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► **To cite this version:**

Raphaël Belliard, Bertrand Eynard, Olivier Marchal. Integrable differential systems of topological type and reconstruction by the topological recursion. 2016. hal-01376169v1

**HAL Id: hal-01376169**

**<https://hal.science/hal-01376169v1>**

Preprint submitted on 4 Oct 2016 (v1), last revised 14 Aug 2017 (v2)

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# Integrable differential systems of topological type and reconstruction by the topological recursion

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**Abstract :** Starting from a  $d \times d$  rational Lax pair system of the form  $\hbar \partial_x \Psi = L \Psi$  and  $\hbar \partial_t \Psi = R \Psi$  we prove that, under certain assumptions (genus 0 spectral curve and additional conditions on  $R$  and  $L$ ), the system satisfies the “topological type property”. A consequence is that the formal  $\hbar$ -WKB expansion of its determinantal correlators, satisfy the topological recursion. This applies in particular to all  $(p, q)$  minimal models reductions of the KP hierarchy, or to the six Painlevé systems.

## 1 Introduction and setting

First, we mention that this article is the generalization of [18, 17] (rank 2 systems) to arbitrary rank. The generalization is not straightforward and requires the new tools of [3, 6] since the loop equations and the spectral curves associated to arbitrary rank systems are far more involved.

## 1.1 Generalities about quantum curves and mirror symmetry

In the past few years, the interest has grown in the notion of “**quantum curves**” related to enumerative geometry problems. In particular the relationship to the topological recursion [11] has raised a specific interest. Indeed, many enumerative geometry problems have two sides related by mirror symmetry (in fact they have a third side, namely integrability) :

- a **B model** side, described in terms of some algebraic manifold, typically a complex plane curve called the “spectral curve” and given by an algebraic equation :

$$E_B(x, y) = 0 \tag{1-1}$$

Many invariants can be associated to a spectral curve, in particular the topological recursion invariants  $W_{g,n}$  of [11]. Many recent papers have dealt with a “quantization” of that spectral curve, into a differential operator :

$$E_B(x, y) \xrightarrow{\text{quantization}} \hat{E}_B(x, \hbar \frac{d}{dx}, \hbar) \tag{1-2}$$

such that  $\hat{E}_B(x, y, 0) = E_B(x, y)$  and such that it annihilates a “wave function”

$$\hat{E}_B(x, \hbar \frac{d}{dx}, \hbar) \cdot \psi_B(x, \hbar) = 0 \tag{1-3}$$

In general, the wave function  $\psi_B(x, \hbar)$  has an  $\hbar$  expansion of WKB type whose coefficients are certain combinations of the  $W_{g,n}$ ’s associated to the spectral curve. In other words in the B-model side, the quantum curve  $\hat{E}_B(x, \hbar \frac{d}{dx}, \hbar)$ , as well as the wave function  $\psi_B(x, \hbar)$ , are built from the classical spectral curve by the Topological Recursion.

- an **A model** side, describing an enumerative geometry problem, typically the enumeration of surfaces of given topologies together with some mapping into some target space. This includes many cohomological field theories, for example Gromov-Witten theory, as well as enumerations of maps, some conformal field theories, computing of knot invariants, random matrices... In all these problems, there is a notion of enumerating geometric objects of a given “genus”, and one can define some generating functions, with a formal parameter called  $\hbar$  (rather denoted  $g_s$  in topological string theory, or  $1/N$  in random matrix theory), by summation over the genus. For example in Gromov-Witten theory, the generating function  $W_{g,n}$  counts the number of holomorphic maps of Riemann surfaces

of genus  $g$  with  $n$  boundaries into a given Calabi-Yau manifold. The genus summation defines a formal series

$$W_n = \sum_{g=0}^{\infty} \hbar^{2g-2+n} W_{g,n}. \quad (1-4)$$

All these formal generating functions  $W_n$  can be put together to define a formal “wave function”  $\psi_A(x, \hbar)$  that encodes all the enumerative geometry.

- **Integrability.** In many such A-models, the geometry implies that the generating functions satisfy some equations (for instance gluing surfaces along their boundaries gives another surface), that can be encoded into an integrable system, such that the wave function  $\psi_A(x, \hbar)$  is its Baker-Akhiezer function. In other words, the geometric properties of the setup imply that the A-model’s wave function  $\psi_A$  has to satisfy some differential system, again typically a quantum curve  $\hat{E}_A(x, \hbar \frac{d}{dx}, \hbar) \cdot \psi_A(x, \hbar) = 0$ . For example the famous Witten-Kontsevich enumerative geometry problem of intersection theory on the moduli space of stable curves is related to the KdV integrable system. The corresponding wave function is simply the Airy function  $\psi_A(x, \hbar) = \text{Ai}(\hbar^{-2/3}x)$  and is annihilated by the operator  $\hat{E}_A = \hbar^2 \frac{d^2}{dx^2} - x$  which is a quantization of the classical spectral curve  $E_A(x, y) = y^2 - x$ . For cohomological field theories, the Dubrovin-Zhang [34, 35] and the Givental [36] formalisms also produce wave functions associated to integrable systems and quantum curves.

- One of the main questions regarding mirror symmetry is then to prove that the A-model and B-model give rise to the same wave function :

$$\psi_A(x, \hbar) \stackrel{?}{=} \psi_B(x, \hbar) \quad , \quad \hat{E}_A(x, \hbar \frac{d}{dx}, \hbar) \stackrel{?}{=} \hat{E}_B(x, \hbar \frac{d}{dx}, \hbar) \quad (1-5)$$

In particular, since  $E_B(x, y) = \hat{E}_B(x, y, 0)$ , it is easy to identify which B-model should be mirror to an A-model if we know  $\hat{E}_A$ . Notice that the equality holds in the sense of formal  $\hbar$ -series, so that we only need to work at the formal level.

- In this article, we shall go from an A-model type integrable system towards a corresponding B-model. In other words, we start from a wave function annihilated by a differential operator in some integrable hierarchy, and prove, under certain assumptions on the differential operators, that its WKB expansion defines some  $W_{g,n}$  differentials that obey the B-model topological recursion.

Our method is a generalization to systems of arbitrary rank of what was done in [18, 17] for  $2 \times 2$  systems. In [2, 6], it was proved that if a differential system satisfies

the so-called **Topological Type (TT)** property, then the corresponding  $W_{g,n}$  will necessarily satisfy the topological recursion.

What we do in this article is therefore to prove that a large class of integrable systems do satisfy the TT property.

We mention that there exists many other articles [3, 23, 24, 25, 26, 27, 28, 29, 30, 31], taking (case by case) the opposite path. Starting from a B-model (i.e. a classical spectral curve), they construct  $\psi_B$  by the topological recursion and prove that there exists a quantum curve of isomonodromic type (and thus related to an integrable system) that annihilates  $\psi_B$ .

Until today, there is no general theorem stating what kind of B-model spectral curve leads to an isomonodromic differential system (a quantum curve), and vice versa, there is no general theorem stating what kind of isomonodromic differential system has a WKB expansion governed by topological recursion. At the moment, all existing articles prove a correspondence within some specific subclasses. Most studied examples are rank 2 systems, which have made it easier.

This article pursues a similar goal, extending a known proof for certain rank 2 systems to higher rank systems. It provides sufficient conditions for a differential system to have a WKB expansion governed by topological recursion. The sufficient conditions are general enough so that they may be applied to many differential systems. In particular, they allow to recover all known cases like [1, 2, 18, 26, 17, 16, 25, 28].

### **Outline :**

- We first introduce the two compatible differential systems, the corresponding Lax equations and some useful notations.
- In section 2, we state the assumptions needed for our result. They rely on describing the algebro-geometric properties of the two underlying spectral curves –the eigenvalue loci of the Lax pair elements. Our assumptions are then that the main spectral curve has genus 0, that the auxiliary one is an embedding without self-intersections, and the requirement that subleading terms are less singular than leading ones in the topological expansions. These assumptions are indeed satisfied for the most famous integrable systems.
- In section 3, we recall the definitions of correlation functions by determinantal formulas and of their formal WKB  $\hbar$ -expansion.
- In section 4, we prove our main result : the correlators built from a Lax pair satisfying our assumptions, are of “topological type” (we recall the definition), which in turn implies that they satisfy topological recursion.

- Section 5 is our summary and conclusion. We mention what generalizations we may expect for non-zero genus spectral curves as well as the issue of the WKB expansion of the wave function in Appendix A.
- In Appendix B, we show examples of classical integrable systems that satisfy our assumptions.

## 1.2 Setting : compatible linear differential systems

Let  $\hbar > 0$  be given (usually called “formal expansion parameter”, “dispersion parameter”, “Planck constant” or just “small parameter”). As in [6], the natural context is the one of a reductive complex Lie algebra  $\mathfrak{g}$  and its associated connected Lie group  $G = e^{\mathfrak{g}}$ , here we will however mostly restrict ourselves to the case  $G = GL_d(\mathbb{C})$  and  $\mathfrak{g} = \mathfrak{gl}_d(\mathbb{C})$  (this is the most common setting in practice) and leave the study with general  $\mathfrak{g}$  for a later work currently under preparation [39].

Instead of a linear differential operator  $\hat{E}(x, \hbar \frac{d}{dx}, \hbar)$  of order  $d$  acting on a scalar wave function  $\psi(x, \hbar)$ , we consider an equivalent (and in fact more general) order one linear matrix differential system. More precisely we consider a time-dependent family of such systems :

- We shall consider a compatible system of linear equations of the form :

$$\begin{aligned} \hbar \partial_x \Psi(x, t, \hbar) &= L(x, t, \hbar) \Psi(x, t, \hbar) \\ \hbar \partial_t \Psi(x, t, \hbar) &= R(x, t, \hbar) \Psi(x, t, \hbar) \end{aligned} \quad (1-6)$$

where the  $d \times d$  matrix  $\Psi(x, t, \hbar) \in GL_d(\mathbb{C})$  is assumed invertible for all  $x$  (in the general setting :  $\Psi(x, t, \hbar) \in G$ ). The  $d \times d$  matrices  $L(x, t, \hbar)$  and  $R(x, t, \hbar)$  (in the general setting  $L, R \in \mathfrak{g}$ ) are assumed to be **rational functions** of  $x$  for any values of  $t$  and  $\hbar$ .  $x$  is usually called a “spectral parameter” and  $t$  a “time parameter”. Note that to shorten notations, **we shall often write only the  $x$  dependence explicitly and drop the  $t$  or  $\hbar$  dependence in the notations** :

$$L(x, t, \hbar) \stackrel{\text{notation}}{\equiv} L(x), \dots \quad (1-7)$$

We shall prove in this article that a set of conditions on  $L$  and  $R$  are sufficient for the system to be of “Topological Type”.

- The compatibility relation of the two equations is called the Lax equation :

$$\hbar \partial_t L(x, t, \hbar) - \hbar \partial_x R(x, t, \hbar) = [R(x, t, \hbar), L(x, t, \hbar)]. \quad (1-8)$$

- Generalization to arbitrary  $G$  : The system of equations (1-6) can be viewed, with  $x \in \Sigma_0$  in a local coordinate patch on a complex curve, as the equation defining a flat section  $\Psi(x, \hbar) \in G$  of a stable principal  $G$ -bundle  $\mathcal{E} = G \rightarrow \Sigma_0$  – with  $G$  a connected reductive Lie group – over  $\Sigma_0$ , equipped with a meromorphic connection  $\nabla = d - \hbar^{-1}L(x, \hbar)dx$ , where  $L(x, \hbar)dx$  is a  $\mathfrak{g}$ -valued meromorphic one form on  $\Sigma_0$ . In this general context,  $x$  is called a spectral parameter,  $\hbar^{-1}L(x, \hbar)dx$  is called a Higgs field, and the pair  $(\mathcal{E}, L)$  is called a Hitchin pair. Here we shall restrict ourselves to the Riemann sphere  $\Sigma_0 = \bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ .
- In the construction of [1, 2], to a solution  $\Psi(x, t, \hbar) \in G$  of the differential system is associated a solution  $M(x, t, \hbar) \in \mathfrak{g}$  of the adjoint system :

$$\begin{aligned} \hbar \partial_x M(x, t, \hbar) &= [L(x, t, \hbar), M(x, t, \hbar)] \\ \hbar \partial_t M(x, t, \hbar) &= [R(x, t, \hbar), M(x, t, \hbar)] \end{aligned} \quad (1-9)$$

whose solutions are of the form

$$\Psi(x, t, \hbar) E \Psi(x, t, \hbar)^{-1} \quad (1-10)$$

where  $E$  is a constant (in the sense  $\partial_x E = 0$ ) element of  $\mathfrak{g}$ . We will therefore denote them as :

$$M(x.E) \stackrel{\text{notation}}{\equiv} M(x.E, t, \hbar) = \Psi(x, t, \hbar) E \Psi(x, t, \hbar)^{-1}, \quad (1-11)$$

often not writing the  $t$  and  $\hbar$  dependence to lighten notations. Notice that any another solution of (1-6) is obtained from  $\Psi$  through the right multiplication  $\Psi(x) \rightarrow \Psi(x)C$  by a constant matrix  $C \in G$ ,  $\partial_x C = 0$ .  $M(x.E)$  then changes to  $M(x.CEC^{-1})$ , i.e. an adjoint transformation of  $E$ . Note also that  $M(x.E)$  depends linearly on  $E \in \mathfrak{g}$ . Since  $\Psi$  has monodromies around the singularities of  $L(x)$ ,  $x$  lives on the universal cover  $\tilde{\Sigma}_0$  of  $\Sigma_0 \setminus \{\text{singularities of } L\}$ .

- Then, still following [1, 2], one associates to the differential system of the form (1-6) or (1-9), a sequence of local  $n$ -forms  $W_n$  on  $(\tilde{\Sigma}_0 \times \mathfrak{g})^n$  usually called the “correlators” :

$$W_n(x_1.E_1, \dots, x_n.E_n) = \begin{cases} \hbar^{-1} \text{Tr } L(x_1) M(x_1.E_1) dx_1 & n = 1 \\ \frac{1}{n} \sum_{\sigma \in \mathfrak{S}_n} \frac{\text{Tr} \prod_{i=1}^n M(x_{\sigma(i)}.E_{\sigma(i)})}{\prod_{i=1}^n (x_{\sigma(i)} - x_{\sigma(i+1)})} \prod_{i=1}^n dx_i & n \geq 2 \end{cases} \quad (1-12)$$

where a choice is made once and for all for writing non-commutative products. In the generalization to a Riemann surface  $\Sigma_0$  instead of  $\bar{\mathbb{C}}$ , the  $\sqrt{dx_i dx_j} / (x_i - x_j)$  terms are replaced by inverses of twisted Fay’s prime forms. In  $G = GL_d(\mathbb{C})$  the

trace of a product is defined in the defining representation (i.e. the usual trace for  $d \times d$  matrices). For other Lie groups, we define the trace by choosing the matrix-trace in a once for all given faithful representation. These correlators  $W_n$  appear naturally in matrix models and in many enumerative problems [1, 2, 6, 10, 32, 33].

- In [6], it was proved that these  $W_n$ 's always satisfy a family of equations called “**loop equations**” (and that are analogous to Virasoro or W-algebra constraints, in the CFT context, see [24]). This is important because loop equations can be solved recursively in terms of  $\hbar$ -expansions.
- The WKB asymptotics are defined as formal  $\hbar$ -series solutions to (1-6) with  $\hbar$  assumed to be small. They take the form :

$$\Psi(x, t, \hbar) \underset{\hbar \rightarrow 0}{\sim} V(x, t) \left( \mathbf{1}_d + \sum_{k=1}^{\infty} \hbar^k \hat{\Psi}^{(k)}(x, t) \right) e^{\frac{1}{\hbar} T(x, t) C} \quad (1-13)$$

that we shall explain below in section 3, let us just mention here that  $T(x, t)$  is diagonal (and more generally belongs to a Cartan subalgebra  $\mathfrak{h} \subset \mathfrak{g}$ ). In turn this implies that if  $CEC^{-1}$  is diagonal ( $\text{Adj}_C E \in \mathfrak{h}$ ), then  $M(x.E, t, \hbar)$  is also an  $\hbar$ -formal series expansion but this time without exponential factors :

$$M(x.E, t, \hbar) = \sum_{k=0}^{\infty} \hbar^k \check{M}^{(k)}(x.E, t), \quad (1-14)$$

Consequently all correlators  $W_n$  defined by (1-12) also admit a  $\hbar$ -formal expansion that we shall denote

$$W_n(x_1.E_1, \dots, x_n.E_n) = \sum_{k=-\delta_{n,1}}^{\infty} \hbar^k W_n^{(k)}(x_1.E_1, \dots, x_n.E_n). \quad (1-15)$$

**The main questions are then the following :**

1. **Is there a general method to compute the coefficients  $W_n^{(k)}$ ?**
  2. **Taking  $k = 2g - 2 + n$ , does  $W_n^{(k)}$  coincide with  $W_{n,g}$  computed from the topological recursion?**
- In [2, 6] some sufficient conditions on the differential systems, known as the “Topological Type” (TT) property, were given to get a positive answer. The goal of this article is to find a large class of Lax pairs  $(L, R)$  satisfying the TT property.



## 2 Assumptions

We shall now describe our assumptions on the pair  $(L(x, t, \hbar), R(x, t, \hbar))$ . These assumptions are described in terms of algebraic geometry and the notion of spectral curve.

### 2.1 Spectral curve(s)

**Assumption 1 ( $\hbar$  expansion)** *We make the assumption that  $L(x, t, \hbar)$  and  $R(x, t, \hbar)$  have a limit at  $\hbar \rightarrow 0$  :*

$$\lim_{\hbar \rightarrow 0} L(x, t, \hbar) = L^{(0)}(x, t) \quad , \quad \lim_{\hbar \rightarrow 0} R(x, t, \hbar) = R^{(0)}(x, t), \quad (2-1)$$

*and that both limits are rational functions of  $x$ . Furthermore, we assume that  $L(x, t, \hbar)$  and  $R(x, t, \hbar)$  have an  $\hbar$  expansion (formal or asymptotic) of the form :*

$$L(x, t, \hbar) = \sum_{k=0}^{\infty} \hbar^k L^{(k)}(x, t) \quad , \quad R(x, t, \hbar) = \sum_{k=0}^{\infty} \hbar^k R^{(k)}(x, t). \quad (2-2)$$

*where all  $L^{(k)}(x, t)$  and  $R^{(k)}(x, t)$  with  $k \geq 0$  are rational functions of  $x$ .*

The spectral curve is defined as the zero locus of the characteristic polynomial of the matrix  $L^{(0)}$ , i.e. the eigenvalue locus, whence the name ‘‘spectral’’ curve. In the general Lie group context, this corresponds to Hitchin’s map.

**Definition 2.1 (Spectral curve)** *The (family of) spectral curve of the differential system is the zero locus of the characteristic polynomial in  $\mathbb{C} \times \mathbb{C}$  :*

$$\mathcal{S} \equiv \mathcal{S}_t = \{(x, y) \in \mathbb{C}^2 \text{ such that } E_t(x, y) = \det(y - L^{(0)}(x, t)) = 0\} \quad (2-3)$$

*This defines an (a family of) algebraic plane curve immersed into  $\mathbb{C} \times \mathbb{C}$ . We define the two meromorphic functions corresponding to the  $x$  and  $y$  projection in  $\mathbb{C} \times \mathbb{C}$  :*

$$x \equiv x_t : \mathcal{S}_t \rightarrow \mathbb{C} \quad , \quad y \equiv y_t : \mathcal{S}_t \rightarrow \mathbb{C} \quad (2-4)$$

$$(x, y) \mapsto x \quad (x, y) \mapsto y. \quad (2-5)$$

*The plane curve can be desingularized. Its desingularization is a smooth compact Riemann surface noted  $\Sigma \equiv \Sigma_t$ , and the functions  $x \equiv x_t$  and  $y \equiv y_t$  can be identified with meromorphic functions  $\Sigma_t \rightarrow \mathbb{C}$ . This allows to redefine the (family of) spectral curve as the triple :*

$$\mathcal{S} \equiv \mathcal{S}_t = (\Sigma_t, x_t, y_t), \quad (2-6)$$

given by a (family of) compact Riemann surface  $\Sigma \equiv \Sigma_t$ , equipped with two meromorphic functions  $x_t : \Sigma_t \rightarrow \mathbb{C}$  and  $y_t : \Sigma_t \rightarrow \mathbb{C}$ . On a compact curve, any two meromorphic functions are related by an algebraic equation :

$$\forall z \in \Sigma_t \quad , \quad E_t(x_t(z), y_t(z)) = 0 \quad \text{where } E \equiv E_t \text{ is a (family of) polynomial} \quad (2-7)$$

thus giving an alternative definition of the spectral curve directly from (2-6). We shall also be interested in the (family of) one-form  $\omega_1^{(0)}$  on  $\Sigma_t$  defined by  $\omega_1^{(0)} = ydx$ , sometimes called the Liouville form, because it is the pullback to the spectral curve, of the tautological form of  $\mathbb{C} \times \mathbb{C}$ , viewed as the cotangent space of  $\mathbb{C}$ .

The  $y$ -degree of the characteristic polynomial is the matrix's size (the rank of  $GL_d$ ) :

$$\deg_y E_t(x, y) = d \quad (2-8)$$

and thus for a given generic  $x \in \mathbb{C}$ , the equation  $E_t(x, y) = 0$  has  $d$  solutions, that are the  $d$  eigenvalues  $Y_1(x, t), \dots, Y_d(x, t)$  of  $L^{(0)}(x, t)$ . They are the images by the function  $y_t$ , of the  $d$  preimages of  $x$  by  $x_t$  :

$$x_t^{-1}(x) = \{z \in \Sigma_t \mid x_t(z) = x\} = \{z^1(x, t), \dots, z^d(x, t)\} \quad (2-9)$$

gives

$$Y_i(x, t) = y_t(z^i(x)). \quad (2-10)$$

Here the ordering of eigenvalues is arbitrary and can always be locally chosen analytical within some open simply connected domain of  $\mathbb{C} \setminus \{\text{Singularities of } x_t \text{ and } x_t^{-1}\}$ . The ordering will turn out to be irrelevant for our purposes, and we assume it chosen within such domain once and for all.

