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HAVIN-MAZYA TYPE UNIQUENESS THEOREM FOR DIRICHLET SPACES

H. BAHAJJI-EL IDRISSE, O. EL-FALLAH, AND K. KELLAY

ABSTRACT. Let μ be a positive finite Borel measure on the unit circle. The associated Dirichlet space $\mathcal{D}(\mu)$ consists of holomorphic functions on the unit disc whose derivatives are square integrable when weighted against the Poisson integral of μ . We give a sufficient condition on a Borel subset E of the unit circle which ensures that E is a uniqueness set for $\mathcal{D}(\mu)$. We also give some examples of positive Borel measures μ and uniqueness sets for $\mathcal{D}(\mu)$.

1. INTRODUCTION

Let μ be a positive finite Borel measure on the unit circle \mathbb{T} , the *harmonically weighted Dirichlet space* $\mathcal{D}^h(\mu)$ is the set of all functions $f \in L^2(\mathbb{T})$ for which

$$\mathcal{D}_\mu(f) = \int_{\mathbb{T}} \mathcal{D}_\xi(f) d\mu(\xi) < \infty,$$

where $\mathcal{D}_\xi(f)$ is the local Dirichlet integral of f at $\xi \in \mathbb{T}$ given by

$$\mathcal{D}_\xi(f) := \int_{\mathbb{T}} \left| \frac{f(\zeta) - f(\xi)}{\zeta - \xi} \right|^2 \frac{|d\zeta|}{2\pi}.$$

The space $\mathcal{D}^h(\mu)$ is endowed with the norm

$$\|f\|_\mu^2 := \|f\|_{L^2(\mathbb{T})}^2 + \mathcal{D}_\mu(f).$$

The analytic weighted Dirichlet space $\mathcal{D}(\mu)$ is defined by

$$\mathcal{D}(\mu) = \{f \in \mathcal{D}^h(\mu) : \widehat{f}(n) = 0, n < 0\}.$$

Let \mathbb{D} be the open unit disc in the complex plane. Let H^2 denote the Hardy space of analytic functions on \mathbb{D} . As usual, we use the identification $H^2 = \{f \in L^2(\mathbb{T}) : \widehat{f}(n) = 0, n < 0\}$. Since $\mathcal{D}(\mu) \subset H^2$, every function $f \in \mathcal{D}(\mu)$ has non-tangential limits almost everywhere on \mathbb{T} . We denote by $f(\zeta)$ the non-tangential limit of f at $\zeta \in \mathbb{T}$ if it exists. Note that

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by Douglas Formula the Dirichlet-type space $\mathcal{D}(\mu)$ is the set of analytic functions $f \in H^2$, such that

$$\int_{\mathbb{D}} |f'(z)|^2 P_\mu(z) dA(z) < \infty,$$

where $dA(z) = dx dy / \pi$ stands for the normalized area measure in \mathbb{D} and P_μ is the Poisson integral of μ given by

$$P_\mu(z) := \int_{\mathbb{T}} \frac{1 - |z|^2}{|\zeta - z|^2} d\mu(\zeta), \quad z \in \mathbb{D}.$$

For a proof of this fact see [4, Theorem 7.2.5] and for numerous results on the Dirichlet-type space and operators acting thereon see [4, 6, 13, 14, 15, 16].

The capacity associated with $\mathcal{D}(\mu)$ is denoted by c_μ and is given by

$$c_\mu(E) := \inf \{ \|f\|_\mu^2 : f \in \mathcal{D}^h(\mu), |f| \geq 1 \text{ a.e. on a neighborhood of } E \}.$$

Since the L^2 norm is dominated by the Dirichlet norm $\|\cdot\|_\mu$, it is obvious that c_μ -capacity 0 implies Lebesgue measure 0. We say that a property holds c_μ -quasi-everywhere (c_μ -q.e.) if it holds everywhere outside a set of zero c_μ capacity. Note that c_μ -q.e. implies a.e. and we have

$$c_\mu(E) = \inf \{ \|f\|_\mu^2 : f \in \mathcal{D}^h(\mu), |f| \geq 1 \text{ } c_\mu\text{-q.e. on } E \}.$$

For more details see [8]. Furthermore every function $f \in \mathcal{D}(\mu)$ has non-tangential limits c_μ -q.e. on \mathbb{T} [3, Theorem 2.1.9]. Note also that if E is a closed subset of \mathbb{T} such that $c_\mu(E) = 0$, then there exists a function $f \in \mathcal{D}(\mu)$ uniformly continuous on \mathbb{T} such that the zero set $\mathcal{Z}(f) := \{\zeta \in \mathbb{T} : f(\zeta) = 0\} = E$ [7, Theorem 1].

