-equivariant for all $\nu = 0, \dots, m-1$.

Proof. (a) To construct (7) we exploit a well ordered blowup construction (5), which starts with a $\mathbb{P}^1$-bundle $\bar\pi_0\colon \bar X_0 \to \bar B$ and finishes with a pseudominimal completion $\bar\pi_m\colon \bar X_m \to \bar B$ of $\pi\colon X \to B$.

For any $\nu = 0, \dots, m$ we let $D_{\nu,\mathrm{ext}}$ (Δ_ν, S_μ, respectively) be the image on $\bar X_\nu$ of the extended divisor $D_{\mathrm{ext}} = D_{m,\mathrm{ext}}$ (the divisor $\Delta = \Delta_m$, the section $S = S_m$, respectively) on $\bar X_m = \bar X$. Let $\Gamma_{\nu,\mathrm{ext}}$ be the weighted dual graph of $D_{\nu,\mathrm{ext}}$ and $\Lambda_{\nu,\max}$ be the union of the fiber components of $\bar\pi_\nu\colon \bar X_\nu \to \bar B$ which correspond to the (extremal) vertices of $\Gamma_{\nu,\mathrm{ext}}$ on maximal distance from S_ν. Let also D_ν be the union of the remaining components of $D_{\nu,\mathrm{ext}}$. Then $\Lambda_{\nu+1,\max}$ is the exceptional divisor of the blowup $\sigma_\nu\colon \bar X_{\nu+1} \to \bar X_\nu$ with center on $\Lambda_{\nu,\max} \setminus D_\nu$.

Consider the open surface $X_\nu = \bar X_\nu \setminus D_\nu$. We claim that X_ν is affine, and $\sigma_\nu(X_{\nu+1}) \subset X_\nu$. Indeed, the latter follows since $\sigma_\nu(\Lambda_{\nu+1,\max} \setminus D_{\nu+1}) \subset \Lambda_{\nu,\max} \setminus D_\nu \subset X_\nu$ due to the above observation. To prove the former we use the Kleiman ampleness criterion (cf. Remark 2.27). We say that a component F of D_{ext} has *level* l if $\mathrm{dist}(F, S) = l + 1$ in Γ_{ext}. For any $\nu \ge l$ we attribute the same level l to the image of F in $\bar X_\nu$; this has the same combinatorial meaning. Choose a sequence of positive integers

$$s_0 \gg a_0 \gg a_1 \gg \ldots \gg a_{m-1} \gg 0\,,$$

and let A_ν be an effective divisor on $\bar X_\nu$ with support D_ν such that a_l is the multiplicity in A_ν of any fiber component of D_ν of level l, $l = 0, \dots, \nu-1$, and s_0 is the multiplicity of S_ν in A_ν. Performing elementary transformations in a fiber over a point $c_i \in \bar B \setminus B$ we may assume that $S_\nu^2 > 0$. Suppose to the contrary that there is an irreducible curve C in $\bar X_\nu$ with $C \cdot A_\nu = 0$. If C is not a component of $D_{\nu,\mathrm{ext}}$ then $\bar\pi(C) \subset B$, hence $\bar\pi|_C = \mathrm{cst}$, which gives a contradiction. If $C = S_\nu$ then clearly $C \cdot A_\nu > 0$, which contradicts our choice of C and A_ν. The same contradiction happens if C is a fiber component of $D_{\nu,\mathrm{ext}}$. Due to the Kleiman criterion, the divisor A_ν with support D_ν is ample. Thus the surface $X_\nu = \bar X_\nu \setminus D_\nu$ is affine.

Letting now $\varrho_{\nu+1} = \sigma_{\nu+1}|_{X_{\nu+1}}\colon X_{\nu+1} \to X_\nu$, $\nu = 0, \dots, m-1$, we obtain a desired sequence (7) of fibered modifications. This proves (a).

To show (b) it suffices to start with a G-equivariant version of sequence (5) constructed in the proof of Lemma 2.20(b). By our construction, σ_ν is G-equivariant and $D_{\nu,\mathrm{ext}}$, D_ν, and X_ν are G-stable. Hence $\varrho_{\nu+1} = \sigma_{\nu+1}|_{X_{\nu+1}}\colon X_{\nu+1} \to X_\nu$ is G-equivariant too for any $\nu = 0, \dots, m-1$. □

The following proposition is an affine analog of the Nagata-Maruyama Theorem about projective ruled surfaces ([66]; see also [55]). It allows to replace the line bundle $X_0 \to B$ in (7) by the trivial bundle $B \times \mathbb{A}^1 \to B$. For the corresponding completions, this amounts to a stratching, which extends feathers by chains of type $[[-1, -2, \dots, -2]]$ in D near S, so loosing the pseudominimality property.

Proposition 2.29. *Let X be the total space of a line bundle $\pi\colon X \to B$ over a smooth affine curve B. Assume that the surface X is affine. Then the following hold.*

(a) *X can be obtained starting with the product $B \times \mathbb{A}^1$ over B via a sequence of fibered modifications*

$$X = Z_M \xrightarrow{\varrho_M} Z_{M-1} \longrightarrow \ldots \longrightarrow Z_1 \xrightarrow{\varrho_1} Z_0 = B \times \mathbb{A}^1 , \tag{8}$$

where for each $i = 0, \ldots, M$ the induced projection $\pi_i \colon Z_i \to B$ yields a line bundle over B.

(b) *If, in addition, $\pi \colon X \to B$ is a marked GDF μ_d-surface, then for $i = 0, \ldots, M$ the morphisms $\varrho_i \colon Z_i \to Z_{i-1}$ in (8) and $\pi_i \colon Z_i \to B$ can be chosen to be μ_d-equivariant with respect to suitable μ_d-actions on the surfaces Z_i and the given μ_d-action on B.*

Proof. (a) Let $A = \mathcal{O}_X(X)$. The natural effective $\mathbb{G}_m$-action along the fibers of π induces a grading $A = \bigoplus_{i \geq 0} A_i$, where $A_0 = \mathcal{O}_B(B)$ and $A_1 \neq \{0\}$ (the latter fact is well known, and can be checked without difficulty). If $u \in A_1$ then the restriction of u to a general fiber of π yields a coordinate on this fiber. It follows that $\psi = (\mathrm{id}_B, u) \colon X \to B \times \mathbb{A}^1$ is a birational morphism of line bundles over B, hence an affine modification (see [54, Thm. 1.1]). Since ψ is $\mathbb{G}_m$-equivariant, its exceptional divisor E, center C, and divisor D are $\mathbb{G}_m$-invariant. Since u is a $\mathbb{G}_m$-quasi-invariant of weight 1, it vanishes along the zero section Z in X with order 1. Thus, we have $u^{-1}(0) = Z \cup F_1 \cup \ldots \cup F_n$, where $F_i = \pi^{-1}(b_i)$, $b_i \in B$, $i = 1, \ldots, n$. Then

$$E = F_1 \cup \ldots \cup F_n, \quad C = \{b_1, \ldots, b_n\} \times \{0\}, \quad \text{and} \quad D = \{b_1, \ldots, b_n\} \times \mathbb{A}^1 \subset B \times \mathbb{A}^1 .$$

So, ψ consists in blowing up a subscheme with support C and deleting the proper transform of D. Therefore, ψ factorizes through the $\mathbb{G}_m$-equivariant fibered modification $\varrho_1 \colon Z_1 \to B \times \mathbb{A}^1$, which consists in blowing up the reduced subscheme C and deleting the proper transform of D. One can factorize in a similar way the resulting birational morphism of line bundles $X \to Z_1$ over B. Proceeding by recursion, after a finite number of steps we get a desired decomposition of ψ into a sequence (8) of fibered modifications. Indeed, this process yields actually a $\mathbb{G}_m$-equivariant resolution of indeterminacies of the inverse birational map $\psi^{-1} \colon B \times \mathbb{A}^1 \dashrightarrow X$, hence it converges. This proves (a).

(b) Under the assumptions of (b), consider the induced μ_d-action on $Z_0 = B \times \mathbb{A}^1$ identical on the second factor. In order that $\psi = (\mathrm{id}_B, u) \colon X \to B \times \mathbb{A}^1$ were μ_d-equivariant it suffices to choose $u \in A_1^{\mu_d}$ being a μ_d-invariant. Since μ_d acts via automorphisms of the line bundle $\pi \colon X \to B$, it normalizes the $\mathbb{G}_m$-action on X. Hence it induced a representation of μ_d via automorphisms of the graded k-algebra $A = \bigoplus_{i \geq 0} A_i$. Let

$$A_1^{(i)} = \{a \in A_1 \,|\, \zeta.a = \zeta^i a \ \forall \zeta \in \mu_d\} .$$

Any element $a \in A_1$ belongs to the μ_d-invariant subspace E spanned by the orbit $\mu_d(a)$. The finite dimensional representation of μ_d in E splits into a sum of one-dimensional representations. Consequently, a can be written as a sum of μ_d-quasi-invariants. It follows that $A_1 = \bigoplus_{i=0}^{d-1} A_1^{(i)}$.

We claim that there exists a nonzero invariant $u \in A_1^{(0)} = A_1^{\mu_d}$. Indeed, for some $i \in \{0, \ldots, d-1\}$ there exists a μ_d-quasi-invariant $h \in A_1^{(i)} \setminus \{0\}$ of weight i. Since by our assumption X is a marked GDF μ_d-surface, there is also a μ_d-quasi-invariant $z \in A_0 = \mathcal{O}_B(B)$ of weight 1 (see Definition 2.1). Then $u = z^{d-i}h \in A_1^{\mu_d}$, as desired.

The resulting birational morphism $\psi = (\mathrm{id}_B, u) \colon X \to B \times \mathbb{A}^1$ over B is μ_d-equivariant. So, this is an affine modification with μ_d-invariant center C and divisor D.

Hence ψ factorizes through the μ_d-equivariant fibered modification $\varrho_1\colon Z_1 \to B\times\mathbb{A}^1$, which consists in blowing up the reduced zero dimensional subvariety $C\subset B\times\{0\}$ on $B\times\mathbb{A}^1$ and deleting the proper transform of D. Continuing by recursion, we arrive as before at a sequence (8) of μ_d-equivariant morphisms. □

Letting in (7) $G=\mu_d$ and extending this sequence on the right by those in (8) with a suitable new enumeration, we arrive at our final sequence of fibered modifications.

Corollary 2.30. *(a) Any GDF surface $\pi\colon X\to B$ can be obtained starting with a product $X_0=B\times\mathbb{A}^1$ via a sequence of fibered modifications*[8]

$$X=X_N\xrightarrow{\varrho_N}X_{N-1}\longrightarrow\ldots\longrightarrow X_1\xrightarrow{\varrho_1}X_0=B\times\mathbb{A}^1 \tag{9}$$

such that the center of ϱ_i is contained in the exceptional divisor of ϱ_{i-1}.

(b) Suppose furthermore that $\pi\colon X\to B$ is a marked GDF μ_d-surface. Then any intermediate surface X_i, $i=0,\ldots,N-1$, comes equipped with induced μ_d-actions so that the morphisms $\varrho_{i+1}\colon X_{i+1}\to X_i$ and $\pi_i\colon X_i\to B$ are μ_d-equivariant.

Remarks 2.31. 1. The morphisms in (9) can be extended to suitable completions yielding a sequence of birational morphisms

$$\hat{X}=\hat{X}_N\xrightarrow{\hat{\varrho}_N}\hat{X}_{N-1}\longrightarrow\ldots\longrightarrow\hat{X}_1\xrightarrow{\hat{\varrho}_1}\hat{X}_0=\bar{B}\times\mathbb{P}^1\,, \tag{10}$$

where $\hat{\pi}_i\colon\hat{X}_i\to\bar{B}$ is a μ_d-equivariant $\mathbb{P}^1$-fibration extending $\pi_i\colon X_i\to\bar{B}$, and $\hat{\varrho}_i\colon\hat{X}_i\to\hat{X}_{i-1}$ is a simultaneous contraction of a μ_d-invariant union of disjoint (-1)-components of $\hat{\pi}_i$-fibers, $i=0,\ldots,N$. Inspecting the proof of Proposition 2.29, we see that on the first M steps certain irreducible fibers of $\mathrm{pr}_1\colon\bar{B}\times\mathbb{P}^1\to\bar{B}$ are replaced by chains of rational curves with sequences of weights of type $[[-1,-2,\ldots,-2,-1]]$. This yields a (non-pseudominimal, in general) completion $\hat{X}_M$ of X_M over $\bar{B}$ with the boundary $\hat{X}_M\smallsetminus X_M$ being a simple normal crossings divisor. The remaining m steps can be done in the same way as in the proof of Proposition 2.28.

2. The section at infinity $\bar{B}\times\{\infty\}$ of $\mathrm{pr}_1\colon\bar{B}\times\mathbb{P}^1\to\bar{B}$ gives rise to a section at infinity S of $\hat{X}=\hat{X}_N\to\bar{B}$ with $S^2=0$. If the line bundle $\bar{X}_0\to\bar{B}$ in (7) is nontrivial, then the completion $(\hat{X},\hat{D})$ of X is different from the pseudominimal completion, say, $(\bar{X},D)$. Indeed, let $\hat{D}_{\mathrm{ext}}$ and D_{ext} be the extended divisor on $\hat{X}$ and on $\bar{X}$, respectively, and let $\hat{D}$ and D be the corresponding boundary (sub)divisors. Then D_{ext} is obtained from $\hat{D}_{\mathrm{ext}}$ by contracting the maximal chains of rational curves of type $[[-1,-2,\ldots,-2]]$ contained in $\hat{D}\ominus S$. Thus, if $\hat{X}\neq\bar{X}$, then the completion $(\hat{X},\hat{D})$ is not pseudominimal.

3. Under our procedure, we have to enlarge our initial set of points $b\in B$ such that the fibers $\pi^{-1}(b)$ are reducible, by the points $b\in B$ with reducible fibers $\hat{\pi}^{-1}(b)$. Since the completion $\hat{X}$ of X is μ_d-equivariant, the latter set is μ_d-stable. We may enlarge this set further to the set of zeros $b_1,\ldots,b_n$ of a μ_d-quasi-invariant function $z\in\mathcal{O}_B(B)$ of weight 1 with only simple zeros. Choosing for each $l=0,\ldots,N-1$ an appropriate ideal $I_l\subset\mathcal{O}_{X_l}(X_l)$, it will be convenient to take the reduced divisor $z^*(0)$ on X_l for the divisor of the modification $\varrho_{l+1}\colon X_{l+1}\to X_l$, cf. Remark 2.27. When mentioning *special fiber components*, we always mean the fiber components of the latter divisor.

Definition 2.32 (*Trivializing completions*)**.** The resolved completion $(\hat{X},\hat{D})$ of a GDF surface X fitting in (10), and the corresponding graph divisor $\mathcal{D}(\hat{\pi})$ will be called *trivializing*. Note that $(\hat{X},\hat{D})$ is obtained from the pseudominimal resolved completion

[8]Notice that there is no direct relation between sequences (5) and (9).

$(\bar{X}, D)$ by introducing additional chains of rational curves next to the section S, which amounts in an $(A, \overline{-1})$-stratching $\mathcal{D}(\bar{\pi}) \rightsquigarrow \mathcal{D}(\hat{\pi})$. In the sequel, when dealing with this kind of completions, we often omit the adjective 'trivializing'.

In the course of the proof of Proposition 2.28 we used a level function on the set of fiber components of $\pi_l\colon X_l \to B$. Let us extend this notion to all surfaces X_l in (9).

Definition 2.33 (*Level function*)**.** We say that a fiber component F of $\pi_m\colon X_m \to B$ *has level* l if it appears for the first time on the surface X_l ($l \le m$) in (9). Thus, any fiber of $\pi_0\colon X_0 \to B$ has level 0, and any fiber component F' of $\pi_l\colon X_l \to B$ has level $\le l$.

Remarks 2.34. 1. Note that a fiber component F of $\pi\colon X \to B$ has level l if and only if the vertex $\bar{F}$ is on distance l from the root v_0 of the fiber tree Γ_b. Hence, on the leaves of Γ_b our level function coincides with the one defined in 2.11. If $\pi\colon X \to B$ is a marked GDF μ_d-surface, then the completion $\hat{X} \to \bar{B}$ is equivariant and the extended divisor $D_{\text{ext}} \subset \hat{X}$ is μ_d-stable. Therefore, in this case the level function is μ_d-invariant.

2. Being defined via the distance function on the extended graph Γ_{ext}, the level function l on the set of fiber components depends on the completion $(\hat{X}, \hat{D})$ of X as in (10).

3. Inspecting our construction of sequence (9) we see that the center of the blowup $\varrho_{l+1}\colon X_{l+1} \to X_l$ is situated on the union of top level (that is, level l) fiber components in X_l.

3. Vector fields and natural coordinates

3.1. Vertical locally nilpotent vector fields.

Lemma 3.1. *Let $\pi\colon X \to B$ be a marked GDF μ_d-surface. Then for any $l = 0, \dots, N$ there is a locally nilpotent regular μ_d-quasi-invariant vertical vector field ∂_l on the surface X_l in (9) non-vanishing on the fiber components of the top level l and vanishing on the fiber components of smaller levels.*

Proof. Consider the locally nilpotent vertical vector field $\partial_0 = \partial/\partial u$ on $B \times \mathbb{A}^1$, where $\mathbb{A}^1 = \operatorname{spec} k[u]$. Clearly, ∂_0 is stable under the μ_d-action on $B \times \mathbb{A}^1$ identical on the second factor. The μ_d-equivariant fibered modification $\varrho_1\colon X_1 \to B \times \mathbb{A}^1$ over B transforms ∂_0 into a μ_d-invariant rational vertical vector field on X_1 with pole of order 1 along the fiber components of level 1. By induction, ∂_0 lifts to a μ_d-invariant rational vertical vector field on X_l with pole of order s on any fiber component of level s, where $s \le l$, and no other pole.

Let $z \in \mathcal{O}_B(B)$ be a μ_d-quasi-invariant regular function on B of weight 1 with simple zeros at the points $b_1, \dots, b_n$. By abuse of notation, we denote by the same letter z its lift to the surface X_l. Then $\partial_l = z^l \partial/\partial u$ generates a regular, locally nilpotent, μ_d-quasi-invariant vertical vector field on X_l of weight l non-vanishing on the fiber components of level l and vanishing on the fiber components of smaller levels. □

3.2. Standard affine charts.

Notation 3.2. Let $\pi\colon X \to B$ be a marked GDF μ_d-surface with a marking $z \in \mathcal{O}_B(B)$, and let $b_1, \dots, b_n \in B$ be the zeros of z. For any $i = 1, \dots, n$ consider in B the affine chart $B_i = B \setminus \{b_1, \dots, \hat{b}_i, \dots, b_n\}$ around the point b_i. So $z|_{B_i}$ vanishes to order 1 at b_i and has no further zero in B_i.

For $l \in \{0,\dots,N\}$ let $F_{i,1},\dots,F_{i,n_i}$ be the components of the fiber $\pi_l^{-1}(b_i)$ in X_l. Consider also the vertical $\mathbb{G}_a$-action H_l on X_l generated by the locally nilpotent regular vertical vector field ∂_l as in Lemma 3.1. In the next proposition we introduce an affine covering on X_l in (9) by affine charts of product type; we call these *standard affine charts*.

Proposition 3.3. *In the notation as above the following hold.*

- *For any $l \in \{0,\dots,N\}$ and any $j \in \{1,\dots,n_i\}$ there is a unique affine chart $U_{i,j} = U_{i,j}^{(l)} \supset F_{i,j}$ in X_l isomorphic to $B_i \times \mathbb{A}^1$ over B_i. All such affine charts form a covering of X_l;*
- *we have $U_{i,j} \cap U_{i,t} = U_{i,j} \setminus F_{i,j} = U_{i,t} \setminus F_{i,t}$ for any $1 \le j,t \le n_i$;*
- *the μ_d-action on X_l induces a μ_d-action by permutations on the collection $(U_{i,j})$;*
- *every affine chart $U_{i,j}$ on X_l is H_l-stable. Furthermore, for any $t \le l$ and any fiber component $F_{i,j}$ of level t on X_l, the H_t-action is well defined and free on $U_{i,j}$;*
- *for any $l,t \in \mathbb{Z}$ with $0 \le t < l \le N$ the composition $\varrho_{l,t} = \varrho_{t+1} \circ \dots \circ \varrho_l \colon X_l \to X_t$ sends the affine chart $U_{i,j}^{(l)}$ around a fiber component $F_{i,j}^{(l)}$ of level t on X_l isomorphically over B_i onto a standard affine chart $U_{i,j'}^{(t)}$ in X_t around the fiber component $F_{i,j'}^{(t)} = \varrho_{l,t}(F_{i,j}^{(l)})$.*

Proof. The assertions are evidently true for the product $X_0 = B \times \mathbb{A}^1$ in (9) with $n_i = 1$ $\forall i$ and $U_{i,1}^{(0)} = \pi_0^{-1}(B_i) = B_i \times \mathbb{A}^1$. Suppose by recursion that they hold for a surface X_{l-1} in (9) and a collection of affine charts $(U_{i,j}^{(l-1)})$ on X_{l-1}. The μ_d-equivariant fibered modification $\varrho_l \colon X_l \to X_{l-1}$ in (9) consists in blowing up with center at a union of μ_d-orbits situated on fiber components of the top level $l-1$ of $\pi_{l-1} \colon X_{l-1} \to B$ and deleting the proper transforms of these fiber components, see Remark 2.34. Let $F = F_{i,j}$ be one of these components on X_{l-1}, and let $U_F = U_{i,j}^{(l-1)}$ be the corresponding standard affine chart in X_{l-1} around F. Then the modification ϱ_l replaces F with new components, say, $F_1,\dots,F_m$ of level l on X_l over the same point $b_i \in B$. Restricted to the chart $U_F \cong B_i \times \mathbb{A}^1$ around F this gives a fibered modification of U_F resulting in a GDF surface over B_i, with the only degenerate fiber over $b_i \in B_i$ consisting of the components $F_1,\dots,F_m$. If instead of blowing up m points, say, $x_1,\dots,x_m$ on F we blow up just one point, say, x_j, then we replace F by just one component, say, F_j. Without loss of generality we may choose a coordinate u in $\mathbb{A}^1$ so that $u(x_j) = 0$, $z(x_j) = 0$, and x_j is the only point in $U_F \cong B_i \times \mathbb{A}^1$ verifying these equations. Then the latter affine modification in the affine chart U_F consists in passing from $\mathcal{O}_{B_i}(B_i)[u]$ to $\mathcal{O}_{B_i}(B_i)[u/z] = \mathcal{O}_{B_i}(B_i)[u']$, where $u' = u/z$. This results again in a product affine chart, say, $U_{i,j'}^{(l)} \cong B_i \times \mathbb{A}^1$ in X_l. In total, we obtain m such charts on X_l over U_F, with intersections as needed. For each fiber component $F_{i,j}$ on X_{l-1} free from the centers of the modification ϱ_l we let $U_{i,j}^{(l)} = \varrho_l^{-1}(U_{i,j}^{(l-1)})$. We leave to the reader to check that the resulting collection of the affine charts $(U_{i,j}^{(l)})$ on X_l still verifies our conditions. □

Remark 3.4. Let $\mathrm{Aut}_B^\circ(X_l)$ be the identity component (in the sense of [69]) of the group of B-automorphisms of X_l. Then clearly any $\alpha \in \mathrm{Aut}_B^\circ(X_l)$ stabilizes each standard affine chart $U_{i,j}$ on X_l. The same is true for the identity component of the group $\mathrm{Aut}_B(\mathcal{X}_l)$ of all B-automorphisms of the cylinder $\mathcal{X}_l = X_l \times \mathbb{A}^1$ and the standard affine charts $U_{i,j} \times \mathbb{A}^1$ on $\mathcal{X}_l$.

3.3. **Natural coordinates.** Let as before $\pi\colon X \to B$ be a marked GDF μ_d-surface with a marking $z \in \mathcal{O}_B(B)$, and let $b_1, \dots, b_n \in B$ be the zeros of z.

Definition 3.5 (*Local coordinates*)**.** We say that an affine chart $U_{i,j}^{(l)}$ around the fiber component $F_{i,j}^{(l)}$ on X_l *has level* t if $F_{i,j}^{(l)}$ is of level t. An isomorphism $U_{i,j}^{(l)} \cong B_i \times \mathbb{A}^1$ provides sections of $\pi_l|_{U_{i,j}^{(l)}}\colon U_{i,j}^{(l)} \to B_i$. Fixing such a section and using the vertical free $\mathbb{G}_a$-action on $U_{i,j}^{(l)}$, we obtain a $\mathbb{G}_a$-equivariant isomorphism $U_{i,j}^{(l)} \cong B_i \times \mathbb{A}^1$, where $\mathbb{G}_a$ acts on the direct product via translations along the second factor. Fixing a coordinate u in $\mathbb{A}^1$ we obtain a coordinate, say, $u_{i,j}^{(l)}$ in $U_{i,j}^{(l)}$.

The restriction $z|_{U_{i,j}^{(l)}}$ vanishes with order 1 along $F_{i,j}^{(l)}$ and has no further zero. Hence the pair $(z, u_{i,j})$ yields local coordinates around the fiber $F_{i,j}^{(l)}$ in $U_{i,j}^{(l)}$. We call them *natural coordinates*. Fixing also a coordinate v in yet another exemplar of the affine line $\mathbb{A}^1$, we get natural local coordinates $(z, u_{i,j}, v)$ in the affine chart $U_{i,j}^{(l)} \times \mathbb{A}^1$ around the affine plane $\mathcal{F}_{i,j}^{(l)} = F_{i,j}^{(l)} \times \mathbb{A}^1 \cong \mathbb{A}^2$ in the cylinder $\mathcal{X}_l = X_l \times \mathbb{A}^1$.

Lemma 3.6. *One can choose the natural local coordinates $(z, u_{i,j})$ in $U_{i,j}^{(l)}$ in such a way that for any $\zeta \in \mu_d$, if $\zeta(U_{i,j}^{(l)}) = U_{i',j'}^{(l)}$ then $\zeta_*(z) = \zeta z$ and $\zeta_*(u_{i,j}) = \zeta^t u_{i',j'}$ for some $t \in \mathbb{Z}$.*

Proof. The assertion is evidently true for $l = 0$. Suppose by recursion that it holds for some $l \in \{0, \dots, N-1\}$. Then it holds for all the affine charts $U_{i,j}^{(l+1)}$ of level $\le l$ on X_{l+1}. Indeed, this follows since the level function is μ_d-invariant and the morphisms $\varrho_{l+1,s}$, $s = 0, \dots, l$, are μ_d-equivariant.

Let now $U_{i,j}^{(l+1)}$ be an affine chart of level $l+1$ on X_{l+1}, and let $\mu_e \subset \mu_d$ (where $e|d$) be the isotropy subgroup of $U_{i,j}^{(l+1)}$ (or, which is the same, of the corresponding fiber component $F_{i,j}^{(l+1)}$ on X_{l+1}). The μ_e-action on $U_{i,j}^{(l+1)}$ induces a μ_e-action on $B_i \times \mathbb{A}^1 \cong U_{i,j}^{(l+1)}$. The sections of $\pi_{l+1}|_{U_{i,j}^{(l+1)}}\colon U_{i,j}^{(l+1)} \to B_i$ are in one-to-one correspondence with the sections of the canonical projection $\mathrm{pr}_1\colon B_i \times \mathbb{A}^1 \to B_i$ and, in turn, with the functions in $\mathcal{O}_{B_i}(B_i)$. Choosing such a section arbitrarily and averaging over its μ_e-orbit we obtain a μ_e-stable section. The latter can be taken for the zero level of a coordinate function $u = u_{i,j}^{(l+1)}$ in $U_{i,j}^{(l+1)}$. There is such a unique function u, which satisfies in addition the relation $\partial_{l+1}(u) = 1$, where ∂_{l+1} is the vertical μ_d-quasi-invariant vector field on X_{l+1} of weight, say, m constructed in Lemma 3.1. For $\zeta \in \mu_s$ the ratio $\zeta.u/u$ does not vanish, hence it is constant along any π_{l+1}-fiber. Thus $\zeta.u = \pi_{l+1}^* f \cdot u$ for some function $f \in \mathcal{O}_{B_i}^\times(B_i)$. From the relations $\partial_{l+1} \circ \zeta = \zeta^m \partial_{l+1}$ and $\partial_{l+1}\pi_{l+1}^* f = 0$ we deduce that $f = \zeta^{-m}$ is a constant, and so, u is a μ_s-quasi-invariant of weight $-m$.

