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HAVIN-MAZYA TYPE UNIQUENESS THEOREM FOR DIRICHLET SPACES

H. BAHAJJI-EL IDRISSE, O. EL-FALLAH, AND K. KELLAY

ABSTRACT. Let μ be a positive finite Borel measure on the unit circle. The associated Dirichlet space $\mathcal{D}(\mu)$ consists of holomorphic functions on the unit disc whose derivatives are square integrable when weighted against the Poisson integral of μ . We give a sufficient condition on a Borel subset E of the unit circle which ensures that E is a uniqueness set for $\mathcal{D}(\mu)$. As application, we give an example of positive measure μ and a countable closed set E such that $\mu(E) = 0$ and E is uniqueness set for $\mathcal{D}(\mu)$.

1. INTRODUCTION

Let μ be a positive finite Borel measure on the unit circle \mathbb{T} , the *harmonically weighted Dirichlet space* $\mathcal{D}^h(\mu)$ is the set of all functions $f \in L^2(\mathbb{T})$ for which

$$\mathcal{D}_\mu(f) = \int_{\mathbb{T}} \mathcal{D}_\xi(f) d\mu(\xi) < \infty,$$

where $\mathcal{D}_\xi(f)$ is the local Dirichlet integral of f at $\xi \in \mathbb{T}$ given by

$$\mathcal{D}_\xi(f) := \int_{\mathbb{T}} \left| \frac{f(\zeta) - f(\xi)}{\zeta - \xi} \right|^2 \frac{|d\zeta|}{2\pi}.$$

The space $\mathcal{D}^h(\mu)$ is endowed with the norm

$$\|f\|_\mu^2 := \|f\|_{L^2(\mathbb{T})}^2 + \mathcal{D}_\mu(f).$$

The analytic weighted Dirichlet space $\mathcal{D}(\mu)$ is defined by

$$\mathcal{D}(\mu) = \{f \in \mathcal{D}^h(\mu) : \widehat{f}(n) = 0, n < 0\}.$$

Let \mathbb{D} be the open unit disc in the complex plane. Let H^2 denote the Hardy space of analytic functions on \mathbb{D} . As usual, we use the identification $H^2 = \{f \in L^2(\mathbb{T}) : \widehat{f}(n) = 0, n < 0\}$. Since $\mathcal{D}(\mu) \subset H^2$, every function $f \in \mathcal{D}(\mu)$ has non-tangential limits almost everywhere on \mathbb{T} . We denote by $f(\zeta)$ the non-tangential limit of f at $\zeta \in \mathbb{T}$ if it exists. Note that by Douglas Formula

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the Dirichlet-type space $\mathcal{D}(\mu)$ is the set of analytic functions $f \in H^2$, such that

$$\mathcal{D}_\mu(f) := \int_{\mathbb{D}} |f'(z)|^2 P_\mu(z) dA(z) < \infty,$$

where $dA(z) = dxdy/\pi$ stands for the normalized area measure in \mathbb{D} and P_μ is the Poisson integral of μ

$$P_\mu(z) := \int_{\mathbb{T}} \frac{1 - |z|^2}{|\zeta - z|^2} d\mu(\zeta), \quad z \in \mathbb{D}.$$

For a proof of this fact see [4, Theorem 7.2.5] and for numerous results on the Dirichlet-type space and operators acting thereon see [4, 12, 13, 14, 15].

The capacity associated with $\mathcal{D}(\mu)$ is denoted by c_μ and is given by

$$c_\mu(E) := \inf \left\{ \|f\|_\mu^2 : f \in \mathcal{D}^h(\mu), |f| \geq 1 \text{ a.e. on a neighborhood of } E \right\}.$$

Since the L^2 norm is dominated by the Dirichlet norm $\|\cdot\|_\mu$, it is obvious that c_μ -capacity 0 implies Lebesgue measure 0. We say that a property holds c_μ -quasi-everywhere (c_μ -q.e.) if it holds everywhere outside a set of zero c_μ capacity. Note that c_μ -q.e. implies a.e. Furthermore every function $f \in \mathcal{D}(\mu)$ has non-tangential limits c_μ -q.e. on \mathbb{T} [3, Theorem 2.1.9]. For more details see [6].