Recall that if $I \subset \mathbb{T}$ be the arc of length $|I| = 1 - \rho$ with midpoint $\zeta \in \mathbb{T}$, then

$$\frac{1}{c_\mu(I)} \asymp 1 + \int_0^\rho \frac{dr}{(1-r)P_\mu(r\zeta) + (1-r)^2}, \quad (1)$$

where the implied constants are absolute. For the proof see [5, Theorem 2].

Let E be a subset of \mathbb{T} , the set E is said to be a uniqueness set for $\mathcal{D}(\mu)$ if, for each $f \in \mathcal{D}(\mu)$ such that its non-tangential limit $f = 0$ c_μ -q.e. on E , we have $f = 0$.

It is well known that for the Hardy space the uniqueness sets coincide with the sets of positive length on \mathbb{T} . Note that if $d\mu = dm$ the normalized arc measure on \mathbb{T} , then the space $\mathcal{D}(\mu)$ coincides with the classical Dirichlet space \mathcal{D} given by

$$\mathcal{D} = \{f \in H^2 : \int_{\mathbb{D}} |f'(z)|^2 dA(z) < \infty\}.$$

In this case, c_m is comparable to the logarithmic capacity and $c_m(I) \asymp |\log |I||^{-1}$ for every arc $I \subset \mathbb{T}$, [12, Theorem 14].

Khavin and Maz'ya proved in [10] that a Borel subset E of \mathbb{T} is a uniqueness set for \mathcal{D} if there exists a family of pairwise disjoint open arcs (I_n) such that

$$\sum_n |I_n| \log \frac{|I_n|}{c_m(E \cap I_n)} = -\infty.$$

For other uniqueness results for \mathcal{D} see also [1, 2, 9, 4, 11].

Our aim in this paper is to extend Khavin and Maz'ya uniqueness theorem to general Dirichlet spaces $\mathcal{D}(\mu)$.

Let $\gamma > 1$ and let $I = (e^{ia}, e^{ib})$. The arc γI is given by

$$\gamma I = (e^{i(a-(\gamma-1)\frac{b-a}{2})}, e^{i(b+(\gamma-1)\frac{b-a}{2})}).$$

The main result of this paper is the following theorem.

Theorem 1.1. *Let E be a Borel subset of \mathbb{T} . Suppose that there exists a family of open arcs (I_n) such that (γI_n) are pairwise disjoint for some $\gamma > 1$ and*

$$\sum_n |I_n| \log \frac{|I_n|}{c_\mu(E \cap I_n)} = -\infty;$$

then E is a uniqueness set for $\mathcal{D}(\mu)$.

The key of the proof of the this theorem is an upper estimate of the average

$$\frac{1}{|I|} \int_I |f(\zeta)| |d\zeta|,$$

for $f \in \mathcal{D}(\mu)$ vanishing on a set $E \subset \mathbb{T}$, in terms of capacity of $E \cap I$, for any open arc I .

The next section is devoted to the proof of Theorem 1.1. In section 3 we give some examples of positive Borel measures μ and uniqueness sets for $\mathcal{D}(\mu)$.

Throughout the paper, we use the following notations: $A \lesssim B$ means that there is an absolute constant C such that $A \leq CB$ and $A \asymp B$ if both $A \lesssim B$ and $B \lesssim A$.

2. PROOF

First, let us introduce some notations which will be useful in the sequel. Let J and L be arcs of \mathbb{T} and let f be a Borel function defined on \mathbb{T} . We set

$$\mathcal{D}_{J,L,\mu}(f) := \int_{\zeta \in J} \int_{\xi \in L} \frac{|f(\zeta) - f(\xi)|^2 |d\zeta|}{|\zeta - \xi|^2} \frac{1}{2\pi} d\mu(\xi),$$

and

$$\langle f \rangle_J := \frac{1}{|J|} \int_J |f(\xi)| |d\xi|.$$

The following lemma is the key in the proof of theorem 1.1.

Lemma 2.1. *Let $\gamma > 1$ and let $f \in \mathcal{D}(\mu)$ such that $f|_E = 0$ for some Borel subset E of \mathbb{T} . Then, for any open arc $I \subset \mathbb{T}$*

$$\langle f \rangle_I^2 \leq \kappa \frac{\mathcal{D}_{\gamma I, \mathbb{T}, \mu}(f) + \int_{\gamma I} |f|^2}{c_\mu(E \cap I)},$$

where κ is a constant depending only on γ .