Choose a generator ξ of the quotient group μ_d/μ_e. For an affine chart $U_{i',j'}^{(l)} \ne U_{i,j}^{(l+1)}$ from the μ_d-orbit of $U_{i,j}^{(l+1)}$, we define the coordinate functions $u_{i',j'}$ on $U_{i',j'}^{(l)}$ to be the ξ^k-image of u with a suitable $k \in \mathbb{Z}$. Applying the same procedure to every orbit of the μ_d-action in the set $(U_{i,j}^{(l+1)})_{l+1}$ of the level $l+1$ affine charts on X_{l+1}, we arrive finally at a desired μ_d-quasi-invariant system of natural coordinates in our collection of affine charts. □

Remarks 3.7. 1. In the natural coordinates (z, u) in a standard affine chart U in X_l of level $t \le l$, the vertical vector field ∂_l on X_l constructed in Lemma 3.1 coincides with $z^{l-t}\partial/\partial u$. In particular, in a top level chart U we have $\partial_l|_U = \partial/\partial u$.

2. If $e > 1$, then the natural coordinates as in the lemma are uniquely defined, while in the case $e = 1$ our choice of a μ_e-stable section is arbitrary, and the coordinate u in the standard affine chart $U_{i,j}$ is defined up to a shear, that is, up to a shift in the vertical direction in the z-fibers. In particular, we may fix our choice so that, if such an affine chart $U_{i,j}$ is of top level and carries a finite number of centers of the blowup $\varrho_{l+1}\colon X_{l+1} \to X_l$, then the natural coordinate u does not vanish in any of these points.

3.4. **Examples of GDF surfaces.** We start with the classical Danielewski example.

Example 3.8 (*Danielewski surfaces*)**.** The Danielewski surface X_1 results from the affine modification $\varrho_1\colon X_1 \to X_0$ of the affine plane $X_0 = \mathbb{A}^2 = \operatorname{Spec} k[z,u]$ with divisor $z = 0$ and center $I = (z, u^2 - 1)$. This consists in blowing up the points $x_1 = (0,1)$ and $x_{-1} = (0,-1)$ in $\mathbb{A}^2$ and deleting the proper transform of the affine line $z = 0$. Letting $A_0 = \mathcal{O}_{X_0}(X_0) = k[z,u]$ and $A_1 = \mathcal{O}_{X_1}(X_1)$ we have

$$A_1 = A_0[(u^2-1)/z] = k[z,u,t_1]/(zt_1 - u^2 + 1)\,,$$

with the projections $\pi_0\colon X_0 \to B = \operatorname{Spec} k[z]$ and $\pi_1\colon X_1 \to B$ given by the inclusions $k[z] \hookrightarrow k[z,u] \hookrightarrow k[z,u,(u^2-1)/z]$. Thus, X_1 is given in $\mathbb{A}^3$ with coordinates (z,u,t_1) by equation

$$zt_1 - u^2 + 1 = 0\,.$$

The unique reducible fiber $\pi_1^*(0)$ of the GDF surface $\pi_1 = z|_{X_1}\colon X_1 \to B = \mathbb{A}^1$ consists of two disjoint affine lines (components of level one)

$$F_1 = \{z = 0, u = 1\} \quad \text{and} \quad F_{-1} = \{z = 0, u = -1\}\,.$$

The complement $X_1 \setminus F_j$ for $j \neq i$ gives a standard affine chart $U_i \cong \mathbb{A}^2$ around F_i. The chart U_1 can be obtained via the affine modification of X_1 along the divisor $z^*(0) = F_1 + F_{-1}$ with center the ideal $\mathbb{V}(F_1) = (z, u-1)$. Thus,

$$\mathcal{O}_{U_1}(U_1) = A_1\,[(u-1)/z] = k[z,u_1], \quad \text{where} \quad u_1 := (u-1)/z = t_1/(u+1)\,.$$

Similarly, an affine chart on X_1 near F_{-1} is

$$U_{-1} = X_1 \setminus F_1 = \operatorname{Spec} A_1[(u+1)/z] = \operatorname{Spec} k[z,u_{-1}] \cong \mathbb{A}^2$$

where z and $u_{-1} = (u+1)/z = t_1/(u-1)$ are natural coordinates in U_{-1}. The locally nilpotent vertical vector field

$$\partial_1 = z\partial/\partial u + 2u\partial/\partial t_1$$

on X_1 restricts to $\partial/\partial u_i$ in U_i, $i = 1, -1$. The phase flow of ∂_1 yields a free $\mathbb{G}_a$-action on X_1. It is sent under ϱ_1 to the field $d\varrho_1(\partial_1) = z\partial_0 = z\partial/\partial u$ on X_0.

The second Danielewski surface X_2 is obtained via the affine modification $\varrho_2\colon X_2 \to X_1$ with divisor $z^*(0)$ on X_1 and center $I = (z, t_1) \subset A_1$. Thus, ϱ_2 consists in blowing up X_1 at the points $x_{1.i} = (0,i,0) \in F_i$, $i = 1, -1$ (the origins of the affine planes $U_i \cong \mathbb{A}^2$, $i = 1, -1$) and deleting the proper transforms of the fiber components F_1 and F_{-1}. Letting $t_2 = t_1/z$ we obtain

$$A_2 := \mathcal{O}_{X_2}(X_2) = A_1[t_1/z] = k[z,u,t_2]/(z^2t_2 - u^2 + 1)\,.$$

Once again, X_2 is a GDF surface with a unique reducible fiber $z^*(0)$ consisting of two components of level 2. Iterating this procedure we arrive at a sequence of Danielewski surfaces

$$X_m = \operatorname{Spec} k[z,u,t_m]/(z^m t_m - u^2 + 1), \quad m = 1, 2, \ldots$$

along with a sequence of affine modifications fitting in (9)

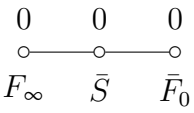
$$\varrho_m\colon X_m \to X_{m-1}, \quad (z,u,t_m) \mapsto (z,u,t_{m-1}), \quad \text{where} \quad t_{m-1} = zt_m\,.$$

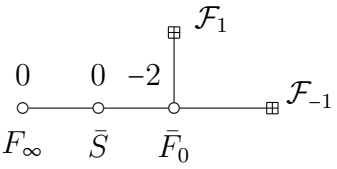
The only special fiber $z^*(0)$ in X_m is reduced and consists of two components of level m. The vector field $z^m\partial/\partial u$ on X_0 lifts to the locally nilpotent vertical vector field on X_m,

$$\partial_m = z^m\partial/\partial u + 2u\partial/\partial t_m\,.$$

Its phase flow defines a free $\mathbb{G}_a$-action on X_m. In each standard affine chart in X_m, the latter action restricts to the standard $\mathbb{G}_a$-action via shifts in the vertical direction.

The extended divisor $D_{\mathrm{ext,m}}$ of a minimal completion $\bar\pi\colon \bar X_m \to \mathbb{P}^1$ has dual graph

(11) $\Gamma_{\mathrm{ext},0}$: 

resp., $\Gamma_{\mathrm{ext,m}}$: 

where $m \geq 1$ and a box stands for the chain $[[-2,\dots,-2,-1]]$ of length m, so that $\mathcal{F}_i$ ends with the (-1)-feather $\bar F_i$ of level m, $i = 1,-1$ (see Example 7.7).

Example 3.9. As an immediate generalization of the preceding example, consider a surface X_m in $\mathbb{A}^3$ with equation $z^m t_m - q(u) = 0$, where $q \in k[u]$ is a polynomial of degree $d \geq 2$ with simple roots. This is a GDF surface with projection $\pi_m = z|_{X_m}\colon X_m \to \mathbb{A}^1$. Letting $X_0 = \mathbb{A}^2$ we have a sequence of affine modifications (9), where $\varrho_i\colon X_i \to X_{i-1}$, $(z,u,t_i) \mapsto (z,u,t_{i-1} = zt_i)$. The vector field $z^m\partial/\partial u$ on $X_0 = \mathbb{A}^2$ lifts to the locally nilpotent vertical vector field on X_m,

$$\partial_m = z^m\partial/\partial u + q'(u)\partial/\partial t_m\,,$$

which generates a free vertical $\mathbb{G}_a$-action on X_m. The dual graph $\Gamma_{\mathrm{ext,n}}$ of the minimal completion $\bar\pi\colon \bar X_m \to \mathbb{P}^1$ differs from the graph in diagram (11) in one aspect: instead of two contractible chains $\mathcal{F}_1$ and $\mathcal{F}_{-1}$, it has d such chains $\mathcal{F}_j$, $j = 1,\dots,d$ of the same length m. The dual graph $\Gamma(\bar\pi_n^{-1}(0))$ is a unibranched rooted tree with the root $\bar F_0$ as a unique possible branching vertex.

Example 3.10. Letting in 3.9 $m = 1$, consider the GDF surface $z|_{X_1}\colon X_1 \to B = \mathbb{A}^1$ given in $\mathbb{A}^3$ by equation $zt_1 - q_1(u) = 0$, where $q = q_1 \in k[u]$ has simple roots $\alpha_1,\dots,\alpha_d$. Its unique special fiber $z = 0$ consists of d disjoint components $F_1,\dots,F_d$ of level 1. The fibered modification $\varrho_1\colon X_1 \to X_0 = \mathbb{A}^2$, $(z,u,t_1) \mapsto (z,u)$, contracts F_i to the point $P_i = (0,\alpha_i) \in X_0$.

Fix a polynomial $q_2 \in k[u,t_1]$ such that, for each $i = 1,\dots,d$, either $q_2(\alpha_i,t_1) \in k[u]$ has $m_i > 0$ simple roots $\beta_{i,1},\dots,\beta_{i,m_i}$, or is equal zero (then we let $m_i = 0$). In $\mathbb{A}^4$ with coordinates (z,u,t_1,t_2) the equations

$$zt_1 - q_1(u) = 0, \quad zt_2 - q_2(u,t_1) = 0$$

define a smooth, irreducible surface X_2 and several disjoint affine planes contained in the hyperplane $z = 0$. The morphism $\sigma_2\colon X_2 \to X_1$, $(z,u,t_1,t_2) \mapsto (z,u,t_1)$, is an affine modification of X_1 along the reduced divisor $z^*(0)$ with center consisting, for each $m_i > 0$, of the reduced zero-dimensional scheme $P_{i,1} + \ldots + P_{i,m_i}$, where $P_{i,j} = (0,\alpha_i,\beta_{i,j}) \in F_i$ and, for each $m_i = 0$, of the whole fiber component F_i. The complement $X_2 \setminus \pi_2^{-1}(0)$

is isomorphic to $\mathbb{A}^1_* \times \mathbb{A}^1$ over $\mathbb{A}^1_*$. The unique special fiber $z = 0$ of the GDF surface $\pi_2 = z|_{X_2}: X_2 \to \mathbb{A}^1$ has $c_2 = m_1 + \ldots + m_d$ components $F_{i,j}$ of level 2 and c_1 components F_i of level 1, where $c_1 = \mathrm{card}\{i \in \{1,\ldots,d\} \,|\, m_i = 0\}$. The dual graph $\Gamma(\bar\pi_2^{-1}(0))$ of the degenerate fiber $\bar\pi_2^{-1}(0)$ of a pseudominimal completion $\bar\pi_2: \bar X_2 \to \bar B = \mathbb{P}^1$ is a rooted tree with a root $\bar F_0$ of level 0, d vertices $\bar F_1, \ldots, \bar F_d$ of level 1, and c_2 vertices $\bar F_{i,j}$, $i = 1,\ldots,d, j = 1,\ldots,m_i$ of level 2, where for $m_i > 0$, $\bar F_{i,j}$ is a neighbor of $\bar F_i$. Proceeding in this way, one can realize any given finite rooted tree Γ of hight 3 as the dual graph $\Gamma(\bar\pi_2^{-1}(0))$, where the weights of the vertices are uniquely determined by Γ.

The vector field $z^2\partial/\partial u$ on $X_0 = \mathbb{A}^2$ lifts to X_2 and extends to a locally nilpotent vector field ∂_2 on $\mathbb{A}^4$, where

$$\partial_2 = z^2\frac{\partial}{\partial u} + zq_1'(u)\frac{\partial}{\partial t_1} + \left(z\frac{\partial q_2}{\partial u}(u,t_1) + \frac{\partial q_2}{\partial t_1}(u,t_1)\right)\frac{\partial}{\partial t_2}\,.$$

The associated vertical $\mathbb{G}_a$-action on X_2 is identical on the fiber components F_i of level 1 (with $m_i = 0$).

Next we chose on each fiber component $F_{i,j}$ of level 2 in X_2 some $m_{i,j}$ distinct points

$$P_{i,j,k} = (0, \alpha_i, \beta_{i,j}, \gamma_{i,j,k}), \quad k = 1,\ldots,m_{i,j}\,,$$

where $m_{i,j} \geq 0$. Let a polynomial $q_3 \in k[u,t_1,t_2]$ be such that, for each (i,j), the polynomial $q_3(\alpha_i,\beta_{i,j},t_2) \in k[t_2]$ has simple roots $\{\gamma_{i,j,k} \,|\, k = 1,\ldots,m_{i,j}\}$ if $m_{i,j} > 0$ and is zero otherwise. In $\mathbb{A}^5$ with coordinates (z,u,t_1,t_2,t_3) the complete intersection with equations

$$zt_1 - q_1(u) = 0, \quad zt_2 - q_2(u,t_1) = 0, \quad zt_3 - q_3(u,t_1,t_2) = 0$$

has a unique smooth, irreducible component X_2 such that $\pi_3 = z|_{X_3}: X_3 \to \mathbb{A}^1$ is a GDF surface, and several planes contained in the hyperplane $z = 0$. The surface X_3 is a fibered affine modification of X_2 with center in the chosen points $P_{i,j,k}$.

Given any GDF surface $\pi_N: X_N \to B = \mathbb{A}^1$ with a unique special fiber $\pi_N^{-1}(0)$ and with the top level N of its components, proceeding as before one recovers the surface X_N along with a (well ordered) chain (9) of affine modifications.

Proposition 3.11. (a) *Any GDF surface $\pi: X \to B = \mathbb{A}^1$ with a unique special fiber $\pi^{-1}(0)$ admits a closed embedding $X \hookrightarrow \mathbb{A}^{N+2}$ onto a surface X_N, which is a unique irreducible component dominating the z-axis of the complete intersection surface in $\mathbb{A}^{N+2}$ given in the coordinates $(z,u,t_1,\ldots,t_N)$ by equations*

$$zt_1 - q_1(u) = 0, \quad zt_2 - q_2(u,t_1) = 0, \quad \ldots, \quad zt_N - q_N(u,t_1,\ldots,t_{N-1}) = 0\,, \tag{12}$$

where the polynomials $q_i \in k[u,t_1,\ldots,t_{i-1}]$, $i = 1,\ldots,N$, satisfy the following conditions:

$(*)$ for every $i = 1,\ldots,N$ and for any solution $P^0 = (u^0, q_1^0, \ldots, q_{i-2}^0)$ of the system

$$q_1(u) = q_2(u,q_1) = \ldots = q_{i-1}(u,q_1,\ldots,q_{i-2}) = 0$$

the polynomial $q_i(P^0,t_{i-1}) \in k[t_{i-1}]$ either is zero, or has simple roots.

(b) *Let $X_0 = \mathbb{A}^2$ and let, for each $i = 1,\ldots,N-1$, X_i be a surface in $\mathbb{A}^{i+2}$ with coordinates $(z,u,t_1,\ldots,t_i)$ dominating the z-axis and verifying the first i equations from (12). Then $\pi_i = z|_{X_i}: X_i \to \mathbb{A}^1$ is a GDF surface with a unique special fiber $\pi_i^{-1}(0)$. Furthermore, the projection $\mathbb{A}^{i+2} \to \mathbb{A}^{i+1}$ ignoring the last coordinate t_i restricts to an affine modification $\varrho_i: X_i \to X_{i-1}$ fitting in (9).*

We will not prove this proposition in full generality; see however Part II for some particular cases.

Example 3.12. For the Danielewski surface X_m as in Example 3.9, system (12) can be chosen as follows:

$$zt_1 - p(u) = 0, \quad zt_2 - t_1 = 0, \quad \dots, \quad zt_m - t_{m-1} = 0\,.$$

Actually, this system reduces to a single equation $z^m t_m - p(z) = 0$ of X_m in $\mathbb{A}^3$.

3.5. **Special μ_d-quasi-invariants.** In the sequel we need μ_d-invariant locally nilpotent derivations on the cylinders over GDF μ_d-surfaces. To this end, in Corollary 3.14 below we construct on such surfaces quasi-invariant functions of prescribed weights. Let us start with the following fact (cf. [53, Lemma 2.12]).

Lemma 3.13. *Suppose we are given a finite group G, a character $\lambda \in G^\vee$, an affine G-variety Y, and a G-stable closed (not necessarily reduced) subscheme Z of Y. Let $f \in \mathcal{O}_Z(Z)$ belongs to λ, that is, $f \circ g = \lambda(g) \cdot f \ \ \forall g \in G$. Then f admits a regular G-quasi-invariant extension to Y which belongs to λ.*

Proof. Letting $A = \mathcal{O}(Y)$ and $B = \mathcal{O}(Z)$, the G-action yields graded decompositions $A = \bigoplus_{\chi\in G^\vee} A_\chi$ and $B = \bigoplus_{\chi\in G^\vee} B_\chi$. The piece A_χ (B_χ, respectively) consists of the G-quasi-invariants in A (in B, respectively) which belong to the character χ. The closed embedding $Z \hookrightarrow Y$ induces a surjection $\varphi : A \to B$ ([48, Thm. III.3.7]). We claim that φ restricts to a surjection $\varphi|_{A_\lambda}\colon A_\lambda \to B_\lambda$ for any $\lambda \in G^\vee$. Indeed, any $f \in B_\lambda$ admits an extension to a regular function $\tilde f \in A$ such that $\varphi(\tilde f) = f$. We have a unique decomposition $\tilde f = \sum_{\chi\in G^\vee} \tilde f_\chi$. Hence $f = \sum_{\chi\in G^\vee} \varphi(\tilde f_\chi)$. Since $f \in B_\lambda$ the summands $\varphi(\tilde f_\chi)$ with $\chi \neq \lambda$ vanish, and so, $f = \varphi(\tilde f_\lambda)$. Hence $\tilde f_\lambda \in A_\lambda$ is a desired G-quasi-invariant extension of f which belongs to λ. □

Corollary 3.14. *Let $\pi\colon X \to B$ be a marked GDF μ_d-surface, let X_l be one of the surfaces in (9), and let F be a fiber component of top level in X_l. Let $\mathcal{F}_0$ be the μ_d-orbit of F in X_l. For a special fiber component F' in X_l, let $U_{F'} = U^{(l)}_{i,j}$ be the standard affine chart around F' in X_l with coordinate $u = u^{(l)}_{i,j}$, Then for any $s \in \mathbb{N}$ one can find a μ_d-quasi-invariant function $\tilde u \in \mathcal{O}_{X_l}(X_l)$ of weight $-l$ such that*

(i) *$\tilde u \equiv u \bmod z^s$ near F' if $F' \subset \mathcal{F}_0$, and*
(ii) *$\tilde u \equiv 0 \bmod z^s$ near F' otherwise.*

Proof. It suffices to apply Lemma 3.13 with $Y = X_l$, $Z = (z^s)^*(0)$ being the sth infinitesimal neighborhood of the union of the special fiber components in X_l, $G = \mu_d$, $\lambda(\zeta) = \zeta^{-l}$ for $\zeta \in \mu_d$, and the function $f \in \mathcal{O}_Z(Z)$ defined in the affine charts $Z \cap U_{F'}$ via $f|_{Z\cap U_{F'}} = u|_{Z\cap U_{F'}}$ for $F' \subset \mathcal{F}_0$ and $f|_{Z\cap U_{F'}} = 0$ otherwise. □

Remark 3.15. In the case of a trivial μ_d-action on X, a function $\tilde u$ satisfying (i) and (ii) exists for any collection $\mathfrak{F}_0$ of (top level) special fiber components in X.

4. Relative flexibility

4.1. Definitions and the main theorem.

Notation 4.1. Let $\pi\colon X \to B$ be a GDF surface, let $\mathcal{X} = X \times \mathbb{A}^1$ be the cylinder over X, and let $(U_{i,j})$ be the system of standard affine charts on X, see Proposition 3.3, along with their natural coordinates $(z, u_{i,j})$, see Definition 3.5. We let $\mathrm{SAut}_B(\mathcal{X})$ be

the subgroup of the group $\mathrm{Aut}^\circ(\mathcal{X})$ generated by all the B-automorphisms[9] of $\mathcal{X}$ that are exponentials of locally nilpotent derivations in $\mathrm{LND}(\mathcal{X})$. Thus,

$$\mathrm{SAut}_B(\mathcal{X}) = \langle \exp(\partial) \,|\, \partial \in \mathrm{LND}(\mathcal{O}(\mathcal{X})),\ \partial(z) = 0 \rangle .$$

Note that any automorphism $\varphi \in \mathrm{SAut}_B(\mathcal{X})$ stabilizes the standard affine chart $U_{i,j}\times\mathbb{A}^1$ in $\mathcal{X}$ with natural coordinates $(z, u = u_{i,j}, v)$ around any special fiber component $F_{i,j}\times\mathbb{A}^1$ in $\mathcal{X}$, see Remark 3.4 and [1, Lemma 4.10]. Furthermore, for any $F = F_{i,j}$, the restriction $\varphi|_{U_F\times\mathbb{A}^1}$ preserves the volume form $\mathrm{d}z\wedge\mathrm{d}u\wedge\mathrm{d}v$ on $U_F\times\mathbb{A}^1$, and the Jacobian determinant of the restriction of φ to any fiber component equals 1.

Definition 4.2 (*Relative flexibility*)**.** We say that the cylinder $\mathcal{X} = X\times\mathbb{A}^1$ is *relatively flexible* (RF, for short) if for any natural $s \geq 1$, any collection $\mathfrak{F}$ of top level special fiber components in X, and any collection of pairs of ordered finite subsets $\Sigma_F = \{x_1, \dots, x_m\}$ and $\Sigma'_F = \{x'_1, \dots, x'_m\}$ in $\mathcal{F} = F\times\mathbb{A}^1$ of the same cardinality $m = m(F)$, where F runs over $\mathfrak{F}$, there exists a B-automorphism $\varphi \in \mathrm{SAut}_B\,\mathcal{X}$ which satisfies the conditions

- (α) $\varphi(\Sigma_F) = \Sigma'_F$ with prescribed (modulo z^s) volume preserving jets at x_ν, $\nu = 1, \dots, m(F)$, provided that these prescribed jets preserve locally the fibration $\mathcal{X} \to B$, for any $F \in \mathfrak{F}$;
- (β) $\varphi|_{U\times\mathbb{A}^1} \equiv \mathrm{id} \mod z^s$ near $F\times\mathbb{A}^1$ for any $F \notin \mathfrak{F}$.

We say that *the condition RF(l, s) holds for X* if the surface X_l in (9) satisfies the above assumptions for a given $s \geq 1$. Clearly, $\mathrm{RF}(l, s) \Rightarrow \mathrm{RF}(l, s-1)$.

Definition 4.3 (*Equivariant relative flexibility*)**.** Let $\pi\colon X \to B$ be a marked GDF μ_d-surface, and let $\mathcal{X} = \mathcal{X}(k)$. Assume that the collection $\mathfrak{F}$ of fiber components as in Definition 4.2 along with the finite sets $\Sigma = \bigcup_{F\in\mathfrak{F}} \Sigma_F$ and $\Sigma' = \bigcup_{F\in\mathfrak{F}} \Sigma'_F$ are μ_d-stable, and the correspondence $\Sigma_F \mapsto \Sigma'_F$ is μ_d-equivariant along with the prescribed jets at the points x_i. We say that $\mathcal{X}(k)$ is *μ_d-relatively flexible* if one can choose a μ_d-equivariant automorphism $\varphi \in \mathrm{SAut}_B\,\mathcal{X}$ as in Definition 4.2. If an intermediate cylinder $\mathcal{X}_l(k)$ satisfies the above assumptions for a given $s > 1$, then we say that *the μ_d-equivariant condition RF(l, k, s) holds for X*. Once again, we have $\mathrm{RF}(l, k, s) \Rightarrow \mathrm{RF}(l, k, s-1)$.

The main result of this section is the following theorem.

Theorem 4.4. *Given any marked GDF μ_d-surface $\pi\colon X \to B$, any $s > 1$ and $l \in \{0, \dots, N\}$, the μ_d-equivariant condition RF$(l, -l, s)$ holds for X.*

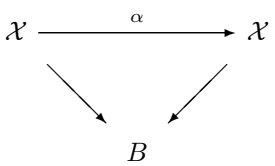
The proof is done at the end of the section.

4.2. Transitive group actions on Veronese cones.

4.5. Given $l, d \in \mathbb{N}$, consider the affine plane $\mathbb{A}^2$ with coordinates (u, v) equipped with the action of the group μ_d of dth roots of unity,

$$\zeta.(u, v) = (\zeta^{-l}u, \zeta^{-l}v) \quad \forall \zeta \in \mu_d .$$

[9] That is, any $\alpha \in \mathrm{SAut}_B(\mathcal{X})$ fits in the commutative diagram

(13)


This action is not effective, in general, but it restricts to an effective action of the subgroup $\mu_e \subset \mu_d$, where $e = d/\gcd(d,l)$. The quotient $\mathbb{A}^2/\mu_d = \mathbb{A}^2/\mu_e$ of the plane by this action is the Veronese cone V_e.

Consider also the locally nilpotent vector fields $\sigma_1 = v\frac{\partial}{\partial u}$ and $\sigma_2 = u\frac{\partial}{\partial v}$ on $\mathbb{A}^2$. The associated one-parameter groups $(u,v) \mapsto (u+tv, v)$ and $(u,v) \mapsto (u, v+tu)$, $t \in k$, generate the $\mathbf{SL}_2$-action on $\mathbb{A}^2$. Notice that σ_1 and σ_2 are μ_d-invariant, and the μ_d-action on $\mathbb{A}^2$ commutes with the $\mathbf{SL}_2$-action. Hence the $\mathbf{SL}_2$-action descends to the Veronese cone V_e.

We use below certain results of [1] based on the notion of a saturated set of locally nilpotent derivations. Let us recall this notion. For a vector field ∂ on a variety X and an automorphism $g \in \operatorname{Aut} X$, we let $\operatorname{Ad}(g)(\partial) = dg(\partial) \circ g^{-1}$.

Definition 4.6 (*Saturation*)**.** ([1, Definition 2.1]) Given an affine variety $X = \operatorname{Spec} A$ over k, a set $\mathcal{N}$ of locally nilpotent regular vector fields on X (that is, of locally nilpotent derivations of the affine k-algebra A) is called *saturated* if

(i) for any $\partial \in \mathcal{N}$ and $a \in \ker \partial$, the *replica* $a\partial \in \mathcal{N}$, and
(ii) $\operatorname{Ad}(g)(\partial) \in \mathcal{N}$ $\forall g \in G$, $\forall \partial \in \mathcal{N}$, where $G = \langle \exp \partial \,|\, \partial \in \mathcal{N} \rangle \subset \operatorname{Aut} A$.

Lemma 4.7. *Given a set $\mathcal{N}$ of locally nilpotent derivations $\mathcal{N} \subset \operatorname{Der} A$ satisfying* (i)*, consider the group $G \subset \operatorname{Aut} A$ as in* (ii) *generated by $\mathcal{N}$. Then the set of locally nilpotent derivations*

$$\mathcal{N}_1 = \{\operatorname{Ad}(g)(\partial) \,|\, g \in G,\ \partial \in \mathcal{N}\}$$

is saturated and generates the same group G.