Recall that if $I \subset \mathbb{T}$ be the arc of length $|I| = 1 - \rho$ with midpoint $\zeta \in \mathbb{T}$, then

$$\frac{1}{c_\mu(I)} \asymp 1 + \int_0^\rho \frac{dr}{(1-r)P_\mu(r\zeta) + (1-r)^2}, \quad (1)$$

where the implied constants are absolute. For the proof see [5, Theorem 2]. Let E be a subset of \mathbb{T} , the set E is said to be a uniqueness set for $\mathcal{D}(\mu)$ if, for each $f \in \mathcal{D}(\mu)$ such that its non-tangential limit $f = 0$ on E , we have $f = 0$.

Note that if $d\mu = dm$ the normalized arc measure on \mathbb{T} , then the space $\mathcal{D}(\mu)$ coincides with the classical Dirichlet space \mathcal{D} given by

$$\mathcal{D} = \left\{ f \in H^2 : \int_{\mathbb{D}} |f'(z)|^2 dA(z) < \infty \right\}.$$

In this case, c_m is comparable to the logarithmic capacity and $c_m(I) \asymp |\log |I||^{-1}$ for every arc $I \subset \mathbb{T}$, [11, Theorem 14].

Khavin and Maz'ya proved in [9] that a Borel subset E of \mathbb{T} is a uniqueness set for \mathcal{D} if there exists a family of pairwise disjoint open arcs (I_n) such that

$$\sum_n |I_n| \log \frac{|I_n|}{c_m(E \cap I_n)} = -\infty.$$

For other uniqueness results for \mathcal{D} see also [1, 2, 7, 4, 8].

Our aim in this paper is to extend Khavin and Maz'ya uniqueness theorem to a general Dirichlet spaces $\mathcal{D}(\mu)$.

Let $\gamma \in (0, 1)$ and let $I = (e^{ia}, e^{ib})$. The arc γI is given by

$$\gamma I = (e^{i(a+(1-\gamma)\frac{b-a}{2})}, e^{i(b-(1-\gamma)\frac{b-a}{2})}).$$

The main result of this paper is the following theorem.

Theorem 1. *Let E be a Borel subset of \mathbb{T} of Lebesgue measure zero. suppose that there exists a family of pairwise disjoint open arcs (I_n) and $\gamma \in (0, 1)$ such that*

$$\sum_n |I_n| \log \frac{|I_n|}{c_\mu(E \cap \gamma I_n)} = -\infty;$$

then E is a uniqueness set for $\mathcal{D}(\mu)$.

As application, we give in section 3, an exemple of positive measure μ and a countable closed set E such that $\mu(E) = 0$ and E is uniqueness set for $\mathcal{D}(\mu)$.

The key of the proof of the main theorem is an upper bound for $\frac{1}{|I|} \int_I |f|$ in terms of capacity of $E \cap I$, for any open arc $|I|$.

The next section is devoted to the proof of Theorem 1. In section 3 we discuss the case of discrete measures.

Throughout the paper, we use the following notations: $A \lesssim B$ means that there is an absolute constant C such that $A \leq CB$ and $A \asymp B$ if both $A \lesssim B$ and $B \lesssim A$.

2. PROOF

First, let us introduce some notations which will be useful in the sequel. Let J and L be arcs of \mathbb{T} and let f be a borelian function defined on \mathbb{T} . We set

$$\mathcal{D}_{J,L,\mu}(f) := \int_{\zeta \in J} \int_{\xi \in L} \frac{|f(\zeta) - f(\xi)|^2}{|\zeta - \xi|^2} \frac{|d\zeta|}{2\pi} d\mu(\xi),$$

and

$$\langle f \rangle_J := \frac{1}{|J|} \int_J |f(\xi)| |d\xi|.$$

The following lemma is the key in the proof of theorem 1.