Proof. Without loss of generality, we can suppose that $I = (e^{-i\theta}, e^{i\theta})$ with $2\theta < \pi$. In this case $\gamma I = (e^{-i\gamma\theta}, e^{i\gamma\theta})$.

Let ϕ be a positive function on \mathbb{T} , $0 \leq \phi \leq 1$, such that $\text{supp } \phi = \frac{1+\gamma}{2}I$, $\phi = 1$ on I and

$$|\phi(\zeta) - \phi(\xi)| \leq \frac{c_1}{|I|} |\zeta - \xi|, \quad \zeta, \xi \in \mathbb{T}.$$

where c_1 depends only on γ . Set $L = \frac{1+\gamma}{2}I$ and consider

$$F(\zeta) := \phi(\zeta) \left| 1 - \frac{|f(\zeta)|}{\langle f \rangle_L} \right|, \quad \zeta \in \mathbb{T}.$$

Hence $F \geq 0$ and $F = 1$ c_μ -q.e on $E \cap I$. Therefore,

$$c_\mu(E \cap I) \leq \|F\|_\mu^2.$$

We claim that

$$\|F\|_\mu^2 \leq \kappa \frac{\mathcal{D}_{\gamma I, \mathbb{T}, \mu}(f) + \int_L |f|^2}{\langle f \rangle_L^2}. \quad (2)$$

where κ depends only on γ .

Indeed, we have

$$\begin{aligned} \|F\|_\mu^2 &= \int_{\mathbb{T}} |F(\zeta)|^2 \frac{|d\zeta|}{2\pi} + \int_{\mathbb{T}} \int_{\mathbb{T}} \frac{|F(\zeta) - F(\xi)|^2}{|\zeta - \xi|^2} \frac{|d\zeta|}{2\pi} d\mu(\xi) \\ &\leq \frac{1}{\langle f \rangle_L^2} \int_L |\langle f \rangle_L - |f(\zeta)||^2 \frac{|d\zeta|}{2\pi} + \int_{\gamma I} \int_{\gamma I} \frac{|F(\zeta) - F(\xi)|^2}{|\zeta - \xi|^2} \frac{|d\zeta|}{2\pi} d\mu(\xi) \\ &\quad + \frac{1}{\langle f \rangle_L^2} \int_{\zeta \in \mathbb{T} \setminus \gamma I} \int_{\xi \in L} \frac{|\langle f \rangle_L - |f(\xi)||^2}{|\zeta - \xi|^2} \frac{|d\zeta|}{2\pi} d\mu(\xi) + \\ &\quad \frac{1}{\langle f \rangle_L^2} \int_{\zeta \in L} \int_{\xi \in \mathbb{T} \setminus \gamma I} \frac{|\langle f \rangle_L - |f(\zeta)||^2}{|\zeta - \xi|^2} \frac{|d\zeta|}{2\pi} d\mu(\xi) \\ &= \frac{A}{2\pi \langle f \rangle_L^2} + \frac{B}{2\pi} + \frac{C}{2\pi \langle f \rangle_L^2} + \frac{D}{2\pi \langle f \rangle_L^2}. \end{aligned} \quad (3)$$

By Cauchy-Schwarz inequality we have

$$|\langle f \rangle_L - |f(\xi)||^2 \leq \frac{1}{|L|} \int_L |f(\eta) - f(\xi)|^2 |d\eta|. \quad (4)$$

Hence

$$\begin{aligned}
A &:= \int_L |\langle f \rangle_L - |f(\zeta)||^2 |d\zeta| \\
&\leq \frac{1}{|L|} \int_L \int_L |f(\eta) - f(\xi)|^2 |d\eta| |d\xi| \\
&\leq \frac{2}{|L|} \int_L \int_L |f(\eta)|^2 |d\eta| |d\xi| + \frac{2}{|L|} \int_L \int_L |f(\xi)|^2 |d\eta| |d\xi| \\
&= 4 \int_L |f(\zeta)|^2 |d\zeta|. \tag{5}
\end{aligned}$$

Next we estimate B . For $(\zeta, \xi) \in \mathbb{T} \times \mathbb{T}$, we have

$$\begin{aligned}
|F(\zeta) - F(\xi)| &= \left| \phi(\zeta) \left(\left| 1 - \frac{|f(\zeta)|}{\langle f \rangle_L} \right| - \left| 1 - \frac{|f(\xi)|}{\langle f \rangle_L} \right| \right) + (\phi(\zeta) - \phi(\xi)) \left| 1 - \frac{|f(\xi)|}{\langle f \rangle_L} \right| \right| \\
&\leq \frac{1}{\langle f \rangle_L} |f(\zeta) - f(\xi)| + \frac{c_1}{\langle f \rangle_L} \frac{|\zeta - \xi|}{|I|} |\langle f \rangle_L - |f(\xi)||. \tag{6}
\end{aligned}$$