Proof. It is not difficult to see that $\mathcal{N}_1$ satisfies (i). Let $G_1 = \langle \exp \partial \,|\, \partial \in \mathcal{N}_1 \rangle$ be the group generated by $\mathcal{N}_1$. We claim that $G_1 = G$, and so, (ii) follows by the chain rule. Indeed, an automorphism $g \in \operatorname{Aut} X$ sends a vector field ∂ on X into the vector field ∂' on X such that $\partial'(g(x)) = dg(\partial(x))$ $\forall x \in X$. Hence $\partial' = \operatorname{Ad}(g)(\partial)$. On the other hand, if ∂ is locally nilpotent with the phase flow $\exp(t\partial) \in \operatorname{Aut} X$, $t \in k$, then the phase flow $\exp(t\partial') \in \operatorname{Aut} X$, $t \in k$, is obtained by the conjugation with g, that is, $\exp(t\partial') = g \circ \exp(t\partial) \circ g^{-1}$. Hence for any $g \in G$ and $\partial \in \mathcal{N}$ one has $\exp(t\partial) \in G$, and so, $\exp(t\partial') = g \circ \exp(t\partial) \circ g^{-1} \in G$. Thus $\exp(t\partial') \in G$ for any $\partial' \in \mathcal{N}_1$. It follows that $G_1 = G$, as claimed. □

Notation 4.8. Given $s \geq 2$, consider the μ_d-invariant replicas $\sigma_{1,f} = v^{ds} f(v^d)\sigma_1$ of σ_1 and $\sigma_{2,g} = u^{ds} g(u^d)\sigma_2$ of σ_2, where $f, g \in k[t]$, along with the associated one-parameter unipotent subgroups Φ_f and Ψ_g of the group $\operatorname{SAut}(\mathbb{A}^2)$. The replicas $\sigma_{1,f}$ vanish modulo v^s on the line $v = 0$, hence Φ_f fixes this line pointwise together with its infinitesimal neighborhood of order s (where $\Phi_f = \mathrm{id}$ for $f = 0$). Let

$$G = \langle \Phi_f,\ \Psi_g \,|\, f, g \in k[t] \rangle \subset \operatorname{SAut}(\mathbb{A}^2)\,. \tag{14}$$

It is easy to see that G is transitive in $\mathbb{A}^2 \setminus \{\bar{0}\}$ and commutes with the μ_d-action in $\mathbb{A}^2$.

Consider also the normal subgroup $H \lhd G$ of all automorphisms $\alpha \in G$ of the from

$$\alpha = \varphi_1 \cdot \psi_1 \cdot \ldots \cdot \varphi_\nu \cdot \psi_\nu\,, \tag{15}$$

where $\varphi_i \in \Phi_{f_i}$ and $\psi_i \in \Psi_{g_i}$, $i = 1, \ldots, \nu$, verifying the condition

$$\varphi_1 \cdot \varphi_2 \cdot \ldots \cdot \varphi_\nu = \mathrm{Id}\,. \tag{16}$$

Proposition 4.9. *Let $s \geq 2$, and let $(O_1, \ldots, O_m)$ and $(O'_1, \ldots, O'_m)$ be two collections of distinct μ_d-orbits in $\mathbb{A}^2$ with $\operatorname{card} O_i = \operatorname{card} O'_i$ for $i = 1, \ldots, m$. For every $i = 1, \ldots, m$ choose a representative $x_i \in O_i$. In the case where d divides l, that is, $e = 1$, suppose in addition that the singletons O_i and O'_i are different from the origin. Then there exists an automorphism $\alpha \in H$ such that*

(i) *$\alpha(O_i) = O'_i$ for $i = 1, \ldots, m$, and*

(ii) *α has prescribed values of volume-preserving r-jets at the points x_i, $i = 1, \ldots, m$, where $r \leq s$, provided that, if $O_i = \{\bar{0}\}$ for some $i \in \{1, \ldots, m\}$ and $e \geq 1$, then this prescribed r-jet at the origin is the r-jet of the identity.* [10]

Proof. Consider the Veronese cone $V_e = \mathbb{A}^2/\mu_d = \operatorname{Spec} k[u, v]^{\mu_d}$, and let $\varrho\colon \mathbb{A}^2 \to V_e$ be the quotient morphism. The cone V_e is smooth outside the vertex $\bar{0} \in V_e$. Since the G-action on $\mathbb{A}^2$ is transitive in $\mathbb{A}^2 \setminus \{\bar{0}\}$ and commutes with the μ_d-action, it descends to a G-action on V_e transitive in $V_e \setminus \{\bar{0}\}$. The μ_d-invariant locally nilpotent vector fields $\sigma_{1,f}$ and $\sigma_{2,g}$ also descends to V_e. The set $\mathcal{N}$ of all these vector fields on V_e satisfies condition (i) of Definition 4.6. Hence by Lemma 4.7, the group $G \subset \operatorname{Aut} V_e$ is generated as well by a bigger saturated set $\mathcal{N}_1$ of locally nilpotent vector fields on the cone V_e. Therefore one can apply Theorems 2.2 and 4.14 from [1].

Suppose first that $\{\bar{0}\}$ is not among the O_i's. By [1, Thm. 2.2], G acts infinitely transitively in $V_e \setminus \{\bar{0}\}$. It follows that there exists $\alpha \in G$ which sends the points $y_i = \varrho(O_i) \in V_e$ into the points $y'_i = \varrho(O'_i)$, $i = 1, \ldots, m$. Acting in $\mathbb{A}^2$, this α transforms the orbit O_i into O'_i for every $i = 1, \ldots, m$. Thus α verifies (i).

By [1, Thm. 4.14] one can find $\alpha \in G$ verifying (i) with a prescribed volume-preserving r-jet at each point $y_i = \varrho(O_i) \in V_e$, $i = 1, \ldots, m$. Since ϱ is a local isomorphism near a chosen point $x_i \in O_i$ over y_i and near its image $\alpha(x_i) \in O'_i$, one may prescribe a volume-preserving r-jet of α at x_i with the given zero term $\alpha(x_i)$.

In the case that $e \geq 2$ and one of the orbits, say, O_m consists of the origin: $O_m = \{\bar{0}\}$, then also $O'_m = \{\bar{0}\}$. Indeed, any μ_e-orbit different from $\{\bar{0}\}$ contains $e > 1$ points. Since $\sigma_{1,f}, \sigma_{2,g} \equiv 0 \mod (u, v)^s$ for any $f, g \in k[t]$, see Notation 4.8, one has $\alpha \equiv \mathrm{id} \mod (u, v)^s$ for any $\alpha \in G$. Thus automatically $\alpha(\bar{0}) = \bar{0}$ and, moreover, the r-jet at the origin of any $\alpha \in G$ is the one of the identity map for any $r \leq s$.

In the case where $e = 1$, we have $V_e = \mathbb{A}^2$ and every orbit O_i and O'_i is a singleton different from $\{\bar{0}\}$ by assumption. Then the argument in the proof works without change.

It remains to find such an automorphism in the subgroup H. Due to the infinite transitivity of G in $V_e \setminus \{\bar{0}\}$ one can find $g \in G$ such that for every $i = 1, \ldots, m$ the image $g(\varrho(O'_i))$ is located in the line $v = 0$ in V_e. Now by the preceding there exists $\alpha \in G$ such that $\alpha \circ g(\varrho(O_i)) = g(\varrho(O'_i))$ for all $i = 1, \ldots, m$, and α has prescribed volume preserving r-jets at these points. Since g is volume-preserving (see [1, Lemma 4.10]) one can find $\alpha_1 = g^{-1} \circ \alpha \circ g\colon \varrho(O_i) \mapsto \varrho(O'_i)$ with prescribed volume preserving r-jets at the points $y_i = \varrho(O_i)$, $i = 1, \ldots, m$.

Decomposing α as in (15) consider the automorphism $\varphi_0 = (\varphi_1 \cdot \ldots \cdot \varphi_\nu)^{-1} \in G$. Since $\varphi_0 \cdot \varphi_1 \cdot \ldots \cdot \varphi_\nu = \mathrm{id}$, replacing φ_1 by $\varphi_0 \circ \varphi_1$ we obtain an automorphism $\alpha' = \varphi_0 \cdot \alpha \in H$. The r-jet of α' at each point $g(\varrho(O_i))$ is the same as the one of α. Indeed, $\varphi_0(g(\varrho(O'_i))) = g(\varrho(O'_i))$, since $g(\varrho(O'_i)) \subset \{v = 0\}$, and φ_0 is identical on the sth

[10] In fact, instead of prescribing the value of an r-jet in a single point of a μ_e-orbit, one might prescribe a μ_e-equivariant system of r-jets on the whole orbit.

infinitesimal neighborhood of this line, where $s \geq r$. Since the subgroup $H \lhd G$ is normal we have $\alpha_1' = g^{-1} \circ \alpha \circ g \in H$, where $\alpha_1' \colon \varrho(O_i) \mapsto \varrho(O_i')$, and α_1' has prescribed volume preserving r-jets at the points $\varrho(O_i)$, $i = 1, \ldots, m$. Thus, $\alpha_1' \in H$ satisfies both (i) and (ii). □

4.3. **Relatively transitive group actions on cylinders.**

Notation 4.10. Let $\pi \colon X \to B$ be a marked GDF μ_d-surface, and let X_l be one of the surfaces in (9). We fix the natural coordinates in the standard affine charts $U_{i,j}$ in X_l so that the convention of Remark 3.7.2 is fulfilled. Let further F be a top level special fiber component in X_l, and let $\mathcal{F}_0$ be the μ_d-orbit of F in X_l. For $s \geq 2$, let $\tilde{u} \in \mathcal{O}_{X_l}(X_l)$ be a μ_d-quasi-invariant function that verify conditions (i) and (ii) of Corollary 3.14. Consider the vertical vector field ∂_l on X_l as in Lemma 3.1 and, for $f, g \in k[t]$, the μ_d-invariant locally nilpotent derivations of the algebra $\mathcal{O}_{X_l}(X_l)$,

$$\tilde{\sigma}_{1,f} = f(v^d)\partial_l \quad \text{and} \quad \tilde{\sigma}_{2,g} = \tilde{u}^{ds+1} g(\tilde{u}^d)\partial/\partial v\,. \tag{17}$$

Letting F run over the set of all top level special fiber components in X_l, the corresponding automorphisms $\tilde{\varphi}_f = \exp(\tilde{\sigma}_{1,f})$, $\tilde{\psi}_g = \exp(\tilde{\sigma}_{2,g}) \in \mathrm{SAut}_B\, \mathcal{X}_l(-l)$ generate a subgroup

$$\tilde{G} = \langle \tilde{\varphi}_f,\ \tilde{\psi}_g \,|\, f, g \in k[t] \rangle \subset \mathrm{SAut}_B\, \mathcal{X}_l(-l)\,. \tag{18}$$

Clearly, $\tilde{G}$ is contained in the centralizer of the cylic subgroup in $\mathrm{Aut}_B\, \mathcal{X}_l(-l)$ induced by the μ_d-action on $\mathcal{X}_l(-l)$.

Consider further the normal subgroup $\tilde{H} \lhd \tilde{G}$, where

$$\tilde{H} = \{\tilde{\alpha} = \tilde{\varphi}_1 \cdot \tilde{\psi}_1 \cdot \ldots \cdot \tilde{\varphi}_\nu \cdot \tilde{\psi}_\nu \in \tilde{G} \,|\, \tilde{\varphi}_1 \cdot \ldots \cdot \tilde{\varphi}_\nu = \mathrm{id}\}\,. \tag{19}$$

Fix a μ_d-stable collection $\mathfrak{F}$ of top level fiber components in X_l. We may restrict in (18) and (19) to the case where F is running over components in $\mathfrak{F}$ only. Let $\tilde{G}_{\mathfrak{F}} \subset \tilde{G}$ and $\tilde{H}_{\mathfrak{F}} \subset \tilde{H}$ be the corresponding subgroups. If $\tilde{u}$ as in Corollary 3.14 is associated to a component $F \in \mathfrak{F}$, then by virtue of condition (ii) of this corollary we have $\tilde{u} \equiv 0 \mod z^s$ in $U_{F'} \times \mathbb{A}^1$ near $\mathcal{F}' = F' \times \mathbb{A}^1$ for any $F' \notin \mathfrak{F}$. Hence $\tilde{\psi}_g \equiv \mathrm{id} \mod z^s$ in $U_{F'} \times \mathbb{A}^1$ near $\mathcal{F}'$ for any $F' \notin \mathfrak{F}$ and any $g \in k[t]$. Due to (19), for any $\tilde{\alpha} \in \tilde{H}_{\mathfrak{F}}$,

$$\tilde{\alpha}|_{U_{F'} \times \mathbb{A}^1} = (\tilde{\varphi}_1 \cdot \tilde{\psi}_1 \cdot \ldots \cdot \tilde{\varphi}_\nu \cdot \tilde{\psi}_\nu)|_{U_{F'} \times \mathbb{A}^1} \equiv \mathrm{id} \mod z^s \quad \forall F' \notin \mathfrak{F}\,. \tag{20}$$

Definition 4.11 (*s-reduced automorphism*)**.** Let F be a special fiber component in X_l, and let U_F be the standard affine chart around F in X_l. Consider the affine chart $U_F \times \mathbb{A}^1$ around the affine plane $\mathcal{F} = F \times \mathbb{A}^1 \simeq \mathbb{A}^2$ in the cylinder $\mathcal{X}_l(-l)$. The group $\tilde{G} \subset \mathrm{SAut}_B\, \mathcal{X}_l(-l)$ preserves every fiber of the $\mathbb{A}^2$-fibration $\mathcal{X}_l(-l) \to B$ and, moreover, every fiber component. Hence any automorphism $\alpha \in \tilde{G}$ preserves the affine chart $U_F \times \mathbb{A}^1$ (cf. Proposition 3.3 and Remark 3.4). The restriction $\alpha|_{U_F \times \mathbb{A}^1}$ can be written in the natural coordinates (z, u, v) in $U_F \times \mathbb{A}^1$ as

$$\alpha|_{U_F \times \mathbb{A}^1} \colon (z, u, v) \mapsto \left(z, \sum_{i=0}^{\infty} z^i f_i(u, v), \sum_{i=0}^{\infty} z^i g_i(u, v) \right).$$

We say that α is *s-reduced* if $f_1 = \ldots = f_s = g_1 = \ldots = g_s = 0$, i.e.

$$\alpha(z, u, v) \equiv (z, f_0(u, v), g_0(u, v)) \bmod (z^s)$$

in any such affine chart $U_F \times \mathbb{A}^1$ in $\mathcal{X}_l(-l)$.

Lemma 4.12. (a) *A composition of s-reduced automorphisms is again s-reduced.*

(b) *Any automorphism $\tilde{\alpha} \in \tilde{H}$ is s-reduced.*

Proof. The proof of (a) is straightforward. To show (b), we let $\tilde{\varphi} = \exp(\tilde{\sigma}_{1,f})$ and $\tilde{\psi} = \exp(\tilde{\sigma}_{2,g})$, where $f, g \in k[t]$. We claim that $\tilde{\varphi}$ and $\tilde{\psi}$ are s-reduced. Indeed, in a standard affine chart U_F of level $t \le l$ in X_l we have $\partial_l|_{U_F} = z^{l-t}\partial/\partial u$ (see Remark 3.7.1). Hence

$$\tilde{\varphi}|_{U_F \times \mathbb{A}^1} = \exp\left(v^{ds+1} f(v^d)\partial/\partial u\right) \colon (z,u,v) \mapsto \left(z, u + z^{l-t} v^{ds+1} f(v^d), v\right). \tag{21}$$

Since $\tilde{u}|_{U_F} \equiv u \mod z^s$ if $t = l$, and $\tilde{u}|_{U_F} \equiv 0 \mod z^s$ otherwise, near the affine plane $\mathcal{F} \subset U_F \times \mathbb{A}^1$ we have

$$\tilde{\psi}|_{U_F \times \mathbb{A}^1} = \exp\left(\tilde{u}^{ds+1} f(\tilde{u}^d)\partial/\partial v\right) \colon (z,u,v) \mapsto \left(z, u, v + u^{ds+1} g(u^d)\right) \mod z^s \tag{22}$$

if $t = l$, and $\tilde{\psi}|_{U_F \times \mathbb{A}^1} \equiv \mathrm{id} \mod z^s$ otherwise. In particular, any $\tilde{\varphi} \in \tilde{G}$ is s-reduced. By (a) any automorphism

$$\tilde{\alpha} = \tilde{\varphi}_1 \cdot \tilde{\psi}_1 \cdot \ldots \cdot \tilde{\varphi}_\nu \cdot \tilde{\psi}_\nu \in \tilde{G}$$

is s-reduced in any top level affine chart $U_F \times \mathbb{A}^1$. If F has level $t < l$, then $\tilde{\psi}_i|_{U_F \times \mathbb{A}^1} \equiv \mathrm{id} \mod z^s \ \forall i = 1, \ldots, \nu$. Hence for any $\tilde{\alpha} \in \tilde{H}$,

$$\tilde{\alpha}|_{U_F \times \mathbb{A}^1} = (\tilde{\varphi}_1 \cdot \ldots \cdot \tilde{\varphi}_\nu)|_{U_F \times \mathbb{A}^1} \equiv \mathrm{id} \mod z^s.$$

This proves (b). □

Proposition 4.13. *Let $\mathfrak{F}$ be a μ_d-stable collection of top level special fiber components F in X_l, and let $\Sigma, \Sigma' \subset \bigcup_{F \in \mathfrak{F}} \mathcal{F}$ be two μ_d-stable finite sets in $\mathcal{X}_l(-l)$, which meet every affine plane $\mathcal{F} = F \times \mathbb{A}^1$, $F \in \mathfrak{F}$, with the same positive cardinality. Assume that, for some $r \le s$, we are given a collection of volume preserving r-jets of automorphisms at the points of Σ, which is μ_d-stable up to multiplication on a μ_d-character and such that the r-jet at $0_F \in \Sigma$ is the r-jet of the identity provided that $e(F) \ge 0$, where $e = d/\gcd(d,l)$ is as in 4.5. Then there exists an automorphism $\tilde{\alpha} \in \tilde{H}_{\mathfrak{F}}$ such that its restriction to any affine chart $U_F \times \mathbb{A}^1$ in $\mathcal{X}_l(-l)$ is s-reduced, and*

(i) $\tilde{\alpha}(\Sigma) = \Sigma'$;
(ii) *$\tilde{\alpha}$ has the prescribed r-jets at the points of Σ;*
(iii) $\tilde{\alpha}|_{U_F \times \mathbb{A}^1} \equiv \mathrm{id} \mod z^s \quad \forall F \notin \mathfrak{F}$.

Proof. Let $F \in \mathfrak{F}$, and let $\mu_d(F)$ be the μ_d-orbit of F in X_l. It suffices to construct such an automorphism $\tilde{\alpha} \in \tilde{H}_{\mathfrak{F}}$ assuming that $\mathfrak{F}$ consists of the components of $\mu_d(F)$. Indeed, then $\tilde{\alpha} \in \tilde{H}_{\mathfrak{F}}$ coincides with the identity modulo z^s near any affine plane $\mathcal{F}'$ in $\mathcal{X}_l(-l)$ for $F' \notin \mathfrak{F}$. Composing such automorphisms $\tilde{\alpha}$ for different top level orbits one obtains a desired automorphism in the general case.

Furthermore, if our conditions (i) and (ii) hold in the affine plane $\mathcal{F}$, then they automatically hold in any affine plane $\mathcal{F}'$ with $F' \in \mathfrak{F}$ due to the μ_d-invariance of the conditions and the μ_d-equivariance of the automorphisms $\tilde{\alpha} \in \tilde{H}_{\mathfrak{F}}$. Hence it suffices to take care just of a particular affine plane $\mathcal{F}$ equipped with two collections of orbits $\{O_i \cap \mathcal{F}\}_{i=1,\ldots,\nu}$ and $\{O'_i \cap \mathcal{F}\}_{i=1,\ldots,\nu}$ of the stabilizer of F in μ_d, cf. Proposition 4.9 [11]. Let U_F be the standard affine chart around F, and let (z,u,v) be the natural coordinates in $U_F \times \mathbb{A}^1$.

[11] The assumption of Proposition 4.9 that $O_i \ne \{\bar{0}\}\ \forall i$ if $e = 1$ holds automatically due to our choice of natural coordinates, see Notation 4.10.

By virtue of (21) and (22), for $f, g \in k[t]$ the automorphisms $\tilde{\varphi}_f, \tilde{\psi}_g \in \tilde{H}_{\mathfrak{F}}$ restrict to

$$\tilde{\varphi}_f|_{\mathcal{F}} = \varphi_f \quad \text{and} \quad \tilde{\psi}_g|_{\mathcal{F}} = \psi_g\,,$$

respectively, where φ_f, ψ_g generate the subgroup $G \subset \mathrm{SAut}(\mathcal{F})$ as in (14). Let $H \lhd G$ be as in Notation 4.8. Applying Proposition 4.9 one can find an automorphism $\alpha = \varphi_1 \cdot \psi_1 \cdot \ldots \cdot \varphi_\nu \cdot \psi_\nu \in H$ satisfying in the affine plane $\mathcal{F} \cong \mathbb{A}^2$ conditions (i) and (ii) of this proposition. Extending every φ_i to $\tilde{\varphi}_i \in H_{\mathfrak{F}}$ and ψ_i to $\tilde{\psi}_i \in H_{\mathfrak{F}}$, we obtain an s-reduced automorphism $\tilde{\alpha} = \tilde{\varphi}_1 \cdot \tilde{\psi}_1 \cdot \ldots \cdot \tilde{\varphi}_\nu \cdot \tilde{\psi}_\nu \in \tilde{H}_{\mathfrak{F}}$, see Lemma 4.12(b). Since $\tilde{\alpha}$ also satisfies (20) in Notation 4.10, then (iii) holds, and so, $\tilde{\alpha}$ is a desired automorphism. □

Proof of Theorem 4.4. Let $\pi\colon X \to B$ be a marked GDF μ_d-surface. We have to show that the μ_d-equivariant condition $\mathrm{RF}(l, -l, s)$ holds for X whatever are $s \geq 2$ and $l \in \{0, \ldots, N\}$. It suffices to show that, given any μ_d-stable collection $\mathfrak{F}$ of top level special fiber components in X_l and any two finite sets $\Sigma, \Sigma' \subset \bigcup_{F \in \mathfrak{F}} \mathcal{F}$ with the same μ_d-orbit structure and with $\operatorname{card} \Sigma_F = \operatorname{card} \Sigma'_F > 0 \ \forall F \in \mathfrak{F}$, where $\Sigma_F = \Sigma \cap \mathcal{F}$, there exists $\varphi \in \mathrm{SAut}_B\, \mathcal{X}_l(-l)$ such that the μ_d-equivariant versions of conditions (α) and (β) in Definition 4.2 are fulfilled.

By Proposition 4.13 one can find $\varphi \in \tilde{H}_{\mathfrak{F}} \subset \mathrm{SAut}_B\, \mathcal{X}_l(-l)$ verifying (i) and (ii) of Proposition 4.9 and condition (20). That is, φ is μ_d-equivariant, s-reduced, verifies (20), sends Σ onto Σ', and has prescribed 2-dimensional r-jets (in the vertical planes) in the chosen points on each μ_d-orbit in Σ. Since φ is s-reduced and $r \leq s$, it also has prescribed 3-dimensional r-jets at given points. Hence φ satisfies condition (α) of Definition 4.2. Due to (20), φ satisfies also condition (β) of this definition. □

4.4. **A parametric Abhyankar-Moh-Suzuki Theorem.** We need in the sequel the following version of the Abhyankar-Moh-Suzuki Epimorphism Theorem.

Proposition 4.14. *Let $\pi\colon X \to B$ be a GDF surface, let $F_1, \ldots, F_t$ be fiber components in X of the top level, and let $\mathcal{F}_i = F_i \times \mathbb{A}^1 \cong \mathbb{A}^2$, $i = 1, \ldots, t$, be the corresponding fiber components of the induced morphism $\mathcal{X} = X \times \mathbb{A}^1 \to B$. For every $i = 1, \ldots, t$ we fix a curve $C_i \subset \mathcal{F}_i$ such that $C_i \cong \mathbb{A}^1$. Then there exists a B-automorphism $\alpha \in \mathrm{SAut}_B(\mathcal{X})$ such that $\alpha(C_i) = F_i \times \{0\}$, $i = 1, \ldots, t$.*

Proof. Choose $i \in \{1, \ldots, t\}$, and let $F = F_i$, $\mathcal{F} = \mathcal{F}_i$, and $C = C_i \subset \mathcal{F}$. Our assertion follows by induction on i from the next claim.

Claim. *There exists a B-automorphism $\beta = \beta_i \in \mathrm{SAut}_B(\mathcal{X})$ such that $\beta(C) = F \times \{0\}$ and $\beta(F' \times \{0\}) = F' \times \{0\}$ for any special fiber component $F' \neq F$.*

Indeed, to deduce the assertion it suffices to let $\alpha = \beta_s \circ \ldots \circ \beta_1$.

Proof of the claim. By Corollary 3.14 one can find $\tilde{u} \in \mathcal{O}(X)$ such that

(i) $\tilde{u}|_F = u_F$, where u_F is an affine coordinate on F;
(ii) $\tilde{u}|_{F'} = 0$ for any $F' \neq F$.

Consider the locally nilpotent derivations on $\mathcal{O}(\mathcal{X})$,

$$\sigma_1 = \partial_l \quad \text{and} \quad \sigma_2 = \tilde{u}\frac{\partial}{\partial v}\,,$$

where l is the highest level of the special fiber components of X, and ∂_l is a vertical locally nilpotent vector field on X as in Lemma 3.1, so that $\partial_l(z) = 0$ and $\partial_l|_F = \partial/\partial u_F$. Consider the replicas

$$\sigma_{1,f} = f(v)\sigma_1 \quad \text{and} \quad \sigma_{2,g} = g(\tilde{u})\sigma_2, \quad \text{where} \quad f, g \in k[t]\,.$$

Their exponentials

$$\varphi_f = \exp(\sigma_{1,f}) \quad \text{and} \quad \psi_g = \exp(\sigma_{2,g}) \in \mathrm{SAut}_B \mathcal{X}$$

generate a subgroup $\mathcal{H}$ of $\mathrm{SAut}_B \mathcal{X}$. In the coordinates (u_F, v) in the affine plane $\mathcal{F} \cong \mathbb{A}^2$ we have

$$\varphi_f|_{\mathcal{F}} : (u_F, v) \mapsto (u_F + f(v), v) \quad \text{and} \quad \psi_g|_{\mathcal{F}} : (u_F, v) \mapsto (u_F, v + u_F g(u_F)) \,.$$

In particular, $\mathcal{H}|_{\mathcal{F}}$ contains all the transvections, hence also the group $\mathbf{SL}(2,k)$. For $F' \neq F$, by virtue of (ii) the group $\mathcal{H}|_{\mathcal{F}'}$ is generated by the shears $\varphi_f|_{\mathcal{F}'}$. It follows that

- $\mathcal{H}|_{\mathcal{F}} = \mathrm{SAut}\, \mathcal{F} \cong \mathrm{SAut}\, \mathbb{A}^2$, and
- the coordinate line $F' \times \{0\} \subset \mathcal{F}'$ stays $\mathcal{H}$-invariant for any $F' \neq F$.

Now the claim follows by the Abhyankar-Moh-Suzuki Theorem. □

The next lemma allows to interchange the u- and v-axes in the top level special fiber components of $\mathcal{X} \to B$.