Lemma 2. *Let $0 < \gamma < 1$ and let $f \in \mathcal{D}(\mu)$ such that $f|_E = 0$ for some borelian subset E of \mathbb{T} . Then, for any open arc $I \subset \mathbb{T}$*

$$\langle f \rangle_{\gamma I}^2 \leq \kappa \frac{\mathcal{D}_{I,\mathbb{T},\mu}(f) + \int_I |f|^2}{C_\mu(E \cap \gamma I)},$$

where κ is a constant depending only on γ .

Proof. Without loss of generality, we can suppose that $I = (e^{-i\theta}, e^{i\theta})$ with $2\theta < \pi$. In this case $\gamma I = (e^{-i\gamma\theta}, e^{i\gamma\theta})$.

Let ϕ be a positive function on \mathbb{T} , $0 \leq \phi \leq 1$, such that $\text{supp } \phi = \frac{1+\gamma}{2}I$, $\phi = 1$ on γI and

$$|\phi(\zeta) - \phi(\xi)| \leq \frac{c_1}{|I|}|\zeta - \xi|, \quad \zeta, \xi \in \mathbb{T}.$$

where c_1 depends only on γ . Set $L = \frac{1+\gamma}{2}I$ and consider

$$F(\zeta) := \phi(\zeta) \left| 1 - \frac{|f(\zeta)|}{\langle f \rangle_L} \right|, \quad \zeta \in \mathbb{T}.$$

Hence $F \geq 0$ and $F = 1$ c_μ -q.e on $E \cap \gamma I$. Therefore,

$$c_\mu(E \cap \gamma I) \leq \|F\|_\mu^2.$$

We claim that

$$\|F\|_\mu^2 \leq \kappa \frac{\mathcal{D}_{I, \mathbb{T}, \mu}(f) + \int_L |f|^2}{\langle f \rangle_L^2}. \quad (2)$$

where κ depends only on γ .

Indeed, we have

$$\begin{aligned} \|F\|_\mu^2 &= \int_{\mathbb{T}} |F(\zeta)|^2 \frac{|d\zeta|}{2\pi} + \int_{\mathbb{T}} \int_{\mathbb{T}} \frac{|F(\zeta) - F(\xi)|^2}{|\zeta - \xi|^2} \frac{|d\zeta|}{2\pi} d\mu(\xi) \\ &\leq \frac{1}{\langle f \rangle_L^2} \int_L |\langle f \rangle_L - |f(\zeta)||^2 \frac{|d\zeta|}{2\pi} + \int_I \int_I \frac{|F(\zeta) - F(\xi)|^2}{|\zeta - \xi|^2} \frac{|d\zeta|}{2\pi} d\mu(\xi) \\ &\quad + \frac{1}{\langle f \rangle_L^2} \int_{\zeta \in \mathbb{T} \setminus I} \int_{\xi \in L} \frac{|\langle f \rangle_L - |f(\xi)||^2}{|\zeta - \xi|^2} \frac{|d\zeta|}{2\pi} d\mu(\xi) \\ &\quad + \frac{1}{\langle f \rangle_L^2} \int_{\zeta \in L} \int_{\xi \in \mathbb{T} \setminus I} \frac{|\langle f \rangle_L - |f(\zeta)||^2}{|\zeta - \xi|^2} \frac{|d\zeta|}{2\pi} d\mu(\xi) \\ &= \frac{A}{2\pi \langle f \rangle_L^2} + \frac{B}{2\pi} + \frac{C}{2\pi \langle f \rangle_L^2} + \frac{D}{2\pi \langle f \rangle_L^2}. \end{aligned} \quad (3)$$