By (6) and (4) we get

$$\begin{aligned}
B &:= \int_{\gamma I} \int_{\gamma I} \frac{|F(\zeta) - F(\xi)|^2}{|\zeta - \xi|^2} |d\zeta| |d\mu(\xi)| \\
&\leq \frac{2}{\langle f \rangle_L^2} \int_{\gamma I} \int_{\gamma I} \frac{|f(\zeta) - f(\xi)|^2}{|\zeta - \xi|^2} |d\zeta| |d\mu(\xi)| + \\
&\quad \frac{2 \times c_1^2}{\langle f \rangle_L^2 |I|^2 |L|} \int_{\gamma I} \int_{\gamma I} \int_L |f(\eta) - f(\xi)|^2 |d\eta| |d\zeta| |d\mu(\xi)| \\
&\leq \frac{2}{\langle f \rangle_L^2} \int_{\gamma I} \int_{\gamma I} \frac{|f(\zeta) - f(\xi)|^2}{|\zeta - \xi|^2} |d\zeta| |d\mu(\xi)| + \\
&\quad \frac{2 \times c_1^2 \times \gamma}{\langle f \rangle_L^2 |I| |L|} \int_{\gamma I} \int_L |f(\eta) - f(\xi)|^2 |d\eta| |d\mu(\xi)| \\
&\leq \frac{2}{\langle f \rangle_L^2} \int_{\gamma I} \int_{\gamma I} \frac{|f(\zeta) - f(\xi)|^2}{|\zeta - \xi|^2} |d\zeta| |d\mu(\xi)| + \\
&\quad \frac{c_2}{\langle f \rangle_L^2} \int_{\gamma I} \int_{\gamma I} \frac{|f(\eta) - f(\xi)|^2}{|\eta - \xi|^2} |d\eta| |d\mu(\xi)| \\
&\leq c_3 \frac{\mathcal{D}_{\gamma I, \gamma I, \mu}(f)}{\langle f \rangle_L^2}. \tag{7}
\end{aligned}$$

where c_2 and c_3 depend only on γ .

Using again (4), we see that

$$\begin{aligned}
C &:= \int_{\zeta \in \mathbb{T} \setminus \gamma I} \int_{\xi \in L} \frac{|\langle f \rangle_L - |f(\xi)||^2}{|\zeta - \xi|^2} |d\zeta| d\mu(\xi) \\
&\leq \int_{\zeta \in \mathbb{T} \setminus \gamma I} \frac{|d\zeta|}{d(\zeta, L)^2} \int_{\xi \in L} |\langle f \rangle_L - |f(\xi)||^2 d\mu(\xi) \\
&\leq \frac{c_4}{|I|} \int_{\xi \in L} |\langle f \rangle_L - |f(\xi)||^2 d\mu(\xi) \\
&\leq \frac{c_5}{|I|^2} \int_{\gamma I} \int_{\gamma I} |f(\eta) - f(\xi)|^2 |d\eta| d\mu(\xi) \\
&\leq c_5 \int_{\gamma I} \int_{\gamma I} \frac{|f(\eta) - f(\xi)|^2}{|\eta - \xi|^2} |d\eta| d\mu(\xi) \\
&= 2\pi c_5 \mathcal{D}_{\gamma I, \gamma I, \mu}(f),
\end{aligned} \tag{8}$$

where c_4 and c_5 depend only on γ .

We have

$$|\zeta - \xi| \geq \frac{\gamma - 1}{4} |I| \quad \text{and} \quad |\eta - \xi| \leq \frac{1 + 2\gamma}{\gamma - 1} |\zeta - \xi| \quad (\xi, \zeta, \eta) \in (\mathbb{T} \setminus \gamma I) \times L \times L.$$