Lemma 4.15. *Let $\pi : X \to B$ be a marked GDF μ_d-surface, and let X_l be one of the surfaces in (9). Given $s > 1$, there exists a μ_d-equivariant automorphism $\tau \in \mathrm{SAut}_B\, \mathcal{X}_l(-l)$ such that, in the natural coordinates in local charts, one has*

- *$\tau|_{U_F \times \mathbb{A}^1} : (z,u,v) \mapsto (z,v,-u) \mod z^s$ for any top level special fiber component F, and*
- *$\tau|_{U_F \times \mathbb{A}^1} = \mathrm{id} \mod z^s$ for any special fiber component F of lower level.*

Proof. Likewise in (17) we let

$$\tilde{\sigma}_1 = v\partial_l \quad \text{and} \quad \tilde{\sigma}_2 = -\tilde{u}\partial/\partial v \,, \tag{23}$$

where ∂_l is the vertical vector field on X_l as in Lemma 3.1 and $\tilde{u} \in \mathcal{O}_{\mathcal{X}_l(-l)}(\mathcal{X}_l(-l))$ is a μ_d-quasi-invariant verifying conditions (i) and (ii) of Corollary 3.14. Letting $\tilde{\varphi} = \exp(\tilde{\sigma}_1)$ and $\tilde{\psi} = \exp(\tilde{\sigma}_2)$, by virtue of (i) and (ii) we obtain

$$\tilde{\varphi}|_{U_F \times \mathbb{A}^1} : (z,u,v) \mapsto (z, u+v, v) \mod z^s$$

and

$$\tilde{\psi}|_{U_F \times \mathbb{A}^1} : (z,u,v) \mapsto (z, u, v-u) \mod z^s$$

if F is of top level, and $\tilde{\psi}|_{U_F \times \mathbb{A}^1} \equiv \mathrm{id} \mod z^s$ otherwise, cf. (21) and (22). Letting $\tau = \tilde{\varphi}\tilde{\psi}\tilde{\varphi}$ we obtain

$$\tau|_{U_F \times \mathbb{A}^1} : (z,u,v) \mapsto (z, v, -u) \mod z^s$$

if F is of top level and $\tau|_{U_F \times \mathbb{A}^1} \equiv \mathrm{id} \mod z^s$ otherwise. □

Using this lemma and Proposition 4.14 we arrive at the following conclusion.

Corollary 4.16. *Under the assumptions of Proposition 4.14 there exists a B-automorphism $\alpha' \in \mathrm{SAut}_B(\mathcal{X})$ such that $\alpha'(C_i) = \{p_i\} \times \mathbb{A}^1$, where $p_i \in F_i$, $i = 1, \ldots, s$.*

We need as well the following versions of Lemma 4.15.

Lemma 4.17. *Under the assumptions of Lemma 4.15, let $\pi_l^{-1}(\Upsilon)$, where $\Upsilon \subset \{b_1, \ldots, b_n\}$, be a μ_d-stable union of special fibers of $\pi_l : X_l \to B$, and let $\mathfrak{F}_\Upsilon$ be the collection of all their fiber components. Consider the subset of top level components*

$$\mathfrak{F}_\Upsilon(l') = \{F \in \mathfrak{F}_\Upsilon \,|\, l(F) = l'\}, \quad \textit{where} \quad l' = l(\mathfrak{F}_\Upsilon) = \max\{l(F) \,|\, F \in \mathfrak{F}_\Upsilon\} \,. \tag{24}$$

Then there is a μ_d-equivariant automorphism $\tau \in \mathrm{SAut}_B\, \mathcal{X}_l(-l)$ such that

$$\tau|_{U_F\times\mathbb{A}^1}\colon (z,u,v)\mapsto (z,v,-u) \mod z^s \tag{25}$$

if $F \in \mathfrak{F}_\Upsilon(l')$ and $\tau|_{U_F\times\mathbb{A}^1} \equiv \mathrm{id} \mod z^s$ otherwise.

Proof. Consider a μ_d-invariant $f \in \pi^*\mathcal{O}_B(B) \subset \mathcal{O}_X(X)$ such that the restriction $f|_{\pi_l^{-1}(b_i)}$ vanishes if $b_i \notin \Upsilon$, and equals 1 otherwise. Note that $\partial_{l'}(f) = 0 = \partial/\partial v(f)$, and $\partial_{l'}$ is regular near the fibers $\pi_l^{-1}(b_i)$, $b_i \in \Upsilon$, and has poles at most along the fibers $\pi_l^{-1}(b_i)$, $b_i \notin \Upsilon$. Hence for any sufficiently large $d > 0$, the replicas

$$\tilde{\sigma}_{f,d} = f^d v\partial_{l'} \quad \text{and} \quad \tilde{\sigma}_{f,d} = -f^d \tilde{u}\partial/\partial v \tag{26}$$

are locally nilpotent derivations on $\mathcal{O}_{\mathcal{X}_l}(\mathcal{X}_l)$. Using these derivations and proceeding as in the proof of Lemma 4.15 yields the result. □

Corollary 4.18. *Let $\pi\colon X \to B$ be a GDF surface with a marking $z \in \mathcal{O}_B(B)$ and with the special fibers $\pi^{-1}(b_i)$, $i = 1,\dots,n$. Assume that for any $i = 1,\dots,n$, all the fiber components in $\pi^{-1}(b_i)$ are of the same level, say, l_i. Then for any $s \geq 1$ there exists a B-automorphism $\tau \in \mathrm{SAut}_B\, \mathcal{X}$ such that for any special fiber component F, in the natural coordinates in the local chart $U_F \times \mathbb{A}^1$ on the cylinder $\mathcal{X} = X \times \mathbb{A}^1$, one has*

$$\tau|_{U_F\times\mathbb{A}^1}\colon (z,u,v)\mapsto (z,v,-u) \mod z^s\,. \tag{27}$$

Furthermore, if $\pi\colon X \to B$ is a GDF μ_d-surface then there is a μ_d-equivariant automorphism $\tau \in \mathrm{SAut}_B\, \mathcal{X}_l(-l)$ such that (27) holds for any special fiber component F.

Proof. For $i \in \{1,\dots,n\}$, letting in Lemma 4.17 $\Upsilon = \Upsilon_i = \{b_i\}$, by our assumption $\mathfrak{F}_\Upsilon(l') = \mathfrak{F}_\Upsilon$ is the set of all fiber components in the fiber $\pi^{-1}(b_i)$, where $l' = l_i$. By Lemma 4.17, there exists $\tau_i \in \mathrm{SAut}_B\, \mathcal{X}$ such that (27) holds for any $F \in \mathfrak{F}_{\Upsilon_i}$, and $\tau|_{U_F\times\mathbb{A}^1} \equiv \mathrm{id} \mod z^s$ otherwise. Then the composition $\tau = \tau_1 \circ \cdots \circ \tau_n \in \mathrm{SAut}_B\, \mathcal{X}$ verifies (27) for any special fiber component F. The proof of the last assertion is similar. □

5. Rigidity of cylinders upon deformation of surfaces

5.1. **Equivariant Asanuma modification.** In the next lemma we introduce an equivariant version of the Asanuma modification. For the reader's convenience, we repeat in (a) the statement of Lemma 1.9.

Lemma 5.1. *Let $\pi\colon X \to B$ be a GDF surface, and let $\varrho\colon X' \to X$ be a fibered modification along a reduced principal divisor $\operatorname{div} f$, where $f \in \pi^*\mathcal{O}_B(B) \setminus \{0\}$ with a reduced center I, see Definition 2.26. Consider the principal divisor $\mathcal{D} = \mathbb{V}(f)\times\mathbb{A}^1$ on the cylinder $\mathcal{X} = X \times \mathbb{A}^1$ and the ideal $J = (I, v) \subset \mathcal{O}_{\mathcal{X}}(\mathcal{X})$ with support contained in $\mathbb{V}(f)\times\{0\}$. Then the following holds.*

(a) *The cylinder $\mathcal{X}' = X' \times \mathbb{A}^1$ is naturally isomorphic to the affine modification Z of $\mathcal{X}$ along divisor $\mathcal{D}$ with center J. This isomorphism fits in the commutative diagram*

$$\begin{array}{ccccc} \mathcal{X}' & \xrightarrow{\cong} & Z & \longrightarrow & \mathcal{X} \\ & \searrow & \downarrow & \swarrow & \\ & & B & & \end{array} \tag{28}$$

where the vertical arrows define $\mathbb{A}^2$-fibrations over B.

(b) *Assume that the modification $\varrho\colon X' \to X$ is equivariant with respect to actions of a finite group G on X, X', and B and, moreover, the ideal I is G-invariant and the function f is G-quasi-invariant and belongs to a character $\chi \in G^\vee$. Define the G-action on the factor $\mathbb{A}^1$ of the cylinder $\mathcal{X} = X \times \mathbb{A}^1$ via the multiplication by a character $\lambda \in G^\vee$. Then all the morphisms in (28) are G-equivariant, where G acts on the factors $\mathbb{A}^1$ of the cylinder $\mathcal{X}' = X' \times \mathbb{A}^1$ via multiplication by λ/χ. In particular, if $G = \mu_d$, $\chi\colon \zeta \mapsto \zeta^t$, and $\lambda\colon \zeta \mapsto \zeta^k$, then $\lambda/\chi\colon \zeta \mapsto \zeta^{k-t}$ $\forall \zeta \in \mu_d$.*

Proof. For the proof of (a), see Lemma 1.9. Statement (b) follows since under its assumptions, the variable v' in the proof of Lemma 1.9 belongs to λ, hence $v = v'/f$ belongs to λ/χ. □

Notation 5.2. Let $\pi_l\colon X_l \to B$ be a marked GDF μ_d-surface as in (9). We let $\mathcal{X}_l(k)$ denote the cylinder $\mathcal{X}_l = X_l \times \mathbb{A}^1$ equipped with a product μ_d-action, where μ_d acts on the second factor via $(\zeta, v) \mapsto \zeta^k v$ for all $v \in \mathbb{A}^1$ and $\zeta \in \mu_d$. By abuse of notation, we still denote by π_l the μ_d-equivariant projection of the induced $\mathbb{A}^2$-fibration $\mathcal{X}_l(k) \to B$, and by z the lift to $\mathcal{X}_l(k)$ of the μ_d-quasi-invariant $z \in \mathcal{O}_B(B)$ of weight 1.

Definition 5.3 (*Asanuma modification*)**.** By Lemma 5.1(b), the upper line of (28) reads as a μ_d-equivariant affine modification of cylinders $\mathcal{X}_{l+1}(k-1) \to \mathcal{X}_l(k)$. The latter modification will be called an *Asanuma modification of the first kind.*

An *Asanuma modification of the second kind* is the affine modification $\mathcal{X}'' \to \mathcal{X}$ of the cylinder $\mathcal{X} = X \times \mathbb{A}^1$ over a marked GDF surface $\pi\colon X \to B$ with divisor $\mathcal{D} = z^*(0)$ on $\mathcal{X}$ and center $I = (z, v) \subset \mathcal{O}_{\mathcal{X}}(\mathcal{X})$. Due to the next lemma, this modification results again in a cylinder.

Lemma 5.4. (a) *There is a B-isomorphism $\mathcal{X}'' \cong_B \mathcal{X}$.*

(b) *If $\pi\colon X \to B$ is a μ_d-surface and $\mathcal{X} = \mathcal{X}(k)$, then $\mathcal{X}'' = \mathcal{X}''(k-1)$.*

Proof. Let $A = \mathcal{O}_X(X)$, $\mathcal{A} = \mathcal{O}_{\mathcal{X}}(\mathcal{X})$, and $\mathcal{A}'' = \mathcal{O}_{\mathcal{X}''}(\mathcal{X}'')$. The modification in (a) amounts to the extension

$$\mathcal{A} = A[v] \hookrightarrow \mathcal{A}'' = \mathcal{A}[v/z] = A[v''] \cong_A A[v] = \mathcal{A}, \text{ where } v'' = v/z\,.$$

This proves (a). Under the assumptions of (b) we have $\zeta.v'' = \zeta^{k-1}v''$ for any $\zeta \in \mu_d$, which yields (b). □

Remark 5.5. Let $\varrho\colon X' \to X$ be a fibered modification as in Lemma 5.1. Consider the product modification of cylinders $\sigma = \varrho \times \mathrm{id}\colon \mathcal{X}' \to \mathcal{X}$ followed by the Asanuma modification of the second type $\mathcal{X}'' \to \mathcal{X}'$. This yields an affine modification $\mathcal{X}'' \to \mathcal{X}$ along divisor $\mathcal{D}$ with a zero-dimensional center, factorized as in Remark 1.4.2. Identifying $\mathcal{X}'$ and $\mathcal{X}''$ via the isomorphism of Lemma 5.4 yields an affine modification $\tilde{\varrho}\colon \mathcal{X}' \to \mathcal{X}$ along divisor $\mathcal{D}$ with a zero-dimensional center such that $\varrho = \tilde{\varrho}|_{X' \times \{0\}}$.

5.2. **Rigidity of cylinders under deformations of GDF surfaces.** Using Lemma 5.1 and Notation 5.2 we deduce the following proposition.

Proposition 5.6. (a) *Given $l \in \{1, \dots, N\}$, consider the μ_d-equivariant fibered modification $\varrho_l\colon X_l \to X_{l-1}$ over B in (9). Then for any $k \in \mathbb{Z}$, ϱ_l induces a μ_d-equivariant affine modification $\tilde{\varrho}_l\colon \mathcal{X}_l(k) \to \mathcal{X}_{l-1}(k+1)$ over B along divisor $\mathcal{D}_l$ with center J_l as defined in Lemma 5.1 and Remark 5.5.*

(b) *Consequently,* (9) *yields a sequence of* μ_d*-equivariant affine modifications*

$$\mathcal{X}_N(-N) \xrightarrow{\tilde{\varrho}_N} \mathcal{X}_{N-1}(-N+1) \longrightarrow \ldots \longrightarrow \mathcal{X}_1(-1) \xrightarrow{\tilde{\varrho}_1} \mathcal{X}_0(0) = (B \times \mathbb{A}^2)(0) \tag{29}$$

with zero-dimensional centers.

Proof. The statement of (a) follows by Lemma 5.1, and (b) is immediate from (a). □

The next theorem is our first main result in this section.

Theorem 5.7. *Let* $\pi\colon X \to B$ *and* $\pi'\colon X' \to B$ *be two marked GDF* μ_d*-surfaces over* B *with the same* μ_d*-quasi-invariant marking* $z \in \mathcal{O}_B(B)$ *of weight 1. Assume that for some trivializing* μ_d*-equivariant completions* $(\hat{X}, \hat{D})$ *and* $(\hat{X}', \hat{D}')$ *of* X *and* X'*, respectively, the graph divisors* $\mathcal{D}(\hat{\pi})$ *and* $\mathcal{D}(\hat{\pi}')$ [12] *are* μ_d*-equivariantly isomorphic. Then for any* $k \in \mathbb{Z}$ *there is a* μ_d*-equivariant* B*-isomorphism of cylinders* $\mathcal{X}(k) \cong_{\mu_d,B} \mathcal{X}'(k)$*. In particular, for* $k = 0$ *the cylinders* $\mathcal{X}(0) = X \times \mathbb{A}^1$ *and* $\mathcal{X}'(0) = X' \times \mathbb{A}^1$ *are* μ_d*-equivariantly* B*-isomorphic.*

Proof. Both sequences (29) associated with the GDF surfaces X and X', respectively, start with the same product $\mathcal{X}_0(0) = (B \times \mathbb{A}^2)(0) = \mathcal{X}'_0(0)$. Using Proposition 5.8 below with $s > N$, it follows by induction that for $l = 0, \ldots, N$ there is a μ_d-equivariant B-isomorphism $\mathcal{X}_l(-l) \cong_{\mu_d,B} \mathcal{X}'_l(-l)$. For $l = N$ we obtain $\mathcal{X}(-N) \cong_{\mu_d,B} \mathcal{X}'(-N)$. By Lemma 5.4, $\mathcal{X}(k) \cong_{\mu_d,B} \mathcal{X}'(k)$ for any $k \in \mathbb{Z}$. □

The following proposition provides the induction step in the proof of Theorem 5.7.

Proposition 5.8. *Under the assumptions of Theorem 5.7, suppose that for some* $l \in \{0, \ldots, N-1\}$ *there exists a* μ_d*-equivariant* B*-isomorphism* $\psi_l\colon \mathcal{X}_l(-l) \to \mathcal{X}'_l(-l)$ *such that*

- (i_l) $\psi_l^*(v') \equiv v \bmod(z^s)$*, where* v *(*v'*, respectively) is the coordinate in the* $\mathbb{A}^1$*-factor of the cylinder* $\mathcal{X}_l(-l)$ *(*$\mathcal{X}'_l(-l)$*, respectively), and*
- (ii_l) *the induced correspondence between the fiber components of* π_l *and* π'_l *is restriction of the isomorphism of graph divisors* $\mathcal{D}(\pi) \xrightarrow{\cong} \mathcal{D}(\pi')$*.*

Then there exists a μ_d*-equivariant* B*-isomorphism* $\psi_{l+1}\colon \mathcal{X}_{l+1}(-l-1) \to \mathcal{X}'_{l+1}(-l-1)$ *such that*

- (i_{l+1}) $\psi_{l+1}^*(v') \equiv v \bmod(z^{s-1})$*, and*
- (ii_{l+1}) *the induced correspondence between the fiber components of* π_{l+1} *and* π'_{l+1} *is restriction of the isomorphism of graph divisors* $\mathcal{D}(\pi) \xrightarrow{\cong} \mathcal{D}(\pi')$*.*

Proof. The morphism $\varrho_{l+1}\colon X_{l+1} \to X_l$ in (9) is an affine modification along reduced divisor $\mathbb{V}(z) = z^*(0)$ with center I, where $\mathbb{V}(I)$ is union of a finite set Σ and the components of $\mathbb{V}(z)$ disjoint from Σ, cf. Remark 2.27. Let $\mathfrak{F}_\Sigma$ be the set of the top level components of $\mathbb{V}(z)$ which meet Σ. By Lemma 5.1, ϱ_{l+1} induces an Asanuma modification $\tilde{\varrho}_{l+1}\colon \mathcal{X}_{l+1}(-l-1) \to \mathcal{X}_l(-l)$ with divisor $\mathbb{V}(z) \times \mathbb{A}^1$ and center $\mathbb{V}(I) \times \{0\}$ in $\mathcal{X}_l(-l)$, which consists of the finite set $\Sigma \times \{0\}$ and the union C of curves isomorphic to $\mathbb{A}^1$ and such that in each component $\mathcal{F} = F \times \mathbb{A}^1 \cong \mathbb{A}^2$ of $\mathbb{V}(z) \times \mathbb{A}^1$ disjoint from $\Sigma \times \{0\}$, $C_\mathcal{F} = C \cap \mathcal{F}$ is given by equation $v = 0$ (thus, $C \subset \{z = v = 0\}$ in $\mathcal{X}_l(-l)$). For any $F \in \mathfrak{F}_\Sigma$ we let

$$\Sigma_\mathcal{F} = \mathcal{F} \cap (\mathbb{V}(I) \times \{0\}) = \mathcal{F} \cap (\Sigma \times \{0\}) = \{x_1, \ldots, x_{m(F)}\}\,. \tag{30}$$

[12] See Definition 2.12.

There is a similar collection of objects related with X' instead of X. In particular, we have a modification $\tilde{\varrho}'_{l+1} : \mathcal{X}'_{l+1}(-l-1) \to \mathcal{X}'_l(-l)$ with divisor $\mathbb{V}(z) \times \mathbb{A}^1$ and center $\mathbb{V}(I') \times \{0\}$ consisting of a finite set $\Sigma' \times \{0\}$ and a union C' of curves $C'_{\mathcal{F}'} \cong \mathbb{A}^1$. By virtue of (i_l) the isomorphism ψ_l sends the pair $(\mathcal{X}_l(-l), \mathbb{V}(z) \times \mathbb{A}^1)$ to the pair $(\mathcal{X}'_l(-l), \mathbb{V}(z) \times \mathbb{A}^1)$ and C to C', but not in general $\Sigma \times \{0\}$ to $\Sigma' \times \{0\}$. However, by virtue of (ii_l) the μ_d-equivariant isomorphism $\mathcal{D}(\pi) \xrightarrow{\cong} \mathcal{D}(\pi')$ of graph divisors yields a one-to-one correspondence $F \rightsquigarrow F'$ between the components in $\mathfrak{F}_\Sigma$ and in $\mathfrak{F}'_{\Sigma'}$, so that

$$m_F = \operatorname{card} \Sigma_{\mathcal{F}} = \operatorname{card} \Sigma_{\mathcal{F}'} = m_{F'}$$

for any $F \in \mathfrak{F}_\Sigma$. One can get a bijection between the centers Σ and Σ' of modifications, replacing ψ_l by a composition $\varphi_l \circ \psi_l$ with a suitable μ_d-equivariant automorphism $\varphi_l \in \mathrm{SAut}_B \mathcal{X}'_l(-l)$ as in Definition 4.3.

Indeed, let in (30), $x_\nu = (0, u_\nu, 0)$ in the natural coordinates (z, u, v) in the standard affine chart $U(F) \times \mathbb{A}^1$ around $\mathcal{F}$. Similarly, for $\mathcal{F}' = \psi_l(\mathcal{F})$ we let

$$\mathcal{F}' \cap (\mathbb{V}(I') \times \{0\}) = \mathcal{F}' \cap (\Sigma' \times \{0\}) = \{x'_1, \dots, x'_{m(F)}\},$$

where $x'_\nu = (0, u'_\nu, 0)$ in the natural coordinates (z, u', v') in the standard affine chart around $\psi_l(\mathcal{F}')$. By (i_l) we obtain

$$\psi_l(x_\nu) = x''_\nu = (0, u''_\nu, 0) \in \psi_l(\mathcal{F}), \quad \nu = 1, \dots, m(F).$$

Due to Theorem 4.4, the surface X' verifies the μ_d-equivariant condition $\mathrm{RF}(l, -l, s)$. Hence one can find a μ_d-equivariant automorphism $\varphi_l \in \mathrm{SAut}_B \mathcal{X}'_l(-l)$ as in Definition 4.3 with suitable prescribed μ_d-equivariant s-jets at the points x''_i chosen so that

(1) up to reordering, $\varphi_l(x''_\nu) = x'_\nu$, $\nu = 1, \dots, m(F)$;
(2) $(\varphi_l \circ \psi_l)^*(z, u', v') = (z, u + c_\nu, v) \bmod (z^s)$ near x'_ν, $\nu = 1, \dots, m(F)$;
(3) $\varphi_l \circ \psi_l(\mathbb{V}(z) \times \mathbb{A}^1) = \mathbb{V}(z) \times \mathbb{A}^1$;
(4) $\varphi_l \circ \psi_l(\mathbb{V}(I) \times \{0\}) = \mathbb{V}(I') \times \{0\}$.

For (2) we use the fact that ψ_l satisfies (i_l), and we choose $c_\nu = u'_\nu - u_\nu$. Due to (3) and (4) the composition $\varphi_l \circ \psi_l$ sends the center and the divisor of $\tilde{\varrho}_{l+1}$ onto the center and the divisor of $\tilde{\varrho}'_{l+1}$. By Lemma 1.5 this composition lifts to a μ_d-equivariant B-isomorphism $\psi_{l+1} \colon \mathcal{X}_{l+1}(-l-1) \to \mathcal{X}'_{l+1}(-l-1)$.

Now condition (i_{l+1}) holds for any special fiber components F in X_{l+1} of level $\le l$. Indeed, let $\psi_l(\mathcal{F}) = \mathcal{F}'$, and let (z, u_l, v_l) ((z, u'_l, v'_l), respectively) be the natural coordinates in a special affine chart $U_F \times \mathbb{A}^1$ in $\mathcal{X}_l(-l)$ around $\mathcal{F}$ (in a special affine chart $U_{F'} \times \mathbb{A}^1$ in $\mathcal{X}'_l(-l)$ around $\mathcal{F}'$, respectively). By (2) we have $(\varphi_l \circ \psi_l)^*(v') \equiv v \bmod z^s$, or, in other words,

$$(\varphi_l \circ \psi_l)^*(v') = v + z^s f(z, u, v) \tag{31}$$

for some $f \in k[z, u, v]$. Since $v_{l+1} = v_l/z$ and $v'_{l+1} = v'_l/z$, we obtain from (31)

$$\psi^*_{l+1}(v'_{l+1}) = v_{l+1} + z^{s-1} f(z, z u_{l+1}, z v_{l+1}),$$

and so, (i_{l+1}) holds.

Let further $\tilde{F}$ be a special fiber component in X_{l+1} of level $l+1$ born as a result of blowing up of a point on F in X_l. Thus the divisor $\tilde{\mathcal{F}}$ in $\mathcal{X}_{l+1}(-l-1)$ is contracted by $\tilde{\varrho}_{l+1}$ to a point $x_\nu = (0, u_\nu, 0) \in U_F \times \mathbb{A}^1 \subset \mathcal{X}_l(-l)$, and its image $\tilde{\mathcal{F}}' = \psi_{l+1}(\tilde{\mathcal{F}})$ to the point $x'_\nu = \varphi_l \circ \psi_l(x_\nu) = (0, u'_\nu, 0) \in U_{F'} \times \mathbb{A}^1 \subset \mathcal{X}'_l(-l)$. We let (z, u_l, v_l) be the natural coordinates in $U_F \times \mathbb{A}^1$, and (z, u'_l, v'_l) be the natural coordinates in $U_{F'} \times \mathbb{A}^1$. Let $U_{\tilde{F}} \times \mathbb{A}^1$ be the standard affine chart in $\mathcal{X}_{l+1}(-l-1)$ around $\tilde{\mathcal{F}}$ with natural coordinates

(z, u_{l+1}, v_{l+1}), and let $U_{\tilde{F}'} \times \mathbb{A}^1$ be the standard affine chart in $\mathcal{X}'_{l+1}(-l-1)$ around $\tilde{\mathcal{F}}'$ with natural coordinates (z, u'_{l+1}, v'_{l+1}). We have

$$(32) \quad (z, u_{l+1}, v_{l+1}) = (z, (u_l - u_\nu)/z, v_l/z) \text{ and } (z, u'_{l+1}, v'_{l+1}) = (z, (u'_l - u'_\nu)/z, v'_l/z)\,.$$

From (32) and condition (2) we deduce

$$\psi_{l+1}\colon (z, u_{l+1}, v_{l+1}) \mapsto (z,\, u'_{l+1},\, v'_{l+1}) \equiv (z,\, (u_l + c_\nu - u'_\nu)/z,\, v'_l/z) \equiv (z, u_{l+1}, v_{l+1}) \bmod z^{s-1}\,.$$

In particular, condition (i_{l+1}) holds for ψ_{l+1}. As to (ii_{l+1}), it holds up to choosing a correct ordering of points in (1). □

5.3. **Rigidity of cylinders under deformations of $\mathbb{A}^1$-fibered surfaces.** Using Theorem 5.7 we obtain our second main result in this section.

Theorem 5.9. *Let $\pi\colon Y \to C$ and $\pi'\colon Y' \to C$ be two $\mathbb{A}^1$-fibered normal affine surfaces over a smooth affine curve C. Let $\hat{Y} \to \hat{C}$ be an SNC completion of the minimal resolution of singularities of Y, and let $\hat{D}_{\mathrm{ext}}$ be the extended divisor of this completion. Let a pair $(\hat{Y}', \hat{D}'_{\mathrm{ext}})$ plays the same role for Y'. Suppose that*

- *the degenerate fibers of π and π' are situated over the same points $p_1, \dots, p_t \in C$;*
- *the corresponding fiber graphs Γ_{p_i} and Γ'_{p_i} are isomorphic for all $i = 1, \dots, t$[13];*
- *making similar contractions in $\hat{D}_{\mathrm{ext}}$ and $\hat{D}'_{\mathrm{ext}}$ we can reduce both $\hat{Y}$ and $\hat{Y}'$ to the direct product $\hat{C} \times \mathbb{P}^1$ with a section $\hat{C} \times \{\infty\}$.*

Then the cylinders $Y \times \mathbb{A}^1$ and $Y' \times \mathbb{A}^1$ are isomorphic over C.