By Cauchy-Schwarz inequality we have

$$|\langle f \rangle_L - |f(\xi)||^2 \leq \frac{1}{|L|} \int_L |f(\eta) - f(\xi)|^2 |d\eta|. \quad (4)$$

Hence

$$\begin{aligned}
A &:= \int_L |\langle f \rangle_L - |f(\zeta)||^2 |d\zeta| \\
&\leq \frac{1}{|L|} \int_L \int_L |f(\eta) - f(\xi)|^2 |d\eta| |d\xi| \\
&\leq \frac{2}{|L|} \int_L \int_L |f(\eta)|^2 |d\eta| |d\xi| + \frac{2}{|L|} \int_L \int_L |f(\xi)|^2 |d\eta| |d\xi| \\
&= 4 \int_L |f(\zeta)|^2 |d\zeta|. \tag{5}
\end{aligned}$$

Next we estimate B . For $(\zeta, \xi) \in \mathbb{T} \times \mathbb{T}$, we have

$$\begin{aligned}
|F(\zeta) - F(\xi)| &= \left| \phi(\zeta) \left(\left| 1 - \frac{|f(\zeta)|}{\langle f \rangle_L} \right| - \left| 1 - \frac{|f(\xi)|}{\langle f \rangle_L} \right| \right) + (\phi(\zeta) - \phi(\xi)) \left| 1 - \frac{|f(\xi)|}{\langle f \rangle_L} \right| \right| \\
&\leq \frac{1}{\langle f \rangle_L} |f(\zeta) - f(\xi)| + \frac{c_1}{\langle f \rangle_L} \frac{|\zeta - \xi|}{|I|} |\langle f \rangle_L - |f(\xi)||. \tag{6}
\end{aligned}$$

By (6) and (4) we get

$$\begin{aligned}
B &:= \int_I \int_I \frac{|F(\zeta) - F(\xi)|^2}{|\zeta - \xi|^2} |d\zeta| |d\mu(\xi)| \\
&\leq \frac{2}{\langle f \rangle_L^2} \int_I \int_I \frac{|f(\zeta) - f(\xi)|^2}{|\zeta - \xi|^2} |d\zeta| |d\mu(\xi)| \\
&\quad + \frac{2 \times c_1^2}{\langle f \rangle_L^2 |I|^2 |L|} \int_I \int_I \int_L |f(\eta) - f(\xi)|^2 |d\eta| |d\zeta| |d\mu(\xi)| \\
&\leq \frac{2}{\langle f \rangle_L^2} \int_I \int_I \frac{|f(\zeta) - f(\xi)|^2}{|\zeta - \xi|^2} |d\zeta| |d\mu(\xi)| \\
&\quad + \frac{2 \times c_1^2}{\langle f \rangle_L^2 |I| |L|} \int_I \int_L |f(\eta) - f(\xi)|^2 |d\eta| |d\mu(\xi)| \\
&\leq \frac{2}{\langle f \rangle_L^2} \int_I \int_I \frac{|f(\zeta) - f(\xi)|^2}{|\zeta - \xi|^2} |d\zeta| |d\mu(\xi)| \\
&\quad + \frac{c_2}{\langle f \rangle_L^2} \int_I \int_I \frac{|f(\eta) - f(\xi)|^2}{|\eta - \xi|^2} |d\eta| |d\mu(\xi)| \\
&\leq c_3 \frac{\mathcal{D}_{I,I,\mu}(f)}{\langle f \rangle_L^2}. \tag{7}
\end{aligned}$$

where c_2 and c_3 depend only on γ .