Therefore, by (4) we obtain

$$\begin{aligned}
D &:= \int_{\xi \in \mathbb{T} \setminus \gamma I} \int_{\zeta \in L} \frac{|\langle f \rangle_L - |f(\zeta)||^2}{|\zeta - \xi|^2} |d\zeta| d\mu(\xi) \\
&\leq \int_{\xi \in \mathbb{T} \setminus \gamma I} \int_{\zeta \in L} \frac{1}{|L|} \int_{\eta \in L} \frac{|f(\eta) - f(\xi) + f(\xi) - f(\zeta)|^2}{|\zeta - \xi|^2} |d\zeta| |d\eta| d\mu(\xi) \\
&\leq 2 \int_{\xi \in \mathbb{T} \setminus \gamma I} \int_{\eta \in L} \int_{\zeta \in L} \frac{|f(\eta) - f(\xi)|^2}{|\zeta - \xi|^2} |d\zeta| |d\eta| d\mu(\xi) + \\
&\quad 2 \int_{\xi \in \mathbb{T} \setminus \gamma I} \int_{\zeta \in L} \frac{|f(\xi) - f(\zeta)|^2}{|\zeta - \xi|^2} |d\zeta| d\mu(\xi) \\
&\leq \frac{c_6}{|I|} \int_{\xi \in \mathbb{T} \setminus \gamma I} \int_{\eta \in L} \int_{\zeta \in L} \frac{|f(\eta) - f(\xi)|^2}{|\eta - \xi|^2} |d\zeta| |d\eta| d\mu(\xi) + \\
&\quad 2 \int_{\xi \in \mathbb{T} \setminus \gamma I} \int_{\zeta \in L} \frac{|f(\xi) - f(\zeta)|^2}{|\zeta - \xi|^2} |d\zeta| d\mu(\xi) \\
&= c_7 \int_{\xi \in \mathbb{T} \setminus \gamma I} \int_{\zeta \in L} \frac{|f(\xi) - f(\zeta)|^2}{|\zeta - \xi|^2} |d\zeta| d\mu(\xi) \\
&\leq 2\pi c_7 \mathcal{D}_{\gamma I, \mathbb{T} \setminus \gamma I, \mu}(f),
\end{aligned} \tag{9}$$

where c_6 and c_7 depend only on γ .

Combining (3), (5), (7), (8) and (9) we get (2) and the proof is complete. \square

Proof of Theorem 1.1. Let $f \in \mathcal{D}(\mu)$ such that $f|_E = 0$ and set $\ell = \sum_n |I_n|$. By Lemma 2.1 and Jensen's inequality it follows that

$$\begin{aligned} \int_{\bigcup I_n} \log |f(\xi)| |d\xi| &= \sum_n |I_n| \frac{1}{|I_n|} \int_{I_n} \log |f(\xi)| |d\xi| \\ &\leq \frac{1}{2} \sum_n |I_n| \log \langle f \rangle_{I_n}^2 \\ &\leq \frac{1}{2} \sum_n |I_n| \log \left(\kappa \frac{\mathcal{D}_{\gamma I_n, \mathbb{T}, \mu}(f) + \int_{\gamma I_n} |f(\zeta)|^2 |d\zeta|}{c_\mu(E \cap I_n)} \right) \\ &= \frac{1}{2} (\ell \log \kappa + \mathcal{I}), \end{aligned}$$

where

$$\mathcal{I} = \sum_n |I_n| \log \frac{|I_n|}{c_\mu(E \cap I_n)} + \underbrace{\sum_n |I_n| \log \left(\frac{\mathcal{D}_{\gamma I_n, \mathbb{T}, \mu}(f) + \int_{\gamma I_n} |f(\zeta)|^2 |d\zeta|}{|I_n|} \right)}_{\mathcal{J}}$$

Using again Jensen inequality and since (γI_n) are pairwise disjoint, we get

$$\begin{aligned} \mathcal{J} &= \ell \sum_n \frac{|I_n|}{\ell} \log \left(\frac{\mathcal{D}_{\gamma I_n, \mathbb{T}, \mu}(f) + \int_{\gamma I_n} |f(\zeta)|^2 |d\zeta|}{|I_n|} \right) \\ &\leq \ell \log \left[\frac{1}{\ell} \left(\sum_n \mathcal{D}_{\gamma I_n, \mathbb{T}, \mu}(f) + \int_{\gamma I_n} |f(\zeta)|^2 |d\zeta| \right) \right] \\ &\leq \ell \log \left[\frac{1}{\ell} (\mathcal{D}_\mu(f) + \|f\|_2^2) \right] \\ &= \ell \log \frac{\|f\|_\mu^2}{\ell}. \end{aligned}$$

Therefore $\mathcal{I} = -\infty$. So by Fatou's Theorem we obtain $f = 0$ and the proof is complete.