**Definition 2.2 (Auxiliary spectral curve)** In the same spirit we define the (family of) auxiliary spectral curve by the zero locus of the characteristic polynomial of  $R^{(0)}$  :

$$\tilde{\mathcal{S}} \equiv \tilde{\mathcal{S}}_t = \{(x, s) \in \mathbb{C}^2 \text{ such that } \tilde{E}_t(x, s) = \det(s - R^{(0)}(x, t)) = 0\} \quad (2-11)$$

that we shall encode as the triple

$$\tilde{\mathcal{S}}_t = (\tilde{\Sigma}_t, \tilde{x}_t, s_t), \quad (2-12)$$

given by a (family of) Riemann surface  $\tilde{\Sigma}_t$ , equipped with two meromorphic functions  $\tilde{x}_t : \tilde{\Sigma}_t \rightarrow \mathbb{C}$  and  $s_t : \tilde{\Sigma}_t \rightarrow \mathbb{C}$ , related by the algebraic equation

$$\forall z \in \tilde{\Sigma}_t \quad , \quad \tilde{E}_t(\tilde{x}_t(z), s_t(z)) = 0. \quad (2-13)$$

Similarly for a given  $x$ , there exist  $d$  solutions noted  $(S_1(x, t), \dots, S_d(x, t))$  of the auxiliary curve  $\tilde{E}_t(x, s) = 0$ . They are the  $d$  eigenvalues of  $R^{(0)}(x, t)$ , and also  $S_i(x, t) = s_t(\tilde{z}_t^i(x))$  with  $\tilde{x}_t(\tilde{z}_t^i(x)) = x$ .

**Lemma 2.1** *The matrices  $L^{(0)}(x, t), R^{(0)}(x, t)$  commute thus they generically have a common basis of eigenvectors and their eigenvalues are not algebraically independent. In particular the spectral curves  $\mathcal{S}_t, \tilde{\mathcal{S}}_t$  have the same desingularization :  $\Sigma_t = \tilde{\Sigma}_t$  and the same  $x$ -projection to  $\bar{\mathbb{C}} : \tilde{x}_t = x_t$ .*

**Proof :**

At order  $\hbar^0$ , the Lax compatibility condition (1-6) reads :

$$[L^{(0)}(x, t), R^{(0)}(x, t)] = 0. \quad (2-14)$$

For generic  $x$ , all the eigenvalues of  $R^{(0)}(x, t)$  are distinct. It implies that the set of matrices commuting with  $R^{(0)}(x, t)$  is the algebra of polynomials of  $R^{(0)}(x, t)$ . Consequently there exists a polynomial  $Q(x, s)$  (the interpolating Lagrange polynomial) such that  $L^{(0)}(x, t) = Q(x, R^{(0)}(x, t))$ , and  $Y_i(x) = Q(x, S_i(x))$ , i.e.  $y(z) = Q(\tilde{x}(z), s(z))$  for all  $z \in \tilde{\Sigma}_t$ . This implies that  $y_t$  is a meromorphic function on  $\tilde{\Sigma}_t$ . Exchanging the roles of  $R^{(0)}$  and  $L^{(0)}$  also shows that  $s_t$  is a meromorphic function on  $\Sigma_t$ . Therefore  $\Sigma_t = \tilde{\Sigma}_t$ , and  $x_t = \tilde{x}_t$ .  $\square$

## 2.2 Geometry of the spectral curve

### 2.2.1 Genus 0 assumption

From now on, we shall assume that our system is such that :

**Assumption 2 (Genus zero Spectral Curve)** *The compact Riemann surface  $\Sigma_t$  has genus equal to 0. This implies that it is isomorphic to the Riemann sphere  $\Sigma_t = \bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\} = \mathbb{C}P^1$  (the complex plane compactified at  $\infty$ ), and that, for any given  $t$  in an open domain, the functions  $x_t, y_t, s_t$ , are rational functions of a variable  $z \in \bar{\mathbb{C}}$  :*

$$x_t(z), y_t(z), s_t(z) \in \mathbb{C}(z) = \{ \text{rational functions of } z \} \quad (2-15)$$

**Remark 2.1** The issue of determining if this genus zero hypothesis can be lifted is mostly open. In fact in the example of matrix models, it is known that the TT property is generically not satisfied when the genus is strictly positive. But a generalization of the TT property can be found by allowing the coefficients in the  $\hbar$  expansion, to be “oscillatory”, i.e. bounded quasi-periodic functions of  $\frac{1}{\hbar}$ . In that case, the oscillatory terms are themselves found by the topological recursion. See [20].

Besides, in knot theory, the TT property happens to hold with spectral curves (A-polynomial) of strictly positive genus. This is due to a miracle that the  $\frac{1}{\hbar}$  term is exactly a period of the oscillatory term, and thus can be treated as a constant coefficient, see [21], and then the TT property holds. So the general situation is still unclear.

**Remark 2.2** The choice of the parameterizing variable  $z$  is arbitrary up to Möbius transformations (automorphisms of the Riemann sphere) :

$$z \mapsto \frac{az + b}{cz + d}. \quad (2-16)$$

In particular, we may chose the coefficients of the Möbius transformation  $a, b, c, d$  to be time dependent :  $a(t), b(t), c(t), d(t)$ .

The functions  $x_t, y_t, s_t$  are functions of both  $z$  and  $t$ , and they are rational in  $z$ . We mention that their dependence on  $t$  does not need to be rational. In many examples they are transcendental functions of  $t$ , like for example solutions of Painlevé equations.

We shall denote for any function  $f(z, t)$  :

$$f'(z, t) = \frac{\partial f}{\partial z} \quad , \quad \dot{f}(z, t) = \frac{\partial f}{\partial t}. \quad (2-17)$$

Note that taking a time derivative at fixed  $x(z, t)$ , following from the chain rule, takes the form of a Poisson bracket  $\{f, x\} = \dot{f} x' - \dot{x} f'$  :

$$\left. \frac{df(z, t)}{dt} \right|_{x(z, t)} = \dot{f} - \dot{x} \frac{f'}{x'} = \frac{\dot{f} x' - \dot{x} f'}{x'} = \frac{1}{x'} \{f, x\}, \quad (2-18)$$

thus reflecting the symplectic structure of  $\mathbb{C} \times \mathbb{C}$  of which the family  $\{\Sigma_t\}_t$  defines a lagrangian foliation.

### 2.2.2 Behavior at poles

**Lemma 2.2** *The poles of the eigenvalues  $y_t(z)$  (resp.  $s_t(z)$ ) are poles of  $L^{(0)}(x(z), t)$  (resp.  $R^{(0)}(x(z), t)$ ) of at least the same order.*

**Proof :**

Let  $\alpha$  be a pole of  $y_t(z)$  of order  $d_\alpha > 0$  so that  $y_t(z) = O((z - \alpha)^{-d_\alpha})$ . Let us assume that  $(z - \alpha)^{d_\alpha} L^{(0)}(x(z), t) = o(1)$ . This would imply that

$$0 = \det(y_t(z)\text{Id} - L^{(0)}(x(z), t)) = y_t(z)^d (1 + o(1)) \quad (2-19)$$

which is a contradiction. This implies that  $L^{(0)}(x(z), t)$  has a pole of order at least  $d_\alpha$ . Obviously, the same holds for  $R^{(0)}(x, t)$ .  $\square$

**Lemma 2.3 (Poisson relation)** *The eigenvalues  $(Y_i(x, t))_{1 \leq i \leq d}$  of  $L^{(0)}(x, t)$  and  $(S_i(x, t))_{1 \leq i \leq d}$  of  $R^{(0)}(x, t)$  are related by :*

$$\frac{\partial Y_i(x, t)}{\partial t} = \frac{\partial S_i(x, t)}{\partial x}. \quad (2-20)$$

Equivalently, the functions  $x_t(z)$ ,  $y_t(z)$ ,  $s_t(z)$  satisfy :

$$\frac{\partial y_t(z)}{\partial t} \frac{\partial x_t(z)}{\partial z} - \frac{\partial x_t(z)}{\partial t} \frac{\partial y_t(z)}{\partial z} = \frac{\partial s_t(z)}{\partial z}, \quad (2-21)$$

or written in the notations of (2-17) :

$$\{y_t, x_t\} = \dot{y}_t x_t' - \dot{x}_t y_t' = s_t'. \quad (2-22)$$

**Proof :**

Since  $L^{(0)}(x, t)$  and  $R^{(0)}(x, t)$  commute, they generically have a common basis of eigenvectors, let us denote  $V(x, t)$  the matrix whose  $i^{\text{th}}$  column is the eigenvector of  $L^{(0)}(x, t)$  with eigenvalue  $Y_i(x, t)$  and of  $R^{(0)}(x, t)$  with eigenvalue  $S_i(x, t)$ . Denoting  $Y(x, t) = \text{diag}(Y_1(x, t), \dots, Y_d(x, t))$  and  $S(x, t) = \text{diag}(S_1(x, t), \dots, S_d(x, t))$ , we have

$$L^{(0)}(x, t) = V(x, t)Y(x, t)V(x, t)^{-1} \quad , \quad R^{(0)}(x, t) = V(x, t)S(x, t)V(x, t)^{-1}. \quad (2-23)$$

Now write the Lax equation to order  $\hbar^1$  and conjugate by  $V(x, t)$  :

$$\begin{aligned} [S(x, t), V(x, t)^{-1}L^{(1)}(x, t)V(x, t)] + [V(x, t)^{-1}R^{(1)}(x, t)V(x, t), Y(x, t)] \\ = \partial_t Y(x, t) - \partial_x S(x, t) \end{aligned} \quad (2-24)$$

The left hand side is a sum of commutators with diagonal matrices, hence has vanishing entries on the diagonal. On the contrary, the right hand side is a diagonal matrix and evaluating its diagonal entries gives the sought result :

$$0 = \partial_t Y(x, t) - \partial_x S(x, t). \quad (2-25)$$

□

As an immediate corollary we get :

**Corollary 2.1** *Finite (i.e at  $x \neq \infty$ ) singularities of  $S$  are also singularities of  $Y$ , of at least the same degree. And if  $S$  has a singularity of order  $d_\infty$  at  $x = \infty$ , then  $Y$  has a singularity at  $x = \infty$  of order at least  $d_\infty + 1$ .*

Note that the converse is not true : some singularities of  $Y$  may be time independent and may not be singularities of  $S$ . In some sense, we can say that  $R^{(0)}$  is less singular than  $L^{(0)}$ .

### 2.2.3 Branchpoints and double points

**Definition 2.3 (Branchpoints)** *We define the branchpoints  $(a_i)_{1 \leq i \leq r}$  as the points of  $\Sigma$  where the map  $z \mapsto x(z)$  is not locally invertible. There may be two kinds of branchpoints :*

- *Finite branchpoints, at which  $x(a_i) \neq \infty$ . They are zeros of the differential  $dx$  :*

$$dx(a_i) = 0. \quad (2-26)$$

*Moreover, they are among the simultaneous solutions of  $E(x, y) = 0$  and  $E_y(x, y) \equiv \partial_y E(x, y) = 0$ .*

- *Branchpoints at poles of  $x$  of order  $\geq 2$ .*

*A branchpoint  $a_i$  of the spectral curve  $\mathcal{S}$  (resp.  $\tilde{\mathcal{S}}$ ) is called regular if it is not a branchpoint of  $y$  (resp.  $s$ ). Generic finite branchpoints of  $x$  have order 2, i.e. are simple zeros of  $dx$ , and regularity means that they are not zeros of  $dy$  (resp.  $ds$ ).*

Note that the branchpoints may depend on time  $t$ . However, the number of branchpoints  $r \geq 1$  does not locally depend on  $t$ . We will also need the following definition :

**Definition 2.4 (Double points (also called self-intersections))** *We define the double points  $((b_i, \bar{b}_i))_{1 \leq i \leq r''}$  (resp.  $((\tilde{b}_i, \tilde{\bar{b}}_i))_{1 \leq i \leq \tilde{r}''}$ ) of the curve  $\mathcal{S} = (\Sigma, x, y)$  (resp. of  $\tilde{\mathcal{S}} = (\Sigma, x, s)$ ), as the pairs  $(b_i, \bar{b}_i) = (z, z')$  (resp.  $(\tilde{b}_i, \tilde{\bar{b}}_i) = (z, z')$ ) solutions of*

$$\left\{ \begin{array}{l} x(z) = x(z') \\ y(z) = y(z') \\ z \neq z' \end{array} \right. , \quad \left( \text{resp.} \left\{ \begin{array}{l} x(z) = x(z') \\ s(z) = s(z') \\ z \neq z' \end{array} \right. \right) \quad (2-27)$$

*These double points  $(x, y) = (x(b_i), y(b_i)) = (x(\bar{b}_i), y(\bar{b}_i)) \in \mathbb{C} \times \mathbb{C}$  of the spectral curve (resp.  $(x, s) = (x(b_i), s(b_i)) = (x(\bar{b}_i), s(\bar{b}_i)) \in \mathbb{C} \times \mathbb{C}$ ), are then solutions of the system*

$$\left\{ \begin{array}{l} E(x, y) = 0 \\ E_y(x, y) = 0 \\ E_x(x, y) = 0 \end{array} \right. , \quad \left( \text{resp.} \left\{ \begin{array}{l} \tilde{E}(x, s) = 0 \\ \tilde{E}_s(x, s) = 0 \\ \tilde{E}_x(x, s) = 0 \end{array} \right. \right) \quad (2-28)$$

We shall make the following assumption regarding the double points of the auxiliary spectral curve :

**Assumption 3 (Regularity of  $\mathcal{S}_t$  and no double points for  $\tilde{\mathcal{S}}_t$ )** *We make the assumption that the auxiliary spectral curve  $\tilde{\mathcal{S}}_t$  is regular and has no double points. In other words,  $\tilde{\mathcal{S}}_t$  is a smooth embedding into  $\mathbb{C} \times \mathbb{C}$  (rather than an immersion) with no self-intersection. Moreover we assume that  $\mathcal{S}_t$  is regular.*

Note that the last assumption does not exclude the possibility that the spectral curve  $\mathcal{S}_t$  admits double points. Moreover, the auxiliary spectral curves  $\tilde{\mathcal{S}}_t = (\tilde{\mathbb{C}}, x_t, s_t)$  satisfying assumptions 2 and 3 are the same as the ones described in [3].

We have the following lemma :

**Lemma 2.4** *The meromorphic one-form*

$$\frac{dx(z)}{E_y(x(z), y(z))} \quad (2-29)$$

is holomorphic at all branchpoints (finite or infinite). It has poles only at double points (generically simple poles at  $b_i$  and  $\bar{b}_i$  with opposite residues) and/or at simple poles of  $x$ .

**Proof :**

This is a classical algebro-geometric result, we refer to [14]. Let us sketch the proof. Near a finite branchpoint  $a$  of order  $k \geq 2$ ,  $z = (x - x(a))^{1/k}$  can be used as a local coordinate. Consider the case  $y(a) \neq \infty$ . Since the branchpoint is regular,  $dy$  does not vanish at that point, i.e.

$$y(z) = y_0 + y_1 z + O(z^2), \quad y_1 \neq 0. \quad (2-30)$$

This gives :

$$E(x, y) = ((y - y_0)^k - y_1^k (x - x(a))) \times (1 + o(1)), \quad (2-31)$$

and

$$\frac{dx}{E_y(x, y)} = \frac{kz^{k-1} dz}{k(y - y_0)^{k-1}} \times (1 + o(1)) \quad (2-32)$$

and thus  $\frac{dx}{E_y(x, y)}$  is analytic at  $z = 0$ , i.e. at  $x = a$ . The other cases where  $x(a) = \infty$  or  $y(a) = \infty$  can be treated similarly in a local variable. In other words, for finite regular branchpoints, both  $dx(z)$  and  $E_y(x(z), y(z))$  vanish, at the same order so that the ratio remains finite.

For double points,  $E_y(x(z), y(z))$  vanishes but not  $dx(z)$ , so that the ratio has a pole. Writing

$$E(x, y) = \prod_{k=1}^d (y - y(z^k(x))) \quad (2-33)$$

we have that when  $z \rightarrow b_i$ , simultaneously  $z' \rightarrow \bar{b}_i$ , and

$$E_y(x(z), y(z)) \sim (y(z) - y(z')) E_{y,y}(x(b_i), y(b_i)) \sim (z - z') \frac{dy(z)}{dz} E_{y,y}(x(b_i), y(b_i)). \quad (2-34)$$

Assuming that the double point is generic, i.e.  $E_{y,y} dy \neq 0$ , we get :

$$\operatorname{Res}_{z \rightarrow b_i} \frac{dx(z)}{E_y(x(z), y(z))} = - \operatorname{Res}_{z \rightarrow \bar{b}_i} \frac{dx(z)}{E_y(x(z), y(z))} = \frac{dx(b_i)}{dy(b_i) E_{y,y}(x(b_i), y(b_i))} \quad (2-35)$$

□

## 2.3 Eigenvectors

Let  $\text{div}_\infty x = \sum_{k=1}^p d_k \alpha_k$  be the divisor of poles of the rational function  $z \mapsto x(z)$ , with  $d_k$  the degree of  $\alpha_k$  ( $\alpha_k$  may depend on  $t$ ). The total degree is the size of the matrix

$$\sum_{k=1}^p d_k = d. \quad (2-36)$$

Up to a Möbius change of variable on  $z$ , we may assume that none of the  $\alpha_k$ 's is at  $\infty$ . We can thus write the rational function  $x(z)$  uniquely as :

$$x(z) = X_{\infty,0} + \sum_{k=1}^p \sum_{l=1}^{d_k} \frac{X_{k,l}}{(z - \alpha_k)^l}. \quad (2-37)$$

where  $d_k \geq 1$ . Moreover, if  $d_k \geq 2$  then  $\alpha_k$  is a (non-finite) branchpoint. Note that if one of the  $\alpha_k$  is at  $\alpha_\infty = \infty$  we would rather write :

$$x(z) = \sum_{l=0}^{d_\infty} X_{\infty,l} z^l + \sum_{k=1}^p \sum_{l=1}^{d_k} \frac{X_{k,l}}{(z - \alpha_k)^l}, \quad (2-38)$$

But to avoid useless notation complications, upon changing  $z$  by a Möbius transformation, we shall assume that all poles of  $x(z)$  are different from  $\infty$ .

### 2.3.1 The generalized Vandermonde matrix $\mathcal{V}(x)$

**Definition 2.5** For generic points  $z \in \Sigma$ , in particular away from the branchpoints, let us define the  $d$ -dimensional vector  $\vec{\mathcal{V}}(z)$  with entries labeled by all possible pairs  $(k, l)$  with  $1 \leq k \leq p$  and  $1 \leq l \leq d_k$  :

$$\vec{\mathcal{V}}(z) = (\mathcal{V}_{k,l}(z))_{k,l} \quad , \quad \text{where } \mathcal{V}_{k,l}(z) = \frac{1}{(z - \alpha_k)^l \sqrt{x'(z)}}. \quad (2-39)$$

In addition we define these matrix entries to be ordered as follows

$$\vec{\mathcal{V}}(z) = (\mathcal{V}_{1,1}(z), \dots, \mathcal{V}_{1,d_1}(z), \dots, \mathcal{V}_{p,1}(z), \dots, \mathcal{V}_{p,d_p}(z)). \quad (2-40)$$

Let  $\mathcal{V}(x)$  be the  $d \times d$  square matrix whose columns are the vectors  $\vec{\mathcal{V}}(z^j(x))$  :

$$\forall 1 \leq k \leq p, 1 \leq l \leq d_k, 1 \leq j \leq d : (\mathcal{V}(x))_{k,l;j} = \mathcal{V}_{k,l}(z^j(x)) \quad (2-41)$$

It is analytic locally in some open simply connected domain, in which the  $z^i$  and the square root are defined.

**Remark 2.3** The sign of the square root, chosen arbitrarily, is well defined and locally analytic within some open simply connected domain – the same domain in which we defined the ordering of  $z^j(x)$ . In fact in all what follows, the square root will almost always appear to the power two, so the sign will eventually be irrelevant.



**Remark 2.4** Note that if  $x$  has only one pole ( $p = 1$  and  $d_1 = d$ ) then the previous matrix is a Vandermonde matrix multiplied by  $\frac{1}{(z-\alpha_1)\sqrt{x'(z)}}$ , hence the name “generalized Vandermonde matrix”.

The matrix  $\mathcal{V}(x)$  satisfies remarkable properties.

**Lemma 2.5** *There exists an invertible  $d \times d$  matrix  $C \equiv C(t)$  (independent of  $x$ ), such that*

$$\mathcal{V}(x)^T C \mathcal{V}(x) = \text{Id}, \quad (2-42)$$

where  $\mathcal{V}(x)^T$  denotes the transpose of the matrix  $\mathcal{V}(x)$ . Its coefficients are given by  $C_{k,l;k',l'} = -\delta_{k,k'} X_{k,l+l'-1}$ .