Proof. By Lemma 2.3, applying a suitable cyclic Galois base change $B \to C$ of order d ramified over the points $p_1, \dots, p_t \in C$, we can replace our $\mathbb{A}^1$-fibered surfaces $\pi\colon Y \to C$ and $\pi'\colon Y' \to C$ by two marked GDF μ_d-surfaces $X \to B$ and $X' \to B$, respectively, using as a marking the same μ_d-quasi-invariant function $z \in \mathcal{O}_B(B)$, see Remark 2.4. Due to our assumptions, the extended graphs of the resulting GDF surfaces X and X' are isomorphic under a μ_d-equivariant isomorphism, which respects the branches corresponding to degenerate fibers. Moreover, these surfaces admit completions verifying the assumptions of Theorem 5.7. Due to this theorem, there is a μ_d-equivariant B-isomorphism $\mathcal{X}(0) \cong_{\mu_d, B} \mathcal{X}'(0)$. Passing to the quotients $\mathcal{X}(0)/\mu_d = Y \times \mathbb{A}^1$ and $\mathcal{X}'(0)/\mu_d = Y' \times \mathbb{A}^1$ yields a desired C-isomorphism $Y \times \mathbb{A}^1 \cong_C Y' \times \mathbb{A}^1$. □

Corollary 5.10. *Let C be a smooth affine curve with marked points $p_1, \dots, p_t \in C$. Consider the set $\mathfrak{H} = \mathfrak{H}(C, p_1, \dots, p_t)$ of all $\mathbb{A}^1$-fibered normal affine surfaces $\pi\colon X \to C$ with degenerate fibers at most over $p_1, \dots, p_t$. Then the set of all C-isomorphism classes of cylinders $\mathcal{X} = X \times \mathbb{A}^1$, where X runs over $\mathfrak{H}$, is at most countable.*

Proof. Indeed, the set of the isomorphism classes of finite trees is countable. The same is true for the set of all ordered t-tuples of such trees as in Theorem 5.9. Now the assertion follows from this theorem. □

Remark 5.11. Let $\pi\colon X \to C$ and $\pi'\colon X' \to C$ be $\mathbb{A}^1$-fibered normal affine surfaces. If $C \notin \mathbb{A}^1$, then any isomorphism of cylinders $\mathcal{X} \cong \mathcal{X}'$ over X and X', respectively, is a C-isomorphism (cf. Lemma 6.11).

[13] This condition ensures that the corresponding fiber components of π and π' have the same multiplicities. Indeed, under our assumptions the isomorphism respects the feathers along with their bridges.

5.4. **Rigidity of line bundles over affine surfaces.** In unpublished notes [9], kindly provided to us by the authors, the study of cylinders over affine surfaces is extended to the total spaces of line bundles over affine surfaces. In Theorem 5.19 below we extends Theorem 5.7 in this wider context. Since we do not use this extension in the subsequent sections, we just indicate the necessary modifications in the proof of Theorem 5.7.

Notation 5.12. Let X be an affine algebraic variety. For a Cartier divisor $T \in \operatorname{CDiv} X$ we let $\pi^T\colon \mathcal{X}^T \to X$ be the associated line bundle on X with a zero section $Z^T \subset \mathcal{X}^T$.

Definition 5.13. Given an effective, reduced Cartier divisor $D \in \operatorname{CDiv} X$, by an *Asanuma modification of the second kind* of $\mathcal{X}^T$ we mean the affine modification $\sigma^D\colon \mathcal{X}^{T,D} \to \mathcal{X}^T$ along the principal divisor $\mathcal{D}^T = (\pi^T)^*(D)$ on $\mathcal{X}^T$ with center $\mathcal{D}^T \cdot Z^T$.

We have the following analogue of Lemma 5.1 (the latter corresponds to the case $T \sim 0$).

Lemma 5.14. *In Notation 5.12, $\pi^{T,D} = \pi^T \circ \sigma^D\colon \mathcal{X}^{T,D} \to \mathcal{X}$ admits a structure of a line bundle such that $\mathcal{X}^{T,D} \cong_X \mathcal{X}^{T-D}$.*

Proof. Choose an open covering $X = \bigcup_i U_i$ such that

- $D \cap U_i = f_i^*(0)$ and $T \cap U_i = \operatorname{div} h_i$, where $f_i \in \mathcal{O}_{U_i}(U_i)$ and $h_i \in \operatorname{Frac} \mathcal{O}_{U_i}(U_i)$.

Then

- $\alpha_{i,j} = f_j/f_i$, $\beta_{i,j} = h_j/h_i \in \mathcal{O}^\times_{U_{i,j}}(U_{i,j})$, where $U_{i,j} = U_i \cap U_j$, are Čech 1-cocycles on X associated with the line bundles $\mathcal{X}^D \to X$ and $\mathcal{X}^T \to X$, respectively.

Letting $V_i = (\pi^T)^{-1}(U_i)$, there are local trivializations $V_i \cong_{U_i} U_i \times \mathbb{A}^1$ of $\pi^T\colon \mathcal{X}^T \to X$, where $\mathbb{A}^1 = \operatorname{Spec} k[v_i]$ with $v_j = \beta_{i,j} v_i$ over $U_{i,j}$. Consider the restriction $V_i' \to V_i$ of the morphism $\sigma\colon \mathcal{X}^{T,D} \to X^T$ over V_i induced by the natural inclusion $\mathcal{O}_{V_i}(V_i) \hookrightarrow \mathcal{O}_{V_i'}(V_i') = \mathcal{O}_{V_i}(V_i)[v_i/f_i]$. We have $V_i' \cong_{U_i} U_i' \times \mathbb{A}^1$, where $\mathbb{A}^1 = \operatorname{Spec} k[v_i']$ with $v_i' = v_i/f_i$. This defines local trivializations of the projection $\pi^{T,D}\colon \mathcal{X}^{T,D} \to X$, hence a structure of a line bundle on $\mathcal{X}^{T,D}$ over X. Note that $v_j' = (\alpha_{i,j}^{-1}\beta_{i,j}) v_i'$, where $\{\alpha_{i,j}^{-1}\beta_{i,j}\}$ is a Čech 1-cocycle on X associated with the line bundle $\pi^{T-D}\colon \mathcal{X}^{T-D} \to X$. □

Notation 5.15. Let X be an affine variety acted by a finite group G, and let $T, D \in \operatorname{Div} X$ be G-invariant divisors, where D is reduced. Then the line bundle $\mathcal{X}^T \to X$ admits a G-linearization, that is, a structure of a G-equivariant line bundle. This structure is not unique, in general. It is defined modulo the multiplication by a character, see, e.g., [65]. Choosing a G-linearization, say, $\mathcal{X}^T(1) \to X$ with the corresponding linear equivariant G-action $\varphi\colon (g, v) \mapsto g.v$ on $\mathcal{X}^T$, for a character $\chi \in G^\vee$ consider a new such action $\varphi^\chi\colon (g, v) \mapsto \chi(g) \cdot g.v$. This yields a new G-linearization denoted by $\mathcal{X}^T(\chi) \to X$.

In the case of a cyclic group $G = \mu_d$, fixing a primitive character χ of μ_d, we write $\mathcal{X}^T(k) \to X$ for the μ_d-linearization on $\mathcal{X}^T \to X$ associated with the character χ^k. With this notation, $\mathcal{X}^T(0) \to X$ corresponds to the given G-linearization. Clearly, the sequence $(\mathcal{X}^T(k))_{k \in \mathbb{Z}}$ is periodic with period d. For any G-invariant divisors $T_1, T_2 \in \operatorname{Div} B$ and any characters $\chi, \lambda \in G^\vee$, there is a G-equivariant isomorphism $\mathcal{X}^{T_1}(\chi) \otimes \mathcal{X}^{T_2}(\lambda) \cong_X \mathcal{X}^{T_1+T_2}(\chi\lambda)$.

In the sequel we need the following simple lemma.

Lemma 5.16. *Let B be a smooth affine curve acted by a finite group G, and let $\xi : L \to B$ be a line bundle over B, which admits a G-linearization. Then for any $b_1, \dots, b_n \in B$*

there are a G-stable open set U containing these points and a G-equivariant trivialization of $\xi|_U$.

Proof. It suffices to find a non-zero G-stable (that is, G-equivariant) rational section $s\colon B \to L$ of ξ, which has neither pole nor zero in $b_1, \dots, b_n$, and take $U = B \setminus \operatorname{supp}(\operatorname{div} s)$. Given any non-zero G-stable rational section $s_0\colon B \to L$ of ξ, one can find a G-invariant rational function $f \neq 0$ on B such that $\operatorname{div} f$ restricts to $\operatorname{div} s$ on $b_1, \dots, b_n$. Then $s = s_0/f$ is a desired G-stable section of ξ. □

Notation 5.17. Let $\pi\colon X \to B$ be a marked GDF μ_d-surface over a smooth affine curve B with a μ_d-quasi-invariant marking $z \in \mathcal{O}_B(B)$ of weight 1. Then the principal divisor $D = z^*(0) \in \operatorname{Div} B$ is μ_d-invariant. Given a μ_d-invariant divisor $T \in \operatorname{Div} B$, consider the line bundle $\mathcal{X}^{T^*} \to X$, where $T^* = \pi^*(T) \in \operatorname{Div} X$. By abuse of notation, we let $\mathcal{X}^T = \mathcal{X}^{T^*}$. If $\xi\colon L \to B$ is the line bundle associated with T, then $\mathcal{X}^T \to X$ is induced by ξ via the morphism $\pi\colon X \to B$. Hence both ξ and $\mathcal{X}^T \to X$ admit μ_d-linearizations such that the natural morphism $\mathcal{X}^T \to L$ is μ_d-equivariant. Choosing such a μ_d-linearization of ξ and the one of $\mathcal{X}^T \to X$, we observe that $L(k)$ naturally corresponds to $\mathcal{X}^T(k)$.

We have the following equivariant version of Lemma 5.14.

Lemma 5.18. *Let things be as in 5.17. Then for any $k \in \mathbb{Z}$ there exists a μ_d-action on $\mathcal{X}^{T,D}$ and a μ_d-equivariant B-isomorphism of line bundles $\mathcal{X}^{T,D} \cong_{\mu_d,B} \mathcal{X}^{T-D}(k-1)$ such that the induced morphism $\sigma^D\colon \mathcal{X}^{T-D}(k-1) \to \mathcal{X}^T(k)$ is μ_d-equivariant.*

Proof. The μ_d-action on $\mathcal{X}^T(k)$ stabilizes the divisor $\mathcal{D}^T = (\pi^T)^*(D) \in \operatorname{Div} \mathcal{X}^T$ and the center $\mathcal{D}^T \cdot Z^T$ of the affine modification $\sigma^D\colon \mathcal{X}^{T,D} \to \mathcal{X}^T(k)$. By [54, Cor. 2.2] (see Lemma 1.5), it lifts to a μ_d-action on $\mathcal{X}^{T,D}$ making σ^D equivariant.

Choose a trivializing open set $U \subset B$ as in Lemma 5.16, and let $V = \pi_X^{-1}(U) \subset X$. Then $\mathcal{X}^T(k) \to X$ admits over V a μ_d-equivariant trivialization $\mathcal{X}^T(k)|_V \cong_{\mu_d,V} (V \times \mathbb{A}^1)(k+m)$, where $\mathbb{A}^1 = \operatorname{Spec} k[v]$, and m is the weight of an equivariant trivialization of $\mathcal{X}^T(0)|_V$. Recall that $D = \operatorname{div} z$, where z has weight 1, and there is a natural isomorphism $\mathcal{X}^{T,D}|_V \cong_V V \times \mathbb{A}^1$, where $\mathbb{A}^1 = \operatorname{Spec} k[v/z]$, compatible with an isomorphism $\mathcal{X}^{T,D} \cong_B \mathcal{X}^{T-D}$ of Lemma 5.14 (see the proof of this lemma). This gives an equivariant trivialization $\mathcal{X}^{T-D}|_V \cong_{\mu_d,V} (V \times \mathbb{A}^1)(k+m-1)$, and shows that the induced μ_d-action on $\mathcal{X}^{T-D}$ has weight $k-1$. Now the assertions follow. □

The following is an analog of Theorem 5.7 in our more general setting.

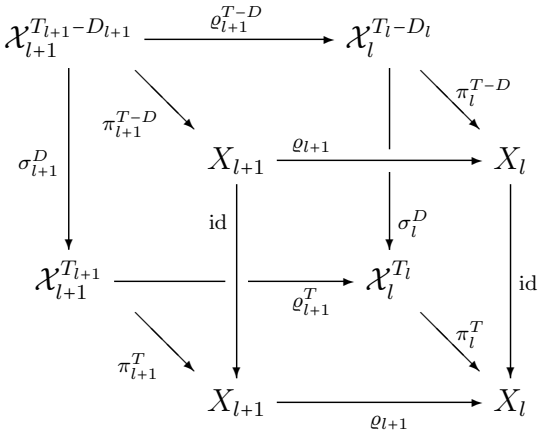
Theorem 5.19. *Let $\pi_X\colon X \to B$ and $\pi_Y\colon Y \to B$ be two marked GDF μ_d-surfaces over B with the same μ_d-quasi-invariant marking $z \in \mathcal{O}_B(B)$ of weight 1. Assume that for some trivializing μ_d-equivariant completions $(\hat{X}, \hat{D}_X)$ and $(\hat{Y}, \hat{D}_Y)$ the graph divisors $\mathcal{D}(\hat{\pi}_X)$ and $\mathcal{D}(\hat{\pi}_Y)$ are μ_d-equivariantly isomorphic. Let $T \in \operatorname{Div} B$ be a μ_d-invariant divisor. Then for any $k \in \mathbb{Z}$ there is a μ_d-equivariant B-isomorphism $\mathcal{X}^T(k) \cong_{\mu_d,B} \mathcal{Y}^T(k)$.*

We need an analog of the Asanuma modification of the first kind for line bundles over surfaces (see Definition 5.21 below). To this end, we use the following notation.

Notation 5.20. Let $z^{-1}(0) = \{b_1, \dots, b_n\} \subset B$. Consider a trivializing sequence (9) of fibered modifications $\varrho_{l+1}\colon X_{l+1} \to X_l$, $l = 0, \dots, N$, along divisors $D_l = z^*(0) \in \operatorname{Div}(X_l)$, where $\pi_l\colon X_l \to B$ are marked GDF μ_d-surfaces over B with a common marking $z \in \mathcal{O}_B(B)$. Given a μ_d-invariant divisor $T \in \operatorname{Pic} B$, replacing T by $T + \operatorname{div} f$, where $f \in \mathcal{O}_B(B)$ is a μ_d-quasi-invariant such that $(\operatorname{div} f)(b_i) = -T(b_i)$, $i = 1, \dots, n$, we may

assume that $b_i \notin \operatorname{supp} T$ $\forall i = 1, \dots, n$. For every $l = 0, \dots, N$ we let $T_l = \pi_l^*(T) \in \operatorname{Div} X_l$. Since $T_{l+1} = \varrho_{l+1}^*(T_l)$, the modification $\varrho_{l+1}\colon X_{l+1} \to X_l$ induces an affine modification $\varrho_{l+1}^T\colon \mathcal{X}_{l+1}^{T_{l+1}} \to \mathcal{X}_l^{T_l}$, which fits in the commutative diagram

(33)



For a fiber component $F_i \subset D_l$ we let C_i be the intersection of F_i with the center of modification $\varrho_{l+1}\colon X_{l+1} \to X_l$. Then $\varrho_{l+1}^T\colon \mathcal{X}_{l+1}^{T_{l+1}} \to \mathcal{X}_l^{T_l}$ is an affine modification along divisor $\mathcal{D}_l = \bigcup_i \mathcal{F}_i$ with center $\mathcal{C}_l = \bigcup_i \mathcal{C}_i$, where $\mathcal{F}_i = (\pi^{T_l})^{-1}(F_i) \cong F_i \times \mathbb{A}^1 \cong \mathbb{A}^2$ and $\mathcal{C}_i \cong C_i \times \mathbb{A}^1 \subset F_i \times \mathbb{A}^1$. There is an alternative: either

(i) C_i is finite, or
(ii) $C_i = F_i$.

In case (i), F_i is a top level component. In case (ii), $X_{l+1} \to X_l$ ($\mathcal{X}_{l+1}^{T_{l+1}} \to \mathcal{X}_l^{T_l}$, respectively) is an isomorphism near F_i (near $\mathcal{F}_i$, respectively).

Definition 5.21. By analogy, we call an *Asanuma modification of the first kind* the birational morphism

$$\kappa_{l+1}\colon \mathcal{X}_{l+1}^{T_{l+1}-D_{l+1}} \to \mathcal{X}_l^{T_l},$$

where $\kappa_{l+1} = \varrho_{l+1}^T \circ \sigma_{l+1}^D = \sigma_l^D \circ \varrho_{l+1}^{T-D}$ is the diagonal composition of morphisms in the back square of (33). Then κ_{l+1} is an affine modification along divisor $\mathcal{D}_l = \bigcup_i \mathcal{F}_i$ on $\mathcal{X}_l^{T_l}$ with center $\bigcup_i (C_i \times \{0\})$. In case (i), $C_i \times \{0\} \subset \mathcal{C}_i \cong \mathbb{A}^2$ is zero-dimensional, while in case (ii) this is just the coordinate axis $v = 0$ in $\mathcal{C}_i \cong \mathbb{A}^2$. Due to Lemma 5.18 and by analogy with (29), we have the following sequence of equivariant Asanuma modifications of the first kind:

$$\mathcal{X}_N^{T_N - N D_N}(-N) \xrightarrow{\tilde{\varrho}_N} \dots \longrightarrow \mathcal{X}_2^{T_2 - 2D_2}(-2) \longrightarrow \mathcal{X}_1^{T_1 - D_1}(-1) \xrightarrow{\tilde{\varrho}_1} \mathcal{X}_0^{T_0}(0). \tag{34}$$

Proof of Theorem 5.19. For the given GDF surfaces $\pi_X\colon X \to B$ and $\pi_Y\colon Y \to B$, consider the corresponding sequences (34) of affine modifications of the line bundles $\mathcal{X}_l^{T_l - lD_l}(-l)$ and $\mathcal{Y}_l^{T_l - lD_l}(-l)$, $l = 0, \dots, N-1$. These sequences start both with the same line bundle $\mathcal{X}_0^{T_0}(0) = (B \times \mathbb{A}^1)^{T_0}(0) = \mathcal{Y}_0^{T_0}(0)$; one may suppose that μ_d acts trivially on the factor $\mathbb{A}^1$. Using Proposition 5.22 below with $s > N$, it follows by induction that for $l = 0, \dots, N$ there is a (non-linear[14]) μ_d-equivariant B-isomorphism

$$\mathcal{X}_l^{T_l - lD_l}(-l) \cong_{\mu_d, B} \mathcal{Y}_l^{T_l - lD_l}(-l),$$

[14]This is not an isomorphism of line bundles, in general.

which sends the zero section $Z(\mathcal{X}_l^{T_l - lD_l}(-l))$ of the first line bundle to such a section of the second one. Replacing T by $T + ND$ we obtain for $l = N$,

$$\mathcal{X}^T(-N) = \mathcal{X}_N^{T_N}(-N) \cong_{\mu_d,B} \mathcal{Y}_N^{T_N}(-N) = \mathcal{Y}^T(-N)$$

via a (μ_d, B)-isomorphism φ respecting the zero sections $Z(\mathcal{X}^T(-N))$ and $Z(\mathcal{Y}^T(-N))$ and the divisors $\mathcal{D}^T(\mathcal{X}^T)$ and $\mathcal{D}^T(\mathcal{Y}^T)$. Hence φ respects also the centers $\mathcal{D}^T(\mathcal{X}^T) \cdot Z(\mathcal{X}^T(-N))$ and $\mathcal{D}^T(\mathcal{Y}^T) \cdot Z(\mathcal{Y}^T(-N))$ of the Asanuma modifications of the second kind. Applying these modifications on both sides, by Lemma 5.18 we decrease by 1 the weights of the μ_d-actions. Due to Lemma 1.5, φ admits a lift to a (μ_d, B)-isomorphism $\tilde{\varphi}$ fitting in the commutative diagram

$$\begin{array}{ccc} \mathcal{X}^{T-D}(-N-1) & \xrightarrow[\cong_{\mu_d,B}]{\tilde{\varphi}} & \mathcal{Y}^{T-D}(-N-1) \\ {\scriptstyle \sigma^D}\downarrow & & \downarrow{\scriptstyle \sigma^D} \\ \mathcal{X}^T(-N) & \xrightarrow[\varphi]{} & \mathcal{Y}^T(-N) \end{array} \tag{35}$$

and respecting the zero sections. Choose $m \geq 1$ such that $-(N+m) \equiv k \mod d$. For $s \gg 1$, after m iterations one arrives at a (μ_d, B)-isomorphism $\mathcal{X}^{T-mD}(k) \cong_{\mu_d,B} \mathcal{Y}^{T-mD}(k)$. This holds for an arbitrary μ_d-stable divisor $T \in \operatorname{Div} B$. Replacing the initial T by $T + mD$, one gets an isomorphism $\mathcal{X}^T(k) \cong_{\mu_d,B} \mathcal{Y}^T(k)$, as required. □

In the proof we used the following analog of Proposition 5.8. By abuse of notation, we let v_i and $\tilde{v}_i$ be the local fiber coordinates of the line bundles $\mathcal{X}_l^T \to X$ and $\mathcal{Y}_l^T \to Y$, respectively.

Proposition 5.22. *Under the assumptions of Theorem 5.19, let*

$$\psi_l \colon \mathcal{X}_l^T(-l) \xrightarrow{\cong_{\mu_d,B}} \mathcal{Y}_l^T(-l)$$

be a μ_d-equivariant B-isomorphism such that

(i_l) $\psi_l^*(\tilde{v}_i) \equiv v_i \mod z^s \ \forall i$.

Then there exists a μ_d-equivariant B-isomorphism

$$\psi_{l+1} \colon \mathcal{X}_{l+1}^{T-D}(-l-1) \xrightarrow{\cong_{\mu_d,B}} \mathcal{Y}_{l+1}^{T-D}(-l-1)$$

such that

(i_{l+1}) $\psi_{l+1}^*(\tilde{v}_i) \equiv v_i \mod z^{s-1} \ \forall i$.

Hint. The proof of Proposition 5.8 goes verbatim modulo the existence of an automorphism φ, which is guaranteed by Theorem 4.4. Thus, it suffices to prove the following analog of Theorem 4.4.

Theorem 5.23. *Let a GDF μ_d-surface $\pi_X \colon X \to B$, $z \in \mathcal{O}_B(B)$, and $T \in \operatorname{Div} B$ be as in Theorem 5.19. Then $\mathcal{X}^T$ satisfies an analog of the μ_d-equivariant condition $RF(l, -l, s)$.*

Proof. It suffices to reproduce mutatis mutandis the proof of Theorem 4.4 (see subsection 4.3). The modifications are as follows.

The coordinate v that was used when working with cylinders, may not exist on the total space of the line bundle $\pi^T \colon \mathcal{X}^T \to X$. Hence, we cannot consider on $\mathcal{X}^T$ the locally nilpotent derivations $\tilde{\sigma}_{1,f}$ and $\tilde{\sigma}_{2,g}$ as in (17). However, one can use instead their analogs, which coinside with these to a given order on any special fiber component $\mathcal{F}_i = (\pi^T)^{-1}(F_i)$ in $\mathcal{X}^T$.

Indeed, let $\xi : L \to B$ be the line bundle associated with T, and let $U \subset B$ be a μ_d-stable dense open subset as in Lemma 5.16, which contains $z^{-1}(0) = \{b_1, \dots, b_n\}$ and such that $\xi|_U$ is trivial as a μ_d-line bundle. Then also the induced line bundle $\pi^T : \mathcal{X}^T \to X$ is trivial over $V = \pi^{-1}(U) \subset X$. Thus, $\mathcal{X}^T|_V \cong_{\mu_d, V} V \times \mathbb{A}^1$, where $\mathbb{A}^1 = \operatorname{Spec} k[v]$. Via this isomorphism, v yields a rational μ_d-quasi-invariant function on $\mathcal{X}^T$, which we denote by the same letter.

Choose a regular μ_d-quasi-invariant function $h \in \mathcal{O}_B(B)$ such that $h - 1 \equiv 0 \mod z^s$ and $h|_{B \setminus U} \equiv 0 \mod z^s$. Consider the lift $\tilde{h} \in \mathcal{O}_{\mathcal{X}^T}(\mathcal{X}^T)$ of h. For $s \gg 1$, the product $\tilde{v} = \tilde{h}v \in \mathcal{O}_{\mathcal{X}^T}(\mathcal{X}^T)$ is a regular μ_d-quasi-invariant, which coinsides with v to order s on any special fiber component $\mathcal{F}_i = (\pi^T)^{-1}(F_i)$ in $\mathcal{X}^T$. Letting $\hat{\sigma}_{1,f} = f(\tilde{v}^d)\partial_l^*$ for $f \in k[t]$, where $\partial_l^* = (\pi^T)^*(\partial_l)$, yields a μ_d-invariant locally nilpotent derivation on $\mathcal{O}_{\mathcal{X}^T}(\mathcal{X}^T)$, which coinsides to order s with $\tilde{\sigma}_{1,f}$ on any fiber component $\mathcal{F}_i$.

Furthermore, for $s \gg 1$ the product $\tilde{h}^{d+1}\partial/\partial v$ is a μ_d-invariant locally nilpotent derivation on $\mathcal{O}_{\mathcal{X}^T}(\mathcal{X}^T)$. Letting $\hat{\sigma}_{2,g} = \tilde{u}^{ds} g(\tilde{u}^d)\tilde{h}^d \partial/\partial v$ for $g \in k[t]$, where $\tilde{u}$ is as defined in 4.10, yields a μ_d-invariant locally nilpotent derivation on $\mathcal{O}_{\mathcal{X}^T}(\mathcal{X}^T)$, which coinsides to order s with $\tilde{\sigma}_{2,g}$ on any fiber component $\mathcal{F}_i$.

Using the locally nilpotent derivations $\hat{\sigma}_{1,f}$ and $\hat{\sigma}_{2,g}$ instead of $\tilde{\sigma}_{1,f}$ and $\tilde{\sigma}_{2,g}$, respectively, the rest of the proof of Theorem 4.4 applies and gives the desired μ_d-equivariant relative flexibility. □

6. Basic examples of Zariski factors

6.1. Line bundles over affine curves.

Proposition 6.1. *Let $\pi : X \to B$ be a line bundle over a smooth affine curve B. Then the surface X is a Zariski factor.*

Proof. If $B \cong \mathbb{A}^1$, then $\pi : X \to B$ is a trivial line bundle, and so, $X \cong \mathbb{A}^2$ is a Zariski factor by the Miyanishi-Sugie-Fujita Theorem ([39, 63]; see also [62, Ch. 3, Thm. 2.3.1]).

We will suppose in the sequel that $B \not\cong \mathbb{A}^1$, and so, any morphism $\mathbb{A}^1 \to B$ is constant. Letting $\mathcal{X} = X \times \mathbb{A}^n$, the natural projection $\tilde{\pi} : \mathcal{X} \to B$ defines on $\mathcal{X}$ a structure of a vector bundle of rank $n+1$ isomorphic to the Whitney sum $\xi \oplus \mathbf{1_n}$, where ξ stands for the given line bundle $\pi : X \to B$ and $\mathbf{1_n}$ for the trivial vector bundle of rank n over B.

Consider a second smooth affine surface X', and let $\mathcal{X}' = X' \times \mathbb{A}^n$. Assume that there is an isomorphism $\varphi : \mathcal{X}' \xrightarrow{\cong} \mathcal{X}$. The vector bundle structure $\tilde{\pi} : \mathcal{X} \to B$ is transferred by φ to such a structure $\tilde{\pi}' = \tilde{\pi} \circ \varphi : \mathcal{X}' \to B$ on $\mathcal{X}'$. It is easily seen that $\tilde{\pi}'$ admits a factorization

$$\tilde{\pi}' : \mathcal{X}' \xrightarrow{\mathrm{pr}_1} X' \xrightarrow{\pi'} B\,, \tag{36}$$

where $\pi' : X' \longrightarrow B$ is an $\mathbb{A}^1$-fibration with reduced and irreducible fibers only, because the fibers of $\tilde{\pi}' : \mathcal{X}' \to B$ are. Therefore, $\pi' : X' \longrightarrow B$ is a line bundle, say, ξ'. Due to (36) the vector bundle $\tilde{\pi}' : \mathcal{X}' \to B$ of rank $n+1$ is isomorphic to the Whitney sum $\xi' \oplus \mathbf{1_n}$.