3. REMARKS AND EXAMPLES

3.1. By (1), see also [5, Theorem 2], we have $c_\mu(\zeta) > 0$, for some $\zeta \in \mathbb{T}$, if and only if

$$\int_0^1 \frac{dr}{(1-r)P_\mu(r\zeta) + (1-r)^2} < \infty.$$

In this case, for every $f \in \mathcal{D}(\mu)$, the non tangential limit, $f(\zeta)$, at ζ exists. We have the following upper estimate

Lemma 3.1. *Let μ be a positive Borel measure on \mathbb{T} and let $\zeta \in \mathbb{T}$ such that $c_\mu(\zeta) > 0$. For $\beta \in (0, 1/2)$ and $f \in \mathcal{D}_\mu$, we have*

$$|f(z) - f(\zeta)|^2 \leq C \left(\int_{S(\zeta, \beta)} |f'(z)|^2 P_\mu(z) dA(z) \right) \left(\int_0^\beta \frac{dx}{x P_\mu((1-x)\zeta) + x^2} \right),$$

where $S(\zeta, \beta) = \{w \in \mathbb{D} : 1 - |w| < \beta, \arg(w\bar{\zeta}) < \beta\}$, $z \in S(\zeta, \beta/4)$ and C is an absolute constant.

Proof. One can use the same idea as in [5, Theorem 2]. \square

3.2. The capacity c_μ is comparable with analytic capacity given by

$$c_\mu^a(E) = \inf\{\|f\|_\mu^2 : f \in \mathcal{D}(\mu) \text{ and } |f| \geq 1 \text{ a.e. on a neighborhood of } E\}.$$

Indeed we have $c_\mu(E) \leq c_\mu^a(E) \leq 4c_\mu(E)$, see [6, Lemma 3.1]. On the other hand If $f \in \mathcal{D}(\mu)$ then $f \in L^2(\mathbb{T}, d\mu)$ and $\|f\|_{L^2(\mathbb{T}, d\mu)} \leq (1 + \mu(\mathbb{T})^{1/2})\|f\|_\mu$, [4, Theorem 8.1.2]. Therefore

$$c_\mu(E) \geq \mu(E)/(1 + \mu(\mathbb{T})^{1/2})^2.$$

We obtain the following result

Corollary 3.2. *Let $\gamma > 1$ and let $f \in \mathcal{D}(\mu)$ such that $f|_E = 0$ for some Borel subset E of \mathbb{T} . Then, for any open arc $I \subset \mathbb{T}$*

$$\langle f \rangle_I^2 \leq \kappa \frac{\mathcal{D}_{\gamma I, \mathbb{T}, \mu}(f) + \int_{\gamma I} |f|^2}{\mu(E \cap I)},$$

where κ is a constant depending only on γ .

3.3. First observe that for the Dirac measure δ_ζ it is known, (see [4]), that $\mathcal{D}(\delta_\zeta) = \mathbb{C} + (z - \zeta)H^2$. Then a Borel set E is a uniqueness set for $\mathcal{D}(\delta_\zeta)$ if and only if the E has a positive Lebesgue measure. This result can be extended to some other discrete measures. For a positive Borel measure μ we will denote by $V_2(\mu)$ the Newtonian potential given by

$$V_2(\mu)(\zeta) = \int_0^{2\pi} \frac{d\mu(e^{it})}{|e^{it} - \zeta|^2}.$$

D. Guillot showed in [9, Theorem 2.1] that if there exists $f \in \mathcal{D}_\mu$ such that $f = 0$ μ a.e on \mathbb{T} then

$$\int_{\mathbb{T}} \log V_2(\mu)(\zeta) |d\zeta| < \infty.$$

He also proved that the converse is true for all discrete measures. The following result is an immediate consequence of [9].

Proposition 3.3. *Consider a positive sequence $(a_n)_{n \geq 1}$ such that $\sum_{n \geq 1} a_n = 1$ and let $\mu = \sum_n a_n \delta_{\zeta_n}$ where $\zeta_n \in \mathbb{T}$. If*

$$\int_{\mathbb{T}} \log V_2(\mu)(\zeta) |d\zeta| < \infty,$$

then a Borel set $E \subset \mathbb{T}$ is a uniqueness set for $\mathcal{D}(\mu)$ if and only if $|E| > 0$.