**Proof :**

From (2-37) we have

$$\frac{x(z) - x(z')}{z - z'} = \sum_{k,l;k',l'} \frac{C_{k,l;k',l'}}{(z - \alpha_k)^l (z' - \alpha_{k'})^{l'}} \quad , \quad C_{k,l;k',l'} = -\delta_{k,k'} X_{k,l+l'-1} \quad (2-43)$$

The matrix  $C$  is made of triangular blocks because  $X_{k,l+l'-1} = 0$  if  $l + l' > d_k + 1$ .  $C$  is invertible because the antidiagonals of each triangular block is  $-C_{k,d_k} \neq 0$  by definition of  $d_k$ . We have :

$$\forall 1 \leq i, j \leq d : (\mathcal{V}(x)^T C \mathcal{V}(x'))_{i,j} = \frac{x - x'}{z^i(x) - z^j(x')} \frac{1}{\sqrt{x'(z^i(x))x'(z^j(x'))}} \quad (2-44)$$

Evaluating at  $x = x'$  we get :

$$\mathcal{V}(x)^T C \mathcal{V}(x) = \text{Id}. \quad (2-45)$$

□

For example in the case  $k = 3$  and  $(d_1, d_2, d_3) = (3, 2, 4)$ , the matrix  $C$  looks like :

$$C = \begin{array}{|cc|ccc|ccc} \hline * & * & . & . & . & . & . & . & . \\ * & . & . & . & . & . & . & . & . \\ \hline . & . & * & * & * & * & . & . & . \\ . & . & * & * & * & . & . & . & . \\ . & . & * & * & . & . & . & . & . \\ . & . & * & . & . & . & . & . & . \\ \hline . & . & . & . & . & . & * & * & * \\ . & . & . & . & . & . & * & * & . \\ . & . & . & . & . & . & * & . & . \\ \hline \end{array} \quad (2-46)$$

Lemma 2.5 implies that :

$$\mathcal{V}(x)^{-1} = \mathcal{V}(x)^T C \quad \text{and} \quad \mathcal{V}(x) \mathcal{V}(x)^T = C^{-1} \quad (2-47)$$

In particular, the matrix  $C$  is always symmetric, and in each block it has the Hankel property : it depends only on  $l + l'$ .

**Corollary 2.2** *The matrix  $\mathcal{V}(x)^{-1}d\mathcal{V}(x)$  is antisymmetric, it is worth zero on its diagonal, and off diagonal entries are given by :*

$$\forall i \neq j : (\mathcal{V}(x)^{-1}d\mathcal{V}(x))_{i,j} = \frac{-\sqrt{dz^i(x)dz^j(x)}}{z^i(x) - z^j(x)} = \frac{-1}{\mathcal{E}(z^i(x), z^j(x))} \quad (2-48)$$

where  $\mathcal{E}(z, z') = \frac{z-z'}{\sqrt{dzdz'}}$  is the prime form on the Riemann sphere.

**Proof :**

Taking the  $x$ -differential of (2-47) and using the fact that  $C$  is independent of  $x$  directly shows that  $\mathcal{V}(x)^{-1}d\mathcal{V}(x)$  is antisymmetric. Starting from (2-44) and differentiating with respect to  $x'$  we get :

$$\begin{aligned} (\mathcal{V}(x)^{-1}d\mathcal{V}(x'))_{i,j} &= \frac{(x - x') dz^j(x')}{(z^i(x) - z^j(x'))^2} \frac{1}{\sqrt{x'(z^i(x))x'(z^j(x'))}} \\ &\quad - \frac{x'(z^j(x')) dz^j(x')}{z^i(x) - z^j(x')} \frac{1}{\sqrt{x'(z^i(x))x'(z^j(x'))}} \\ &\quad - \frac{1}{2} \frac{x''(z^j(x')) dz^j(x')}{x'(z^j(x'))} \frac{x - x'}{z^i(x) - z^j(x')} \frac{1}{\sqrt{x'(z^i(x))x'(z^j(x'))}} \end{aligned} \quad (2-49)$$

We now take the limit  $x \rightarrow x'$ . Since  $x = x(z^i(x)) = x(z^j(x))$  we get the equalities  $dx = x'(z^i(x))dz^i(x) = x'(z^j(x))dz^j(x)$ . When  $i \neq j$ , the denominator does not vanish and only the terms without  $x - x'$  in the numerator survive thus giving the claimed result. When  $i = j$ , the first two terms are computed by Taylor expansion up to the second order, i.e. involve the second derivative of  $x$ , which is exactly canceled by the last term.  $\square$

**Corollary 2.3** *With  $G = GL_d(\mathbb{C})$ , with Cartan subalgebra  $\mathfrak{h}$  the set of diagonal matrices, and defining the canonical basis of  $\mathfrak{h}$  :  $e_i = \text{diag}(0, \dots, 0, \overset{i}{1}, 0, \dots, 0)$ , we have the identity :*

$$\frac{\text{Tr } \mathcal{V}(x)e_i\mathcal{V}(x)^{-1}\mathcal{V}(x')e_j\mathcal{V}(x')^{-1}}{(x - x')^2} dx dx' = \frac{dz^i(x)dz^j(x')}{(z^i(x) - z^j(x'))^2} = B(z^i(x), z^j(x')), \quad (2-50)$$

where  $B(z, z') = \frac{dzdz'}{(z-z')^2}$  is the fundamental 2<sup>nd</sup> kind bi-differential of the Riemann sphere.

Moreover, we get the following property :

**Proposition 2.1** *The matrix  $x \mapsto \mathcal{V}(x)e_i\mathcal{V}(x)^{-1}$  is a rational function of  $z^i(x)$ . It is only singular when  $z^i(x)$  is at the branchpoints (i.e. finite branchpoints where  $x'(z) = 0$  and poles of  $x(z)$  of degree at least 2).*

**Proof :**

Use  $\mathcal{V}(x)^{-1} = \mathcal{V}(x)^T C$  and the definition of  $\mathcal{V}(x)$  :

$$(\mathcal{V}(x)e_i\mathcal{V}(x)^T)_{(k,l),(k',l')} = \frac{1}{(z^i(x) - \alpha_k)^l (z^i(x) - \alpha_{k'})^{l'} x'(z^i(x))} \quad (2-51)$$

This function has poles when  $x'(z^i(x))$  vanishes, i.e. at branchpoints, and also possibly at the punctures  $z^i(x) = \alpha_k$ .

If  $\alpha_j$  is a puncture (i.e. a pole of  $x(z)$ ) but not a branchpoint we must have  $d_j = 1$  and thus  $l = 1$ . We get that :

$$(\mathcal{V}(x)e_i\mathcal{V}(x)^T)_{(k,l),(k',l')} = O\left((z^i(x) - \alpha_j)^{d_j+1-l\delta_{k,j}-l'\delta_{k',j}}\right). \quad (2-52)$$

The worst case happens when  $k = k' = j$  implying  $l = l' = 1$ , in which case the exponent is 0, showing that  $\mathcal{V}(x)e_i\mathcal{V}(x)^T$  has no pole.  $\square$

We will now use the matrix  $\mathcal{V}(x)$  and its properties to formulate our next assumption.

## 2.4 Decomposition of the matrix of eigenvectors at order $\hbar^0$

**Assumption 4 (Eigenvector decomposition)** *We assume that there exists an invertible  $d \times d$  matrix  $v(t)$ , independent of  $x$ , such that*

$$V(x, t) = v(t)\mathcal{V}(x) \quad (2-53)$$

*is an invertible matrix whose columns are the eigenvectors of  $L^{(0)}$  (and thus of  $R^{(0)}$ ). Consequently we have (not writing the  $t$  dependence to lighten notations) :*

$$L^{(0)}(x) = v\mathcal{V}(x)Y(x)\mathcal{V}(x)^T C v^{-1}, \quad (2-54)$$

$$R^{(0)}(x) = v\mathcal{V}(x)S(x)\mathcal{V}(x)^T C v^{-1}. \quad (2-55)$$

*In coordinates it is equivalent to :*

$$\begin{aligned} (L^{(0)}(x))_{i,j} &= \sum_{k,l,k',l',l'',m} \frac{-y(z^m(x))v_{i;k,l}X_{k',l'+l''-1}(v^{-1})_{k',l'';j}}{(z^m(x) - \alpha_k)^l (z^m(x) - \alpha_{k'})^{l'} x'(z^m(x))} \\ (R^{(0)}(x))_{i,j} &= \sum_{k,l,k',l',l'',m} \frac{-s(z^m(x))v_{i;k,l}X_{k',l'+l''-1}(v^{-1})_{k',l'';j}}{(z^m(x) - \alpha_k)^l (z^m(x) - \alpha_{k'})^{l'} x'(z^m(x))} \end{aligned} \quad (2-56)$$

Notice that the last assumption implies that :

$$v(t)^{-1}L^{(0)}(x, t)v(t)C(t)^{-1} \quad \text{and} \quad v(t)^{-1}R^{(0)}(x, t)v(t)C(t)^{-1} \quad (2-57)$$

are symmetric matrices.

**Remark 2.5** This is a very strong assumption on  $L^{(0)}(x, t)$  and  $R^{(0)}(x, t)$ . It implies that the  $x$ -dependent part of  $L^{(0)}(x, t)$  (resp.  $R^{(0)}(x, t)$ ) has in fact only  $\frac{d(d+1)}{2}$  degrees of freedom, rather than  $d^2$ . In other words it imposes  $\frac{d(d-1)}{2}$  constraints on  $L^{(0)}(x, t)$  (resp.  $R^{(0)}(x, t)$ ).

**Remark 2.6** The purpose of assumption 4 is to match the (defined below) correlator  $W_2^{(0)}$  with the fundamental  $2^{\text{nd}}$  kind bi-differential  $B(z_1, z_2)$ , defined in Corollary 2.3, as it is necessary for the system to satisfy the topological type property.

This assumption may look too restrictive on the matrices  $L^{(0)}$  and  $R^{(0)}$  but the set of matrices which satisfy it is far from empty. In fact most (if not all) well-known integrable systems satisfy it and examples of Painlevé systems and  $(p, q)$  minimal models are given in appendix B.

## 2.5 Classification of admissible systems

From (2-56) we must have :

$$\begin{aligned}
(v^{-1}L^{(0)}(x)vC^{-1})_{k,l;k',l'} &= \sum_{j=1}^d \mathcal{V}_{k,l}(z^j(x))\mathcal{V}_{k',l'}(z^j(x))y(z^j(x)) \\
&= \sum_{j=1}^d \frac{1}{(z^j(x) - \alpha_k)^l} \frac{1}{(z^j(x) - \alpha_{k'})^{l'}} \frac{y(z^j(x))}{x'(z^j(x))} \\
&= \sum_{j=1}^d \operatorname{Res}_{z \rightarrow z^j(x)} \frac{1}{(z - \alpha_k)^l} \frac{1}{(z - \alpha_{k'})^{l'}} \frac{y(z)}{x(z) - x} \\
&= - \sum_{p \in \{\text{poles of } x \text{ and } y\}} \operatorname{Res}_{z \rightarrow p} \frac{1}{(z - \alpha_k)^l} \frac{1}{(z - \alpha_{k'})^{l'}} \frac{y(z)}{x(z) - x}
\end{aligned} \tag{2-58}$$

The pole at  $z = \alpha_i$  gives a polynomial of  $x$  of degree lower or equal to  $\frac{l\delta_{k,i} + l'\delta_{k',i} - 2d_i + \deg_{\alpha_i} y}{d_i}$ . Thus if  $y$  has no pole at  $\alpha_i$ , this gives at most an  $x$  independent term, and only for  $k = k' = i, l = l' = d_i$ .

If  $p$  is a pole of  $y$  which is not a pole of  $x$ , we get a pole  $(x(p) - x)^m$  with  $m \leq \frac{\deg_p y}{1 + \operatorname{ord}_p x'}$ .

### 2.5.1 Decomposition on $z^r$

Any rational function  $y(z)$  can be uniquely written as

$$y(z) = \sum_{r=0}^{d-1} z^r f_r(x(z)). \tag{2-59}$$

where  $f_r(x)$  is a rational function of  $x$ . Since functions of  $x$  go through (2-58), it is sufficient to study the cases  $y(z) = z^r$ .

So let us substitute  $y(z) \rightarrow z^r$  in (2-58), with  $0 \leq r \leq d-1$ , and we assume (up to a Möbius transformation of  $z$ ) that  $x$  is regular at  $z = \infty$  (i.e. none of the  $\alpha_i$ 's is located at  $\infty$ ). The contribution to (2-58) of poles at  $\alpha_i$ 's is a constant matrix  $\hat{A}_{i,r}$ , which is a triangular block of size  $d_i$ , which we denote :

$$A_{r,0} = \sum_i \tilde{A}_{i,r} \quad , \quad (\tilde{A}_{i,r})_{k,l;k',l'} = \delta_{k,i} \delta_{k',i} \tilde{A}_{i,r,l+l'} \quad (2-60)$$

that is non vanishing only if  $l+l' \geq d_i+1$ . On the anti-diagonal we get :

$$\tilde{A}_{i,r,d_i+1} = \frac{-\alpha_i^r}{X_{i,d_i}}. \quad (2-61)$$

Example :

$$\tilde{A}_{2,r} = \begin{array}{|c|c|c|} \hline \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & * & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & * & * & * & \cdot & \cdot & \cdot \\ \hline \cdot & * & * & * & * & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline \end{array} \quad (2-62)$$

The contribution of the pole at  $z = \infty$  takes the form :

$$\sum_{m=1}^r \frac{\hat{A}_{r,m}}{(x - x(\infty))^m} \quad (2-63)$$

and we have that :

$$(\hat{A}_{r,m})_{k,l;k',l'} = 0 \quad \text{if} \quad l+l'-2 > r-m. \quad (2-64)$$

For example if  $r = 1$  we have  $(\hat{A}_{1,1})_{k,l;k',l'} = \delta_{l,1} \delta_{l',1}$  :

$$\hat{A}_{1,1} = \begin{array}{|c|c|c|} \hline 1 & \cdot & 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline 1 & \cdot & 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline 1 & \cdot & 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline \end{array} \quad (2-65)$$

Finally :

$$v^{-1}L^{(0)}(x)vC^{-1} = \sum_{r=0}^{d-1} \sum_{m=0}^r \frac{f_r(x)}{(x - x(\infty))^m} \hat{A}_{r,m}, \quad (2-66)$$

we end up with a matrix  $L^{(0)}(x)$  that, up to some left/right multiplications by  $x$ -independent matrices ( $v$  on the left and  $Cv^{-1}$  on the right) of a very restrictive form.

### 2.5.2 Decomposition on $(z - \alpha_i)^{-r}$

A better decomposition is the following : any function  $y(z)$  can be uniquely written as

$$y(z) = \sum_i \sum_{r=1}^{d_i} \frac{\mathcal{Y}_{i,r}(x(z))}{(z - \alpha_i)^r}, \quad (2-67)$$

where each  $\mathcal{Y}_{i,r}(x)$  is a rational function of  $x$ , given by

$$\mathcal{Y}_{i,r}(x) = - \sum_j \frac{y(z^j(x))}{x'(z^j(x))} \sum_{l=r}^{d_i} X_{i,l}(z^j(x) - \alpha_i)^{r-l-1} \quad (2-68)$$

This gives

$$v^{-1}L^{(0)}(x)vC^{-1} = \sum_{i,r} \mathcal{Y}_{i,r}(x)A_{i,r}(x) \quad (2-69)$$

where the matrices  $A_{i,r}(x)$  are computed using  $y(z) = (z - \alpha_i)^{-r}$  with  $1 \leq r \leq d_i$ . Using (2-58), we find that each  $A_{i,r}(x)$  is a polynomial of  $x$  of degree at most 1

$$A_{i,r}(x) = xA'_{i,r} + A_{i,r}, \quad (2-70)$$

where the matrices  $A'_{i,r}$  and  $A_{i,r}$  have the following block shape :

$$r = 1 \quad \rightarrow \quad A_{2,1} = \begin{array}{|c|c|c|} \hline \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot \\ \hline \end{array}, \quad A'_{2,1} = \begin{array}{|c|c|c|} \hline \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot \\ \hline \end{array} \quad (2-71)$$

$$r = 2 \quad \rightarrow \quad A_{2,2} = \begin{array}{|c|c|c|} \hline \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot \\ \hline \end{array}, \quad A'_{2,2} = \begin{array}{|c|c|c|} \hline \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot \\ \hline \end{array} \quad (2-72)$$

and so on, for higher  $r$ , the non-vanishing off-diagonal blocks have size  $r \times d_i$ , and the non-vanishing entries are some universal functions of the  $X_{i,k}$ 's. Eventually we have

$$v^{-1}L^{(0)}(x)vC^{-1} = \sum_{i,r} \mathcal{Y}_{i,r}(x)(xA'_{i,r} + A_{i,r}) \quad (2-73)$$

Again we obtain a very restrictive class of matrices  $L^{(0)}(x)$ .

### 2.5.3 Classification of $R^{(0)}(x)$

The previous results also hold for  $R^{(0)}$  with  $y$  replaced by  $s$ . However due to the requirement that the auxiliary curve does not have any double points, the interesting cases are even more restrictive.

We may uniquely write

$$s(z) = \sum_{j=0}^m f_j(x(z)) z^j \quad , \quad m \leq d-1. \quad (2-74)$$

If  $m = 1$ , then it is obvious that there can be no double points, in that case

$$s(z) = f_0(x(z)) + f_1(x(z))z. \quad (2-75)$$

In other words,  $R^{(0)}(x, t)$

$$\begin{aligned} R^{(0)}(x, t) &= f_0(x, t)v(t)\hat{A}_{0,0}(t)C(t)v(t)^{-1} \\ &+ f_1(x, t)v(t)\left(\hat{A}_{1,0}(t)C(t) + \frac{\hat{A}_{1,1}(t)C(t)}{x - x(\infty, t)}\right)v(t)^{-1}. \end{aligned} \quad (2-76)$$

Up to a Möbius transformation on  $z$  we could have chosen  $z = \infty$  to be a pole of  $x$ , and then we would have obtained

$$R^{(0)}(x, t) = f_\infty(x, t)v(t)(A_{\infty,1}(t)C(t) + xA'_{\infty,1}(t)C(t))v(t)^{-1}. \quad (2-77)$$

Remark that all  $(p, q)$  minimal models, as well as Painlevé systems are indeed like that.

Notice that if  $d_\infty > 1$  then  $A'_{\infty,1}(t)C(t)$  is a nilpotent matrix :

$$(A'_{\infty,1}(t)C(t))_{k,l;k',l'} = \frac{1}{X_{\infty, d_\infty}} \delta_{k,\infty} \delta_{k',\infty} \delta_{l, d_\infty} \delta_{l',1}. \quad (2-78)$$

## 2.6 Assumptions regarding the $\hbar$ higher orders

In order to prove the topological type property and in addition to assumptions 2 and 4, we make the following sufficient assumptions regarding the spectral curve and the possible singularities of the system. We shall need the notion that  $L^{(k \geq 1)}$  has to be “less singular” than  $L^{(0)}$  – symbolically denoted  $L^{(k)} \prec L^{(0)}$  –. Our precise statement is the following :

**Assumption 5 (Analytic behavior  $L^{(k)} \prec L^{(0)}$ )** *We assume that :*

- for every  $k \geq 1$ , all poles of  $L^{(k)}(x, t)$  are among the poles of  $L^{(0)}(x, t)$ .

- for any matrix  $\tilde{C}$ , and any generic distinct  $x_0, x_1$ , the following  $\hbar$ -formal series whose coefficients are bi-rational functions of  $x$  and  $y$  :

$$\frac{\det \left( y - L(x) - \frac{\tilde{C}}{(x-x_0)(x-x_1)} \right) - \det (y - L^{(0)}(x))}{E_y(x, y)} dx \quad (2-79)$$

is, when restricted to the spectral curve, a one-form  $\Omega(z)$  that is analytic (at each order in  $\hbar$ ) at all singularities of  $L^{(0)}(x)$ .

Equivalently, its only singularities can either be poles over  $x = x_0$  and  $x = x_1$ , due to the  $\frac{\tilde{C}}{(x-x_0)(x-x_1)}$  term, or at double points of  $\mathcal{S} : (b_i, \tilde{b}_i)$ .

$$\begin{aligned} \Omega(z) &= \det \left( y(z) - L(x(z)) - \frac{\tilde{C}}{(x(z) - x_0)(x(z) - x_1)} \right) \frac{dx(z)}{E_y(x(z), y(z))} \\ &= \sum_i \beta_i \left( \frac{dz}{z - b_i} - \frac{dz}{z - \tilde{b}_i} \right) + \sum_{i \in \{0,1\}} \sum_{j=1}^d \sum_{k=1}^d \frac{c_{i,j,k} dz}{(z - z^j(x_i))^k} \end{aligned} \quad (2-80)$$

where the coefficients  $\beta_i, c_{i,j,k}$  are formal power series of  $\hbar$ , starting at  $O(\hbar)$ .

In other words, the  $\hbar$  corrections do not change the Newton's polygon of  $E(x, y)$ . They may only change the interior coefficients, as well as possibly adding poles over  $x = x_0$  or  $x = x_1$ .

Evaluating this one-form at  $z^i(x)$ , inserting and subtracting the diagonal term  $Y(x) = V(x)^{-1}L^{(0)}(x)V(x)$  and then expanding the determinant, we get after simplification that it is equal to

$$\Omega(z^i(x)) = dx \sum_{I \subset \{1, \dots, d\}, i \in I} \frac{\det_{I \times I} \left( V(x)^{-1} \left( L(x) - L^{(0)}(x) + \frac{\tilde{C}}{(x-x_0)(x-x_1)} \right) V(x) \right)}{\prod_{j \in I, j \neq i} (y(z^i(x)) - y(z^j(x)))}. \quad (2-81)$$

In particular, to order  $\hbar$  we must have  $\forall i$  :

$$dx \left( V(x)^{-1} L^{(1)}(x) V(x) \right)_{i,i} = 0 \quad (2-82)$$

(which implies  $W_1^{(0)}(x.e_i) = 0$ , as we will see below). This is equivalent to say that  $L^{(1)}(x)$  must be derived from  $L^{(0)}(x)$ , i.e.  $\exists \tilde{L}^{(1)}(x)$  such that

$$L^{(1)}(x) = [\tilde{L}^{(1)}(x), L^{(0)}(x)]. \quad (2-83)$$

At order  $\hbar^2$  we get that

$$dx \left( V(x)^{-1} L^{(2)}(x) V(x) \right)_{i,i} - \sum_{j \neq i} dx \frac{\left( V(x)^{-1} L^{(1)}(x) V(x) \right)_{i,j} \left( V(x)^{-1} L^{(1)}(x) V(x) \right)_{j,i}}{(y(z^i(x)) - y(z^j(x)))} \quad (2-84)$$

is analytic at all poles of  $y$ .