The fiberwise biregular map of the total spaces

$$\varphi : \mathcal{X}' = \operatorname{tot}(\xi' \oplus \mathbf{1_n}) \to \mathcal{X} = \operatorname{tot}(\xi \oplus \mathbf{1_n})$$

identical on the base B sends the zero section $Z' \cong B$ of $\xi' \oplus \mathbf{1_n}$ onto a section, say, Z of $\xi \oplus \mathbf{1_n}$. Composing φ with a shift by $-Z$ in $\xi \oplus \mathbf{1_n}$ we may assume that Z is the zero section of $\xi \oplus \mathbf{1_n}$. The differential $d\varphi|_Z$ yields an isomorphism of the normal bundles

$\mathcal{N}'$ of Z' in $\mathrm{tot}(\xi' \oplus \mathbf{1_n})$ and $\mathcal{N}$ of Z in $\mathrm{tot}(\xi \oplus \mathbf{1_n})$. Since $\mathcal{N}' \cong \xi' \oplus \mathbf{1_n}$ and $\mathcal{N} \cong \xi \oplus \mathbf{1_n}$ as vector bundles, we have $\xi' \oplus \mathbf{1_n} \cong \xi \oplus \mathbf{1_n}$, that is, the line bundles ξ and ξ' are stably equivalent. In fact, these are equivalent. Indeed, by [74, §8, Corollary] one has

$$\xi \cong \det(\xi \oplus \mathbf{1_n}) \cong \det(\xi' \oplus \mathbf{1_n}) \cong \xi' .$$

So, $X' \cong_B X$, as required. □

As for an equivariant version of this proposition, see Lemma 6.9.

Remark 6.2. In [40, 9.10.1] Fujita asks whether the cylinder over the surface $(\mathbb{A}^1 \setminus \{k \text{ points}\}) \times \mathbb{A}^1$ depends essentially on the isomorphism type of the factor $\mathbb{A}^1 \setminus \{k \text{ points}\}$. Proposition 6.1 answers this question in affirmative.

Let us give a first application of this proposition.

Remark 6.3. Let $\pi\colon X \to B$ be a normal $\mathbb{A}^1$-fibered surface over a smooth affine curve B. Let $B^* = B \setminus \{b_1, \dots, b_s\}$ be the maximal Zariski open subset of B over which π is locally trivial. Letting $X^* = \pi^{-1}(B^*)$ we consider the line bundle $\xi_{X^*} = (\pi|_{X^*}\colon X^* \to B^*)$. Let $[\xi_{X^*}]$ be the class of ξ_{X^*} in the Picard group $\mathrm{Pic}(B^*)$.

Let $z \in \mathcal{O}_B(B)$ be a marking, that is, a regular function on B with simple zeros $b_1, \dots, b_n$ vanishing in the points b_j, $j = 1, \dots, s$, and also in all points of B over which the fibers of the trivializing resolved completion $(\hat{X}, \hat{D})$ of X as in (10) are reducible. The fiber graph Γ_{b_j} over a point b_j with $j > s$ is a chain, say, L_j of hight $a_j = \mathrm{ht}\,(L_j) \geq 0$. The divisor $\sum_{j=s+1}^{n} a_j b_j$ on B^* belongs to the class $[\xi_{X^*}]$ in $\mathrm{Pic}\, B^*$; it has support contained in the principal reduced divisor $\mathrm{div}(z|_{B^*})$.

Lemma 6.4. *The class $[\xi_{X^*}]$ is cancellation invariant.*

Proof. Let $\pi_X\colon X \to B$ and $\pi_Y\colon Y \to C$ be two normal $\mathbb{A}^1$-fibered surfaces, and let ξ_X and ξ_Y be the associated line bundles as in Definition 6.3. Suppose that for some $m \geq 1$ there is an isomorphism of cylinders

$$\varphi\colon \mathcal{X} = X \times \mathbb{A}^m \xrightarrow{\cong} \mathcal{Y} = Y \times \mathbb{A}^m .$$

We have to show that then $[\xi_{X^*}] = [\xi_{Y^*}]$.

If at least one of the curves B and C is not isomorphic to $\mathbb{A}^1$, then φ induces an isomorphism $C \cong B$ (cf. Remark 5.11). Identifying B and C via this isomorphism, we may assume that $C = B$ and φ is a B-isomorphism.

In particular, B and C are simultaneously rational or not. If they are, then $\mathrm{Pic}\, X^* = \mathrm{Pic}\, Y^* = 0$, and so, $[\xi_{X^*}] = 0 = [\xi_{Y^*}]$, as stated. In the other case, assuming that φ is a B-isomorphism, it induces a B-isomorphism between the total spaces of the vector bundles $\xi_{X^*} \oplus \mathbf{1}_m$ and $\xi_{Y^*} \oplus \mathbf{1}_m$. As in the proof of Proposition 6.1, this implies an isomorphism of line bundles $\xi_{X^*} \cong \xi_{X^*}$, and so, the equality $[\xi_{X^*}] = [\xi_{Y^*}]$. □

6.2. **Parabolic $\mathbb{G}_m$-surfaces: an overview.**

Definitions 6.5 (*DPD presentation for parabolic $\mathbb{G}_m$-surfaces*)**.** ([31]) A *parabolic $\mathbb{G}_m$-surface* is a normal affine surface X endowed with an effective $\mathbb{G}_m$-action along the fibers of an $\mathbb{A}^1$-fibration $\pi\colon X \to C$ over a smooth affine curve C. The $\mathbb{G}_m$-action on X defines a grading

$$\mathcal{O}_X(X) = \bigoplus_{n \geq 0} A_n, \quad \text{where} \quad A_n = H^0(C, \mathcal{O}_C(\lfloor nD_X \rfloor)) \ \ \forall n \geq 0$$

for a $\mathbb{Q}$-divisor D_X on C. This is called a *Dolgachev-Pinkham-Demazure presentation*, or a *DPD presentation* for short, see [31, Thm. 3.2]. The $\mathbb{Q}$-divisor D_X on C is uniquely defined by $\pi\colon X \to C$ up to a linear equivalence. Any fiber $\pi^*(p)$, $p \in C$, is irreducible of multiplicity m, where $D_X(p) = (e/m)[p]$ with coprime $e, m \in \mathbb{Z}$. The reduced fiber $\pi^{-1}(p)$ is smooth and isomorphic to $\mathbb{A}^1$ ([32, Rem. 3.13(iii)]). The projection $\pi\colon X \to C$ admits a section consisting of the fixed points of the $\mathbb{G}_m$-action on X. The singularities of X are the fixed points in the multiple fibers of π. More precisely, if $D_X(p) = (e/m)[p]$, where $m > 1$ and e, m are coprime, then the unique fixed point x_p over p is a cyclic quotient singularity of type (m, e'), where $e' \in \{1, \dots, m-1\}$ and $e' \equiv e \mod m$, see [31, Prop. 3.8].

Lemma 6.6. *Given a parabolic $\mathbb{G}_m$-surface $\pi\colon X \to C$ and a branched covering $\mu\colon B \to C$, let $\pi'\colon X' \to B$ be obtained from the cross-product $B \times_C X$ via normalization. Then $\pi'\colon X' \to B$ is again a parabolic $\mathbb{G}_m$-surface, and the $\mathbb{Q}$-divisors D_X on C and $D_{X'}$ on B in the corresponding DPD presentations are related via $D_{X'} = \mu^* D_X$.*

Proof. The projection $\pi\colon X \to C$ is the orbit morphism of a parabolic $\mathbb{G}_m$-action, say, Λ on X, with only smooth, irreducible fibers. Hence the fibers of $\pi'\colon X' \to B$ are also irreducible, and Λ lifts to a parabolic $\mathbb{G}_m$-action Λ' on the cross-product $B \times_C X$, where $\lambda\colon (b, x) \mapsto (b, \lambda.x)$ $\forall \lambda \in \mathbb{G}_m$. This lifted action survives in the normalization $X' \to B \times_C X$. Thus Λ lifts to a parabolic $\mathbb{G}_m$-action Λ' on X' such that $\pi'\colon X' \to B$ is the orbit morphism, and the induced morphism $\mu'\colon X' \to X$ is $\mathbb{G}_m$-equivariant.

On the other hand, consider the $\mathbb{Q}$-divisor $D_{X''} = \mu^* D_X$ on B and the corresponding parabolic $\mathbb{G}_m$-surface $\pi''\colon X'' \to B$ with the DPD presentation related to the pair $(B, D_{X''})$. For any $n \geq 0$ there is a natural embedding $A_n = H^0(C, \mathcal{O}_C(\lfloor nD_X \rfloor)) \hookrightarrow \hat{A}_n = H^0(B, \mathcal{O}_B(\lfloor nD_{X''} \rfloor))$. This yields a monomorphism of graded rings

$$\mathcal{O}_X(X) = \bigoplus_{n \geq 0} A_n \hookrightarrow \mathcal{O}_{X''}(X'') = \bigoplus_{n \geq 0} A''_n$$

and the induced $\mathbb{G}_m$-equivariant surjection $\mu''\colon X'' \to X$ that fits in the commutative diagram

$$\begin{array}{ccc} X'' & \xrightarrow{\mu''} & X \\ {\scriptstyle \pi''}\downarrow & & \downarrow{\scriptstyle \pi} \\ B & \xrightarrow[\mu]{} & C \end{array}$$

By the universal property of the cross-product, μ'' can be factorized as

$$\mu''\colon X'' \to B \times_C X \xrightarrow{\pi} X\,.$$

Since X'' is normal, we have as well a factorization $\mu''\colon X'' \xrightarrow{\psi} X' \xrightarrow{\pi} X$, where ψ is a $\mathbb{G}_m$-equivariant B-surjection[15]. Since all fibers of a parabolic $\mathbb{G}_m$-surface are irreducible ([32, Rem. 3.13(iii)]), ψ is a bijection. Due to the normality of both X'' and X', this is an isomorphism. Now the conclusion follows. □

[15] That is, ψ fits in the commutative diagram

$$\begin{array}{ccc} X'' & \xrightarrow{\psi} & X' \\ & {\scriptstyle \pi''}\searrow \quad \swarrow{\scriptstyle \pi'} & \\ & B & \end{array}\,.$$

It is known that a Gizatullin $\mathbb{G}_m$-surface X is toric if and only if the associated extended graph Γ_{ext} is linear, see [33, Lem. 2.20]. A similar criterion is available for parabolic $\mathbb{G}_m$-surfaces, since these are locally infinitesimally toric over the base.

A finite rooted tree Γ will be called a *star* if any branch of Γ at the root vertex is a linear chain.

Proposition 6.7. *Let $\pi\colon X \to C$ be an $\mathbb{A}^1$-fibration on a normal affine surface X over a smooth affine curve C, and let $\pi'\colon X' \to B$ be a marked GDF μ_d-surface obtained from $\pi\colon X \to C$ via a suitable cyclic base change with the Galois group μ_d after a subsequent normalization, see Lemma 2.3. Then the following are equivalent.*

(i) *$\pi'\colon X' \to B$ is a line bundle;*
(ii) *$\pi\colon X \to C$ is a parabolic $\mathbb{G}_m$-surface;*
(iii) *the extended graph Γ_{ext} of a resolved minimal completion $\bar\pi\colon \bar X \to \bar C$ is star-shaped with the root vertex S as the center, that is, any fiber tree of a special fiber in X is a chain.*

Proof. (i)⇒(ii). Suppose that $\pi'\colon X' \to B$ is a line bundle. In this case all fibers of the $\mathbb{A}^1$-fibration $X \to C$ are irreducible. The μ_d-action on X' preserves the fibration $\partial' i\colon X' \to B$. Hence it sends sections of π' to sections. Taking the fiberwise barycentre Z of the μ_d-orbit of the zero section yields a μ_d-invariant section of π' (cf. the proof of Lemma 3.6). The shift by Z in the vertical direction on X' conjugates the standard parabolic $\mathbb{G}_m$-action on X' along the fibers of π' with the zero section as the fixed point set to such an action, say, Λ' with Z as the fixed point set. The new $\mathbb{G}_m$-action Λ' commutes with the μ_d-action on X', hence descends to a parabolic $\mathbb{G}_m$-action Λ on X along the fibers of $\pi\colon X \to C$, so that C is the orbit space for this action. Thus Λ converts X into a parabolic $\mathbb{G}_m$-surface over C. This proves (i)⇒(ii).

(ii)⇒(i). Conversely, suppose that $\pi\colon X \to C$ is the orbit morphism of a parabolic $\mathbb{G}_m$-action Λ on X. Then Λ lifts to a parabolic $\mathbb{G}_m$-action $\tilde\Lambda$ on the cross-product $B \times_C X$, where $\lambda\colon (b, x) \mapsto (b, \lambda.x)\ \forall \lambda \in \mathbb{G}_m$. This lifted action survives in the normalization $\tilde X \to B \times_C X$. Thus Λ lifts through the branching covering construction $X' \to X$ as in 2.2. In this way the GDF surface $\pi'\colon X' \to B$ inquires a parabolic $\mathbb{G}_m$-action along the fibers of π'. Hence all these fibers are reduced and irreducible, see [32, Rem. 3.13(iii)]. It follows that $\pi'\colon X' \to B$ is a line bundle. This proves the equivalence (ii)⇔(i).[16]

For the implication (ii)⇒(iii) we send the reader to [33, Prop. 3.22]. To end the proof, it is enough to show the implication (iii)⇒(i).

(iii)⇒(i). Suppose that the extended graph Γ_{ext} with S is a star with center at the root vertex S. We claim that in this case any fiber $\pi^{-1}(p)$, $p \in C$, is irreducible. Indeed, let $\pi^{-1}(p)$ be a special fiber. The branch $\mathcal{B}$ of Γ_{ext} at the root S, which is the dual graph of $\pi^{-1}(p)$, is linear. Hence the components $F_1, \dots, F_s$ of the fiber $\pi^{-1}(p)$ can be ordered according to the distance from S of the corresponding feather components $\bar F_1, \dots, \bar F_s$ of $\mathcal{B}$. Let $\bar F_1$ meet the boundary divisor D, and $\bar F_2$ be the feather component of $\mathcal{B}$ closest to $\bar F_1$. The components of $cB \ominus D$ different form $\bar F_1, \dots, \bar F_s$ are contracted to singular points of X. Assuming that $s \geq 2$ we deduce that F_1 meets F_2 in X, which is impossible due to the fact that the fiber components of an $\mathbb{A}^1$-fibration are disjoint, see [62, Ch. 3, Lem. 1.4.1(1)]. This proves the claim.

[16] The $\mathbb{Q}$-divisor $\mathcal{D}'$ on B in the corresponding DPD presentation is integral and represents the class of the line bundle $\pi'\colon X' \to B$ in the Picard group $\operatorname{Pic} B$, cf. Lemma 6.6.

Under the branch covering construction, the morphism $B \to C$ is ramified to full order d over each point $p \in C$ such that the fiber $\pi^*(p) = m_p\pi^{-1}(p)$ is multiple. Furthermore, the multiplicity m_p divides d. It follows that on the GDF surface $\tilde{\pi}\colon \tilde{X} \to B$, all the fibers of $\tilde{\pi}$ are reduced and irreducible. Hence $\tilde{\pi}\colon \tilde{X} \to B$ is a line bundle. Thus, (i) holds. □

6.3. **Parabolic $\mathbb{G}_m$-surfaces as Zariski factors.**

Theorem 6.8. *Any parabolic $\mathbb{G}_m$-surface is a Zariski factor.*

The proof is divided into several steps, see Lemmas 6.9-6.19 below.

Lemma 6.9. *Let $\pi\colon X \to C$ and $\pi'\colon X' \to C'$ be $\mathbb{A}^1$-fibered normal affine surfaces over smooth affine curves, where $\pi\colon X \to C$ is a parabolic $\mathbb{G}_m$-surface. Suppose that for some $n \geq 1$ there is an isomorphism $\varphi\colon X' \times \mathbb{A}^n \xrightarrow{\cong} X \times \mathbb{A}^n$ sending the induced $\mathbb{A}^{n+1}$-fibration $X' \times \mathbb{A}^n \to C'$ into the one $X \times \mathbb{A}^n \to C$. Then the surfaces X and X' are isomorphic.*

Proof. By abuse of notation, we let $\pi\colon \mathcal{X} \to C$ and $\pi'\colon \mathcal{X}' \to C'$ be the induced $\mathbb{A}^{n+1}$-fibrations on the n-cylinders $\mathcal{X} = X \times \mathbb{A}^n$ and $\mathcal{X}' = X' \times \mathbb{A}^n$, respectively. Clearly, the isomorphism φ fits in the commutative diagram

(37)
$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow{\varphi} & \mathcal{X} \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ C' & \xrightarrow[\cong]{} & C \end{array}$$

Identifying C' and C in this diagram one can deduce that the special fibers of both $\mathbb{A}^1$-fibrations $\pi'\colon X' \to C$ and $\pi\colon X \to C$ are irreducible multiple fibers situated over the same points and having the same multiplicities. Hence applying to both surfaces the branched covering construction with the same cyclic base change $B \to C$, one gets the GDF μ_d-surfaces $\tilde{X}'$ and $\tilde{X}$ with cyclic coverings $\tilde{X}' \to X'$ and $\tilde{X} \to X$. By Propositiona 6.7, $\tilde{\pi}\colon \tilde{X} \to B$ is a line bundle, and so, $\tilde{X}$ is a Zariski factor by virtue of Proposition 6.1. The same branched covering construction applied to the n-cylinders $\mathcal{X}'$ and $\mathcal{X}$ (which are isomorphic over C) gives μ_d-equivariantly isomorphic (over B) n-cylinders $\tilde{\mathcal{X}}' \cong \tilde{\mathcal{X}}$. It follows that $\tilde{X}' \cong \tilde{X}$, hence $\tilde{X}'$ inherits a μ_d-equivariant structure of a line bundle over B.

Assume first that this line bundle structure on $\tilde{X}'$ is compatible with the original structure of a GDF surface $\tilde{\pi}'\colon \tilde{X}' \to B$. Then the μ_d-quotients $\pi'\colon X' \to C$ and $\pi'\colon X' \to C$ are isomorphic as parabolic $\mathbb{G}_m$-surfaces over C. In particular, $X' \cong X$, as required.

Otherwise, $\tilde{X}'$ (and then also $\tilde{X} \cong \tilde{X}'$) possesses two different $\mathbb{A}^1$-fibrations over affine bases. Then $\tilde{X}$ is a Gizatullin surface, $B \cong \mathbb{A}^1$, and $\tilde{X} \cong \mathbb{A}^2$ with $\tilde{\pi}\colon \tilde{X} \to B$ being the standard linear projection $\mathbb{A}^2 \to \mathbb{A}^1$. This morphism is μ_d-equivariant, hence the induced μ_d-action on $\mathbb{A}^2$ is equivalent in the natural coordinates on $\tilde{X} \cong \mathbb{A}^2$ to a diagonal action $\zeta\colon (x, y) \mapsto (\zeta x, \zeta^e y)$, where $\zeta \in \mu_d$ and $e \in \{0, \dots, d-1\}$. So $X = \tilde{X}/\mu_d \cong \mathbb{A}^2/\mu_d$ is a non-degenerate affine toric surface.

With the same reasoning, $\tilde{\pi}'\colon \tilde{X}' \to B$ is equivalent to the standard linear projection $\mathbb{A}^2 \to \mathbb{A}^1$, and $X' = \tilde{X}'/\mu_d \cong \mathbb{A}^2/\mu_d$ is a non-degenerate affine toric surface, where the μ_d-action on $\mathbb{A}^2$ is $\zeta\colon (x, y) \mapsto (\zeta x, \zeta^{e'} y)$ with $\zeta \in \mu_d$ and $e' \in \{0, \dots, d-1\}$. The induced linear diagonal μ_d-actions on $\tilde{\mathcal{X}} \cong \mathbb{A}^{n+2}$ and on $\tilde{\mathcal{X}}' \cong \mathbb{A}^{n+2}$ have weights $(1, e, 0, \dots, 0)$ and $(1, e', 0, \dots, 0)$, respectively. Since the n-cylinders $\tilde{\mathcal{X}}$ and $\tilde{\mathcal{X}}'$ are μ_d-equivariantly

isomorphic, these sequences of weights coincide; in particular, $e = e'$. It follows that $X' \cong X$. □

6.10. In the next lemma we use the following terminology. An affine variety is called $\mathbb{A}^1$-*uniruled* if a general point of X belongs to the image of a nonconstant morphsim $\mathbb{A}^1 \to X$. Given a morphism $\pi\colon X \to C$ onto a smooth curve C, the fiber over a point $p \in C$ is called *multiple* if $d > 1$, where d is the greatest common divisor of the multiplicities of the components of the divisor $\pi^*(p)$.

One can find in the literature different versions of the following results, see, e.g., [70] and [46, Thm. 4.1].

Lemma 6.11. *Let X be an affine variety, and let $\pi\colon X \to B$ be a morphism onto a smooth affine curve B. Assume that one of the following conditions is fulfilled.*

(i) $B \not\cong \mathbb{A}^1$;
(ii) $B \cong \mathbb{A}^1$ *and π has at least two multiple fibers.*

Then the following hold.

(a) *If general fibers of π are $\mathbb{A}^1$-uniruled, then any automorphism $\alpha \in \operatorname{Aut} X$ preserves the fibration $\pi\colon X \to B$, that is, sends the fibers onto fibers.*
(b) *If X is normal, then there is no surjective morphism $\mathbb{A}^n \to X$ with finite fibers.*

Proof. (a) In case (i) any morphism $\mathbb{A}^1 \to B$ is constant, hence the assertion follows. Assuming (ii), suppose on the contrary that there exists $\alpha \in \operatorname{Aut} X$ which does not preserve the fibration $\pi\colon X \to B$. In this case there is a morphism $\varphi\colon \mathbb{A}^1 \to X$ such that the composition $f = \pi \circ \varphi \in k[t]$ is a nonconstant polynomial with at least two multiple fibers, say, $f^*(0)$ and $f^*(1)$. Thus $f = p^r = 1 - q^s$, where $p^r + q^s = 1$, $r, s \geq 2$, $p, q \in k[t]$, and $\deg p = d/r$, $\deg q = d/s$, where $d = \deg f$. The derivative f' vanishes to order $r-1$ at any root of p and to order $s-1$ at any root of q. This yields the inequality

$$(r-1)/r + (s-1)/s \leq (d-1)/d\,.$$

Since $r, s \geq 2$ this implies in turn

$$1 \leq \left(1 - \frac{1}{r}\right) + \left(1 - \frac{1}{s}\right) \leq \left(1 - \frac{1}{d}\right),$$

which give a contradiction.

(b) Assuming that $\nu\colon \mathbb{A}^n \to X$ is a surjective morphism with finite fibers, consider the restriction of ν to a general line $l \cong \mathbb{A}^1$ in $\mathbb{A}^n$. Since l is general, X is normal, and ν is finite, the image $\nu(l)$ does not meet the singular locus of X and $\pi \circ \nu\colon l \to B$ is dominant. Hence $B \cong \mathbb{A}^1$. Now the same argument as before applies and gives the result. □

In the proof of Theorem 6.8 we use the following fact.

Lemma 6.12. *Let $\pi\colon X \to \mathbb{P}^1$ be an $\mathbb{A}^1$-fibration on a normal affine surface X. Assume that the group $\operatorname{Pic} X$ is finite and $\pi(X) \supset \mathbb{A}^1$, where $\mathbb{A}^1 = \mathbb{P}^1 \setminus \{\infty\}$. Then $\pi(X) = \mathbb{A}^1$, all fibers of π are irreducible, and the divisor class group $\operatorname{Cl} X$ is generated by the classes of the multiple fibers of π.*

Proof. Let $\bar\pi\colon \bar X \to \mathbb{P}^1$ be a resolved completion of $\pi\colon X \to \mathbb{P}^1$ with extended graph Γ_{ext}. Then Γ_{ext} is a rooted tree with the horizonthal section S as a root. Let $\mathcal{B}_1, \dots, \mathcal{B}_m$ be the degenerate fibers of $\bar\pi$ over the points $b_i \in \mathbb{P}^1$, $i = 1, \dots, m$, respectively. Suppose

that $\mathcal{B}_i$ cosists of $n_i + m_i$ components, where n_i is the number of components of $\mathcal{B}_i$ which are at the same time components of the boundary divisor $D = \bar{X} \setminus X_{\text{resolved}}$ or of the exceptional divisor E of the resolution of singularities of X, and m_i is the number of components of the fiber $\pi^{-1}(b_i)$. Suppose to the contrary that the map $\pi\colon X \to \mathbb{P}^1$ is surjective. Then $m_i > 0$ $\forall i = 1, \dots, m$. Contracting subsequently (-1)-fiber components we arrive finally at a Hirzebruch surface $\mathbb{F}_s$. Note that we contracted in this way $n_i + m_i - 1$ components of $\mathcal{B}_i$, $i = 1, \dots, m$. Since $\operatorname{rk}\operatorname{Pic}\mathbb{F}_s = 2$, we have $\operatorname{rk}\operatorname{Pic}\bar{X} = 2 + \sum_{i=1}^{m}(n_i + m_i - 1)$. Letting $\natural P$ be the number of components of a divisor P, we obtain

$$\operatorname{rk}\operatorname{Pic} X = \operatorname{rk}\operatorname{Pic}\bar{X} - \natural(D + E) =$$
$$\left(2 + \sum_{i=1}^{m}(n_i + m_i - 1)\right) - \left(1 + \sum_{i=1}^{m} n_i\right) = 1 + \sum_{i=1}^{m}(m_i - 1) \geq 1\,,$$

which contradicts our assumption on finiteness of $\operatorname{Pic} X$. Thus, $\pi(X) = \mathbb{A}^1$.

Letting $b_1 = \infty$ we obtain that $m_1 = 0$. Assuming further that the fiber, say, $\pi^{-1}(b_2)$ has at least 2 components, that is, $m_2 > 1$, the same computation yields again the inequality $\operatorname{rk}\operatorname{Pic} X \geq 1$, a contradiction.

Let $\omega \subset \mathbb{A}^1$ be a Zariski open dense subset such that $U = \pi^{-1}(\omega)$ is isomorphic over ω to the cylinder $\omega \times \mathbb{A}^1$, and so, $\operatorname{Cl} U = 0$. For $\mathcal{D} = X \setminus U$ one has the exact sequence $\operatorname{Div}\mathcal{D} \to \operatorname{Cl} X \to \operatorname{Cl} U \to 0$, where $\operatorname{Div}\mathcal{D}$ is the subgroup of Weil divisors on X supported by $\mathcal{D}$, see [62, p. 206]. Thus $\operatorname{Cl} X$ is generated by the fibers of π contained in $\mathcal{D}$. Any reduced fiber of π represents the zero class in $\operatorname{Cl} X$. Hence $\operatorname{Cl} X$ is generated by the classes of the multiple fibers of π. □

Lemma 6.13. *Let $\pi\colon X \to C$ be a parabolic $\mathbb{G}_m$-surface, and let X' be a normal affine surface. Suppose that there is an isomorphism $\varphi\colon \mathcal{X}' \xrightarrow{\cong} \mathcal{X}$ of the n-cylinders $\mathcal{X} = X \times \mathbb{A}^n$ and $\mathcal{X}' = X' \times \mathbb{A}^n$. If the induced $\mathbb{A}^{n+1}$-fibration $\pi\colon \mathcal{X} \to C$ satisfies one of conditions* (i) *and* (ii) *of Lemma 6.11, then $X' \cong X$.*

Proof. Extending the standard n-torus action on $\mathbb{A}^n$ by the identity on the factor X' we obtain a $\mathbb{G}_m^n$-action on the n-cylinder $\mathcal{X}' = X' \times \mathbb{A}^n$ and as well the φ-conjugate $\mathbb{G}_m^n$-action on $\mathcal{X} \cong \mathcal{X}'$. By Lemma 6.11(a) this action respects the fibration $\pi\colon \mathcal{X} \to C$, and so, induces a $\mathbb{G}_m^n$-action on C. The latter is fixed-pointed, since the fiber of π through any fixed point in $\mathcal{X}$ is $\mathbb{G}_m^n$-stable, and the multiple fibers of π are as well. This implies that under any one of the conditions (i) and (ii) the induced $\mathbb{G}_m^n$-action on C is identical. Thus, for any $c \in C$ the $\mathbb{G}_m^n$-action on $\mathcal{X}$ restricts to a $\mathbb{G}_m^n$-action on the fiber $F_c = \pi^{-1}(c) \cong \mathbb{A}^{n+1}$. The latter action is effective on a general fiber, and so, by a theorem of Białynicki-Birula [10], is equivalent to a linear $\mathbb{G}_m^n$-action on $\mathbb{A}^{n+1}$. After such a linearization the fixed point set in $\mathbb{A}^{n+1}$ becomes a linear subspace. We claim that this subspace is one-dimensional.