Using again (4), we see that

$$\begin{aligned}
C &:= \int_{\zeta \in \mathbb{T} \setminus I} \int_{\xi \in L} \frac{|\langle f \rangle_L - |f(\xi)||^2}{|\zeta - \xi|^2} |d\zeta| d\mu(\xi) \\
&\leq \int_{\zeta \in \mathbb{T} \setminus I} \frac{|d\zeta|}{d(\zeta, L)^2} \int_{\xi \in L} |\langle f \rangle_L - |f(\xi)||^2 d\mu(\xi) \\
&\leq \frac{c_4}{|I|} \int_{\xi \in L} |\langle f \rangle_L - |f(\xi)||^2 d\mu(\xi) \\
&\leq \frac{c_5}{|I|^2} \int_I \int_I |f(\eta) - f(\xi)|^2 |d\eta| d\mu(\xi) \\
&\leq c_5 \int_I \int_I \frac{|f(\eta) - f(\xi)|^2}{|\eta - \xi|^2} |d\eta| d\mu(\xi) \\
&= 2\pi c_5 \mathcal{D}_{I, I, \mu}(f), \tag{8}
\end{aligned}$$

where c_4 and c_5 depend only on γ .

Obviously, we have

$$|\zeta - \xi| \geq \frac{1-\gamma}{4} \quad \text{and} \quad |\eta - \xi| \leq \frac{4}{1-\gamma} |\zeta - \xi| \quad (\xi, \zeta, \eta) \in \mathbb{T} \setminus I \times L \times L.$$

Therefore, by (4) we obtain

$$\begin{aligned}
D &:= \int_{\xi \in \mathbb{T} \setminus I} \int_{\zeta \in L} \frac{|\langle f \rangle_L - |f(\zeta)||^2}{|\zeta - \xi|^2} |d\zeta| d\mu(\xi) \\
&\leq \int_{\xi \in \mathbb{T} \setminus I} \int_{\zeta \in L} \frac{1}{|L|} \int_{\eta \in L} \frac{|f(\eta) - f(\xi) + f(\xi) - f(\zeta)|^2}{|\zeta - \xi|^2} |d\zeta| |d\eta| d\mu(\xi) \\
&\leq 2 \int_{\xi \in \mathbb{T} \setminus I} \int_{\eta \in L} \int_{\zeta \in L} \frac{|f(\eta) - f(\xi)|^2}{|\zeta - \xi|^2} |d\zeta| |d\eta| d\mu(\xi) \\
&\quad + 2 \int_{\xi \in \mathbb{T} \setminus I} \int_{\zeta \in L} \frac{|f(\xi) - f(\zeta)|^2}{|\zeta - \xi|^2} |d\zeta| d\mu(\xi) \\
&\leq \frac{c_6}{|I|} \int_{\xi \in \mathbb{T} \setminus I} \int_{\eta \in L} \int_{\zeta \in L} \frac{|f(\eta) - f(\xi)|^2}{|\eta - \xi|^2} |d\zeta| |d\eta| d\mu(\xi) \\
&\quad + 2 \int_{\xi \in \mathbb{T} \setminus I} \int_{\zeta \in L} \frac{|f(\xi) - f(\zeta)|^2}{|\zeta - \xi|^2} |d\zeta| d\mu(\xi) \\
&= c_7 \int_{\xi \in \mathbb{T} \setminus I} \int_{\zeta \in L} \frac{|f(\xi) - f(\zeta)|^2}{|\zeta - \xi|^2} |d\zeta| d\mu(\xi) \\
&\leq 2\pi c_7 \mathcal{D}_{I, \mathbb{T} \setminus I, \mu}(f), \tag{9}
\end{aligned}$$

where c_6 and c_7 depend only on γ .

Combining (3), (5), (7), (8) and (9) we get (2) and the proof is complete. \square