Proof. Since $\mathcal{D}(\mu) \subset H^2$, it is obvious that the condition $|E| > 0$ is sufficient. Conversely, Let $E \subset \mathbb{T}$ be a Borel set such that $|E| = 0$, there exists a function $\varphi \in H^\infty \setminus \{0\}$ such that φ has non-tangential boundary limits 0 on E . By [8], there exists $f \in \mathcal{D}(\mu) \cap H^\infty \setminus \{0\}$ such that $f(\zeta_n) = 0$, ($n \geq 1$). Now it suffices to prove that $\varphi f \in \mathcal{D}(\mu)$. Indeed,

$$\begin{aligned}
 \mathcal{D}_\mu(\varphi f) &= \int_{\mathbb{T}} \int_{\mathbb{T}} \left| \frac{\varphi f(\zeta) - \varphi f(\zeta')}{\zeta - \zeta'} \right|^2 |d\zeta| d\mu(\zeta') \\
 &= \int_{\mathbb{T}} \int_{\mathbb{T}} \left| \frac{\varphi f(\zeta)}{\zeta - \zeta'} \right|^2 |d\zeta| d\mu(\zeta') \\
 &\leq \|\varphi\|_\infty^2 \int_{\mathbb{T}} \int_{\mathbb{T}} \left| \frac{f(\zeta)}{\zeta - \zeta'} \right|^2 |d\zeta| d\mu(\zeta') \\
 &= \|\varphi\|_\infty^2 \int_{\mathbb{T}} \int_{\mathbb{T}} \left| \frac{f(\zeta) - f(\zeta')}{\zeta - \zeta'} \right|^2 |d\zeta| d\mu(\zeta') \\
 &= \|\varphi\|_\infty^2 \mathcal{D}_\mu(f)
 \end{aligned}$$

which completes the proof. \square

Note that for a positive Borel measures $\mu = \sum_n a_n \delta_{\zeta_n}$ where $\zeta_n \in \mathbb{T}$ such that

$$\int_{\mathbb{T}} \log V_2(\mu)(\zeta) |d\zeta| = \infty, \tag{10}$$

we have no complete characterization for uniqueness sets for $\mathcal{D}(\mu)$. In this case, by [9] the countable set $\{\zeta \in \mathbb{T} : \mu(\zeta) \neq 0\}$ is a uniqueness set for $\mathcal{D}(\mu)$. The situation is more complicated than in the previous case. In fact, using theorem 1.1, we will construct smaller uniqueness set for $\mathcal{D}(\mu)$. Before stating this result, we give an example of discrete measures μ satisfying (10). A closed set $E \subset \mathbb{T}$ is said to be a Carleson set if

$$\int_{\mathbb{T}} \log \text{dist}(\zeta, E) |d\zeta| > -\infty.$$

This condition is equivalent to $|E| = 0$ and

$$\sum_n |\ell_n| \log |\ell_n| > -\infty,$$

where ℓ_n are the complementary intervals of E . For more informations on Carleson sets, see e.g. [4, §4.4].

Proposition 3.4. *Let E be a countable closed set of the unit circle such that E is not Carleson set. Then there exists a discrete measure μ on the unit circle such that $E = \{\zeta \in \mathbb{T} : \mu(\{\zeta\}) \neq 0\}$ and (10) is satisfied.*

Proof. Let $\mathbb{T} \setminus E = \bigcup (a_n, b_n)$ and let

$$\mu = \sum_{n \geq 1} (b_n - a_n) (\delta_{a_n} + \delta_{b_n}).$$

We have

$$\begin{aligned}
\int_{\mathbb{T}} \log V_2(\mu)(\zeta) |d\zeta| &\geq \sum_{n \geq 1} \int_{a_n}^{b_n} \log \frac{b_n - a_n}{\text{dist}^2(e^{it}, \{a_n, b_n\})} dt \\
&\geq \sum_{n \geq 1} \int_{a_n}^{b_n} \log \frac{b_n - a_n}{(b_n - a_n)^2} dt \\
&= \sum_{n \geq 1} (b_n - a_n) \log \frac{1}{b_n - a_n} = +\infty.
\end{aligned}$$

□

3.4. Here we give an example of positive measure μ and a countable closed set E such that $\mu(E) = 0$ and E is uniqueness set for $\mathcal{D}(\mu)$ (see Corollary 3.6). This result was easily obtain from Theorem 1.1. We can also obtain this result by more direct method using Lemma 3.1 and easy estimates of harmonic measures for some suitably domains.