**Remark 2.7** Assumption 5 may appear technical but it can be proved easily in many cases. For example :

1. Assumption 5 is trivially verified if for all  $k \geq 1$ ,  $L^{(k)}$  is independent of  $x$ . (This happens for the Airy Lax pair for example).
2. Assumption 5 is verified if  $L(x, \hbar)$  is a Fuchsian system, i.e. has only simple poles  $c_i(t)$  independent of  $\hbar$ , and residues  $C_i(t, \hbar)$  whose eigenvalues are independent of  $\hbar$

$$L(x, t, \hbar) = \sum_{i=1}^p \frac{C_i(t, \hbar)}{x - c_i(t)} \quad (2-85)$$

Indeed, in that case the poles of  $L^{(k)}$  are the same as those of  $L^{(0)}$ . The eigenvalues of  $L(x, t, \hbar)$  have only simple poles above  $x = c_i(t)$ , with residues the eigenvalues of  $C_i(t, \hbar)$ , and thus all the singular behavior of the eigenvalues of  $L(x, t, \hbar)$ , is independent of  $\hbar$ , showing that the characteristic polynomials of  $L(x, t, \hbar)$  and  $L^{(0)}(x, t)$  can differ only by the interior of their Newton's polygon.

## 2.7 Parity Assumption

In order to prove sufficient conditions for the topological type property, we need (as proposed in [2]) another assumption :

**Assumption 6 (Parity)** *We assume that there exists a matrix  $\Gamma(t, \hbar) = \sum_{k=0}^{\infty} \hbar^k \Gamma^{(k)}(t)$ , independent of  $x$ , such that*

$$L(x, t, -\hbar) = \Gamma(t, \hbar)^{-1} L(x, t, \hbar)^T \Gamma(t, \hbar). \quad (2-86)$$

with

$$\Gamma^{(0)} = (v^T(t))^{-1} C v(t)^{-1} = \Gamma^{(0)T}. \quad (2-87)$$

Again this assumption is not empty and it is satisfied for many well-known integrable systems. For example it is satisfied for the Painlevé Lax pairs and the  $(p, q)$  minimal models. Also, to leading order in  $\hbar$  this assumption is a consequence of assumption 4.

This assumption was made in [2] and automatically gives one of the TT properties. This assumption is not know to be necessary, but so far we have not found any counter example.

Notice that we have

$$\Gamma(t, -\hbar) = \Gamma(t, \hbar)^T, \quad (2-88)$$

i.e. for all  $k \geq 0$  :

$$\Gamma^{(k)}(t) = (-1)^k \Gamma^{(k)}(t)^T. \quad (2-89)$$

In other words, the coefficients of the matrices appearing in the series expansion of  $\Gamma(t, \hbar)$  are either symmetric or antisymmetric matrices depending on the parity of their index.

### 3 The matrix $M(x.E, t)$ and the correlators $W_n$

Following the works of [1, 2] we now define the following quantities (we omit to explicit the  $t$ -dependence for clarity) :

**Definition 3.1** For any solution  $\Psi(x)$  of the system (1-6), and any constant  $d \times d$  matrix  $E \in \mathfrak{g}$ , we define

$$M(x.E) = \Psi(x)E\Psi(x)^{-1}. \quad (3-1)$$

It satisfies the adjoint system to (1-6)

$$\begin{aligned} \hbar\partial_x M(x.E) &= [L(x), M(x.E)] \\ \hbar\partial_t M(x.E) &= [R(x), M(x.E)]. \end{aligned} \quad (3-2)$$

In other words, at fixed  $E$ , the map  $x \mapsto M(x.E)$  is a flat section of the adjoint connection on the adjoint bundle.

**Remark 3.1** Equations (1-6) are isospectral, i.e. they imply that the eigenvalues of  $M(x.E)$  are independent of  $x$  and of  $t$ .

Most often we shall choose  $E$  in a Cartan subalgebra  $\mathfrak{h} \subset \mathfrak{gl}_d(\mathbb{C})$ , i.e. a diagonal matrix, and thus define  $e_a$  the basis of rank one diagonal projectors :

$$e_a = \text{diag}(0, \dots, 0, \overset{a}{1}, 0, \dots, 0). \quad (3-3)$$

In that case, since  $e_a$  is a rank one projector, then so is  $M(x.e_a)$  :

$$M(x.e_a, t, \hbar)^2 = M(x.e_a, t, \hbar) \quad , \quad \text{Tr} M(x.e_a, t, \hbar) = 1. \quad (3-4)$$

#### 3.1 WKB expansion

WKB expansions are usually defined for wave functions  $\Psi(x)$ , but here we shall use the adjoint version of WKB for  $M(x.E)$ . In the end, the two versions are equivalent, as explained in remark 3.2 below.

Note that WKB expansions are defined only within sectors,  $x \in$  an open simply connected domain containing no singularity of  $x, y, s$  neither any branchpoints. In such sectors, a consistent analytic ordering of preimages  $z^1(x), \dots, z^d(x)$  is well defined, as well as the sign of the square-root  $\sqrt{x'(z^i(x))}$ .

The system (3-2) admits an  $\hbar$  formal series solution :

**Theorem 3.1 ( $\hbar$  expansion of  $M$ )** *There exists a unique  $\hbar$ -formal series*

$$M(x.e_a, t, \hbar) = V(x, t)e_aV(x, t)^{-1} + \sum_{k=1}^{\infty} M^{(k)}(z^a(x), t)\hbar^k \quad (3-5)$$

that satisfies the differential system :

$$\begin{aligned}\hbar\partial_x M(x.e_a, t, \hbar) &= [L(x, t, \hbar), M(x.e_a, t, \hbar)] \\ \hbar\partial_t M(x.e_a, t, \hbar) &= [R(x, t, \hbar), M(x.e_a, t, \hbar)]\end{aligned}\quad (3-6)$$

and such that  $M(x.e_a, t, \hbar)$  is a rank one projector :

$$M(x.e_a, t, \hbar)^2 = M(x.e_a, t, \hbar) \quad , \quad \text{Tr } M(x.e_a, t, \hbar) = 1. \quad (3-7)$$

Moreover, the coefficients  $(M^{(k)}(z, t))_{i,j}$  are **rational functions of  $z$** .

**Remark 3.2** The  $\hbar$ -expansion of  $M$  is equivalent to the WKB expansion for  $\Psi$  given by :

$$\Psi^{\text{WKB}}(x, t, \hbar) = V(x, t) \left( \mathbf{1}_d + \sum_{k \geq 1} \hbar^k \Psi^{(k)}(x, t) \right) e^{\hbar^{-1}T(x,t)} \quad (3-8)$$

where  $T(x, t) = \text{diag}(T_1(x, t), \dots, T_d(x, t))$  with

$$\partial_x T_i(x, t) = y_i(x, t) \quad , \quad \partial_t T_i(x, t) = s_i(x, t). \quad (3-9)$$

Indeed, if one chooses  $E = e_a$  diagonal, the exponential terms disappear in the product  $M = \Psi E \Psi^{-1}$  and one finds an  $\hbar$  expansion for  $M$  without exponential terms.

Vice-versa,  $\Psi$  is recovered from  $M$  by the formula (proved in appendix A)

$$\Psi(x, t, \hbar)_{i,a} = M_{i,1}(x.e_a) e^{\frac{1}{\hbar} \int^x dx' \frac{\sum_{k=1}^d M_{1,k}(x'.e_a) L(x')_{k1}}{M_{1,1}(x'.e_a)}} \quad (3-10)$$

and an  $\hbar$ -expansion for  $M$  of the form (3-5) leads to a WKB type expansion for  $\Psi$ .

**Proof :**

First notice that the property of being a rank one projector is compatible with the flows (in  $x$  and  $t$ ) of the differential system, the flows are isospectral, meaning that the eigenvalues of  $M(x.E)$  are conserved.

Let us start by studying the formal expansion of  $M$  conjugated by  $V$ , i.e.  $\tilde{M}(x.e_a, t, \hbar) = V(x, t)^{-1} M(x.e_a, t, \hbar) V(x, t)$ . Its  $\hbar$ -expansion is of the form :

$$\tilde{M}(x.e_a, t, \hbar) = e_a + \sum_{k=1}^{\infty} \hbar^k \tilde{M}^{(k)}(x.e_a, t). \quad (3-11)$$

It satisfies the differential system :

$$\begin{aligned}\hbar\partial_t \tilde{M}(x.e_a, t) &= \left[ V^{-1}(x, t) R(x, t) V(x, t) - V(x, t)^{-1} (\hbar\partial_t V(x, t)), \tilde{M}(x.e_a, t) \right] \\ \hbar\partial_x \tilde{M}(x.e_a, t) &= \left[ V^{-1}(x, t) L(x, t) V(x, t) - V(x, t)^{-1} (\hbar\partial_x V(x, t)), \tilde{M}(x.e_a, t) \right]\end{aligned}\quad (3-12)$$

The differential system (3-12) for  $\tilde{M}$  is sufficient to determine recursively the coefficients  $\tilde{M}^{(k)}(x.e_a, t)$  of the expansion. Let us first denote

$$\tilde{U}(x) = V(x, t)^{-1} \partial_t V(x, t) = \mathcal{V}^T C v^{-1} \partial_t v \mathcal{V} + \mathcal{V}^{-1} \partial_t \mathcal{V}, \quad (3-13)$$

and

$$\tilde{U}(x)_{i,j} = \tilde{u}(z^i(x), z^j(x)), i \neq j \quad , \quad \tilde{U}(x)_{i,i} = \tilde{u}_{\text{diag}}(z^i(x)) \quad (3-14)$$

where  $\tilde{u}(z, z') \sqrt{x'(z)x'(z')}$  is a rational function of  $z$  and  $z'$ , and  $\tilde{u}_{\text{diag}}(z)x'(z)$  is a rational function of  $z$ . Similarly we define :

$$U(x) = V(x, t)^{-1} \partial_x V(x, t) = \mathcal{V}^T C \partial_x \mathcal{V}, \quad U(x)_{i,j} = u(z^i(x), z^j(x)), i \neq j \quad (3-15)$$

According to corollary 2.2, we have  $U(x)_{i,i} = 0$  and if  $i \neq j$

$$U(x)_{i,j} = \frac{-1}{(z^i(x) - z^j(x)) \sqrt{x'(z^i(x))x'(z^j(x))}}. \quad (3-16)$$

The first step of the proof is to show by recursion on  $k$  that :

- if  $i \neq j$ ,  $\tilde{M}^{(k)}(x.e_a, t)_{i,j} = m_k(z^a(x), z^i(x), z^j(x))_{i,j}$ , where  $m_k(z, z', z'')_{i,j} \sqrt{x'(z')x'(z'')}$  is a rational function of all its arguments.
- if  $i = j$ ,  $\tilde{M}^{(k)}(x.e_a, t)_{i,i} = m_k(z^a(x), z^i(x))_{i,i}$  where  $m_k(z, z')_{i,i} x'(z')$  is a rational function of all its arguments. For convenience in the notations, we shall write  $m_k(z, z')_{i,j} = m_k(z, z', z')_{i,i}$ .

To leading order in  $\hbar$ , equations (3-12) reduce to :

$$0 = [S, \tilde{M}^{(0)}] = [Y, \tilde{M}^{(0)}], \quad (3-17)$$

Thus,  $\tilde{M}^{(0)}(x.e_a) = e_a$  satisfies the last equations and moreover we have  $m_0(z, z', z'')_{i,j} = 0$  if  $i \neq j$  and  $m_0(z, z', z')_{i,i} = \delta_{i,a}$  if  $i = j$ . Consequently the induction is initialized for  $k = 0$ .

Let us look at  $k \geq 1$ . Looking at the first equation of (3-12) at order  $\hbar^k$  provides :

$$\partial_t \tilde{M}^{(k-1)} = \left[ S, \tilde{M}^{(k)} \right] + \sum_{l=1}^k \left[ V^{-1} R^{(l)} V, \tilde{M}^{(k-l)} \right] + \left[ \tilde{M}^{(k-1)}, V^{-1} \partial_t V \right] \quad (3-18)$$

In other words for  $i \neq j$  we get :

$$\begin{aligned} & \left( \tilde{M}^{(k)}(x.e_a, t) \right)_{i,j} \\ &= \frac{1}{s_i(x) - s_j(x)} \left( \partial_t \left( \tilde{M}^{(k-1)} \right) \right)_{i,j} \end{aligned}$$

$$\begin{aligned}
& - \sum_{l=1}^k \left[ V^{-1} R^{(l)} V, \tilde{M}^{(k-l)} \right]_{i,j} + \left[ \tilde{M}^{(k-1)}, V^{-1} \partial_t V \right]_{i,j} \\
= & \frac{1}{s(z^i(x)) - s(z^j(x))} \left( \partial_t m_{k-1}(z^a(x), z^i(x), z^j(x))_{i,j} \right. \\
& \left. - \sum_{l=1}^k \left[ \mathcal{V}^T v^T C R^{(l)} v \mathcal{V}, \tilde{M}^{(k-l)} \right]_{i,j} + \left[ \tilde{M}^{(k-1)}, \tilde{U}(x) \right]_{i,j} \right) \\
= & \frac{1}{s(z^i(x)) - s(z^j(x))} \left( \partial_t m_{k-1}(z^a(x), z^i(x), z^j(x))_{i,j} \right. \\
& - \sum_{l=1}^k \sum_{p,q,r} \mathcal{V}_p(z^i(x)) (v^T C R^{(l)}(x) v)_{p,q} \mathcal{V}_q(z^r(x)) m_{k-l}(z^a(x), z^r(x), z^j(x))_{r,j} \\
& + \sum_{l=1}^k \sum_{p,q,r} m_{k-l}(z^a(x), z^i(x), z^r(x))_{i,r} \mathcal{V}_p(z^r(x)) (v^T C R^{(l)}(x) v)_{p,q} \mathcal{V}_q(z^j(x)) \\
& - \sum_r m_{k-1}(z^a(x), z^i(x), z^r(x))_{i,r} \tilde{u}(z^r(x), z^j(x)) \\
& \left. + \sum_r \tilde{u}(z^i(x), z^r(x)) m_{k-1}(z^a(x), z^r(x), z^j(x))_{r,j} \right) \tag{3-19}
\end{aligned}$$

To see that, upon multiplying by  $\sqrt{x'(z^i(x))x'(z^j(x))}$  it is a rational function of  $z^a(x), z^i(x), z^j(x)$ , notice that :

- The square roots contained in  $\mathcal{V}_q(z^r)$  get multiplied by square roots from the  $m_{k-l}(z^r)$ , and so all square roots come by pairs providing rational functions of  $z$ .
- We can replace  $x$  in  $R^{(l)}(x)$  by  $R^{(l)}(x(z^a(x)))$  which is a rational function of  $z^a(x)$ .
- For any rational function  $f(z)$ , the sum  $\sum_r f(z^r(x))$  is a symmetric function of  $z^1(x), \dots, z^d(x)$ , therefore it is a rational function of their elementary symmetric polynomials, i.e. of the coefficients of  $x(z) - x$ . Consequently it is a rational function of  $x$ , hence it is a rational function of  $x(z^a(x))$ . Thus we end up with a rational function of  $z^a(x)$ .

For the diagonal entries  $i = j$ , we use the fact that  $M^2 = M$  thus implying that  $\tilde{M}^2 = \tilde{M}$  too. Consequently we have  $\tilde{M}_{i,i} = (\tilde{M})_{i,i}^2 + \sum_{j \neq i} \tilde{M}_{i,j} \tilde{M}_{j,i}$  and thus looking at order  $\hbar^k$  we get :

$$\left( 1 - 2(\tilde{M}^{(0)})_{i,i} \right) \left( \tilde{M}^{(k)} \right)_{i,i} = \sum_{l=0}^k \sum_{i \neq j} (\tilde{M}^{(l)})_{i,j} (\tilde{M}^{(k-l)})_{j,i} - \sum_{l=1}^{k-1} (\tilde{M}^{(l)})_{i,i} (\tilde{M}^{(k-l)})_{i,i} \tag{3-20}$$

By definition we have  $(\tilde{M}^{(0)})_{i,i} = \delta_{i,a} \in \{0, 1\}$ , thus we have  $1 - 2(\tilde{M}^{(0)})_{i,i} = \pm 1$  and :

$$\begin{aligned}
\left( \tilde{M}^{(k)}(x.e_a, t) \right)_{i,i} &= \frac{1}{1 - 2\delta_{i,a}} \left( \sum_{l=0}^k \sum_{i \neq j} (\tilde{M}^{(l)}(x.e_a, t))_{i,j} (\tilde{M}^{(k-l)}(x.e_a, t))_{j,i} \right. \\
& \left. - \sum_{l=1}^{k-1} (\tilde{M}^{(l)}(x.e_a, t))_{i,i} (\tilde{M}^{(k-l)}(x.e_a, t))_{i,i} \right)
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{1 - 2\delta_{i,a}} \left( \sum_{l=0}^k \sum_{i \neq j} \sum_{r=1}^d m_l(z^a, z^i, z^r)_{i,r} m_{k-l}(z^a, z^r, z^j)_{r,j} \right) \\
&\quad - \sum_{l=1}^{k-1} \sum_{r=1}^d m_l(z^a, z^i, z^i)_{i,i} m_{k-l}(z^a, z^i, z^i)_{i,i} \quad (3-21)
\end{aligned}$$

which is also a rational function of  $z^a, z^i$ . This proves the recursion for  $\tilde{M}$ . We shall now use this result to prove that the coefficients of  $M^{(k)}$  are rational functions of  $z$ . Conjugating the last result by  $V(x, t)$  gives :

$$M_{i,j}^{(k)} = \sum_{p,q,n,r=1}^d v_{i,n} \mathcal{V}_n(z^p(x)) m_k(z^a, z^p, z^q)_{p,q} \mathcal{V}_r(z^q) (Cv)_{r,j} \quad (3-22)$$

The sum over  $p$  and  $q$  yields a rational function of  $z^a(x)$  and thus the coefficients of  $M^{(k)}$  are rational functions of  $z$ .  $\square$

Note that similar computations with the  $x$ -differential equation lead to :

$$\begin{aligned}
\left( \tilde{M}^{(k)}(x.E, t) \right)_{i,j} &= \frac{1}{y_i(x) - y_j(x)} \left( \partial_x \left( \tilde{M}^{(k-1)} \right)_{i,j} \right. \\
&\quad \left. - \sum_{l=1}^k \left[ V^{-1} L^{(l)} V, \tilde{M}^{(k-l)} \right]_{i,j} + \left[ \tilde{M}^{(k-1)}, V^{-1} \partial_x V \right]_{i,j} \right) \\
\left( \tilde{M}^{(k)}(x.E, t) \right)_{i,i} &= \frac{1}{1 - 2\delta_{i,a}} \left( \sum_{l=0}^k \sum_{j \neq i} \left( \tilde{M}^{(l)}(x.E, t) \right)_{i,j} \left( \tilde{M}^{(k-l)}(x.E, t) \right)_{j,i} \right. \\
&\quad \left. - \sum_{l=1}^{k-1} \left( \tilde{M}^{(l)}(x.E, t) \right)_{i,i} \left( \tilde{M}^{(k-l)}(x.E, t) \right)_{i,i} \right) \quad (3-23)
\end{aligned}$$

We will use these results to analyze the possible singularities of the matrices  $M^{(k)}(z, t)$ . The results are presented in the following section.

## 3.2 Singularity structure of $M$

We have the following theorem :

**Theorem 3.2 (Singularity structure of  $M$ )** *The matrices  $(M^{(k)}(z))_{k \geq 0}$  may only have poles at the branchpoints or at the poles of  $L^{(0)}(x)$ . In particular, they are regular at the double points of  $\mathcal{S}$ .*

**Proof :**

Let us prove the theorem by recursion on  $k$ . For  $k = 0$  we have :

$$\left( v^{-1} M^{(0)}(z) v C^{-1} \right)_{i,l;j,l'} = \frac{1}{(z - \alpha_i)^l (z - \alpha_j)^{l'} x'(z)} \quad (3-24)$$

Thus,  $M^{(0)}(z)$  may only have poles at the zeros of  $x'(z)$  (that are branchpoints) or at the  $\alpha_i$ 's. In particular  $(v^{-1}M^{(0)}(z)vC^{-1})_{i,l;j,l'}$  has a pole at  $z = \alpha_m$  of degree equal to  $l\delta_{m,i} + l'\delta_{m,j} - d_m - 1$ . The only case for which  $\alpha_m$  is not a branchpoint corresponds to  $d_m = 1$ . In that case, we necessarily have  $l = l' = 1$  and the degree of the pole is zero. In other words if  $\alpha_m$  is not a branchpoint,  $M^{(0)}(z)$  has no pole at  $z = \alpha_m$ . Therefore  $M^{(0)}(z)$  has poles only at branchpoints.