Proof of Theorem 1. Let $f \in \mathcal{D}(\mu)$ such that $f|_E = 0$ and set $\ell = \sum_n |I_n|$. By Lemma 2 and Jensen's inequality it follows that

$$\begin{aligned} \int_{\bigcup \gamma I_n} \log |f(\xi)| |d\xi| &= \sum_n \gamma |I_n| \frac{1}{\gamma |I_n|} \int_{\gamma I_n} \log |f(\xi)| |d\xi| \\ &\leq \frac{\gamma}{2} \sum_n |I_n| \log \langle f \rangle_{\gamma I_n}^2 \\ &\leq \frac{\gamma}{2} \sum_n |I_n| \log \left(\kappa \frac{\mathcal{D}_{I_n, \mathbb{T}, \mu}(f) + \int_{I_n} |f(\zeta)|^2 |d\zeta|}{C_\mu(E \cap \gamma I_n)} \right) \\ &= \frac{\gamma}{2} \ell \log \kappa + \frac{\gamma}{2} \mathcal{I}, \end{aligned}$$

where

$$\mathcal{I} = \sum_n |I_n| \log \frac{|I_n|}{C_\mu(E \cap \gamma I_n)} + \sum_n |I_n| \log \left(\frac{\mathcal{D}_{I_n, \mathbb{T}, \mu}(f) + \int_{I_n} |f(\zeta)|^2 |d\zeta|}{|I_n|} \right)$$

Using again Jensen inequality, we get

$$\begin{aligned} \ell \sum_n \frac{|I_n|}{\ell} \log \left(\frac{\mathcal{D}_{I_n, \mathbb{T}, \mu}(f) + \int_{I_n} |f(\zeta)|^2 |d\zeta|}{|I_n|} \right) &\leq \ell \log \left[\frac{1}{\ell} \left(\sum_n \mathcal{D}_{I_n, \mathbb{T}, \mu}(f) + \int_{I_n} |f(\zeta)|^2 |d\zeta| \right) \right] \\ &\leq \ell \log \left[\frac{1}{\ell} (\mathcal{D}_\mu(f) + \|f\|_\mu^2) \right] \\ &= \ell \log \frac{\|f\|_\mu^2}{\ell}. \end{aligned}$$

Therefore $\mathcal{I} = -\infty$. So by Fatou's Theorem we obtain $f = 0$ and the proof is complete.

3. COUNTABLE SUM OF DIRAC MEASURES

First observe that for the Dirac measure δ_ζ it is known, (see [4]), that $\mathcal{D}(\delta_\zeta) = \mathbb{C} + (z - \zeta)H^2$. Then a Borel set E is a uniqueness set for $\mathcal{D}(\delta_\zeta)$ if and only if the E has a positive Lebesgue measure. This result can be extended to some other discrete measures. For a positive Borel measure μ we will denote by $V_2(\mu)$ the Newtonian potential given by

$$V_2(\mu)(\zeta) = \int_0^{2\pi} \frac{d\mu(e^{it})}{|e^{it} - \zeta|^2}.$$

D. Guillot showed in [7, Theorem 2.1] that if there exists $f \in \mathcal{D}_\mu$ such that $f = 0$ μ a.e on \mathbb{T} then

$$\int_{\mathbb{T}} \log V_2(\mu)(\zeta) |d\zeta| < \infty.$$

He also proved that the converse is true for all discrete measures. The following result is an immediate consequence of [7].

Proposition 3. Consider a positive sequence $(a_n)_{n \geq 1}$ such that $\sum_{n \geq 1} a_n = 1$ and let $\mu = \sum_n a_n \delta_{\zeta_n}$ where $\zeta_n \in \mathbb{T}$. If

$$\int_{\mathbb{T}} \log V_2(\mu)(\zeta) |d\zeta| < \infty,$$

then a Borel set $E \subset \mathbb{T}$ is a uniqueness set for $\mathcal{D}(\mu)$ if and only if $|E| > 0$.