Indeed, the fixed point set of our $\mathbb{G}_m^n$-action on $\mathcal{X}$ is the surface $X'' = \varphi(X' \times \{0\}) \subset \mathcal{X}$. The weights of the tangent action at any fixed point are non-negative, and the fixed point subspace at a smooth point of X'' has dimension 2. Thus, for a general $c \in C$, the fixed point set $\varphi(X' \times \{0\}) \cap F_c$ of the $\mathbb{G}_m^n$-action on the fiber $F_c \cong \mathbb{A}^{n+1}$ is a smooth curve $l_c \cong \mathbb{A}^1$.

Letting $X'' = \varphi(X' \times \{0\}) \cong X'$ the restriction $\pi|_{X''}\colon X'' \to C$ yields an $\mathbb{A}^1$-fibration on X'' and, in turn, an $\mathbb{A}^{n+1}$-fibration $\pi'\colon \mathcal{X}' \to C$, where $\pi' = \pi \circ \varphi$. It follows by Lemma 6.9 that $X \cong X'$. □

6.14. Thus, in order to prove Theorem 6.8 we may suppose in the sequel that, under the assumptions of Lemma 6.13, neither (i) nor (ii) of Lemma 6.11 holds, that is, $C = \mathbb{A}^1$ and the fibration $\pi\colon X \to \mathbb{A}^1$ has at most one multiple fiber. By virtue of Proposition 6.1 we may suppose as well that the parabolic $\mathbb{G}_m$-surface $\pi\colon X \to \mathbb{A}^1$ has exactly one multiple fiber $\pi^{-1}(0)$ of multiplicity $d > 0$, and so, X has a unique singular point, say, x, which is the unique fixed point of the $\mathbb{G}_m$-action on the multiple fiber $\pi^{-1}(0)$ and a cyclic quotient singularity. Then X' has as well a unique singular point, say, x'.

Lemma 6.15. *In the setup of 6.14, the germs (X, x) and (X', x') of surface singularities are isomorphic.*

Proof. Let $\sigma_1\colon X_1 \to X$ be the blowup of the maximal ideal of the unique singular point $x \in X$ followed by a normalization. The induced morphism of n-cylinders $\sigma_1 \times \mathrm{id}\colon \mathcal{X}_1 \to \mathcal{X}$ consists in the blowup of the ideal of the singular ruling $\operatorname{sing}\mathcal{X} = \{x\} \times \mathbb{A}^n$ and a subsequent normalization. By a theorem of Zariski ([80]; see also [57]), a sequence of blowups in maximal ideals and subsequent normalizations

$$X_m \xrightarrow{\sigma_m} X_{m-1} \to \ldots \to X_1 \xrightarrow{\sigma_1} X_0 = X\,,$$

resolves the singularity of X. It induces a similar sequence of blowups in rulings of our n-cylinders and subsequent normalizations

$$\mathcal{X}_m \xrightarrow{\sigma_m} \mathcal{X}_{m-1} \to \ldots \to \mathcal{X}_1 \xrightarrow{\sigma_1} \mathcal{X}_0 = \mathcal{X}$$

resulting in a resolution of singularity of the variety $\mathcal{X} \cong \mathcal{X}'$. The exceptional divisor of the resolution $\mathcal{X}_m \to \mathcal{X}$ is $\mathcal{E} = E \times \mathbb{A}^n$, where E is the exceptional divisor of the resolution $X_m \to X$.

Let further $\sigma_1'\colon X_1' \to X'$ be the blowup of the maximal ideal of the unique singular point $x' \in X'$ followed by a normalization. Then $\sigma_1' \times \mathrm{id}\colon \mathcal{X}_1' \to \mathcal{X}'$ is the blowup of the ideal of the singular ruling $\operatorname{sing}\mathcal{X}' = \{x'\} \times \mathbb{A}^n$ followed by a normalization. Under an isomorphism $\varphi\colon \mathcal{X}' \xrightarrow{\cong} \mathcal{X}$ this ruling goes to the ruling $\operatorname{sing}\mathcal{X} = \{x\} \times \mathbb{A}^n$. Hence φ lifts to an isomorphism $\varphi_1\colon \mathcal{X}_1' \xrightarrow{\cong} \mathcal{X}_1$. Continuing in this way, we arrive at a resolution $\mathcal{X}_m' \to \mathcal{X}'$, where $\mathcal{X}_m' = X_m' \times \mathbb{A}^n \cong \mathcal{X}_m$, with exceptional divisor $\mathcal{E}' = E' \times \mathbb{A}^n \cong \mathcal{E} = E \times \mathbb{A}^n$, where E' is the exceptional divisor of the induced resolution of singularity $X_m' \to X'$. Under this procedure, the singularities of the embedded surfaces $X \times \{0\} \subset \mathcal{X}$ and $X' \times \{0\} \subset \mathcal{X}'$ are simultaneously resolved, and there is an isomorphism $\varphi_m\colon \mathcal{X}_m' \xrightarrow{\cong} \mathcal{X}_m$ such that $\varphi_m(\mathcal{E}') = \mathcal{E}$. The only irreducible complete curves in $\mathcal{E}$ (in $\mathcal{E}'$, respectively) are of the form $E_i \times \{v\}$ ($E_i' \times \{v'\}$, respectively), where E_i and E_i' are components of E and E', respectively, and $v, v' \in \mathbb{A}^n$. Given such a curve $E_i' \times \{v'\}$ there is a curve $E_{\sigma(i)} \times \{v\}$ such that $\varphi(E_i' \times \{v'\}) = E_{\sigma(i)} \times \{v\}$. It follows that $\varphi(E_i' \times \mathbb{A}^n) = E_{\sigma(i)} \times \mathbb{A}^n$. The image $\varphi(X_m' \times \{v'\})$ is a smooth surface in $\mathcal{X}_m$ which meets the exceptional divisor $\mathcal{E} \subset \mathcal{X}_m$ transversely along the curve $\varphi(E' \times \{v'\}) = E \times \{v\} \subset X_m \times \{v\}$. The same is true for $X_m \times \{v\}$, namely, this is a smooth surface in $\mathcal{X}_m$ which meets $\mathcal{E}$ transversely along the same curve $E \times \{v\}$. Projecting both surfaces to X_m via the canonical projection $\mathcal{X}_m \to X_m$ yields a local isomorphism of the surface germs $(\varphi(X_m' \times \{v'\}), E \times \{v\})$ and $(X_m \times \{v\}, E \times \{v\})$ near the common exceptional divisor $E \times \{v\}$. Contracting the divisor $E \times \{v\}$ yields an isomorphism between the singular germs (X, x) and (X', x'). □

Remind that a toric variety X is called *non-degenerate* if $\mathcal{O}_X(X)^\times = k^*$.

Lemma 6.16. *Under the assumptions of 6.14, X is a non-degenerate toric affine surface.*

Proof. The branch covering construction for X done via the base change $\mathbb{A}^1 \to \mathbb{A}^1$, $z \mapsto z^d$, leads to a GDF μ_d-surface $\tilde{\pi}\colon \tilde{X} \to \mathbb{A}^1$. By Proposition 6.7, $\tilde{\pi}$ is a line bundle over $\mathbb{A}^1$. This bundle is trivial since $\operatorname{Pic}\mathbb{A}^1 = 0$. Taking as a zero section a μ_d-invariant section of $\tilde{\pi}$, one obtains a new μ_d-equivariant line bundle structure. A trivialization gives a μ_d-equivariant isomorphism $\tilde{X} \cong \mathbb{A}^2$, where the cyclic group acts on $\mathbb{A}^2$ via $\zeta\colon (z,u) \mapsto (\zeta z, \zeta^e u)$ $\forall \zeta \in \mu_d$ for some $e \in \{0,\dots,d-1\}$ coprime with d, and with the standard projection $\mathbb{A}^2 \to \mathbb{A}^1$, $(z,u) \mapsto z$. Thus $X \cong \mathbb{A}^2/\mu_d$ is a toric affine surface of type (d,e). □

In the proof of the next lemma we use the notion of the Cox ring (see [2], [15]). Let us recall a simple version of this notion adapted to our particular setting.

Definition 6.17 (*Cox ring*)**.** Let X be a normal affine variety with a cyclic divisor class group $\operatorname{Cl} X' \cong \mathbb{Z}/d\mathbb{Z}$. Let $F \in \operatorname{WDiv} X$ be a Weil divisor such that its class generates $\operatorname{Cl} X$. Consider the $(\mathbb{Z}/d\mathbb{Z})$-graded *Cox ring*

$$\operatorname{Cox}\mathcal{O}_X(X) = \bigoplus_{j=0}^{d-1} H^0(X, \mathcal{O}_X(jF))\zeta^j\,,$$

where ζ is a primitive dth root of unity in $k^\times$. Then $\widetilde{X} = \operatorname{Spec}\operatorname{Cox}\mathcal{O}_X(X)$ is a normal affine surface equipped with a μ_d-action defined by the $(\mathbb{Z}/d\mathbb{Z})$-grading on $\mathcal{O}_{\widetilde{X}}(\widetilde{X}) = \operatorname{Cox}\mathcal{O}_X(X)$. The embedding $\mathcal{O}_X(X) \subset \operatorname{Cox}\mathcal{O}_X(X)$ onto the subalgebra of μ_d-invariants yields the quotient morphism $\widetilde{X} \to X = \widetilde{X}/\mu_d$. We call this a *Cox covering construction*.

If $\mathcal{X} = X \times \mathbb{A}^n$ is the n-cylinder over X, then the divisor class group $\operatorname{Cl}\mathcal{X} \cong \operatorname{Cl} X \cong \mathbb{Z}/d\mathbb{Z}$ is generated by the class of the Weil divisor $\mathcal{F} = F \times \mathbb{A}^n$ on $\mathcal{X}$, see [37, Thm. 8.1]. The Cox covering construction applied to $(\mathcal{X}, \mathcal{F})$ yields the n-cylinder $\widetilde{\mathcal{X}} = \widetilde{X} \times \mathbb{A}^n$ over $\widetilde{X}$.

Lemma 6.18. *Under the assumptions of 6.14, X' is a non-degenerate toric affine surface.*

Proof. Consider the Weil divisor $F_0 = \pi^{-1}(0)$ on X. Its class generates the group $\operatorname{Cl} X \cong \mathbb{Z}/d\mathbb{Z}$. By [3, Thm. 3.1] the branch covering $\mathbb{A}^2 \to X = \mathbb{A}^2/\mu_d$ as in Lemma 6.16 coincides with the Cox covering defined by the pair (X, F_0), see Definition 6.17. Letting $\mathcal{F}_0 = F_0 \times \mathbb{A}^n$ and applying the cyclic covering Cox construction to the pair $(\mathcal{X}, \mathcal{F}_0)$ we obtain the n-cylinder $\widetilde{\mathcal{X}} = \mathbb{A}^2 \times \mathbb{A}^n = \mathbb{A}^{n+2}$. Recall (see [37, Thm. 8.1]; cf. [39, (9.9.8)]) that $\operatorname{Cl} X$ is a cancellation invariant. Hence there are isomorphisms

$$\operatorname{Cl} X' \cong \operatorname{Cl}\mathcal{X}' \cong \operatorname{Cl}\mathcal{X} \cong \operatorname{Cl} X \cong \mathbb{Z}/d\mathbb{Z}\,. \tag{38}$$

Choose a Weil divisor F_0' on X' whose class in $\operatorname{Cl} X'$ gives the class of F_0 in $\operatorname{Cl} X$ via the sequence of isomorphisms (38). Applying the Cox covering construction to the pair (X', F_0') leads to a cyclic μ_d-covering $\widetilde{X}' \to X'$. Letting $\mathcal{F}_0' = F_0' \times \mathbb{A}^n$ and applying the Cox covering construction to the pair $(\mathcal{X}', \mathcal{F}_0')$ yields the n-cylinder $\widetilde{\mathcal{X}}' = \widetilde{X}' \times \mathbb{A}^n$. We claim that $\widetilde{\mathcal{X}}'$ is isomorphic to $\widetilde{\mathcal{X}} \cong \mathbb{A}^{n+2}$. Indeed, we have $\mathcal{F}_0' \sim (\varphi^{-1})^*\mathcal{F}_0$ on $\mathcal{X}'$. However, up to an isomorphism, the Cox covering construction does not depend on the choice of a divisor in the class generating the group $\operatorname{Cl}\mathcal{X}' \cong \mathbb{Z}/d\mathbb{Z}$. Hence, this construction applied to the pair $(\mathcal{X}', (\varphi^{-1})^*\mathcal{F})$ yields a variety isomorphic to $\widetilde{\mathcal{X}}'$. On

the other hand, this variety is also isomorphic to $\widetilde{\mathcal{X}}$, because it is obtained by a similar construction applied to the pair $(\mathcal{X}, \mathcal{F})$.

It follows that $\widetilde{X}' \times \mathbb{A}^n \cong \mathbb{A}^{n+2}$. By the Miyanishi-Sugie-Fujita Theorem ([62, Ch. 3, Thm. 2.3.1]), $\widetilde{X}' \cong \mathbb{A}^2$. Thus, $X' \cong \mathbb{A}^2/\mu_d$. Since any action of a finite group on the affine plane can be linearized (see, e.g., [39, Thm. 2] and the references therein), the result follows. □

The next lemma completes the proof of Theorem 6.8.

Lemma 6.19. *Under the assumptions of* 6.14 *one has* $X' \cong X$.

Proof. Two non-degenerate toric affine surfaces are isomorphic if and only if their singularities are. By Lemma 6.15, the singularities (X, x) and (X', x') are isomorphic. Now the assertion follows from Lemmas 6.16 and 6.18. □

Remark 6.20. It is worth mentioning that there is a somewhat longer proof of Lemma 6.19, which avoids the reference to the difficult Miyanishi-Sugie-Fujita theorem. Namely, one has first to establish an isomorphism of germs (X, x) and (X', x') (see Lemma 6.15), and then consider the multiplicities d and d' of singular fibers of natural projections $X \to \mathbb{A}^1$ and $X' \to \mathbb{A}^1$, respectively. If X' is toric then the local isomorphism of singularities implies that $d' = d$ and, therefore, $X \cong X'$. However, in the case of X' being non-toric, using the local isomorphism of singularities one can show that $d' > d$ (more precisely, X' is an affine modification of a toric surface with a singular fiber of multiplicity $k \geq 2$, and d' is divisible by dk). This implies, as before, that $\operatorname{Cl} X' \cong \operatorname{Cl} \mathcal{X}' \cong \mathbb{Z}/d'\mathbb{Z}$, while $\operatorname{Cl} \mathcal{X} \cong \operatorname{Cl} X \cong \mathbb{Z}/d\mathbb{Z}$. In particular, the cylinders $\mathcal{X}'$ and $\mathcal{X}$ cannot be isomorphic, which yields the desired contradiction.

7. Zariski 1-factors

7.1. **Stretching and rigidity of cylinders.** We use in the sequel the following auxiliary fact.

Lemma 7.1. *Let X be a normal affine surface, that admits an $\mathbb{A}^1$-fibration $X \to C$ over a smooth affine curve C, and let $\bar{X} \to \bar{C}$ be a pseudominimal completion with the corresponding extended graph Γ_{ext}. Then the number $v(\Gamma_{\mathrm{ext}})$ of vertices of Γ_{ext} does not depend on the choice of an $\mathbb{A}^1$-fibration on X over an affine base. So, $v(X) := v(\Gamma_{\mathrm{ext}})$ is an invariant of X.*

Proof. Recall (see [36, Def. 2.16]) that every feather component F of the extended divisor D_{ext} is born under a blowup at a smooth point of the boundary divisor $D = \bar{X} \setminus X$. The unique component D_i of D containing the center of this blowup is called the *mother component* of F. The *normalization procedure* as defined in [36, Def. 3.2] replaces Γ_{ext} by the *normalized extended graph* $\Gamma_{\mathrm{ext,norm}}$, such that any feather component F in Γ_{ext} becomes an extremal (-1)-vertex in $\Gamma_{\mathrm{ext,norm}}$ attached at its mother component D_i. Under this procedure the total number of vertices remains the same: $v(\Gamma_{\mathrm{ext,norm}}) = v(\Gamma_{\mathrm{ext}})$. Furthermore, these graphs are assumed to be *standard*; the standartization procedure does not affect the number of vertices, see [36, §1]. By [36, Thm. 3.5] the standard normalized extended graph $\Gamma_{\mathrm{ext,norm}}$ of X is unique (i.e., does not depend on the choice of an $\mathbb{A}^1$-fibration on X over an affine base) unless X is a *Gizatullin surface*. The latter means that the minimal dual graph Γ of D is linear. In the case of a Gizatullin

surface, $\Gamma_{\mathrm{ext,norm}}$ is unique up to a *reversion* $\Gamma_{\mathrm{ext,norm}} \rightsquigarrow \Gamma^{\vee}_{\mathrm{ext,norm}}$. However, the reversion neither changes the number of vertices in Γ, nor does it in $\Gamma_{\mathrm{ext,norm}}$. The latter is due to the *Matching Principle* ([36, Thm. 3.11]), which establishes a one-to-one correspondence between the feather components of $\Gamma_{\mathrm{ext,norm}}$ and $\Gamma^{\vee}_{\mathrm{ext,norm}}$ along with their mother components. In conclusion, $v(\Gamma_{\mathrm{ext}}) = v(\Gamma_{\mathrm{ext,norm}})$ is an invariant of the surface X. □

Definition 7.2 (*Geometric stretching*)**.** Given an effective divisor $A = \sum_{i=1}^{n} a_i b_i \in \operatorname{Div} B$ and an integer vector $\bar{m} \in \mathbb{Z}^n_{\geq 0}$, a fibered modification $\theta : X' \to X$ between two GDF surfaces $\pi_{X'}\colon X' \to B$ and $\pi_X\colon X \to B$ over the same base will be called a (geometric) $(A, \bar{m})$-*stretching* if its effect on the graph divisors $\mathcal{D}(\pi) \rightsquigarrow \mathcal{D}(\pi')$ is a (combinatorial) $(A, \bar{m})$-stretching as in Definition 2.16.

An $(A, \bar{m})$-stretching will be called a *principal top-level stretching* if it satisfies the conditions

(i) $A = \operatorname{div} f$ is a principal effective divisor, where $f \in \mathcal{O}_B(B)$ is nonzero, and
(ii) $m_i = \operatorname{ht}(\Gamma_{b_i})$ $\forall i = 1, \dots, n$.

Remark 7.3. The action of an effective principal divisor $A = \operatorname{div} f = \sum_{i=1}^{n} a_i b_i \in \operatorname{Div}(B)$, where $f \in \mathcal{O}_B(B)$, on a type divisor $\operatorname{tp}(\mathcal{D}(\pi_X))$ via an $(A, \bar{m})$-stretching with $m_i = -1$ $\forall i = 1, \dots, n$ amounts to asserting in the fiber Γ_{b_i} the chain $[[-1, -2, \dots, -2]]$ of length a_i below the root, so that in the new extended divisor Γ'_{ext}, the (-1)-vertex is a neighbor of the section S. This operation does not affect the GDF surface X, but only its special completion $(\hat{X}, \hat{D})$ as in (10). Indeed, this amounts to perform in (5) an affine (Asanuma-type) modification $L \to X_0 = B \times \mathbb{A}^1$ with divisor $(f \circ \pi)^*(0)$ and center $f^*(0) \times \{0\}$. The latter leads in turn to a trivial line bundle $L \cong_B B \times \mathbb{A}^1$, because, for $\mathbb{A}^1 = \operatorname{Spec} k[u]$,

$$\mathcal{O}_L(L) = \mathcal{O}_B(B)[u/f] = \mathcal{O}_B(B)[u'], \quad \text{where} \quad u' = u/f\,.$$

Thus, performing the remaining fibered modifications in (5) gives again the same surface $X = X_m$.

Lemma 7.4. *Let $\pi\colon X \to B$ be a GDF surface with a graph divisor $\mathcal{D}(\pi) = \sum_{b \in B} \Gamma_b b$. Consider also a pair $(A = \operatorname{div} f, \bar{m})$ satisfying conditions* (i) *and* (ii) *of Definition 7.2. Let $\mathfrak{F}$ be the set of all the top level m_i fiber components $F \subset \pi^{-1}(b_i)$ for $i = 1, \dots, n$. Choose a function $\tilde{u} \in \mathcal{O}_X(X)$ such that*

(i) *$\tilde{u}|_F$ is an affine coordinate on $F \cong \mathbb{A}^1$ for any $F \in \mathfrak{F}$, and*
(ii) *$\tilde{u}|_F = 0$ for any $F \notin \mathfrak{F}$.*

Define $X' = \operatorname{Spec} \mathcal{O}_X(X)[\tilde{u}/f]$. Then $X' \to B$ is a GDF surface, and the morphism $X' \to X$ induced by the embedding $\mathcal{O}_X(X) \hookrightarrow \mathcal{O}_{X'}(X')$ is a principal top-level $(A, \bar{m})$-stretching.

Proof. If $F \notin \mathfrak{F}$, then, clearly, $X' \to X$ is a B-isomorphism over the standard neighborhood U_F in X. For $F \in \mathfrak{F}$ with $\pi(F) = b_i$, in the standard coordinate chart (z, u) in U_F we may assume that $\tilde{u} \equiv u \mod z$ and $f \circ \pi \equiv z^{a_i}$ near F. In this chart $X' \to X$ amounts to an a_i-iterated affine modification with a reduced divisor F and center in the maximal ideal (u, z) and its infinitesimally near points. This corresponds to joining the left end of the chain $L_i = [[-2, \dots, -2, -1]]$ of length a_i to the feather $\bar{F}$ (a tip of Γ_{b_i} on the top level m_i), while decreasing the weight of $\bar{F}$ by 1. Now the assertion follows. □

The following proposition shows that a principal top-level stretching does not affect the B-isomorphism type of the cylinder.

Proposition 7.5. *Let $\pi\colon X \to B$ be a marked GDF surface with a marking $z \in \mathcal{O}_B(B)$, where $z^{-1}(0) = \{b_1, \dots, b_n\}$. Suppose that*

(α) *all the fiber components of $z^{-1}(b_i)$ are of the same (top) level m_i for $i = 1, \dots, n$.*

Let $\theta\colon X' \to X$ be a principal top-level $(A, \bar{m})$-stretching as in Definition 7.2, where $A = \operatorname{div} z^d$, $d \geq 0$. Then the following hold.

(a) *If $\pi\colon X \to B$ is not a line bundle, then $X' \not\cong X$.*

(b) *Given $s > 1$, there is a B-isomorphism of cylinders*

$$\varphi\colon \mathcal{X} \xrightarrow{\cong_B} \mathcal{X}'$$

over X and X', respectively, such that, for any special fiber components F in X and F' in X' with $\varphi(F \times \mathbb{A}^1) = F' \times \mathbb{A}^1$, one has

$$(\varphi|_{U_F \times \mathbb{A}^1})_*\colon (z, u, v) \mapsto (z, u', v') \mod z^{s-1} \tag{39}$$

in the natural coordinates (z, u, v) and (z, u', v') in the standard affine charts $U_F \times \mathbb{A}^1 \subset X \times \mathbb{A}^1$ and $U_{F'} \times \mathbb{A}^1 \subset X' \times \mathbb{A}^1$, respectively.

Proof. (a) Assuming that $\pi\colon X \to B$ is not a line bundle, let us show that $X \not\cong X'$. By Lemma 7.4 the pseudominimal extended graphs Γ_{ext} and Γ'_{ext} of the (minimal) completions $\bar{\pi}\colon \bar{X} \to \bar{B}$ and $\bar{\pi}'\colon \bar{X}' \to \bar{B}$ of X and X', respectively, have different number of vertices. By Lemma 7.1 the latter number is an invariant of the surface. It follows that $X \not\cong X'$, as claimed.

(b) Applying induction on d, it suffices to establish the assertion for $d = 1$. Let $\kappa\colon \mathcal{X}'' \to \mathcal{X}$ be the Asanuma modification of the second kind (see Definition 5.3). By Lemma 5.4(a) there is a B-isomorphism $\mathcal{X} \cong_B \mathcal{X}''$. Thus, it is enough to establish the existence of a B-isomorphism $\mathcal{X}' \cong_B \mathcal{X}''$. Let

$$A = \mathcal{O}_X(X), \quad A' = \mathcal{O}_{X'}(X'), \quad \mathcal{A} = \mathcal{O}_{\mathcal{X}}(\mathcal{X}), \quad \text{and} \quad \mathcal{A}'' = \mathcal{O}_{\mathcal{X}''}(\mathcal{X}''),$$

where $A' = A[\tilde{u}/z]$ and $\mathcal{A}'' = \mathcal{A}[v/z]$. Since

$$\mathcal{O}_{\mathcal{X}'}(\mathcal{X}') = A[\tilde{u}/z][v] = \mathcal{A}[\tilde{u}/z],$$

it suffices to show that $\mathcal{A}[v/z] \cong \mathcal{A}[\tilde{u}/z]$.

By our assumption, all special fiber components in X are of top level in their fibers. Hence by Corollary 4.18 there exists an automorphism $\tau \in \mathrm{SAut}_B\, \mathcal{X}$ such that

$$(\tau|_{U_F \times \mathbb{A}^1})_*\colon (z, u, v) \mapsto (z, v, -u) \mod z^s$$

for any special fiber component F in X, see (27). Since $\tilde{u} \equiv u \mod z^s$ in U_F, we have

$$\tau_*\colon (z, \tilde{u}, v) \mapsto (z, v, -\tilde{u}) \mod z^s. \tag{40}$$

Therefore, τ_* sends the ideal $I = (z, v) \subset \mathcal{A}$ onto the ideal $I' = (z, \tilde{u}) \subset \mathcal{A}$ preserving the principal ideal $(z) = I \cap I'$. By Lemma 1.5, τ_* induces a B-isomorphism of modifications $\mathcal{A}[v/z] \cong_{\mathcal{O}_B(B)} \mathcal{A}[\tilde{u}/z]$, as desired.

Letting $\tilde{u}_1 = \tilde{u}/z$ and $v_1 = v/z$ we obtain by (40)

$$\tau_*\colon (z, \tilde{u}_1, v) \mapsto (z, v_1, -\tilde{u}) \mod z^{s-1}.$$

Let

$$\beta\colon \mathcal{X} \xrightarrow{\cong_B} \mathcal{X}'', \quad (z, \tilde{u}, v) \mapsto (z, \tilde{u}, v_1),$$

be a B-isomorphism as in Lemma 5.4. Letting $\widetilde{\varphi} = \tau^{-1} \circ \beta \colon \mathcal{X} \xrightarrow{\cong_B} \mathcal{X}'$ yields a B-isomorphism of cylinders such that

$$\widetilde{\varphi}_* \colon (z, \tilde{u}, v) \mapsto (z, -v, \tilde{u}_1) \mod z^{s-1}\,.$$

Since all the special fiber components in X' are of top level in their fibers, by Corollary 4.18 there is a B-automorphism $\tau' \in \mathrm{SAut}_B(\mathcal{X}')$ preserving every special affine plane $\mathcal{F}' = F' \times \mathbb{A}^1$ in $\mathcal{X}'$ and such that

$$\tau'_* \colon (z, u', v') \mapsto (z, -v', u') \mod z^{s-1}$$

in the natural coordinates (z, u', v') in the standard affine chart $U_{F'} \times \mathbb{A}^1$. The composition

$$\varphi = (\tau')^3 \circ \widetilde{\varphi} \colon \mathcal{X} \xrightarrow{\cong_B} \mathcal{X}'$$

sends the natural coordinates (z, u, v) in any standard affine chart $U_F \times \mathbb{A}^1 \subset \mathcal{X}$ into the natural coordinates (z, u', v') modulo z^{s-1} in $U_{F'} \times \mathbb{A}^1 \subset \mathcal{X}'$, where $\mathcal{F}' = \varphi(\mathcal{F})$. □

Remark 7.6. The proof in (b) of the fact that $\mathcal{X} \cong_B \mathcal{X}'$ applies actually to a top-level stretching $X' \to X$ defined by an arbitrary principal effective divisor $A = \operatorname{div} f$, $f \in \mathcal{O}_B(B) \setminus \{0\}$, instead of the divisor $A = \operatorname{div} z^d$ defined by a marking z.