Then assume that  $M^{(k')}(z)$  has poles only at branchpoints and/or poles of  $L^{(0)}(x)$  for all  $k' < k$ . Let us assume that  $M^{(k)}(z)$  has a pole at a point  $p$  of some order  $r \geq 1$  where  $p$  is not a branchpoint nor a pole of  $L^{(0)}$ . We write

$$M^{(k)}(z) \stackrel{z \rightarrow p}{\sim} \frac{M^{(k),r}}{(z-p)^r} + O((z-p)^{1-r}) \quad (3-25)$$

The polar part at  $z = p$  of the equation  $M = M^2$  at order  $\hbar^k$ , is :

$$M^{(k),r} = M^{(k),r}(z)M^{(0)}(p) + M^{(0)}(p)M^{(k),r}. \quad (3-26)$$

Notice that  $M^{(0)}(p)$  is a rank one matrix of the form

$$v^{-1}M^{(0)}(p)v = uu^T C \quad , \quad u^T C u = e_a \text{ with } u = \mathcal{V}(x)e_a = v^{-1}V e_a \quad (3-27)$$

Moreover,  $vu = V e_a$  (resp.  $e_a u^T C v^{-1}$ ) is a right (resp. left) eigenvalue of  $R^{(0)}(x(p))$  of eigenvalue  $s(p^a)$  :

$$\begin{aligned} v^{-1}R^{(0)}(x(p))vu &= s(p^a)vu \\ e_a u^T C v^{-1}R^{(0)}(x(p))v &= s(p^a)e_a u^T C \end{aligned} \quad (3-28)$$

Indeed we have  $V e_a = (\mathbf{0}, \dots, \mathbf{0}, \mathbf{v}_a^a, \mathbf{0}, \dots, \mathbf{0})$  where  $\mathbf{0}$  is the  $d$ -dimensional zero vector and  $\mathbf{v}_a^a$  is the  $a^{\text{th}}$  eigenvector of  $\mathcal{R}^{(0)}(x(p))$ . Consequently,  $\mathcal{R}^{(0)}(x(p))V e_a = s(p^a)V e_a$  and inserting  $vu = V e_a$  provides the first identity. The second identity follows from  $SV^{-1} = V^{-1}\mathcal{R}^{(0)}$  which is equivalent to  $S\mathcal{V}^T C v^{-1} = \mathcal{V}^T C v^{-1}\mathcal{R}^{(0)}$ . Multiplying on the left by  $e_a = e_a^T$  and observing that  $e_a S = S e_a^T = s(p^a)e_a$  and  $u^T = e_a^T \mathcal{V}^T$  gives the second identity.

Let us denote

$$H = v^{-1}M^{(k),r}v. \quad (3-29)$$

We must have from (3-26) :

$$H = H u u^T C + u u^T C H = H u u^T C + v^{-1}V e_a u^T C H. \quad (3-30)$$

multiplying on the right by  $u$  gives  $Hu = Hu(u^T C u) + u(u^T C H u) = H u e_a + u(u^T C H u)$  and thus after a multiplication on the right by  $e_a$  :

$$u^T C H u e_a = 0. \quad (3-31)$$

Moreover, using the fact that  $k$  is minimal, the polar part at order  $\hbar^k$  of  $\hbar\partial_t M = [R, M]$  at  $z = p$  implies that

$$[H, v^{-1}R^{(0)}(x(p))v] = 0. \quad (3-32)$$

This last equation implies that  $vHu$  and  $e_a u^T C H v^{-1}$  are respectively right and left eigenvectors of  $R^{(0)}(x(p))$  for the eigenvalue  $s(p^a)$ . Indeed from the last equation and  $u = v^{-1}V e_a$  :

$$v^{-1}\mathcal{R}^{(0)}(x(p))vHu = Hv^{-1}\mathcal{R}^{(0)}(x(p))V e_a = s(p^a)Hv^{-1}V e_a = s(p^a)Hu \quad (3-33)$$

Similarly, using that  $e_a u^T C v^{-1}R^{(0)}(x(p)) = s(p^a)e_a u^T C v^{-1}$  we get :

$$e_a u^T C H v^{-1}\mathcal{R}^{(0)}(x(p))v = e_a u^T C v^{-1}\mathcal{R}^{(0)}(x(p))vH = s(p^a)e_a u^T C H \quad (3-34)$$

Since  $p$  is neither a pole of  $L^{(0)}$  (and thus not a pole of  $R^{(0)}$ ) nor a branchpoint,  $s(p^a)$  is not degenerate, i.e. the eigenspace has dimension one, and therefore, there exist some scalars  $\mu, \tilde{\mu}$  such that

$$vHu = \mu u \quad , \quad e_a u^T C H v^{-1} = \tilde{\mu} e_a u^T C v^{-1}. \quad (3-35)$$

which is equivalent to

$$Hu = \mu u \text{ and } e_a u^T C H = \tilde{\mu} e_a u^T \quad (3-36)$$

In particular, using (3-31), we get

$$0 = u^T C H u e_a = \mu u^t C u e_a = \mu e_a, \quad (3-37)$$

and thus  $\mu = 0$  and  $Hu = 0$ . Similarly,

$$0 = e_a u^T C H u e_a = \tilde{\mu} e_a u^T C u e_a = \tilde{\mu} e_a \quad (3-38)$$

Hence  $\tilde{\mu} = 0$  and  $e_a u^T C H = 0$ . Finally we insert the last results into (3-30), this gives  $H = 0$ , and thus  $M^{(k),r} = 0$ , which contradicts our polar assumption. Therefore  $M^{(k)}(z)$  has no pole at  $z = p$ .  $\square$

### 3.3 Correlators

From the matrices  $M(x.E, t, \hbar)$ , we define the *connected* correlators as in [1, 2] :

**Definition 3.2 (Connected correlators)** We define for  $n \geq 1$

$$W_n(x_1.E_1, \dots, x_n.E_n) = \begin{cases} \hbar^{-1} \text{Tr} L(x_1)M(x_1.E_1)dx_1 & n = 1 \\ \frac{(-1)^{n-1}}{n} \sum_{\sigma \in \mathfrak{S}_n} \frac{\text{Tr} \prod_{i=1}^n M(x_{\sigma(i)}.E_{\sigma(i)})}{\prod_{i=1}^n (x_{\sigma(i)} - x_{\sigma(i+1)})} \prod_{i=1}^n dx_i & n \geq 2 \end{cases} \quad (3-39)$$



These correlators are symmetric  $n$ -forms on  $(\mathbb{C} \times \mathfrak{g})^n$  and they are linear in each  $E_i$ . They appear naturally in matrix models and in many enumerative problems [1, 2, 6, 10, 32, 33]. From the CFT point of view,  $W_n(x_1.E_1, \dots, x_n.E_n)$  is the correlation function corresponding to the insertions of  $n$  currents  $J(x_i.E_i)$  [24].

Note that like  $M(x.E, t, \hbar)$ , the  $W_n$ 's are defined as  $\hbar$ -formal series. We will describe more precisely the  $\hbar$  expansion and its properties below. We will also need the non-connected version of the correlators :

**Definition 3.3 (Non-connected correlators)** *The correlators (non connected) are defined from the connected ones by summing over partitions. Denoting  $X_i = x_i.E_i$  we define :*

$$\tilde{W}_n(X_1, \dots, X_n; t, \hbar) = \sum_{\mu \vdash \{X_1, \dots, X_n\}} \prod_{i=1}^{\ell(\mu)} W_{|\mu_i|}(\mu_i; t, \hbar) \quad (3-40)$$

where we sum over all partitions of the set  $\{X_1, \dots, X_n\}$  of  $n$  points. For example

$$\tilde{W}_1(X_1; t) = W_1(X_1; t), \quad (3-41)$$

$$\tilde{W}_2(X_1, X_2; t) = W_1(X_1; t)W_1(X_2; t) + W_2(X_1, X_2; t) \quad (3-42)$$

$$\begin{aligned} \tilde{W}_3(X_1, X_2, X_3; t) = & W_1(X_1; t)W_1(X_2; t)W_1(X_3; t) + W_1(X_1; t)W_2(X_2, X_3; t) \\ & + W_1(X_2; t)W_2(X_1, X_3; t) + W_1(X_3; t)W_2(X_1, X_2; t) \\ & + W_3(X_1, X_2, X_3; t) \end{aligned} \quad (3-43)$$

and so on...

**Remark 3.3** One often says that the connected correlators are the ‘‘cumulants’’ of the non-connected ones.

### 3.4 Tau function

We also recall for bookkeeping (indeed we shall not use it in this article) the definition of the Tau-function by Miwa-Jimbo [37, 38]. Let  $T(x) = \text{diag}(T_1(x), \dots, T_i(x))$  (it is the exponential term of the WKB expansion (3-8)), such that

$$\frac{\partial T_i(x)}{\partial x} = y_i(x) \quad , \quad \frac{\partial T_i(x)}{\partial t} = s_i(x), \quad (3-44)$$

In other words :

$$\frac{\partial T(x)}{\partial t} = \sum_{i=1}^d s_i(x) e_i \quad \text{with} \quad e_i = \text{diag}(0, \dots, 0, \overset{i}{1}, 0, \dots, 0). \quad (3-45)$$

The Miwa-Jimbo-Ueno-Takasaki definition of the Tau function is [37, 38]

$$\hbar \frac{\partial \ln \mathfrak{T}}{\partial t} = \sum_{q=\text{poles of } x, s} \text{Res}_{x \rightarrow q} \text{Tr} \left( \frac{\partial T(x)}{\partial t} \Psi(x)^{-1} \frac{\partial \Psi(x)}{\partial x} \right) dx. \quad (3-46)$$

Let us rewrite it in our notations, using  $W_1$  :

$$\begin{aligned} \hbar \frac{\partial \ln \mathfrak{T}}{\partial t} &= \sum_{q=\text{poles of } x, s} \text{Res}_{x \rightarrow q} \text{Tr} \left( \frac{\partial T(x)}{\partial t} \Psi(x)^{-1} \frac{\partial \Psi(x)}{\partial x} \right) dx \\ &= \hbar^{-1} \sum_{q=\text{poles of } x, s} \sum_{i=1}^d \text{Res}_{x \rightarrow q} s_i(x) \text{Tr} (e_i \Psi(x)^{-1} L(x) \Psi(x)) dx \\ &= \hbar^{-1} \sum_{q=\text{poles of } x, s} \sum_{i=1}^d \text{Res}_{x \rightarrow q} s_i(x) \text{Tr} (L(x) \Psi(x) e_i \Psi(x)^{-1}) dx \\ &= \sum_{q=\text{poles of } x, s} \sum_{i=1}^d \text{Res}_{x \rightarrow q} s_i(x) W_1(x, e_i). \end{aligned} \quad (3-47)$$

In particular it explains why the one-form  $W_1(x, E)$  is so useful. For  $n \geq 2$ , the notation  $W_n$  follows the definition of correlation functions arising in topological recursion and in the study of random Hermitian matrices. In fact it has been shown recently in a series of papers [1, 2] that under suitable conditions, known as Topological Type property, the correlation functions presented in definition 3.3 can be reconstructed from the application of the topological recursion to the spectral curve  $E(x, y) = 0$  attached to the differential system. The precise statement can be found in [2] and will be summarized in the next section for our purposes.

## 4 Topological Type property : definition and proof

### 4.1 Topological Type property

Here we recall the definition 3.3 of [2], specialized to the case of a genus 0 spectral curve (thus skipping many unnecessary geometric technicalities appearing only when the genus is strictly positive) :

**Proposition 4.1 (Definition 3.3 of [2])** *A sequence of differential forms  $W_n$  is said to have an expansion of topological type (TT property) when :*

1. Existence of an expansion in  $\hbar$  : The  $W_n$ 's are formal series of  $\hbar$  of the form :

$$W_n(X_1, \dots, X_n; t, \hbar) = \sum_{k=-\delta_{n,1}}^{\infty} W_n^{(k)}(X_1, \dots, X_n; t) \hbar^k \quad (4-1)$$

whose coefficients, denoted

$$\omega_n^{(k)}(z^{a_1}(x_1), \dots, z^{a_n}(x_n)) = W_n^{(k)}(x_1.e_{a_1}, \dots, x_n.e_{a_n}; t) dx_1 \dots dx_n \quad (4-2)$$

are rational functions of their arguments (this is where we use the fact that the genus is zero). Moreover, the one-form  $\omega_1^{(0)}(z)$  is required to be the Liouville form :

$$\omega_1^{(0)}(z) = y(z)x'(z)dz = y(z)dx(z). \quad (4-3)$$

and the bi-differential form  $\omega_2^{(0)}(z_1, z_2)$  is required to correspond to the fundamental 2<sup>nd</sup> kind differential (“Bergmann-Schiffer-Klein kernel”) of the Riemann sphere (also specific to genus 0 curve) :

$$\omega_2^{(0)}(z_1, z_2) = B(z_1, z_2) = \frac{dz_1 dz_2}{(z_1 - z_2)^2}. \quad (4-4)$$

2. Loop equations : The  $W_n$ 's satisfy loop equations, i.e. for all  $1 \leq k \leq d$ , for all  $n \geq 0$  and for all  $X_1, \dots, X_n$  with  $X_i = x_i.E_i$  ( $E_i \in \mathfrak{h}$ ) the following quantity

$$\sum_{1 \leq i_1 < i_2 < \dots < i_k \leq d} \tilde{W}_{k+n}(x.e_{i_1}, \dots, x.e_{i_k}, X_1, \dots, X_n) = \tilde{P}_{k,n}(x; X_1, \dots, X_n) \quad (4-5)$$

is well defined when  $x$  and all  $x_i$  are distinct. Moreover it is a rational function of  $x$  such that the meromorphic one-form of  $z$  :

$$\eta_n(z; X_1, \dots, X_n) = \left( \frac{y^d + \sum_{k=1}^d (-1)^k y^{d-k} \tilde{P}_{k,n}(x; X_1, \dots, X_n)}{E_y(x, y)} dx \right)_{x=x(z), y=y(z)} \quad (4-6)$$

has no pole at the poles of  $x$  and  $y$  neither at branchpoints. Its only poles may be at double points (zeros of  $E_y(x, y)$ ) and/or at coinciding points  $x(z) = x_i$  for some  $i$ .

3. Pole property : For  $(n, k) \notin \{(1, -1), (2, 0)\}$ , the rational differential forms  $\omega_n^{(k)}(z_1, \dots, z_n)$  may only have poles at the branchpoints of the spectral curve. In particular they must have no pole at double points of the spectral curve, nor at coinciding points  $z_i = z_j$  with  $i \neq j$ .
4. Parity property : Under the change  $\hbar \leftrightarrow -\hbar$  the correlation functions satisfy  $(W_n)_{-\hbar} = (-1)^n (W_n)_{\hbar}$ . This is equivalent to say that the  $\hbar$ -expansions of the  $W_n$  only contain even (resp. odd) exponents in  $\hbar$  when  $n$  is even (resp. odd).
5. Leading order property : For  $n \geq 1$  we have  $W_n = O(\hbar^{n-2})$ .

Note that combining the existence of an expansion in  $\hbar$ , the parity property and the leading order property is equivalent to say that :

$$\forall n \geq 1 : W_n = \sum_{g=0}^{\infty} \hbar^{2g-2+n} W_{g,n} \quad , \quad W_{g,n} = W_n^{(n+2g-2)}, \quad \omega_{g,n} = \omega_n^{(n+2g-2)}. \quad (4-7)$$

It was proved in [6] that the TT property and the loop equations (here obtained by construction for the  $(W_n)_{n \geq 1}$ 's [2]) imply that the coefficients  $\omega_{g,n}$  satisfy the topological recursion.

We now claim that our assumptions 1, 2, 3, 4, 5 and 6 imply that the Topological Type property is satisfied for the set of correlators  $(W_n)_{n \geq 1}$ . Notice that assumption 1 is usually automatically verified in the formal context while assumptions 2, 3 and 4 can be verified from only  $\hbar^0$  computations. Eventually only assumptions 5 and 6 require general properties (location and number of poles, Hamiltonian structure, etc.) of the Lax system.

## 4.2 Proof of condition 1 of the TT property : Existence of an $\hbar$ expansion for correlators

The existence of an  $\hbar$  expansion for the correlation functions is an immediate corollary of theorem 3.1 for  $M(x.E, t, \hbar)$ . Indeed, inserting the series expansion of  $M(x.E, t)$  in definition 3.2 provide the wanted  $\hbar$  expansion, whose coefficients are indeed rational functions of the  $z^i(x)$ . Therefore only the explicit computations of  $\omega_1^{(0)}(z)$  and  $\omega_2^{(0)}(z_1, z_2)$  remain to prove.

- The computation of  $\omega_1^{(0)}(z)$  is straightforward from the definition :

$$\begin{aligned} W_1^{(0)}(x.e_a) &= \text{Tr } M^{(0)}(x.e_a)L^{(0)}(x) = \text{Tr } V(x)e_aV(x)^{-1}V(x)Y(x)V(x)^{-1} \\ &= \text{Tr } e_aY(x) = Y_a(x) = y(z^a(x)). \end{aligned} \quad (4-8)$$

Eventually it gives :

$$\omega_1^{(0)}(z) = y(z)x'(z)dz. \quad (4-9)$$

- The computation of  $\omega_2^{(0)}(z_1, z_2)$  is a direct consequence of corollary 2.3. Hence condition 1 of the TT property is proved.

## 4.3 Proof of condition 2 of the TT property : Loop equations

The proof is already done in [5], and we shall also use a rewriting as in [3].

### 4.3.1 Rewriting of the loop equations

Recall that  $e_1, \dots, e_d$  span a Cartan subalgebra  $\mathfrak{h}$  when the Lie group is taken to be  $G = GL_d(\mathbb{C})$ . Using this basis, the Casimirs of  $\mathfrak{g} = \mathfrak{gl}_d(\mathbb{C})$  are

$$C_k = \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq d} e_{i_1} \otimes \dots \otimes e_{i_k}. \quad (4-10)$$

Consequently, the loop equations derived in [5] are :

**Theorem 4.1 (From [5])** *If  $L(x)$  is a rational function of  $x$ , then the non-connected correlators  $\tilde{W}_n$ 's are such that for all  $1 \leq k \leq d$ , for all  $n \geq 0$ , and for all  $X_1, \dots, X_n$  with  $X_i = x_i.E_i$  where  $E_i \in \mathfrak{h}$ , the following quantity :*

$$\sum_{1 \leq i_1 < i_2 < \dots < i_k \leq d} \tilde{W}_{k+n}(x.e_{i_1}, \dots, x.e_{i_k}, X_1, \dots, X_n) = \tilde{P}_{k,n}(x; X_1, \dots, X_n) \quad (4-11)$$

is well defined for  $x$  and all  $x_i$  distinct. Moreover, it is a rational function of  $x$ , with only possible poles at the poles as  $L(x)$  and at coinciding points  $x = x_i$  for some  $i$ .

Saying that it is well defined is not obvious, because  $\tilde{W}_n$  has poles at coinciding points, due to the presence of  $W_2$  factors. However,  $W_2(x.e_{i_1}, x'.e_{i_2})$  has no pole on the diagonal  $x = x'$  if  $i_1 \neq i_2$ , and the summation in (4-11) is only on distinct indices  $i_1 \neq i_2 \neq \dots \neq i_k$ .

Another version of the same theorem can be written after partially decomposing the non-connected correlators  $\tilde{W}_{k+n}$ 's into the connected ones  $W_i$ . It is given by :

**Corollary 4.1** *For all  $1 \leq k \leq d$ , for all  $x$ , for all  $n \geq 0$ , and for all  $X_i = x_i.E_i$  (with  $x$  and all  $x_i$ 's are distinct), the following quantity :*

$$\sum_{1 \leq i_1 < i_2 < \dots < i_k \leq d} \mathcal{W}_{k,n}(x.e_{i_1}, \dots, x.e_{i_k}, X_1, \dots, X_n) = P_{k,n}(x; X_1, \dots, X_n) \quad (4-12)$$

is a rational function of  $x$ . In this formula  $\mathcal{W}_{k,n}(x.e_{i_1}, \dots, x.e_{i_k}, X_1, \dots, X_n)$  stands for

$$\mathcal{W}_{k,n}(K; A) = \sum_{\ell} \sum_{(I_1, \dots, I_\ell) \vdash K; J_1 \uplus J_2 \uplus \dots \uplus J_\ell = A} \prod_{i=1}^{\ell} W_{|I_i|+|J_i|}(I_i \cup J_i) \quad (4-13)$$

where we denoted the ensembles  $A = \{X_1, \dots, X_n\}$  and  $K = \{x.e_{i_1}, \dots, x.e_{i_k}\}$ . In other words, we sum over all partitions of  $K$  into non-empty parts, and we associate to each part  $I_i$  of  $K$  a (possibly empty) part  $J_i$  of  $A$ , in all possible ways.

In short, every  $I_i \cup J_i$ , which is a part of  $K \cup A$ , contains at least one element of  $K$  but  $J_i$  may be empty. This is nearly the same definition as  $\tilde{W}_{k+n}$ , but the latter may contain parts without elements of  $K$  (i.e.  $I_i = \emptyset$ ). For example we have :

$$\mathcal{W}_{2,1}(X, X'; X_1) = W_1(X)W_2(X', X_1) + W_1(X')W_2(X, X_1) + W_2(X, X', X_1) \quad (4-14)$$

which differs from

$$\begin{aligned} \tilde{W}_3(X, X', X_1) &= W_1(X)W_2(X', X_1) + W_1(X')W_2(X, X_1) + W_2(X, X', X_1) \\ &\quad + W_1(X_1) (W_1(X)W_1(X') + W_2(X, X')) \\ &= \mathcal{W}_{2,1}(X, X'; X_1) + W_1(X_1) \mathcal{W}_{2,0}(X, X'). \end{aligned} \quad (4-15)$$

Let us reformulate again the loop equations by summing over  $k$  and making a generating series with a formal variable  $y$  :

**Corollary 4.2** *The following quantity :*

$$\begin{aligned} Q_n(x, y; X_1, \dots, X_n) &= y^d \delta_{n,0} \\ &\quad + \sum_{k=1}^d (-1)^k \hbar^k y^{d-k} \sum_{1 \leq i_1 < \dots < i_k \leq d} \mathcal{W}_{k,n}(x.e_{i_1}, \dots, x.e_{i_k}, X_1, \dots, X_n) \end{aligned} \quad (4-16)$$

is a polynomial of  $y$  of degree  $\leq d$ . It is also a rational function of  $x$  with only possible poles at the poles of  $L(x)$  and at coinciding points  $x = x_i$ .