Proof. Since $\mathcal{D}(\mu) \subset H^2$, it is obvious that the condition $|E| > 0$ is sufficient. Conversely, Let $E \subset \mathbb{T}$ be a Borel set such that $|E| = 0$, there exists a function $\varphi \in H^\infty \setminus \{0\}$ such that $\varphi|_E = 0$. By [6], there exists $f \in \mathcal{D}(\mu) \cap H^\infty \setminus \{0\}$ such that $f(\zeta_n) = 0$, ($n \geq 1$). Now it suffices to prove that $\varphi f \in \mathcal{D}(\mu)$. Indeed,

$$\begin{aligned} \mathcal{D}_\mu(\varphi f) &= \int_{\mathbb{T}} \int_{\mathbb{T}} \left| \frac{\varphi f(\zeta) - \varphi f(\zeta')}{\zeta - \zeta'} \right|^2 |d\zeta| d\mu(\zeta') \\ &= \int_{\mathbb{T}} \int_{\mathbb{T}} \left| \frac{\varphi f(\zeta)}{\zeta - \zeta'} \right|^2 |d\zeta| d\mu(\zeta') \\ &\leq \|\varphi\|_\infty^2 \int_{\mathbb{T}} \int_{\mathbb{T}} \left| \frac{f(\zeta)}{\zeta - \zeta'} \right|^2 |d\zeta| d\mu(\zeta') \\ &= \|\varphi\|_\infty^2 \int_{\mathbb{T}} \int_{\mathbb{T}} \left| \frac{f(\zeta) - f(\zeta')}{\zeta - \zeta'} \right|^2 |d\zeta| d\mu(\zeta') \\ &= \|\varphi\|_\infty^2 \mathcal{D}_\mu(f). \end{aligned}$$

Which complete the proof. \square

Note that for a positive Borel measures $\mu = \sum_n a_n \delta_{\zeta_n}$ where $\zeta_n \in \mathbb{T}$ such that

$$\int_{\mathbb{T}} \log V_2(\mu)(\zeta) |d\zeta| = \infty, \quad (10)$$

we have no complete characterization for uniqueness sets for $\mathcal{D}(\mu)$. In this case, by [7] the countable set $\{\zeta \in \mathbb{T} : \mu(\zeta) \neq 0\}$ is a uniqueness set for $\mathcal{D}(\mu)$. The situation is more complicated than the previous case. In fact, Using theorem 1, we will construct more smaller uniqueness set for $\mathcal{D}(\mu)$. Before stating this result, we give an example of discrete measures μ satisfying (10). A closed set $E \subset \mathbb{T}$ is said to be a Carleson set if

$$\int_{\mathbb{T}} \log \text{dist}(\zeta, E) |d\zeta| > -\infty.$$

This condition is equivalent to $|E| = 0$ and

$$\sum_n |\ell_n| \log |\ell_n| > -\infty,$$

where ℓ_n are the complementary intervals of E , For more informations on Carleson sets, see e.g. [4, §4.4].

Proposition 4. *Let E be a countable closed set of the unit circle such that E is non Carleson set. Then there exists a discrete measure μ on the unit circle such that $E = \{\zeta \in \mathbb{T} : \mu(\zeta) \neq 0\}$ and (10) is satisfied.*

Proof. Let $\mathbb{T} \setminus E = \bigcup (a_n, b_n)$ and let $\mu = \sum_{n \geq 1} c_n (\delta_{a_n} + \delta_{b_n})$, with $c_n = b_n - a_n$.

We have

$$\begin{aligned} \int_{\mathbb{T}} \log V_2(\mu)(\zeta) |d\zeta| &\geq \sum_{n \geq 1} \int_{a_n}^{b_n} \log \frac{c_n}{\text{dist}^2(e^{it}, \{a_n, b_n\})} dt \\ &\geq \sum_{n \geq 1} \int_{a_n}^{b_n} \log \frac{c_n}{(b_n - a_n)^2} dt \\ &= \sum_{n \geq 1} (b_n - a_n) \log \frac{1}{b_n - a_n} = +\infty. \end{aligned}$$

□

Theorem 5. *There exists a discrete probability measure μ on the unit circle and a countable closed set $E \subset \mathbb{T}$, such that $\mu(E) = 0$ and E is a set of uniqueness for $\mathcal{D}(\mu)$.*

To prove this result we need the following lemma.