Let us illustrate Proposition 7.5 on the example of the Danielewski surfaces.

Example 7.7. Recall that the nth Danielewski surface X_n is given in $\mathbb{A}^3$ with coordinates (z, u, t_n) by equation $z^n t_n - u^2 + 1 = 0$, see Example 3.8. The function $t_n = t_0/z^n$ gives (modulo z) the natural coordinate on each component of the special fiber $z = 0$. Letting $\tilde{u} = t_n$ the morphism $\varrho_n \colon X_n \to X_{n-1}$, $(z, u, t_n) \mapsto (z, u, t_{n-1} = z t_n)$, becomes a stretching, and the composition $X_n \to X_1$ an iterated stretching. Proposition 7.5 provides an alternative proof of Danielewski's theorem [17], which says that the cylinders $X_n \times \mathbb{A}^1$, $n \in \mathbb{N}$, are all isomorphic. The extended graph $\Gamma_{\mathrm{ext,n}}$ as in (11) of a minimal completion $\bar{\pi}_n \colon \bar{X}_n \to \bar{B}$ can be recovered starting with $\Gamma_{\mathrm{ext},0}$ and using Lemma 7.4.

The next result is an equivariant version of Proposition 7.5.

Proposition 7.8. *Let $\pi \colon X \to B$ be a marked GDF μ_d-surface with a μ_d-quasi-invariant marking $z \in \mathcal{O}_B(B)$ of weight 1, where $z^{-1}(0) = \{b_1, \dots, b_n\}$. Suppose that condition (α) of Proposition 7.5 holds, and let $\bar{m} = (m_1, \dots, m_n)$, where $m_i = \mathrm{ht}\,\Gamma_{b_i}$, $i = 1, \dots, n$. Consider a principal top-level μ_d-equivariant $(A, \bar{m})$-stretching $\varrho \colon X^{(d)} \to X$, where $A = \operatorname{div} z^d$, and for any surface X_l in (9), $l = 0, \dots, N$, a similar stretching $X_l^{(d)} \to X_l$. Then for any $k \in \mathbb{Z}$ there exists a μ_d-equivariant B-isomorphism of cylinders $\varphi \colon \mathcal{X}_l(k) \xrightarrow{\cong_B} \mathcal{X}_l^{(d)}(k)$ satisfying (39). Furthermore, $X_l^{(d)} \not\cong X_l$ provided that $\pi_l \colon X_l \to B$ is not a line bundle.*

Proof. Recall that the sequence $(\mathcal{X}_l(k))_{k \in \mathbb{Z}}$ is periodic with period d, where a μ_d-equivariant B-isomorphism $\eta \colon \mathcal{X}_l(k-d) \xrightarrow{\cong_B} \mathcal{X}_l(k)$ is provided by the iterated Asanuma modification of the second type

$$\mathcal{A}_l(k) \hookrightarrow \mathcal{A}_l(k)[v/z^d] \cong_B \mathcal{A}_l(k-d), \quad \text{where} \quad \mathcal{A}_l(k) = \mathcal{O}_{\mathcal{X}_l(k)}(\mathcal{X}_l(k))\,,$$

see Lemma 5.4(b). By Corollary 4.18 one can choose an μ_d-equivariant automorphism $\tau \in \mathrm{SAut}\,\mathcal{X}_l(-l)$ interchanging $\tilde{u}$ and v modulo z^s (up to a sign) and leaving z invariant, where $\tilde{u} \in A_l = \mathcal{O}_{X_l}(X_l)$ is a μ_d-quasi-invariant function of weight $-l$, which yields (modulo z^s) the natural coordinate u on any special fiber component F in X_l. Then

the argument in the proof of Proposition 7.5 gives the existence of a μ_d-equivariant isomorphism

$$\mathcal{A}_l(-l)[v/z^d] \cong \mathcal{A}_l(-l)[\tilde{u}/z^d]\,,$$

where

$$\operatorname{Spec}\mathcal{A}_l(-l)[v/z^d] = \mathcal{X}_l(-l-d) \cong_{\mu_d} \mathcal{X}_l(-l)$$

and

$$\operatorname{Spec}\mathcal{A}_l(-l)[\tilde{u}/z^d] = \operatorname{Spec}A_l[\tilde{u}/z^d][v](-l) = \mathcal{X}_l^{(d)}(-l)\,.$$

It follows that

$$\mathcal{X}_l(-l) \cong_{\mu_d} \mathcal{X}_l^{(d)}(-l)\,.$$

Let $n \in \mathbb{N}$ be sufficiently large so that $nd > k + l$. Applying on the both sides the $(nd-k-l)$-iterated Asanuma modification of the second kind, by Lemma 5.4(b) we obtain finally a desired μ_d-equivariant isomorphism

$$\mathcal{X}_l(k) \cong_{\mu_d} \mathcal{X}_l^{(d)}(k)\,.$$

The non-isomorphism $X_l^{(d)} \not\cong X_l$ was established in Proposition 7.5(a). □

7.2. **Non-cancellation for GDF surfaces.** The main result of this section is the following theorem.

Theorem 7.9. *Let $\pi\colon X \to B$ be a GDF surface. Then the surface X is a Zariski 1-factor if and only if $\pi\colon X \to B$ is a line bundle.*

Proof. The 'if' part follows from Proposition 6.1, and the 'only if' part from the next version of Proposition 7.5, which does not assume condition (α). □

Proposition 7.10. *Let $\pi\colon X \to B$ be a marked GDF surface with a marking $z \in \mathcal{O}_B(B)$, where $z^{-1}(0) = \{b_1, \dots, b_n\}$. Suppose that $\pi\colon X \to B$ is not a line bundle, and $\pi_1\colon X_1 \to B$ in (7) is not a line bundle either*[17]*. Let $\theta\colon X_1' \to X_1$ be a principal top-level $(A, \bar{m})$-stretching, where $A = \operatorname{div} z^d$, $d \geq 1$, and $\bar{m} = (m_1, \dots, m_n)$ with $m_i = \operatorname{ht}\Gamma_{b_i}(\hat{X}_1)$. Then there exists a sequence of GDF surfaces*

$$X_m' \xrightarrow{\varrho_m'} X_{m-1}' \longrightarrow \dots \longrightarrow X_2' \xrightarrow{\varrho_2'} X_1' \tag{41}$$

similar to (7) without its first term X_0 such that the cylinders $\mathcal{X}_i$ and $\mathcal{X}_i'$ over X_i and X_i', respectively, are B-isomorphic, and $\operatorname{tp}(\mathcal{D}(\pi_{X_i'})) = A \cdot \operatorname{tp}(\mathcal{D}(\pi_{X_i}))$, while $X_i' \not\cong X_i$ for any $i = 1, \dots, m$.

Proof. The GDF surface $\pi_0\colon X_0 \to B$ in (7) is a line bundle. So, the corresponding graph divisor $\mathcal{D}(\pi_0)$ is a chain divisor. It is easily seen that $\pi_1\colon X_1 \to B$ satisfies condition (α), while $\mathcal{D}(\pi_1)$ is not a chain divisor. If the fiber graph Γ_{b_i} is not a chain, then its minimization is not a chain either. Hence the pseudominimal extended divisors $\Gamma_{\mathrm{ext}}(\hat{X}_1)$ and $\Gamma_{\mathrm{ext}}(\hat{X}_1')$ have different number of vertices. By Lemma 7.1, the surfaces X_1 and X_1' are not isomorphic. By Proposition 7.5, the cylinders $\mathcal{X}_1$ and $\mathcal{X}_1'$ over X_1 and X_1', respectively, are B-isomorphic. Given $s \gg 1$, let $\varphi\colon \mathcal{X}_1 \xrightarrow{\cong_B} \mathcal{X}_1'$ be a B-isomorphism as in Proposition 7.5. By (39), φ sends the special fiber components of $X_1 \times \{0\}$ to such components of $X_1' \times \{0\}$. If $F_1, \dots, F_m$ are the special fiber components in X_1, then for any $i = 1, \dots, m$ there is a unique special fiber component F_i' in X_1' born under the

[17] Otherwise (7) can be shorten.

stretching $\theta\colon X_1' \to X_1$ as the result of a d-iterated blowing up with center over a point of F_i. Then $\varphi(F_i \times \{0\}) = F'_{\sigma(i)} \times \{0\}$ for some permutation σ of the indices $\{1, \dots, m\}$.

Proceeding by recursion, we will construct a sequence (41).

Let $\Sigma_i \subset F_i$ be the set of centers of blowups on F_i in the affine modification $\varrho_2\colon X_2 \to X_1$ in (7), let $\Sigma = \bigcup_i \Sigma_i$, $\mathcal{F} = \bigcup_{\Sigma_i \neq \varnothing} F_i$, and $\mathcal{Z} = \Sigma \times \{0\} \subset \mathcal{F} \times \{0\}$. Let $\Sigma'_{\sigma(i)} \times \{0\} = \varphi(\Sigma_i \times \{0\}) \subset F'_{\sigma(i)} \times \{0\}$, $i = 1, \dots, m$, and let $\mathcal{Z}' = \varphi(\mathcal{Z}) = \Sigma' \times \{0\}$, where $\Sigma' \subset \mathcal{F}' := \bigcup_{F_i \subset \mathcal{F}} F'_{\sigma(i)} \subset X_1'$.

Let $\varrho_2'\colon X_2' \to X_1'$ be the geometric affine modification with center $\Sigma' \subset \mathcal{F}'$ and divisor $\mathcal{F}'$, see Remark 1.4.3. For an algebraic modification which yields ϱ_2', one may take the affine modification with center $\Sigma' \cup \bigcup_{F_i' \not\subset \mathcal{F}'} F_i'$ and divisor $z^*(0) \subset X_1'$. Let $\tilde{\varrho}_2\colon \mathcal{X}_2 \to \mathcal{X}_1$ and $\tilde{\varrho}_2'\colon \mathcal{X}_2' \to \mathcal{X}_1'$ be the Asanuma modifications of the first kind as in Lemma 5.1(a), which correspond to ϱ_2 and ϱ_2', respectively. Since the isomorphism $\varphi_1 := \varphi\colon \mathcal{X}_1 \longrightarrow \mathcal{X}_1'$ sends the center $\mathcal{Z}$ and the divisor $\mathcal{D}$ of the affine modification $\tilde{\varrho}_2$ to the center $\mathcal{Z}'$ and the divisor $\mathcal{D}'$ of $\tilde{\varrho}_2'$, by Lemma 1.5 it lifts to an isomorphism $\varphi_2\colon \mathcal{X}_2 \longrightarrow \mathcal{X}_2'$ of the cylinders over X_2 and X_2', respectively. By Lemma 7.1, the surfaces X_2 and X_2' are not isomorphic, since the numbers of vertices in the corresponding pseudominimal extended graphs Γ_{ext} and Γ'_{ext} are different. Inspecting the proof of Lemma 1.9 shows that φ_2 satisfies again (39) with the exponent $s-1$ replaced by $s-2$. Now we can apply the same argument to the isomorphism $\varphi_2\colon \mathcal{X}_2 \longrightarrow \mathcal{X}_2'$ instead of $\varphi_1\colon \mathcal{X}_1 \longrightarrow \mathcal{X}_1'$. By recursion, we arrive at a sequence (41) with desired properties. □

Next we give an equivariant version of Proposition 7.10.

Proposition 7.11. *Let $\pi\colon X \to B$ be a marked GDF μ_d-surface which is not a line bundle. Then there exists a sequence of pairwise non-isomorphic, marked GDF μ_d-surfaces $X^{(nd)}$ with μ_d-equivariantly B-isomorphic cylinders: $X \times \mathbb{A}^1 \cong_{\mu_d, B} X^{(nd)} \times \mathbb{A}^1$ $\forall n \in \mathbb{N}$.*

Proof. Following the lines of the proof of Proposition 7.10, consider sequence (7) with $X_m = X$ and the marked GDF μ_d-surface $\pi_1\colon X_1 \to B$ in (7). Proceeding as in the proof of Proposition 7.10, for $n \in \mathbb{N}$ we let $X_1^{(nd)}$ be the GDF μ_d-surface obtained from X_1 via a principal top-level $(A, \bar{m})$-stretching, where $A = \operatorname{div} z^{nd}$. By Lemma 7.4, the number of vertices of the corresponding pseudominimal extended graph $\Gamma^{(nd)}_{\mathrm{ext},1}$ strictly grows to infinity with n. Hence, by Lemma 7.1, the GDF surfaces $X_1^{(nd)}$, $n = 1, 2, \dots$, are pairwise non-isomorphic.

By Proposition 7.8, given $s > 1$ there is a μ_d-equivariant isomorphism of cylinders $\varphi\colon \mathcal{X}_1(-1) \xrightarrow{\cong_B} \mathcal{X}_1^{(nd)}(-1)$ satisfying an analog of (39). Lemma 4.15 provides a μ_d-equivariant automorphism $\alpha' \in \mathrm{SAut}_B\, \mathcal{X}_1^{(nd)}(-1)$ such that the B-isomorphism $\psi_1 = \alpha' \circ \varphi_1\colon \mathcal{X}_1(-1) \xrightarrow{\cong_B} \mathcal{X}_1^{(nd)}(-1)$ sends the natural coordinates (z, u, v) in any standard affine chart $U_F \times \mathbb{A}^1 \subset \mathcal{X}_1(-1)$ into the natural coordinates (z, u', v') modulo z^{s-1} in $U_{F'} \times \mathbb{A}^1 \subset \mathcal{X}_1^{(nd)}(-1)$, where $F' \times \mathbb{A}^1 = \varphi_1(F \times \mathbb{A}^1)$. It follows that $F' = \psi_1(F)$. Repeating for any $n \in \mathbb{N}$ in a μ_d-equivariant fashion the construction from the proof of Proposition 7.10, we arrive at a sequence of pairwise non-isomorphic marked GDF μ_d-surfaces $X^{(nd)} = X_m^{(nd)}$ with μ_d-equivariantly B-isomorphic cylinders $\mathcal{X}_m^{(nd)}(-m) \cong_B \mathcal{X}_m(-m)$. Applying now the $(Nd - m)$-iterated Asanuma modification of the second kind with $N \in \mathbb{N}$ sufficiently large, by Lemma 5.4(b) we obtain finally a desired μ_d-equivariant B-isomorphism $\mathcal{X}(0) \cong_{(\mu_d, B)} \mathcal{X}^{(nd)}(0)$. □

7.3. **Extended graphs of Gizatullin surfaces.** Here we recall some facts on Gizatullin surfaces, which will be used in the sequel. We start by introducing the notation for the covering trick extended to completions.

Notation 7.12. Let $\pi_Y\colon Y \to C$ be an $\mathbb{A}^1$-fibration over an affine curve C, $\bar{Y} \to \bar{C}$ be a completion of $\pi_Y\colon Y \to C$, and $\bar{\pi}_Y\colon \bar{Y}_{\mathrm{resolved}} \to \bar{C}$ be a pseudominimal resolved completion of $\pi_Y\colon Y \to C$ with associated extended graph $\Gamma = \Gamma_{\mathrm{ext}}$. Contracting the exceptional divisor E of the minimal resolution of singularities of Y yields a birational morphism $\sigma\colon \bar{Y}_{\mathrm{resolved}} \to \bar{Y}$. Given a branched covering $B \to C$ as in 2.2, consider the normalizations of the cross-products $\bar{Y}_{\mathrm{resolved}} \times_{\bar{C}} \bar{B}$ and $\bar{Y} \times_{\bar{C}} \bar{B}$, the respective minimal desingularizations $\hat{X}_{\mathrm{resolved}} \to (\bar{Y}_{\mathrm{resolved}} \times_{\bar{C}} \bar{B})_{\mathrm{norm}}$ and $\hat{X} \to (\bar{Y} \times_{\bar{C}} \bar{B})_{\mathrm{norm}}$, and the induced $\mathbb{P}^1$-fibrations $\hat{X}_{\mathrm{resolved}} \to \bar{B}$ and $\hat{X} \to \bar{B}$. Recall that a similar branch covering construction starting with $\pi_Y\colon Y \to C$ leads to the GDF surface $\pi_X\colon X \to B$ as in Definition 2.2, where X is a smooth surface by Lemma 2.24(b). Hence, $\hat{X} \to \bar{B}$ is a completion of $X \to B$ dominated by $\hat{X}_{\mathrm{resolved}} \to \bar{B}$. The induced morphism $\hat{\sigma}\colon \hat{X}_{\mathrm{resolved}} \to \hat{X}$ contracts the preimage of the exceptional divisor E of $\sigma\colon \bar{Y}_{\mathrm{resolved}} \to \bar{Y}$.

Definition 7.13 (*Gizatullin surfaces*)**.** Recall (see e.g., [36]) that a Gizatullin surface X is a normal affine surface of class (ML_0) [18]. Such a surface X admits at least two $\mathbb{A}^1$-fibrations over the affine line. Actually, the base C of any $\mathbb{A}^1$-fibration $\pi\colon X \to C$ is isomorphic to $\mathbb{A}^1$, provided C is a smooth affine curve. Such a fibration has at most one degenerate fiber. One may assume that this is the fiber $\pi^{-1}(0)$.

In the proof of Theorem 7.16 we use the following fact.

Lemma 7.14. *Given a Gizatullin surface X, the following hold.*

(a) *The set $\Omega(X)$ of isomorphism classes of the pseudominimal extended graphs Γ_{ext} of all possible $\mathbb{A}^1$-fibrations $X \to \mathbb{A}^1$ is a finite set. Furthermore, there exists $d \in \mathbb{N}$ such that the multiplicities of the fiber components in any such fibration divide d.*

(b) *For any $\mathbb{A}^1$-fibration $X \to C = \mathbb{A}^1$, consider a branch covering construction with a cyclic Galois base change $B = \mathbb{A}^1 \to C$ of degree d with d as in (a), the resulting $\mathbb{A}^1$-fibration $\tilde{X} \to B$, and the associated pseudominimal extended graph $\tilde{\Gamma}_{\mathrm{ext}}$. Then the set $\tilde{\Omega}(X, d)$ of isomorphism classes of all such graphs is finite.*

Proof. By Lemma 7.1 all graphs in $\Omega(X)$ have the same number $v(X)$ of vertices. However, the number of non-isomorphic graphs on a given set of vertices is finite. Furthermore, given an $\mathbb{A}^1$-fibration $\pi\colon X \to \mathbb{A}^1$, the multiplicities of the fiber components of π can be deduced in a combinatorial way from the associated extended graph Γ_{ext}. Since the number of all such graphs in $\Omega(X)$ is finite, there is $d \in \mathbb{N}$ divisible by all such multiplicities for all possible $\mathbb{A}^1$-fibrations $\pi\colon X \to \mathbb{A}^1$. This yields (a).

To show (b) it suffices, by virtue of (a), to restrict just to the $\mathbb{A}^1$-fibrations on X with a fixed pseudominimal extended graph $\Gamma_{\mathrm{ext}} = \Gamma$. Let $\pi\colon X \to C = \mathbb{A}^1$ be such an $\mathbb{A}^1$-fibration. Recall that π has at most one degenerate fiber. Up to a choice of a coordinate z in $\mathbb{A}^1$ one may assume that this fiber is $\pi^{-1}(0)$ and the base change $B \to C$ is $z \mapsto z^d$.

In notation 7.12, the extended graph $\hat{\Gamma}_{\mathrm{ext}}$ is dominated by $\hat{\Gamma}_{\mathrm{ext,resolved}}$. In turn, the pseudominimal extended graph $\tilde{\Gamma}_{\mathrm{ext}}$ in $\tilde{\Omega}(X, d)$ is dominated by $\hat{\Gamma}_{\mathrm{ext}}$ and also by

[18] Thus, in our definition X is different from $\mathbb{A}^1_* \times \mathbb{A}^1$, were $\mathbb{A}^1_* = \mathbb{A}^1 \setminus \{0\}$.

$\hat{\Gamma}_{\mathrm{ext,resolved}}$. This yields an upper bound $v(\tilde{\Gamma}_{\mathrm{ext}}) \le v(\hat{\Gamma}_{\mathrm{ext,resolved}})$ for the number of vertices. We claim that $v(\hat{\Gamma}_{\mathrm{ext,resolved}})$ is bounded above by a function depending only on d and on Γ_{ext}, and so, only on X, as desired.

To show the claim, note that for any vertex of $\Gamma_{\mathrm{ext}} = \Gamma$ there is at most d vertices of $\hat{\Gamma}_{\mathrm{ext,resolved}}$ such that the corresponding curves in $\hat{X}_{\mathrm{resolved}}$ dominate the one in $\bar{X}_{\mathrm{resolved}}$. Hence it suffices to find an upper bound on the number of remaining vertices of $\hat{\Gamma}_{\mathrm{ext,resolved}}$, which correspond to the curves in $\hat{X}_{\mathrm{resolved}}$ contracted in $\bar{X}_{\mathrm{resolved}}$.

Let E' and E'' be two fiber components of the extended divisor D_{ext} that meet in $\bar{X}$, with respective multiplicities m' and m''. Let (x, y) be local coordinates in an analytic chart in $\bar{X}$ centered at the intersection point $E' \cap E'' = \{p\}$ with E' and E'' as the axes. Then the local equation of a germ of the cross-product $\bar{X} \times_{\bar{C}} \bar{B}$ near p is given by equation $z^d - x^{m'}y^{m''} = 0$. Likewise, letting $E'' = S$ the germ of $\bar{X} \times_{\bar{C}} \bar{B}$ near p is given by equation $z^d = zx^{m'}$. In both cases, normalizing such a surface germ produces several (anyway, $\le d$) cyclic quotient singularities of type uniquely determined by d, m', and m''. The resolution graphs of these singular points, that are Hirzebruch-Jung strings, are also uniquely determined by d and Γ. It follows that the number of vertices in the total preimage in $\hat{\Gamma}_{\mathrm{ext,resolved}}$ of the edge $[E, E']$ of Γ_{ext} is bounded above in terms of d and Γ. Finally, $v(\hat{\Gamma}_{\mathrm{ext,resolved}})$ is bounded above by a function of d and Γ, as claimed. □

Remarks 7.15. 1. There is a remarkable sequence $(X_n)_{n\in\mathbb{N}}$ of Gizatullin surfaces, called the *Danilov-Gizatullin surfaces*. In [35, Thms. 1.0.1, 1.0.5, and Ex. 6.3.21] we constructed deformation families of pairwise non-equivalent (modulo the $\mathrm{Aut}\, X_n$-action) $\mathbb{A}^1$-fibrations $X_n \to \mathbb{A}^1$ on a given Danilov-Gizatullin surface X_n with bases of dimensions growing with n. We wonder whether there exists such a collection of families with bases of infinitely growing dimensions on some fixed Gizatullin surface X. Lemma 7.14 indicates that the negative answer is more plausible.

2. See Part II for a more thorough analysis of the extended graph $\hat{\Gamma}_{ext,resolved}$.

7.4. **Zariski 1-factors and affine $\mathbb{A}^1$-fibered surfaces.** The following is one of the main results of this section.

Theorem 7.16. *Let $\pi\colon X \to C$ be a normal $\mathbb{A}^1$-fibered affine surface over a smooth affine curve C. Then X is a Zariski 1-factor if and only if $\pi\colon X \to C$ is a parabolic $\mathbb{G}_m$-surface.*

The "if" part follows from Theorem 6.8 (see also Proposition 6.1 for the smooth case). The "only if" part is proven in the next proposition.

Proposition 7.17. *Let $\pi\colon X \to C$ be an $\mathbb{A}^1$-fibration on a normal affine surface X over a smooth affine curve C. If X is a Zariski 1-factor, then $\pi\colon X \to C$ admits a structure of a parabolic $\mathbb{G}_m$-surface.*

Proof. Consider all possible $\mathbb{A}^1$-fibrations $X \to Z$ on X over smooth affine curves, along with the corresponding pseudominimal extended graphs Γ_{ext}. We claim that the set $\Omega(X)$ of the isomorphism classes of such graphs is finite. Indeed, if there are at least two different such fibrations, then X is a Gizatullin surface and $Z \cong \mathbb{A}^1$. Now the claim follows by Lemma 7.14(a).

Let d be the least common multiple of the multiplicities of the multiple fibers in the $\mathbb{A}^1$-fibrations $X \to Z$. This number can be deduced in a combinatorial way from the extended graphs Γ_{ext} in $\Omega(X)$, hence it is finite.

By Lemma 2.3, applying to the given $\mathbb{A}^1$-fibration $X \to C$ a suitable Galois base change $B \to C$ with the Galois group μ_d and a subsequent normalization, we obtain a marked GDF μ_d-surface $\tilde{X} \to B$. Suppose that $X \to C$ is not a parabolic $\mathbb{G}_m$-surface. Then by Proposition 6.7, $\tilde{X} \to B$ is not a line bundle. We are going to construct an infinite sequence of normal affine surfaces $X^{(nd)}$ non-isomorphic to X such that the cylinders $X^{(nd)} \times \mathbb{A}^1$ and $X \times \mathbb{A}^1$ are isomorphic, thus showing that X cannot be a Zariski 1-factor.

Indeed, by Proposition 7.11 there is a sequence of pairwise non-isomorphic marked GDF μ_d-surfaces $\tilde{X}^{(nd)} \to B$ such that for all $n \in \mathbb{N}$ the cylinders $\tilde{\mathcal{X}}^{(nd)}(0)$ and $\tilde{\mathcal{X}}(0)$ are μ_d-equivariantly B-isomorphic, while $v(\tilde{X}^{(nd)}) \to \infty$ as $n \to \infty$. Passing to the quotients under the μ_d-action yields a sequence of $\mathbb{A}^1$-fibered normal affine surfaces $X^{(nd)} = \tilde{X}^{(nd)}/\mu_d \to C$ with isomorphic over C cylinders:

$$X^{(nd)} \times \mathbb{A}^1 = \tilde{\mathcal{X}}^{(nd)}(0)/\mu_d \cong_C \tilde{\mathcal{X}}(0)/\mu_d = X \times \mathbb{A}^1, \quad \forall n \in \mathbb{N}.$$

We claim that under our assumptions, for all $n \in \mathbb{N}$ sufficiently large, the surfaces $X^{(nd)}$ are not isomorphic to X.

To show the claim, suppose to the contrary that $X^{(nd)} \cong X$ for an infinite set, say, I of values of $n > 1$. Then X admits at least two different $\mathbb{A}^1$-fibrations over affine bases, hence, $X \in (\mathrm{ML}_0)$ is a Gizatullin surface. Indeed, otherwise any isomorphism $\varphi\colon X^{(nd)} \xrightarrow{\cong} X$ sends the $\mathbb{A}^1$-fibration $X^{(nd)} \to C$ to the unique $\mathbb{A}^1$-fibration $X \to C$. So, after the base change $B \to C$ it can be lifted to an isomorphism of the normalized cyclic coverings $\tilde{\varphi}\colon \tilde{X}^{(nd)} \xrightarrow{\cong} \tilde{X}$. This gives a contradiction, since $v(\tilde{X}^{(nd)}) > v(\tilde{X})$.

Thus, under our assumptions X and also $X^{(nd)} \cong X$, $n \in I$, are Gizatullin surfaces different from $\mathbb{A}^1_* \times \mathbb{A}^1$. Hence $C \cong \mathbb{A}^1$. One may suppose that the covering $B \to C = \mathbb{A}^1$ is unramified off the origin, and so, $B \cong \mathbb{A}^1$ too. By Lemma 7.14 the set $\tilde{\Omega}(X, d) = \tilde{\Omega}(X^{(nd)}, d)$, $n \in I$, is finite. In particular, for any $n \in I$ the pseudominimal extended graph $\tilde{\Gamma}^{(nd)}_{\mathrm{ext}}$ associated with the GDF surface $\tilde{\pi}\colon \tilde{X}^{(nd)} \to B$ (which is a cyclic cover over $X^{(nd)}$) belongs to the finite set $\tilde{\Omega}(X^{(nd)}, d) = \tilde{\Omega}(X, d)$. Since the set $I \subset \mathbb{N}$ is infinite, this contradicts the fact that $v(\tilde{\Gamma}^{(nd)}_{\mathrm{ext}}) = v(\tilde{X}^{(nd)}) \to \infty$ as $n \to \infty$. Hence $X \ncong X^{(nd)}$ for all $n \gg 1$, as required. □

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