Eventually, it is useful to separate the leading term of the 1-point function  $W_1(x.e_a) = \hbar^{-1} y(z^a(x)) + O(\hbar)$  from the subleading ones. We thus introduce the following :

**Definition 4.1** *We define :*

$$\begin{aligned} \hat{W}_1(x.e_a) &= W_1(x.e_a) - \hbar^{-1} y(z^a(x)) \text{ for } n = 1 \\ \hat{W}_n &= W_n \text{ for } n > 1. \end{aligned} \quad (4-17)$$

Then we define the hat-disconnected correlators :

$$\hat{W}_n(A) = \sum_{\ell} \sum_{(I_1, \dots, I_\ell) \vdash A} \prod_{i=1}^{\ell} \hat{W}_{|I_i|}(I_i), \quad (4-18)$$

and the hat-partially disconnected correlators :

$$\hat{W}_{k,n}(K; A) = \sum_{\ell} \sum_{(I_1, \dots, I_\ell) \vdash K; J_1 \uplus J_2 \uplus \dots \uplus J_\ell = A} \prod_{i=1}^{\ell} \hat{W}_{|I_i|+|J_i|}(I_i \cup J_i) \quad (4-19)$$

In other words, we have the same definition as  $\tilde{W}_n$  and  $\mathcal{W}_{k,n}$  respectively but with the factors  $W_j$ 's replaced by  $\hat{W}_j$ 's.

**Remark 4.1** Notice that  $\hat{W}_{k+n}$  and  $\hat{W}_{k,n}$  are related :

$$\hat{W}_{k+n}(K \cup A) = \sum_{A' \subset A} \hat{W}_{k,|A'|}(K; A') \hat{W}_{|A|-|A'|}(A \setminus A'). \quad (4-20)$$

i.e.  $\hat{W}_n$ 's are linear combinations of  $\hat{W}_{k,n}$ 's with coefficients independent of  $K$ . This is particularly convenient since it means that every statement about the analytic structure of the  $\hat{W}_n$ 's is immediately transmitted to  $\hat{W}_{k,n}$  and vice-versa.

Eventually, the loop equations can be reformulated (see [3]) in another way :

**Corollary 4.3 (Loop equation, version of [3])** *The following quantities :*

$$\begin{aligned} \tilde{P}_n(x, y; X_1, \dots, X_n) &= \sum_{I \subset \{x.e_1, \dots, x.e_d\}} \hbar^{|I|} \hat{W}_{|I|,n}(I; X_1, \dots, X_n) \prod_{a \notin I} (y - y(z^a(x))) \\ &= E(x, y) \sum_{I \subset \{x.e_1, \dots, x.e_d\}} \hbar^{|I|} \frac{\hat{W}_{|I|,n}(I; X_1, \dots, X_n)}{\prod_{a \in I} (y - y(z^a(x)))}, \end{aligned} \quad (4-21)$$

and

$$\begin{aligned} P_n(x, y; X_1, \dots, X_n) &= \sum_{I \subset \{x.e_1, \dots, x.e_d\}} \hbar^{|I|} \hat{W}_{|I|,n}(I; X_1, \dots, X_n) \prod_{a \notin I} (y - y(z^a(x))) \\ &= E(x, y) \sum_{I \subset \{x.e_1, \dots, x.e_d\}} \hbar^{|I|} \frac{\hat{W}_{|I|,n}(I; X_1, \dots, X_n)}{\prod_{a \in I} (y - y(z^a(x)))} \end{aligned} \quad (4-22)$$

are polynomials of  $y$  of degree  $\leq d$ . Moreover they are rational functions of  $x$  with only possible poles at the poles of  $L(x)$  and at coinciding points  $x = x_i$ .

In fact in [6] the explicit expression of  $\tilde{P}_n(x, y; X_1, \dots, X_n)$  was derived :

**Theorem 4.2 (From [6])** *We have :*

$$\tilde{P}_n(x, y; X_1, \dots, X_n) = [\epsilon_1 \epsilon_2 \dots \epsilon_n] \det (y \text{Id}_d - L(x) - F_\epsilon(X_1, \dots, X_n)) \quad (4-23)$$

where

$$\begin{aligned} F_\epsilon(X_1, \dots, X_n) &= \sum_{i=1}^n \frac{\epsilon_i M(X_i)}{(x - x_i)(x_i - x)} + \sum_{i \neq j} \frac{\epsilon_i \epsilon_j M(X_i) M(X_j)}{(x - x_i)(x_i - x_j)(x_j - x)} \\ &+ \sum_{k=3}^n \sum_{i_1 \neq i_2 \neq \dots \neq i_k} \frac{\epsilon_{i_1} \dots \epsilon_{i_k} M(X_{i_1}) \dots M(X_{i_k})}{(x - x_{i_1})(x_{i_1} - x_{i_2}) \dots (x_{i_k} - x)} \end{aligned} \quad (4-24)$$

and where the notation is such that  $[\epsilon_1^{k_1} \epsilon_2^{k_2} \dots \epsilon_n^{k_n}] f(\epsilon)$  is the coefficient of  $\epsilon_1^{k_1} \epsilon_2^{k_2} \dots \epsilon_n^{k_n}$  of the polynomial  $f(\epsilon)$ .

### 4.3.2 $\hbar$ expansion of the loop equations

Since the right hand side of the loop equations (4-23) has an  $\hbar$  expansion (from assumption 1 and theorem 3.1), so does the left hand side. Thus we write :

$$\begin{aligned} P_n(x, y; X_1, \dots, X_n) &= \sum_{j \geq 0} \hbar^j P_n^{(j)}(x, y; X_1, \dots, X_n) \\ \tilde{P}_n(x, y; X_1, \dots, X_n) &= \sum_{j \geq 0} \hbar^j \tilde{P}_n^{(j)}(x, y; X_1, \dots, X_n). \end{aligned} \quad (4-25)$$

where each  $P_n^{(j)}(x, y; X_1, \dots, X_n)$  and  $\tilde{P}_n^{(j)}(x, y; X_1, \dots, X_n)$  is a polynomial of  $y$  of degree  $\leq d$  and a rational function of  $x$ .

### 4.3.3 Specializing the loop equations on the spectral curve

Let us denote :

$$D = \{x.e_1, \dots, x.e_d\} \quad , \quad A = \{X_1, \dots, X_n\}. \quad (4-26)$$

We isolate terms  $I = \emptyset$  and  $|I| = 1$  in Corollary (4.3). We find :

$$\begin{aligned} P_n(x, y; A) &= \hat{W}_n(A)E(x, y) + \sum_{i=1}^d \hbar \hat{W}_{1,n}(x.e_i; A) \prod_{j \neq i} (y - y(z^j(x))) \\ &+ \sum_{I \subset D / |I| \geq 2} \hbar^{|I|} \hat{W}_{|I|,n}(I; A) \frac{E(x, y)}{\prod_{i \in I} (y - y(z^i(x)))} \end{aligned} \quad (4-27)$$

Specializing at  $y = y(z^{i_0}(x))$  for a given  $i_0$  we observe that the first term vanishes while the second term only restricts to  $i = i_0$ . We find :

$$\begin{aligned} P_n(x, y(z^{i_0}(x)); A) &= \hbar \hat{W}_{1,n}(x.e_{i_0}; A) E_y(x, y(z^{i_0}(x))) \\ &+ \sum_{\{i_0\} \subset I \subset D / |I| \geq 2} \hbar^{|I|} \hat{W}_{|I|,n}(I; A) \prod_{i \notin I} (y(z^{i_0}(x)) - y(z^i(x))). \end{aligned} \quad (4-28)$$

We recognize here  $E_y(x, y) = \partial_y E(x, y) = \sum_{i=1}^d \prod_{j \neq i} (y - y(z^j(x)))$  evaluated at the point  $y = y(z^{i_0}(x))$ . Moreover  $\hat{W}_{1,n}(x.e_{i_0}; A) = W_{n+1}(x.e_{i_0}, A)$ . Indeed, in the definition of  $\hat{W}_{1,n}(x.e_{i_0}; A)$ , all parts must contain  $x.e_{i_0}$  and thus there must be exactly one part. In the end we have :

$$\begin{aligned} P_n(x, y(z^{i_0}(x)); A) &= \hbar W_{n+1}(x.e_{i_0}, A) E_y(x, y(z^{i_0}(x))) \\ &+ \sum_{\{i_0\} \subset I \subset D / |I| \geq 2} \hbar^{|I|} \hat{W}_{|I|,n}(I; A) \prod_{i \notin I} (y(z^{i_0}(x)) - y(z^i(x))) \end{aligned} \quad (4-29)$$



#### 4.3.4 Poles of $P_n^{(k)}$

Theorem 4.2 together with assumption 5 imply :

**Corollary 4.4** *For every  $(k, n) \in \mathbb{N}^2 \setminus \{(0, 0)\}$  and for every generic  $X_1, \dots, X_n$ , the function*

$$P_n^{(k)}(x(z), y(z); X_1, \dots, X_n) \frac{dx(z)}{E_y(x(z), y(z))} \quad (4-30)$$

*is a one-form on the Riemann sphere, whose poles may be only at coinciding points or at double points. Thanks to remark 4.1, the same applies to  $\tilde{P}_n^{(k)}$ .*

### 4.4 Proof of condition 3 of the TT property : The pole structure

We want to prove that for  $(k, n) \notin \{(-1, 1), (0, 2)\}$ , the only poles of  $\omega_n^{(k)}(z, z_2, \dots, z_n)$  may be at branchpoints. By definition, the only possible singularities may arise at branchpoints, double points, coinciding points and punctures (i.e. simple poles of  $x$  or poles of  $y$ ).

- *No poles at double points* : We have proved that the  $\hbar^k$  term  $M^{(k)}$  is a rational function of  $z^a(x)$  without poles at double points, so by definition all  $W_n$  cannot have poles at double points. This imply that the  $\omega_n^{(g)}$  are regular at the double poles.
- *No poles at coinciding points* : By definition, the  $W_n$ 's involve denominators  $\frac{1}{x_i - x_j}$  that may lead to poles at coinciding points. However, for  $n > 2$ , the poles at coinciding points may be at most simple poles and the residue is a sum over permutations, that contains both each permutation and its inverse having opposite residues. Therefore the total residue vanishes and there is no pole at coinciding points. For  $n = 2$ , the pole at coinciding points can be a double pole. More precisely, the coefficient of the double pole is

$$\lim_{x_2 \rightarrow x_1} \text{Tr } M(x_1.e_a)M(x_2.e_b) = \text{Tr } e_a e_b = \delta_{a,b}. \quad (4-31)$$

which is independent of  $\hbar$  and thus  $W_2^{(k)}$  has no double pole for  $k > 0$ . Eventually, there could be a simple pole in  $W_2^{(k)}(x_1, x_2)$  at  $x_1 = x_2$ , but the symmetry  $W_2^{(k)}(x_1, x_2) = W_2^{(k)}(x_2, x_1)$  implies that the residue must vanish. Therefore  $W_2^{(k)}(x_1, x_2)$  has no pole at coinciding points, for  $k > 0$ . Consequently all differentials  $\omega_n^{(k)}$  with  $(k, n) \notin \{(-1, 1), (0, 2)\}$  are regular at coinciding points.

- *No poles at punctures (i.e. simple poles of x or y)* : In principle,  $M^{(k)}(z)$  may have poles at poles of x and y (poles of  $L^{(0)}$ ), so the  $\omega_n^{(k)}$  may also have such poles. We shall prove by recursion on  $k + n$  that for  $k + n \geq 0$ ,  $\omega_n^{(k)}$  has no pole at the punctures.

- This is clearly true for  $\omega_2^{(0)}$  from Corollary 2.3.

- This is also true for  $\omega_1^{(0)}$ . Indeed from (4-29) with  $n = 0$  and  $k = 1$  we get :

$$\omega_1^{(0)}(z) = \frac{P_0^{(1)}(x(z), y(z)) dx(z)}{E_y(x(z), y(z))} \quad (4-32)$$

From corollary 4.4, the right hand side cannot have poles at the poles of x or y. Note that this implies that  $\omega_1^{(0)}(z)$  has no pole at all, and therefore  $\omega_1^{(0)}(z) = 0$ .

- Let us assume that  $\omega_{n'}^{(k')}$  have no pole at poles of x or y for all  $k' + n' < k + n$ . Writing (4-29) with  $A = \{X_2, \dots, X_n\}$  we get :

$$\begin{aligned} \omega_n^{(k)}(z; A) &= \frac{P_{n-1}^{(k)}(x(z), y(z); A) dx(z)}{E_y(x(z), y(z))} \\ &- \sum_{\{i_0\} \subset I \subset D / |I| \geq 2} \frac{\hat{\mathcal{W}}_{|I|, n-1}^{(k-|I|)}(I; A) (dx(z))^{|I|}}{\prod_{i \in I \setminus \{i_0\}} (y(z^{i_0}(x)) - y(z^i(x)))} dx(z)} \end{aligned} \quad (4-33)$$

The term on the first line has no pole at the punctures from corollary 4.4. The numerator on the second line only involves  $k - |I| + |I| + n - 1 < k + n$ , and so by recursion hypothesis, the numerator has no pole at punctures. The denominator also does not vanish at the punctures (notice that it vanishes only at branchpoints and double points). Therefore we prove the property for  $k + n$  and we conclude by induction.

## 4.5 Proof of condition 4 of the TT property : The parity property

It was proved in [2] that assumption 6 is a sufficient condition to get the parity property and we shall not redo the (easy) proof of [2] here. We just mention that we do not know if the converse is true : is assumption 6 also a necessary condition to get the parity property? At the moment we do not know any counter-example and all known integrable systems that we have been looked at satisfy assumption 6.

## 4.6 Proof of condition 5 of the TT property : The leading order property

This condition is the hardest to prove. In [1, 2] a method called “loop insertion operator” was used, and part of the proof was missing (this can be fixed for instance as

in chapter 5 of [10], for Painlevé I hierarchy, i.e.  $(p, 2)$  minimal models). We shall not pursue this complicated method.

Instead, in [18], another proof, based on loop equations, was presented, but only for  $2 \times 2$  systems. Here we extend this loop equation method to higher rank systems. The generalization is not straightforward, because loop equations are much more involved in higher rank. It is obvious that the proof can be done for  $W_n$  or  $\omega_n$  equivalently since they are related by a multiplication by  $dx_1 \dots dx_n$  which does not depend on  $\hbar$ .

We shall prove by induction on  $k \geq 1$  that :

**Theorem 4.3** *The following proposition  $\mathcal{P}_k$  holds for  $k \geq 1$  :*

$$\mathcal{P}_k : \text{For all } j \geq k : W_j = O(\hbar^{k-2}).$$

**Proof :** We first observe that  $\mathcal{P}_1$  and  $\mathcal{P}_2$  are trivially verified. Indeed, by definition (see the  $\hbar$  expansion of  $M(x.E)$  in theorem 3.1)  $W_1(x_1.E_1)$  is of order  $\hbar^{-1}$  while all other correlation functions  $W_n(x_1.E_1, \dots, x_n.E_n)$  with  $n \geq 2$  are at least of order  $\hbar^0$ .

Let us now assume that for a given  $n \geq 2$ , propositions  $\mathcal{P}_1$  up to  $\mathcal{P}_n$  are verified. We now need to control the order of the last term of (4-29) and thus of

$$\hat{W}_{|I|,n}(I; A) = \sum_{(I_1, \dots, I_l) \vdash I; A_1 \sqcup \dots \sqcup A_l = A} \prod_{i=1}^l \hat{W}_{|I_i|+|A_i|}(I_i, A_i) \quad (4-34)$$

where we recall that none of the parts  $I_i$  can be empty.

There are 3 different cases :

1.  $|I_i| = 1$  and  $A_i = \emptyset$ . In that case we get  $\hat{W}_1(x.e_i)$  which is of order at least  $\hbar^0$  (because we have a hat version of  $W_1$  whose  $\hbar^{-1}$  term is removed).
2.  $1 < |I_i| + |A_i| \leq n$ . In that case we can apply  $\mathcal{P}_{|I_i|+|A_i|}$  and thus we get an order of  $\hbar^{|I_i|+|A_i|-2}$ .
3.  $|I_i| + |A_i| > n$ . In that case, we can only apply  $\mathcal{P}_n$  and thus we get an order of  $\hbar^{n-2}$ .

Consequently we get :

$$\hat{W}_{|I_i|+|A_i|}(I_i, A_i) = O(\hbar^{\text{Min}(n, |I_i|+|A_i|)-2+\delta_{|I_i|+|A_i|=1}}) \quad (4-35)$$

Thus we obtain a term of order :

$$\hbar^{|I|} \prod_{i=1}^l \hat{W}_{|I_i|+|A_i|}(I_i, A_i) = O\left(\hbar^{\sum_{i=1}^l (\text{Min}(n, |I_i|+|A_i|)-2+\delta_{|I_i|+|A_i|=1})+|I|}\right) \quad (4-36)$$

We shall prove the following inequality :

$$\sum_{i=1}^l (\text{Min}(n, |I_i| + |A_i|) - 2 + \delta_{|I_i|+|A_i|=1}) + |I| - n \geq 0 \quad (4-37)$$

with  $\sum_{i=1}^l |A_i| = n$ ,  $|I_i| \geq 1$  and  $\sum_{i=1}^l |I_i| = |I|$ . In the case when  $l = 1$ , we have  $|A_1| = n$  and  $|I_1| = |I| \geq 1$  so we end up with  $n - 2 + 1 + |I| - n = |I| - 1 \geq 0$  since  $I$  cannot be empty. For our future discussion, we will denote  $L_1$  the set of indexes for which  $1 < |I_i| + |A_i| \leq n$  and  $L_2$  the set of indexes for which  $|I_i| + |A_i| > n$ . We will denote respectively  $l_1 = |L_1|$  and  $l_2 = |L_2|$  satisfying  $l_1 + l_2 = l$ . The case where  $l_2 = 0$  i.e. the minimum is always equal to  $|I_i| + |A_i|$  is trivial since in that case we end up at least with  $|I| + n - 2l + |I| - n = 2(|I| - l) \geq 0$  since all  $I_i$  have at least one element  $|I_i| \geq 1$ .

Let us now consider the general case where  $l \geq 2$  and  $l_2 \geq 1$ . We first observe :

$$\begin{aligned} |I| &= \sum_{i \in L_1} |I_i| + \sum_{i \in L_2} |I_i| \\ &\geq l_1 + \sum_{i \in L_2} |I_i| = l_1 + \sum_{i \in L_2} (|I_i| + |A_i|) - \sum_{i \in L_2} |A_i| \\ &\geq l_1 + l_2(n+1) - \sum_{i \in L_2} |A_i| \\ &\geq l_1 + l_2(n+1) - n \end{aligned} \quad (4-38)$$

Inserting (4-38) into (4-37) we obtain :

$$\begin{aligned} &\sum_{i=1}^l (\text{Min}(n, |I_i| + |A_i|) - 2 + \delta_{|I_i|+|A_i|=1}) + |I| - n \\ &= \sum_{i \in L_1} (|I_i| + |A_i| - 2 + \delta_{|I_i|+|A_i|=1}) + (n-2)l_2 + |I| - n \\ &\geq \sum_{i \in L_1} (|I_i| + |A_i| - 2 + \delta_{|I_i|+|A_i|=1}) + (n-2)l_2 + l_1 + l_2(n+1) - n - n \\ &= \sum_{i \in L_1} (|I_i| + |A_i| - 2 + \delta_{|I_i|+|A_i|=1}) + 2(nl_2 - n - l_2) + l_1 + l_2 \\ &= \sum_{i \in L_1} (|I_i| + |A_i| - 2 + \delta_{|I_i|+|A_i|=1}) + 2(n-1)(l_2-1) + l - 2 \end{aligned} \quad (4-39)$$

The terms in the first sum are non-negative. Then, since  $n \geq 2$ , and  $l_2 \geq 1$ ,  $(n-1)(l_2-1)$  is always non-negative. Then, since  $l \geq 2$ , the last term is also non-negative, which concludes the proof of inequality (4-37).

Going back to (4-36) and inserting inequality (4-37), we obtain that the second line of (4-29) is at least of order  $O(\hbar^n)$ . Thus for any  $k > 0$ , evaluating at order  $\hbar^{n-k}$  in (4-29) we get :

$$P_n^{(n-k-1)}(x, y^{i_0}(x); A) = W_{n+1}^{(n-k)}(x.e_{i_0}, A)E_y(x, y(z^{i_0}(x))) \quad (4-40)$$

From  $\mathcal{P}_n$  we know that the right hand side vanishes for  $k > 1$ . Therefore the only possibly non-vanishing term is

$$W_{n+1}^{(n-2)}(x.e_{i_0}, A)dx = \frac{P_n^{(n-1)}(x, y(z^{i_0}(x)); A)}{E_y(x, y(z^{i_0}(x)))}dx \quad (4-41)$$

From the study of the pole structure (see section 4.4) we know that  $W_{n+1}^{(n-2)}(x.E_{i_0}, A)$  has no pole at coinciding points neither at double points, whereas from corollary 4.4, the right hand side may have poles only there. This implies that  $W_{n+1}^{(n-2)}(x.E_{i_0}, A)dx$  is a meromorphic one-form on the Riemann sphere without any poles. There is no meromorphic differential on the Riemann sphere without poles, except 0 so that we get :

$$W_{n+1}^{(n-2)}(x.E_{i_0}, A) = 0. \quad (4-42)$$

Therefore we conclude that  $W_{n+1}^{(n-2)}(x.E_{i_0}, A) = 0$  i.e. that  $W_{n+1}(x.E_{i_0}, A)$  is at least of order  $\hbar^{n-1}$ .