Lemma 6. *Let $\zeta \in \mathbb{T}$, $a \in (0, 1/2)$. Let μ be a probability measure on \mathbb{T} such that $\mu \geq \sum_{k \geq 2} a k^{-2} \delta_{\zeta e^{iak}}$, then $c_\mu(\{\zeta\}) \gtrsim \sqrt{a}$.*

Proof. Let

$$\nu = \sum_{k \geq 2} \frac{a}{k^2} \delta_{\zeta e^{iak}}.$$

By (1), see also [5, Theorem 2], we have

$$\begin{aligned} \frac{1}{c_\mu(\zeta)} &\asymp 1 + \int_0^1 \frac{dr}{(1-r)P_\mu(r\zeta) + (1-r)^2} \\ &\lesssim 1 + \int_0^1 \frac{dr}{(1-r)P_\nu(r\zeta) + (1-r)^2} \end{aligned}$$

For $a \leq 1 - r$, we have

$$\begin{aligned} (1-r)P_\nu(r\zeta) &\asymp \sum_{k \geq 2} \frac{a(1-r)^2}{k^2((1-r)^2 + a^{2k})} \\ &\asymp \sum_{k \geq 2} \frac{a}{k^2} \\ &\asymp a. \end{aligned}$$

For $1 - r < a$, let k_r be a real number such that $1 - r = a^{k_r}$. Since $x \mapsto x^2 a^{2x}$; $x \geq 1$, is a decreasing function, we have

$$\begin{aligned} (1-r)P_\nu(r\zeta) &\asymp \sum_{k \geq 2} \frac{a(1-r)^2}{k^2((1-r)^2 + a^{2k})} \\ &= \sum_{k \geq k_r} \frac{a}{k^2} + \sum_{k < k_r} \frac{a(1-r)^2}{k^2 a^{2k}} \\ &\asymp \frac{a}{k_r} \\ &= \frac{a \log(1/a)}{\log(1/1-r)}. \end{aligned}$$

Then we obtain

$$\begin{aligned} \frac{1}{c_\mu(\zeta)} &\leq 1 + \int_0^{1-a} \frac{dr}{a + (1-r)^2} + \int_{1-a}^1 \frac{dr}{\frac{a \log(1/a)}{\log(1/1-r)} + (1-r)^2} \\ &\leq 1 + \frac{1}{\sqrt{a}} \int_0^{\frac{1}{\sqrt{a}}} \frac{du}{1+u^2} + \int_{1-a}^1 \frac{\log(1/1-r) dr}{a \log(1/a)} \\ &\asymp \frac{1}{\sqrt{a}} \end{aligned}$$

and the lemma is proved. \square

Proof of Theorem 5. Let E be the set define by

$$E := \left\{ \zeta_n = e^{i\theta_n} : n \geq 3 \right\} \cup \{1\},$$

with $\theta_n := 1/\log(n)$. Let $a_n = \varepsilon(\theta_n - \theta_{n+1})$ for some small $\varepsilon > 0$. Consider the sequence $\zeta_{n,k} = e^{i\theta_{n,k}}$ where

$$\theta_{n,k} := \theta_n + a_n^k \quad k \geq 2.$$

The measure μ is given by

$$\mu := \sum_{n \geq 3} \sum_{k \geq 2} \frac{a_n}{k^2} \delta_{\zeta_{n,k}}.$$

To prove that E is a uniqueness set for $\mathcal{D}(\mu)$, we use theorem 1. Let $I_n = (\theta_n - a_n, \theta_n + a_n)$. By lemma 6; we have $c_\mu(\zeta_n) \geq \sqrt{a_n}$. Then

$$\begin{aligned} \sum_n |I_n| \log \left(\frac{c_\mu(I_n \cap E)}{|I_n|} \right) &= \sum_n |I_n| \log \left(\frac{c_\mu(\zeta_n)}{|I_n|} \right) \\ &\geq \sum_n a_n \log \left(\frac{1}{a_n} \right) = +\infty \end{aligned}$$

and the proof is complete.

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