



**HAL**  
open science

# Runge-Kutta Theory and Constraint Programming

Julien Alexandre Dit Sandretto, Alexandre Chapoutot

► **To cite this version:**

Julien Alexandre Dit Sandretto, Alexandre Chapoutot. Runge-Kutta Theory and Constraint Programming. International Symposium on Scientific Computing, Computer Arithmetics and Verified Numerics, Sep 2016, uppsala, Sweden. hal-01376456

**HAL Id: hal-01376456**

**<https://hal.science/hal-01376456v1>**

Submitted on 4 Oct 2016

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Runge-Kutta Theory and Constraint Programming

Julien Alexandre dit Sandretto and Alexandre Chapoutot

U2IS, ENSTA ParisTech, Université Paris-Saclay,  
828 bd des Maréchaux, 91762 Palaiseau cedex France  
alexandre@ensta.fr

**Keywords:** Runge-Kutta, Butcher theory, Validated simulation

## Introduction

There exist many Runge-Kutta methods (explicit or implicit), more or less adapted to a given class of problems. Some of them have interesting properties such as  $A$ -stability for stiff problems or symplecticity for problems with energy conservation. Defining a new method, adapted to a given class of problems, has become a challenge. Indeed, the number of stages and the order don't stop to increase. This race to the "best" method is interesting but forgot an important problem. More precisely, the coefficients of a Runge-Kutta method are more and more difficult to compute and the result is often given in floating-point numbers, which may lead to violate their definition rules. We propose a method using interval analysis tools to compute Runge-Kutta coefficients by using a solver based on guaranteed constraint programming. Moreover, with a global optimization process and a well chosen cost function, we propose a way to define some novel optimal Runge-Kutta methods.

One step of a Runge-Kutta integration scheme, applied on an ordinary differential equation  $\dot{y} = f(t, y)$ , is obtained with

$$y_{n+1} = y_n + h \sum_{i=1}^s b_i k_i, \text{ where } k_i = f \left( t_0 + c_i h, y_0 + h \sum_{j=1}^s a_{ij} k_j \right). \quad (1)$$

The coefficients  $c_i$ ,  $a_{ij}$  and  $b_i$ , for  $i, j = 1, \dots, s$ , fully characterize the Runge-Kutta methods and they are usually synthesized in a *Butcher*

*tableau* [1] of the form:

$$\begin{array}{c|ccc}
 c_1 & a_{11} & \dots & a_{1s} \\
 \vdots & \vdots & \ddots & \vdots \\
 c_s & a_{s1} & \dots & a_{ss} \\
 \hline
 & b_1 & \dots & b_s
 \end{array}$$

## Main idea

Our approach consists on the generation of constraints defined by the Butcher theory in order to build a Runge-Kutta method. Then we solve these constraints with a Branch&Prune algorithm. We also propose a Branch&Bound approach to define new optimal methods w.r.t. an easy to obtain cost function.

### Interval coefficients preserve properties

In a preliminary stage, we can verify that a Runge-Kutta method with interval coefficients preserves the Butcher rule and then that a method given for an order  $p$  has really a local truncature error in  $O(h^{p+1})$ . We also propose three methods using interval tools to check the linear stability, algebraically stability and symplecticity properties.

## Main results

First, our Branch&Prune based approach is used to find existing methods and by the way re-discover the Runge-Kutta theory such as i) Gauss-Legendre is the only 2-stages 4-order method; ii) there is no 2-stages 5-order method; etc. Our Branch&Bound method finds the same results as Ralston [2]. Second, both of our methods is used to define new validated Runge-Kutta methods.

### References:

- [1] BUTCHER, JOHN C., Coefficients for the study of Runge-Kutta integration processes, *Journal of the Australian Mathematical Society*, 5 (1963), No. 3, pp. 185–201.
- [2] RALSTON, ANTHONY, Runge-Kutta methods with minimum error bounds, *Mathematics of computation*, (1962), pp. 431-437.