We now need to extend the previous result to higher correlators  $W_{n+p}$  with  $p > 1$ . For  $m \geq n + 1$ , we define the property  $\tilde{\mathcal{P}}_{n,m}$  :

$$\tilde{\mathcal{P}}_{n,m} : W_m = O(\hbar^{n-1})$$

We want to prove it by induction on  $m \geq n + 1$ .

The last result (4-42) implies that  $\tilde{\mathcal{P}}_{n,n+1}$  is verified so that initialization of the second induction is done.

Let  $m \geq n + 1$  and assume that  $\tilde{\mathcal{P}}_{n,n+1}, \dots, \tilde{\mathcal{P}}_{n,m}$  hold. Let  $A = \{X_1, \dots, X_m\}$  a set of distinct points of size  $m$ , and use (4-29) :

$$\begin{aligned} P_m(x, y(z^{i_0}(x)); A) &= \hbar W_{m+1}(x.e_{i_0}, A)E_y(x, y(z^{i_0}(x))) \\ &+ \sum_{\{i_0\} \subset I \subset D, |I| \geq 2} \hbar^{|I|} \hat{\mathcal{W}}_{|I|,m}(I; A) \prod_{i \notin I} (y(z^{i_0}(x)) - y(z^i(x))) \end{aligned} \quad (4-43)$$

In the decomposition of definition 4.1 of  $\hat{\mathcal{W}}_{|I|,m}(I; A)$ , consider 4 cases :

1.  $|I_i| = 1$  and  $A_i = \emptyset$  : In that case we get  $\hat{W}_1(x.E_i)$  which is of order at least  $\hbar^0$  (because we have a hat version of  $W_1$  whose  $\hbar^{-1}$  term is removed).
2.  $1 < |I_i| + |A_i| \leq n$  : In that case we can apply  $\mathcal{P}_{|I_i|+|A_i|}$  and thus we get an order of  $\hbar^{|I_i|+|A_i|-2}$
3.  $n < |I_i| + |A_i| \leq m$  : In that case, we can apply  $\tilde{\mathcal{P}}_{n,|I_i|+|A_i|}$  and thus we get an order of  $\hbar^{n-1}$

4.  $|I_i| + |A_i| > m$  : In that case we can only apply  $\mathcal{P}_n$  and thus we get an order of  $\hbar^{n-2}$

We will denote  $L_1$  the set of indexes for which  $1 < |I_i| + |A_i| \leq n$ ,  $L_2$  the set of indexes for which  $n < |I_i| + |A_i| \leq m$  and finally  $L_3$  the set of indexes for which  $|I_i| + |A_i| > m$ . We will also denote  $l_1 = |L_1|$ ,  $l_2 = |L_2|$  and  $l_3 = |L_3|$ . These non-negative integers satisfy  $l_1 + l_2 + l_3 = l$ . Putting it all together, we obtain that  $\hat{\mathcal{W}}_{|I|,m}(I; A)$  is of order :

$$\hbar^{|I|} \prod_{i=1}^l \hat{W}_{|I_i|+|A_i|}(I_i, A_i) = O\left(\hbar^{\sum_{i \in L_1} (|I_i|+|A_i|-2+\delta_{|I_i|+|A_i|=1}) + \sum_{i \in L_2} (n-1) + \sum_{i \in L_3} (n-2)+|I|}\right) \quad (4-44)$$

Therefore we need to prove the following inequality :

$$\sum_{i \in L_1} (|I_i| + |A_i| - 2 + \delta_{|I_i|+|A_i|=1}) + l_2(n-1) + l_3(n-2) + |I| - n \geq 0 \quad (4-45)$$

with  $\sum_{i=1}^l |A_i| = m$ ,  $|I_i| \geq 1$ ,  $\sum_{i=1}^l |I_i| = |I|$  and  $l_1 + l_2 + l_3 = l$ . The case  $l_2 = l_3 = 0$  is trivial because we find in (4-45) at least  $|I| + m - 2l + |I| - n = 2(|I| - l) + m - n \geq 0$ .

In the case  $l_2 + l_3 \geq 0$  we can use the following identity :

$$\begin{aligned} |I| &= \sum_{i \in L_1} |I_i| + \sum_{i \in L_2} |I_i| + \sum_{i \in L_3} |I_i| \\ &= \sum_{i \in L_1} |I_i| + \sum_{i \in L_2} (|I_i| + |A_i|) + \sum_{i \in L_3} (|I_i| + |A_i|) - \sum_{i \in L_2 \cup L_3} |A_i| \\ &\geq l_1 + (n+1)l_2 + l_3(m+1) - m \end{aligned} \quad (4-46)$$

Inserting this inequality back into (4-45) we find :

$$\begin{aligned} &\sum_{i \in L_1} (|I_i| + |A_i| - 2 + \delta_{|I_i|+|A_i|=1}) + l_2(n-1) + l_3(n-2) + |I| - n \\ &\geq l_2(n-1) + l_3(n-2) + l_1 + (n+1)l_2 + l_3(m+1) - m - n \\ &\geq 2nl_2 + l_3(n+m-1) - n - m + l_1 \\ &\geq 2nl_2 + (l_3-1)(n+m-1) + l_1 - 1 \\ &\geq 2n(l_2 + l_3 - 1) + l_1 - 1 \end{aligned} \quad (4-47)$$

If  $l_2 + l_3 > 1$  or  $l_1 > 0$ , this is clearly non-negative. The only problematic case could be when  $l_2 + l_3 = 1$  and  $l_1 = 0$ , and thus there can be only one part. This implies that  $|A_i| = m$ , and  $|A_i| + |I_i| = m + |I| > m$  and thus we are in the case  $l_3 = 1$  and  $l_2 = 0$ . In this case, the inequality (4-45) amounts to  $n - 2 + |I| - n = |I| - 2 \geq 0$ . It is obviously true because the terms with  $|I| \leq 1$  are the first line of (4-43) and have been put aside. Consequently, inequality (4-45) is proved.

Inserting (4-45) into (4-44), we deduce that  $\hbar^{|I|}\hat{W}_{|I|,j_0}(I; A)$  is at least of order  $\hbar^n$ . Since  $\hat{\mathcal{P}}_{n,m}$  holds, we know that  $W_{m+1}^{(n-2)}(x.e_{i_0}, A)$  is at most of order  $O(\hbar^{n-2})$ . Writing (4-43) at order  $\hbar^{n-1}$  gives :

$$P_m^{(n-1)}(x, y(z^{i_0}(x)); A) \frac{dx}{E_y(x, y(z^{i_0}(x)))} = W_{m+1}^{(n-2)}(x.e_{i_0}, A) dx \quad (4-48)$$

Then, the same argument used for (4-41) (i.e. the r.h.s. and the l.h.s. are meromorphic one-forms on the Riemann sphere without any common poles, so they identically vanish) concludes that  $W_{m+1}^{(n-2)}(x.e_{i_0}, A) = 0$ . This is where we need the genus zero assumption !

Thus we have proved that if  $\tilde{\mathcal{P}}_{n,j}$  is valid for all  $n+1 \leq j \leq m$  then  $\tilde{\mathcal{P}}_{n,m+1}$  is verified. Since we have proved that  $\tilde{\mathcal{P}}_{n,n+1}$  is also verified we conclude by induction on  $m$  that for all  $m \geq n+1$ ,  $\tilde{\mathcal{P}}_{n,m}$  is valid. In other words, for all  $m \geq n+1$  :  $W_{m+1}(x.e_{i_0}, A) = 0$  is at least of order  $\hbar^{n-1}$ . This is precisely proposition  $\mathcal{P}_{n+1}$ .

We finally conclude by induction on  $n$  that proposition  $\mathcal{P}_n$  is valid for all  $n \geq 1$ , i.e. that the correlation functions  $W_n$  are at least of order  $\hbar^{n-2}$ .  $\square$

## 5 Conclusion

### 5.1 Summary of the results

We have generalized the proof of [17, 18] to higher rank systems. We showed that all Lax pairs obeying some assumptions satisfy the TT property, and thus their correlators  $W_n$  have an  $\hbar$ -expansion given by the topological recursion. This result typically lies in a mirror symmetry statement : showing that the A-model correlation functions coincide with the B-model.

We expect that the assumptions we made to prove the TT property, are in fact satisfied by most integrable systems. Our strongest assumption is probably the genus zero assumption but among integrable systems that have a genus zero spectral curve we do not know for the moment any example that does not satisfy our assumptions.

If the genus of the spectral curve happens to be strictly positive, then the notions of TT property, WKB expansion and of topological recursion would fail all together. However, allowing oscillatory terms like in [20] should cure the problem and should give a generalization of the present article. The precise statement of the conjecture is made in [4] and the conjecture is strongly supported by the fact that it correctly gives the Jones polynomials to the first few orders in  $\hbar$ , as verified in [21].

## 5.2 Conjecture for the reconstruction of $\Psi$ via the topological recursion

So far, we have proved that the determinantal correlation functions  $W_n$ s built from a solution  $\Psi$  of the differential system, satisfy the topological recursion.

The next question of interest in quantum curve theory is to ask for the following : how can we recover  $\Psi$  from the topological recursion correlation functions  $W_{g,n}$ 's? The formulas conjectured in [11, 4] are the following :

**Conjecture 5.1 (Exponential formula)** *We should have the following WKB expansion :*

$$\left( \frac{\Psi(x')^{-1}\Psi(x)}{x-x'} \sqrt{dx dx'} \right)_{j,i} = \frac{e^{\frac{1}{\hbar} \int_{z^j(x')}^{z^i(x)} \omega_{0,1}}}{E(z^i(x), z^j(x'))} e^{\sum_{2g-2+n>0} \frac{\hbar^{2g-2+n}}{n!} \int_{z^j(x')}^{z^i(x)} \dots \int_{z^j(x')}^{z^i(x)} \omega_{g,n}} \quad (5-1)$$

where  $E(z, z') = \frac{(z-z')}{\sqrt{dz dz'}}$  is the Riemann sphere's prime form.

Consequently, the WKB expansion of  $\Psi$  should be :

$$\Psi(x)_{k,1;i} = \frac{e^{\frac{1}{\hbar} \Phi_k(z^i(x))}}{z^i(x) - \alpha_k} e^{\sum_{2g-2+n>0} \frac{\hbar^{2g-2+n}}{n!} \int_{\alpha_k}^{z^i(x)} \dots \int_{\alpha_k}^{z^i(x)} \omega_{g,n}} \quad (5-2)$$

and if  $1 \leq j \leq d_k$  :

$$(v^{-1}\Psi(x))_{k,j;i} = \frac{d^{j-1}}{dz'^{j-1}} \left( \frac{e^{\frac{1}{\hbar} \Phi_k(z^i(x))}}{z^i(x) - z'} e^{\sum_{2g-2+n>0} \frac{\hbar^{2g-2+n}}{n!} \int_{z'}^{z^i(x)} \dots \int_{z'}^{z^i(x)} \omega_{g,n}} \right)_{z'=\alpha_k} \quad (5-3)$$

where  $\Phi_k(z)$  is a regularized version of  $\int_{\alpha_k}^z \omega_{0,1}$  (which is divergent), defined by

$$\begin{aligned} V_k(z) &= \operatorname{Res}_{z' \rightarrow \alpha_k} \omega_{0,1}(z') \ln \left( 1 - \frac{x(z)^{1/d_k}}{x(z')^{1/d_k}} \right), & t_k &= \operatorname{Res}_{\alpha_k} \omega_{0,1} \\ \Phi_k(z) &= \int_{\alpha_k}^z \left( \omega_{0,1} - dV_k + \frac{t_k}{d_k} \frac{dx}{x} \right) + V_k(z) - \frac{t_k}{d_k} \ln x(z) \end{aligned} \quad (5-4)$$

In other words we define  $V_k$  and  $t_k$  as the polar part of  $\omega_{0,1}$ , so that  $\omega_{0,1} - dV_k + \frac{t_k dx}{d_k x}$  has no pole at  $\alpha_k$ , we integrate it from  $\alpha_k$  to  $z$ , and add back the term we have subtracted.

**Remark 5.1** Note that for any generic point  $q$  in a neighborhood of  $\alpha_k$ ,  $\Phi_k(z)$  is a regularization of  $\int^z \omega_{0,1}$ , by adding a constant :

$$\Phi_k(z) = C_{q,\alpha_k} + \int_q^z \omega_{0,1} \quad (5-5)$$

where  $C_{q,\alpha_k}$  is a constant independent of  $z$ , it depends only on  $q$  and  $\alpha_k$ .

**Remark 5.2** Those formula are of course to be understood in the sense of formal  $\hbar$ -series.



**Remark 5.3** As usual, those formulas make sense only within open domains where the  $z^i(x)$ 's, the square roots  $\sqrt{x'(z)}$  and the branches of  $\ln x(z)$  and  $x(z)^{1/d_k}$  are defined. These domains can be called “Stokes sectors”. They are not global, as is usual with WKB asymptotics implying the Stokes phenomenon : they change when changing sector.

**Remark 5.4** Those formulas should hold only for spectral curves of genus zero, as argued in [11, 4]. The higher genus formulas are given in [4, 20], and [21] is a check for the Jones polynomials of some knots.

It is easy to see that the first few orders in  $\hbar$  of those formulas are the right ones. In [1, 4], it was verified that the conjecture is true in general to order  $O(\hbar^2)$  (i.e. the third non trivial order, since the leading order is  $\hbar^{-1}$ ).

The main question is to prove that the whole series is indeed correct formally at all order. The conjecture has been proved to hold for a number of examples : the Airy case proved in [1], the Catalan case in [27], and many other cases in [23, 24, 29, 30]. Recently, a larger class of examples or rank greater than two was proved in [3]. What is missing at the moment is a general proof that could tackle all orders in a sufficiently generic way.

## Acknowledgments

B.E. thanks Centre de Recherches Mathématiques de Montréal, the FQRNT grant from the Québec government. O.M. would like to thank Université de Lyon, Université Jean Monnet and Institut Camille Jordan for financial support. This work is also supported by the ERC Starting Grant no. 335739 “Quantum fields and knot homologies” funded by the European Research Council under the European Union’s Seventh Framework Programme. B.E. also thanks Piotr Sułkowski. The authors would also like to thank the organizers of the “Moduli spaces, integrable systems, and topological recursions” workshop in Montréal where the first part of this work was realized. They also thank the AMS and the organizers of the “Von Neumann symposium” in North Carolina 2016 where some of this work was also realized.

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## A Recovering $\Psi$ from $M$

By definition,  $M(x.e_a) = \Psi(x)e_a\Psi(x)^{-1}$  satisfies the ODE :

$$\hbar\partial_x M(x) = [L(x), M(x)], \tag{A.1}$$

Moreover,  $M(x.e_a)$  is a rank one projector. Let us denote  $\Psi(x)^{-1} = \Phi(x)^T$ . We have :

$$M_{i,j}(x.e_a) = \Psi_{i,a}(x)\Phi_{j,a}(x) \tag{A.2}$$

and thus,  $\forall 1 \leq k \leq d$  :

$$\Psi_{i,a}(x) = M_{i,k}(x.e_a) f_{k,a}(x) \text{ where } f_{k,a}(x) = \frac{1}{\Phi_{k,a}(x)}. \quad (\text{A.3})$$

Let us insert this into the ODE for  $\Psi_{i,a}(x)$  :

$$\hbar \partial_x \Psi_{i,a}(x) = \sum_{j=1}^d L_{i,j} \Psi_{j,a}(x) \quad (\text{A.4})$$

This gives inserting (A.3) into (A.4) and using (A.1) for the derivative of  $M$  :

$$[L(x), M(x.e_a)]_{i,k} f_{k,a}(x) + M_{i,k}(x.e_a) \hbar \partial_x f_{k,a}(x) = (L(x)M(x.e_a))_{i,k} f_{k,a}(x) \quad (\text{A.5})$$

and thus

$$\hbar \frac{\partial_x f_{k,a}(x)}{f_{k,a}(x)} = \frac{(M(x.e_a)L(x))_{i,k}}{M_{i,k}(x.e_a)}, \quad (\text{A.6})$$

where we notice that the l.h.s. is independent of  $i$ . Therefore,  $\forall 1 \leq i' \leq d$  :

$$f_{k,a}(x) = e^{\frac{1}{\hbar} \int^x \sum_{j=1}^d \frac{M_{i',j}(x.e_a) L_{j,k}(x)}{M_{i',k}(x.e_a)}} \quad (\text{A.7})$$

and  $\forall 1 \leq i, a, k, i' \leq d$  :

$$\Psi_{i,a}(x) = M_{i,k}(x.e_a) e^{\frac{1}{\hbar} \int^x \sum_{j=1}^d \frac{M_{i',j}(x.e_a) L_{j,k}(x)}{M_{i',k}(x.e_a)}}. \quad (\text{A.8})$$

For example we could chose  $i' = k = 1$  :

$$\Psi_{i,a}(x) = M_{i,1}(x.e_a) e^{\frac{1}{\hbar} \int^x \sum_{j=1}^d \frac{M_{1,j}(x.e_a) L_{j,1}(x)}{M_{1,1}(x.e_a)}}. \quad (\text{A.9})$$

This can also be written

$$\Psi_{i,a}(x) = \frac{M_{i,k}(x.e_a)}{M_{i',k}(x.e_a)} e^{\frac{1}{\hbar} \int^x \sum_{j=1}^d L_{i',j}(x) \frac{M_{j,k}(x.e_a)}{M_{i',k}(x.e_a)}}. \quad (\text{A.10})$$

In particular, choosing  $i' = i$  :

$$\Psi_{i,a}(x) = e^{\frac{1}{\hbar} \int^x \sum_{j=1}^d L_{i,j}(x) \frac{M_{j,k}(x.e_a)}{M_{i,k}(x.e_a)}}. \quad (\text{A.11})$$

Similarly

$$\Phi_{i,a}(x) = e^{\frac{-1}{\hbar} \int^x \sum_{j=1}^d L_{j,i}(x) \frac{M_{k,j}(x.e_a)}{M_{k,i}(x.e_a)}}. \quad (\text{A.12})$$

In conclusion, if both  $M(x.e_a)$  and  $L(x)$  have a formal  $\hbar$  power series expansion, then  $\Psi$  has a WKB expansion. Note that the converse is also true from theorem 3.1 : if  $L(x)$  has a power series expansion in  $\hbar$  and  $\Psi(x)$  has a WKB expansion then  $M(x.e_a)$  has a power series expansion in  $\hbar$ .

## B Examples : Painlevé and $(p, q)$ minimal models

In this appendix, we present various cases in which our method can be applied. The first one deals with  $(p, q)$  minimal models that were studied in [2]. The second one deals with the Painlevé Lax pairs and was developed in [18]. For clarity we will only focus on the Painlevé VI case though all other Painlevé systems can be treated similarly (details can be found in [18]). The purpose of this section is also to give interesting examples for which all assumptions presented in this paper are satisfied.

### B.1 $(p, q)$ minimal models

They were studied with the topological recursion in [2]. However the proof presented in [2] was incomplete since the proof of the leading order property used an insertion operator that has been proved to be ill-defined later. This new proof uses our general method. We will here follow the standard notations of [2] taking in particular  $q = d$ .

In  $(p, q)$  minimal models ( $p$  and  $q$  are coprime strictly positive integers, see [2] for details),  $R(x, t, \hbar)$  is a  $q \times q$  companion matrix :

$$R(x, t, \hbar) = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ & & 0 & 1 & \vdots \\ \vdots & & & & \ddots & 0 \\ 0 & \dots & 0 & & & 1 \\ u_{d-1}(t, \hbar) & \dots & u_1(t, \hbar) & & & u_0(t, \hbar) - x \end{pmatrix} \quad (\text{B.1})$$

The matrix  $\Psi(x, t)$ , described in [2], is given by :

$$\Psi(x, t, \hbar) = \begin{pmatrix} \psi_1(x, t) & \dots & \psi_q(x, t) \\ (\hbar\partial_t)\psi_1(x, t) & \dots & (\hbar\partial_t)\psi_q(x, t) \\ \vdots & & \vdots \\ (\hbar\partial_t)^{q-1}\psi_1(x, t) & \dots & (\hbar\partial_t)^{q-1}\psi_q(x, t) \end{pmatrix} \quad (\text{B.2})$$

where  $(\psi_i)_{1 \leq i \leq q}$  are linearly independent solutions of the system :

$$x\psi(x, t) = Q\psi(x, t) , \quad \hbar\partial_t\psi(x, t) = -P\psi(x, t) \quad \text{and} \quad [P, Q] = \hbar \quad (\text{B.3})$$

where the operator  $(P, Q)$  are of the form :

$$\begin{aligned} P &= \sum_{k=0}^p v_k(t)(\hbar\partial_t)^k, \quad v_p = 1, v_{p-1} = 0, v_{p-2} = -pu(t) \\ Q &= \sum_{l=0}^q u_l(t)(\hbar\partial_t)^l, \quad u_q = 1, u_{q-1} = 0, u_{q-2} = -qu(t) \end{aligned} \quad (\text{B.4})$$

In particular, the condition  $[P, Q] = \hbar$  determines all functions  $(v_i)_{1 \leq i \leq p}$  and  $(u_i)_{1 \leq i \leq q}$  in terms of  $u(t)$  and its derivatives. The  $L(x, t) = (L_{k,j}(x, t))_{1 \leq k, j \leq q}$  matrix is determined by decomposing the operators  $(L_k)_{k \geq 0}$  on the basis  $((\hbar \partial_t)^i)_{i \geq 0}$  :

$$L_k(x, t) = \sum_{j=0}^q L_{k,j}(x, t) (\hbar \partial_t)^j \quad (\text{B.5})$$

where the operators  $(L_k)_{k \geq 0}$  can be defined recursively as :

$$L_0(x, t) = - \sum_{l=0}^p v_l(t) F_l(x, t) , \quad L_{k+1}(x, t) = (\hbar \partial_t) L_k(x, t) + L_{k,q-1}(x, t)(x - Q) \quad (\text{B.6})$$

with  $F_l(x, t) = \sum_{j \geq 0} F_{l,j}(x, t) (\hbar \partial_t)^j$  defined recursively by :

$$F_0(x, t) = 1 , \quad F_{l+1}(x, t) = (\hbar \partial_t) F_l(x, t) + F_{l,q-1}(x, t)(x - Q) \quad (\text{B.7})$$

In particular, it is obvious from the definitions to see that  $L(x, t)$  is a polynomial in  $x$ .