Lemma 3.5. *Let $\zeta \in \mathbb{T}$, $a \in (0, 1/2)$. Let μ be a probability measure on \mathbb{T} such that $\mu \geq \sum_{k \geq 2} ak^{-2} \delta_{\zeta e^{iak}}$, then $c_\mu(\{\zeta\}) \gtrsim \sqrt{a}$.*

Proof. Let

$$\nu = \sum_{k \geq 2} \frac{a}{k^2} \delta_{\zeta e^{iak}}.$$

By (1), we have

$$\frac{1}{c_\mu(\{\zeta\})} \lesssim 1 + \int_0^1 \frac{dr}{(1-r)P_\nu(r\zeta) + (1-r)^2}.$$

For $a \leq 1-r$, we have

$$\begin{aligned}
(1-r)P_\nu(r\zeta) &\asymp \sum_{k \geq 2} \frac{a(1-r)^2}{k^2((1-r)^2 + a^{2k})} \\
&\asymp \sum_{k \geq 2} \frac{a}{k^2} \asymp a.
\end{aligned}$$

For $1-r < a$, let k_r be a real number such that $1-r = a^{k_r}$. Since $x \mapsto x^2 a^{2x}$; $x \geq 1$, is a decreasing function, we have

$$\begin{aligned}
(1-r)P_\nu(r\zeta) &\asymp \sum_{k \geq 2} \frac{a(1-r)^2}{k^2((1-r)^2 + a^{2k})} \\
&= \sum_{k \geq k_r} \frac{a}{k^2} + \sum_{k < k_r} \frac{a(1-r)^2}{k^2 a^{2k}} \\
&\asymp \frac{a \log(1/a)}{\log(1/1-r)}.
\end{aligned}$$

Then we obtain

$$\frac{1}{c_\mu(\{\zeta\})} \leq 1 + \int_0^{1-a} \frac{dr}{a + (1-r)^2} + \int_{1-a}^1 \frac{dr}{\frac{a \log(1/a)}{\log(1/(1-r))} + (1-r)^2} \lesssim \frac{1}{\sqrt{a}}$$

and the lemma is proved. \square

Corollary 3.6. *There exists a discrete probability measure μ on the unit circle and a countable closed set $E \subset \mathbb{T}$, such that $\mu(E) = 0$ and E is a set of uniqueness for $\mathcal{D}(\mu)$.*

Proof. Let E be the set define by

$$E := \{\zeta_n = e^{i\theta_n} : n \geq 3\} \cup \{1\},$$

with $\theta_n := 1/\log(n)$. Let $a_n = \varepsilon(\theta_n - \theta_{n+1})$ for some small $\varepsilon > 0$. Consider the sequence $\zeta_{n,k} = e^{i\theta_{n,k}}$ where $\theta_{n,k} := \theta_n + a_n^k$, $k \geq 2$. The measure μ is given by

$$\mu := \sum_{n \geq 3} \sum_{k \geq 2} \frac{a_n}{k^2} \delta_{\zeta_{n,k}}.$$

To prove that E is a uniqueness set for $\mathcal{D}(\mu)$, we use Theorem 1.1. Let $I_n = (\theta_n - a_n, \theta_n + a_n)$. By lemma 3.5; we have $c_\mu(\zeta_n) \geq \sqrt{a_n}$. Then

$$\begin{aligned} \sum_n |I_n| \log \left(\frac{c_\mu(I_n \cap E)}{|I_n|} \right) &= \sum_n |I_n| \log \left(\frac{c_\mu(\zeta_n)}{|I_n|} \right) \\ &\geq \frac{1}{2} \sum_n a_n \log \left(\frac{1}{a_n} \right) = +\infty \end{aligned}$$

and the proof is complete. \square

3.5. Now we give an example of positive measure μ on \mathbb{T} without atoms and a perfect closed set E such that E is uniqueness set for $\mathcal{D}(\mu)$. Let ℓ_n be a positive sequence and let K be the associated generalized Cantor set. Let $\alpha \in (0, 1)$ and consider the measure $d\mu(\zeta) = \text{dist}(\zeta, K)^\alpha dm(\zeta)$. By [5, Theorem 2] and [7, Theorem 2]

$$c_\mu(K) = 0 \iff \sum_n 2^{-n} \ell_n^{-\alpha} = \infty.$$

Let $I_n = (e^{i(\log(n+1))^{-1}}, e^{i(\log n)^{-1}})$. Note that

$$\sum_{n \geq 2} |I_n| \log |I_n| = -\infty.$$

We reproduce the generalized Cantor set in each I_n , denoted by K_n . We have

$$c_\mu(K_n \cap I_n) \asymp c_\mu(K) |I_n|^\alpha.$$

Consider now $E = \{1\} \cup_n K_n$, we have $c_\mu(E) = 0$ and we get

$$\sum_n |I_n| \log \frac{|I_n|}{c_\mu(E \cap I_n)} = -\infty.$$

So by Theorem 1.1, E is uniqueness set for $\mathcal{D}(\mu)$.

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