In the context of  $(p, q)$  minimal models, one is interested in formal expansion in  $\hbar$ . Since the functions  $(u_i(t, \hbar))_{i \geq 0}$  and  $(v_i(t, \hbar))_{i \geq 0}$  admit a formal expansion in  $\hbar$ , we get that assumption 1 is verified. Moreover, the spectral curve is of genus 0, so assumption 2 is verified. It is given by (see Proposition 5.2 of [2]) :

$$\begin{aligned} x_t(z) &= \sum_{k=0}^q u_k^{(0)}(t) z^k \\ y_t(z) &= \sum_{l=0}^p v_l^{(0)}(t) z^l \end{aligned} \quad (\text{B.8})$$

The auxiliary spectral curve is given by the characteristic polynomial of the companion matrix  $R(x, t, \hbar)$  :

$$\tilde{E}(x, s, t, \hbar = 0) = \det(s - R^{(0)}(x, t)) = x_t(s) - x \quad (\text{B.9})$$

The set of solutions of  $\tilde{E}_t(x, s; t, 0) = 0$  is thus the set of all  $(x_t(z), z)$  for  $z \in \Sigma = \bar{\mathbb{C}}$ . Therefore the auxiliary spectral curve is equivalent to the triple :

$$\tilde{\mathcal{S}}_t = (\bar{\mathbb{C}}, x_t, s_t) \quad (\text{B.10})$$

with the function  $s_t$  is the identity map  $s_t : z \mapsto z$ . Obviously the auxiliary spectral curve does not admit any double points and the spectral curve (B.8) is regular so assumption 3 is verified. Note that in our setting, the poles of the  $x_t$  function correspond to  $k = 1$ ,  $d_1 = q$  and  $\alpha_1 = \infty$ . In other words,  $z \mapsto x_t(z)$  has only one pole at infinity of order  $q$  (in the general theory developed above, the point  $z = \infty$  was assumed not to be

a pole of  $x$ . This means that some of the above formulas require some basic adaptations to accommodate this particular case). Since the  $R(x, t)$  matrix is a companion matrix, its eigenvectors are given by a Vandermonde-like matrix and we obtain :

$$V(x, t) = \mathcal{V}(x) \Rightarrow v(t) = I_q \quad (\text{B.11})$$

In particular, assumption 4 is trivially satisfied.

Notice that by definition,  $L(x, t)$  is a polynomial in  $x$  whose coefficients admit an  $\hbar$ -expansion. Thus, assumption 5 is satisfied.

Assumption 6 was partly proved in [2]. Indeed, the authors proved that the matrix  $\Gamma(t)$  given by (note that there is a change of convention in [2] where the  $\Gamma$  matrix is defined as the inverse of our present matrix and with a global  $(-1)^{q-1}$  constant) :

$$\Gamma(t) = \gamma(t)^{-1} \text{ with } \gamma(t) = (-1)^{q-1} \Phi(x, t) \Psi(x, t)^T \quad (\text{B.12})$$

satisfy (2-86) (See [2] for a precise definition of  $\Phi(x, t)$ ). In particular Theorem 5.2 of [2] proves that the matrix  $\gamma(t)$  does not depend on  $x$ . Therefore the only remaining issue to prove assumption 6 is to match  $\Gamma^{(0)}$  with  $(v(t)^T)^{-1} C v(t)^{-1}$  to satisfy (2-87). We observe that by definition, the generalized Vandermonde matrix  $\mathcal{V}(x)$  leads to :

$$C = \begin{pmatrix} u_1^{(0)}(t) & u_2^{(0)}(t) & \dots & u_{q-2}^{(0)}(t) & 0 & 1 \\ u_2^{(0)}(t) & \ddots & \ddots & \ddots & 1 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ u_{q-2}^{(0)}(t) & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 1 & \ddots & \ddots & \ddots & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 \end{pmatrix} \quad (\text{B.13})$$

In other words :  $C_{i,j} = 0$  if  $i + j > q + 1$  and  $C_{i,j} = u^{(0)}(t)_{i+j-1}$  if  $i + j \leq q + 1$ . Its inverse is given by :

$$C^{-1} = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 & 1 \\ 0 & \ddots & \ddots & \ddots & 1 & a_2 \\ \vdots & \ddots & \ddots & \ddots & \ddots & a_3 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 1 & \ddots & \ddots & \ddots & a_{q-1} \\ 1 & a_2 & a_3 & \dots & a_{q-1} & a_q \end{pmatrix} \quad (\text{B.14})$$

In other words,  $(C^{-1})_{i,j} = 0$  if  $i + j < q + 1$  and  $(C^{-1})_{i,j} = a_{i+j-q}$  if  $i + j \geq q + 1$ . The coefficients  $(a_i)_{1 \leq i \leq q}$  are determined by the following recursion (obtained by looking at



the term  $(C^{-1}C)_{i,1} = \delta_{i,1}$  with  $1 \leq i \leq q$  :

$$a_1 = 1, \quad a_2 = 0 \quad \text{and} \quad a_{i+1} = - \sum_{j=1}^{i-1} a_j u_{j+q-i+1}^{(0)}(t) \quad \text{for } 2 \leq i \leq q-1 \quad (\text{B.15})$$

Since  $v(t) = I_q$ , condition (2-87) is equivalent to prove that  $C^{-1} = \gamma^{(0)}(t)$ . The matrix  $\gamma(t)$  (unfortunately denoted  $\mathbf{C}$  with entries labeled from 0 to  $q-1$  in [2]) is described in equations 5.77, 5.78 and 5.79 of [2]. It satisfies  $\gamma_{i,j} = 0$  if  $i+j < q+1$  and

$$\begin{aligned} \gamma_{1,j} &= \delta_{j,q} \quad \text{for } 1 \leq j \leq q \\ \hbar \partial_t \gamma_{i,j} &= \gamma_{i,j+1} - \gamma_{i+1,j} \quad \text{for } 1 \leq i, j \leq q-1 \\ \hbar \partial_t \gamma_{i,q-1} &= -\gamma_{i,q} - \sum_{l=0}^{q-2} u_l(t) \gamma_{i,l+1} \quad \text{for } 1 \leq i \leq q-1 \end{aligned} \quad (\text{B.16})$$

Let us denote for clarity  $B = \gamma^{(0)}$ . Projecting the last set of equations at order  $\hbar^0$  gives  $B_{i,j} = 0$  if  $i+j < q+1$  and :

$$\begin{aligned} B_{1,j} &= \delta_{j,q} \quad \text{for } 1 \leq j \leq q \\ B_{i+1,j} &= B_{i,j+1} \quad \text{for } 1 \leq i, j \leq q-1 \\ B_{i,q} &= - \sum_{l=0}^{q-2} u_l^{(0)}(t) B_{i,l+1} \quad \text{for } 1 \leq i \leq q-1 \end{aligned} \quad (\text{B.17})$$

The second equation is equivalent to say that  $B$  is a Hankel matrix of the same form as  $C^{-1}$ . In other words,  $B_{i,j} = 0$  if  $i+j < q+1$  and  $B_{i,j} = b_{i+j-q}$  if  $i+j \geq q+1$ . The coefficients  $(b_i)_{1 \leq i \leq q}$  are determined by the first and last equations of (B.17). We get :

$$b_1 = 1, \quad b_2 = 0 \quad \text{and} \quad b_{i+1} = - \sum_{l=1}^{q-2} b_l u_l^{(0)}(t) \quad \text{for } 2 \leq i \leq q-1 \quad (\text{B.18})$$

Hence we recover the same recursion as (B.15). This finally proves that  $C^{-1} = \gamma^{(0)}$  so that assumption 6 is verified.

In conclusion, we have proved all required assumptions for the  $(p, q)$  minimal models that therefore satisfy the Topological Type property.

## B.2 Painlevé VI case

Painlevé equations were studied with the topological recursion in [17] (Painlevé II) and [18] (all six Painlevé equations). A simpler method (only valid in the case  $d=2$ ) was used to prove that the Painlevé Lax pairs satisfy the topological type property. We propose here to show that our generalization also applies directly to these cases. We will only carry out the Painlevé VI case (which is the most difficult) in details

but all results presented here can be easily adapted to the other Painlevé cases using computations presented in [18].

In the Painlevé 6 system we have :

$$L_{\text{VI}}(x, t, \hbar) = \frac{A_0(t)}{x} + \frac{A_1(t)}{x-1} + \frac{A_t(t)}{x-t}, \quad R_{\text{VI}}(x, t, \hbar) = -\frac{A_t(t)}{x-t} - \frac{(q-t)(\theta_\infty - \hbar)}{2t(t-1)}\sigma_3 \quad (\text{B.19})$$

$$A_0 = \begin{pmatrix} z_0 + \frac{\theta_0}{2} & -\frac{q}{t} \\ \frac{tz_0(z_0 + \theta_0)}{q} & -(z_0 + \frac{\theta_0}{2}) \end{pmatrix}, \quad A_1 = \begin{pmatrix} z_1 + \frac{\theta_1}{2} & \frac{q-1}{t-1} \\ -\frac{(t-1)z_1(z_1 + \theta_1)}{q-1} & -(z_1 + \frac{\theta_1}{2}) \end{pmatrix}$$

$$A_t = \begin{pmatrix} z_t + \frac{\theta_t}{2} & -\frac{q-t}{t(t-1)} \\ \frac{t(t-1)z_t(z_t + \theta_t)}{q-t} & -(z_t + \frac{\theta_t}{2}) \end{pmatrix}, \quad A_\infty = \begin{pmatrix} \frac{\theta_\infty}{2} & 0 \\ 0 & -\frac{\theta_\infty}{2} \end{pmatrix} = -(A_0 + A_1 + A_t)$$

Here,  $z_0(t)$ ,  $z_1(t)$  and  $z_t(t)$  are auxiliary functions of  $t$  that can be expressed in terms  $q(t)$  and a function  $p(t)$  defined by :

$$p = \frac{z_0 + \theta_0}{q} + \frac{z_1 + \theta_1}{q-1} + \frac{z_t + \theta_t}{q-t} \quad (\text{B.20})$$

The explicit expression for  $z_0$ ,  $z_1$  and  $z_t$  in terms of  $q$  can be found in [18] where  $q(t)$  is shown to satisfy a  $\hbar$ -deformed version of the Painlevé 6 equation (see [18] for details). Note that the matrix form  $L(x, t)dx$  has simple poles at  $x \in \{0, 1, \infty, t\}$  while  $R(x, t)dx$  only has simple poles at  $x \in \{\infty, t\}$ . Existence of an  $\hbar$ -expansion is discussed in [18] where assumption 1 is proved. At first order in  $\hbar$  it is shown in [18] that the spectral and auxiliary curves are of genus 0 :

$$y^2 = \frac{\theta_\infty^2(x - q_0)^2 P_2(x)}{4x^2(x-1)^2(x-t)^2}$$

$$s^2 = \frac{(q_0 - t)^2 \theta_\infty^2 P_2(x)}{4t^2(t-1)^2(x-t)^2} \quad (\text{B.21})$$

where  $P_2(x) = x^2 + \left(-1 - \frac{\theta_0^2 t^2}{\theta_\infty^2 q_0^2} + \frac{\theta_1^2 (t-1)^2}{\theta_\infty^2 (q_0-1)^2}\right)x + \frac{\theta_0^2 t^2}{\theta_\infty^2 q_0^2}$  that can be written equivalently  $P_2(x) = x^2 + \left(-\frac{\theta_0^2 t(t+1)}{\theta_\infty^2 q_0^2} + \frac{\theta_1^2 t(t-1)}{\theta_\infty^2 (q_0-1)^2} - \frac{\theta_t^2 t(t-1)}{\theta_\infty^2 (q_0-t)^2}\right)x + \frac{\theta_0^2 t^2}{\theta_\infty^2 q_0^2}$ . Here  $q_0$  stands for  $q^{(0)}(t)$  the leading order in  $\hbar$  of  $q(t)$ . It satisfies an algebraic equation of degree 6 that can be found explicitly in [18]. Inserting this result in the definition of  $R^{(0)}(x, t)$ , we get an expression of  $z_t^{(0)}$  and  $q_0$  in terms of  $a, b$  and  $t$  (as well as the monodromy parameters) :

$$(q_0 - t) = \pm \frac{t(t-1)\theta_t}{\theta_\infty \sqrt{(t-a)(t-b)}}$$

$$z_t^{(0)} = -\frac{\theta_t}{2} + \frac{1}{4} + \frac{\theta_\infty(q_0 - t)(t - \frac{a+b}{2})}{2t(t-1)} = -\frac{\theta_t}{2} \pm \frac{t - \frac{a+b}{2}}{2\sqrt{(t-a)(t-b)}}$$

(B.22)

so that we get :

$$\begin{aligned}
R^{(0)}(x, t) &= \pm \begin{pmatrix} -\frac{\theta_t(x-\frac{a+b}{2})}{2(x-t)\sqrt{(t-a)(t-b)}} & \frac{\theta_t}{\theta_\infty(x-t)\sqrt{(t-a)(t-b)}} \\ -\frac{(b-a)^2\theta_t\theta_\infty}{16(x-t)\sqrt{(t-a)(t-b)}} & \frac{\theta_t(x-\frac{a+b}{2})}{2(x-t)\sqrt{(t-a)(t-b)}} \end{pmatrix} \\
L^{(0)}(x, t) &= \frac{(x-q_0)t(t-1)}{x(x-1)(q_0-t)} R^{(0)}(x, t)
\end{aligned} \tag{B.23}$$

The spectral curve (B.21) is of genus 0 with two finite branchpoints at  $x \in \{a, b\}$  zeros of the polynomial  $P_2(x)$  thus assumption 2 is satisfied. Note that there is also a double point at  $x = q_0$  for the spectral curve but it is absent in the auxiliary curve. Since the spectral curve is of genus 0, it can be parametrized globally and we choose a parametrization suitable with the convention that  $z = \infty$  is not a pole of  $x(z)$  (so that it slightly differs from the usual Zhukovski parametrization of [18]). We take :

$$\begin{aligned}
x(z) &= \frac{a+b}{2} + \frac{b-a}{2} \left( 1 + \frac{1}{z-1} - \frac{1}{z+1} \right) = b + \frac{b-a}{(z+1)(z-1)} \\
y(z) &= \frac{\theta_\infty(x(z)-q_0)(b-a)z}{2(z-1)(z+1)x(z)(x(z)-1)(x(z)-t)} \\
s(z) &= \frac{(q_0-t)\theta_\infty(b-a)z}{2(z-1)(z+1)t(t-1)(x(z)-t)} = \pm \frac{(b-a)z\theta_t}{2(z-1)(z+1)(x(z)-t)\sqrt{(t-a)(t-b)}}
\end{aligned} \tag{B.24}$$

Note that  $x'(z) = -\frac{2z(b-a)}{(z+1)^2(z-1)^2}$ . In the  $z$  variable, the two branchpoints are located at  $z = 0$  and  $z = \infty$  while the poles are located at  $z = \pm 1$ . The involution (corresponding to  $x(z) = x(\bar{z})$ ) is given by  $\bar{z} = -z$ . Inverting the relation between  $x$  and  $z$  leads to :

$$z^1(x) = \sqrt{\frac{x-a}{x-b}} \text{ and } z^2(x) = -\sqrt{\frac{x-a}{x-b}} \tag{B.25}$$

so that

$$\begin{aligned}
S_1(x) &= \frac{\theta_\infty(q_0-t)\sqrt{(x-a)(x-b)}}{t(t-1)(x-t)} = \pm \frac{\theta_t}{(x-t)} \sqrt{\frac{(x-a)(x-b)}{(t-a)(t-b)}} \\
S_2(x) &= -\frac{\theta_\infty(q_0-t)\sqrt{(x-a)(x-b)}}{t(t-1)(x-t)} = \mp \frac{\theta_t}{(x-t)} \sqrt{\frac{(x-a)(x-b)}{(t-a)(t-b)}} \\
Y_1(x) &= \frac{\theta_\infty(x-q_0)\sqrt{(x-a)(x-b)}}{x(x-1)(x-t)} \\
Y_2(x) &= -\frac{\theta_\infty(x-q_0)\sqrt{(x-a)(x-b)}}{x(x-1)(x-t)}
\end{aligned} \tag{B.26}$$

In particular, from the last identities it is straightforward to verify that the auxiliary curve has no double points, i.e. that assumption 3 is satisfied. Moreover, application of the previous formulas leads to :

$$\vec{\mathcal{V}}(z) = \left( -\frac{i(z+1)}{\sqrt{2(b-a)}\sqrt{z}}, -\frac{i(z-1)}{\sqrt{2(b-a)}\sqrt{z}} \right) \quad (\text{B.27})$$

and thus :

$$\begin{aligned} \mathcal{V}(x) &= \begin{pmatrix} -\frac{i(z^1(x)+1)}{\sqrt{2(b-a)}} \left(\frac{x-b}{x-a}\right)^{\frac{1}{4}} & -\frac{(z^2(x)+1)}{\sqrt{2(b-a)}} \left(\frac{x-b}{x-a}\right)^{\frac{1}{4}} \\ -\frac{i(z^1(x)-1)}{\sqrt{2(b-a)}} \left(\frac{x-b}{x-a}\right)^{\frac{1}{4}} & -\frac{(z^2(x)-1)}{\sqrt{2(b-a)}} \left(\frac{x-b}{x-a}\right)^{\frac{1}{4}} \end{pmatrix} \\ &= \begin{pmatrix} -\frac{i}{\sqrt{2(b-a)}} \frac{\sqrt{x-a}+\sqrt{x-b}}{((x-a)(x-b))^{\frac{1}{4}}} & -\frac{1}{\sqrt{2(b-a)}} \frac{\sqrt{x-b}-\sqrt{x-a}}{((x-a)(x-b))^{\frac{1}{4}}} \\ -\frac{i}{\sqrt{2(b-a)}} \frac{\sqrt{x-a}-\sqrt{x-b}}{((x-a)(x-b))^{\frac{1}{4}}} & \frac{1}{\sqrt{2(b-a)}} \frac{\sqrt{x-b}+\sqrt{x-a}}{((x-a)(x-b))^{\frac{1}{4}}} \end{pmatrix} \\ &= \frac{1}{\sqrt{2(b-a)}} \begin{pmatrix} -i \left( \left(\frac{x-a}{x-b}\right)^{\frac{1}{4}} + \left(\frac{x-b}{x-a}\right)^{\frac{1}{4}} \right) & \left(\frac{x-a}{x-b}\right)^{\frac{1}{4}} - \left(\frac{x-b}{x-a}\right)^{\frac{1}{4}} \\ i \left( \left(\frac{x-b}{x-a}\right)^{\frac{1}{4}} - \left(\frac{x-a}{x-b}\right)^{\frac{1}{4}} \right) & \left(\frac{x-a}{x-b}\right)^{\frac{1}{4}} + \left(\frac{x-b}{x-a}\right)^{\frac{1}{4}} \end{pmatrix} \end{aligned} \quad (\text{B.28})$$

It is then straightforward to verify that :

$$\begin{aligned} \mathcal{V}(x)\mathcal{V}(x)^T &= \begin{pmatrix} \frac{(z^2(x)-z^1(x))}{(b-a)} \left(\frac{x-b}{x-a}\right)^{\frac{1}{2}} & \frac{((z^2(x))^2-(z^1(x))^2)}{(b-a)} \left(\frac{x-b}{x-a}\right)^{\frac{1}{2}} \\ \frac{((z^2(x))^2-(z^1(x))^2)}{(b-a)} \left(\frac{x-b}{x-a}\right)^{\frac{1}{2}} & -\frac{(z^2(x)-z^1(x))}{(b-a)} \left(\frac{x-b}{x-a}\right)^{\frac{1}{2}} \end{pmatrix} \\ &= \begin{pmatrix} -\frac{2}{b-a} & 0 \\ 0 & \frac{2}{b-a} \end{pmatrix} \end{aligned} \quad (\text{B.29})$$

Hence we get  $C = \frac{b-a}{2} \text{diag}(-1, 1)$  as claimed from (2-43). Note that we also get :

$$\begin{aligned} \mathcal{V}(x, t) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathcal{V}(x, t)^T &= \begin{pmatrix} -\frac{\sqrt{\frac{x-a}{x-b}+\sqrt{\frac{x-b}{x-a}+2}}}{2(b-a)} & \frac{\sqrt{\frac{x-b}{x-a}-\sqrt{\frac{x-a}{x-b}}}}{2(b-a)} \\ \frac{\sqrt{\frac{x-b}{x-a}-\sqrt{\frac{x-a}{x-b}}}}{2(b-a)} & -\frac{\sqrt{\frac{x-a}{x-b}+\sqrt{\frac{x-b}{x-a}-2}}}{2(b-a)} \end{pmatrix} \\ \mathcal{V}(x, t) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathcal{V}(x, t)^T &= \begin{pmatrix} \frac{\sqrt{\frac{x-a}{x-b}+\sqrt{\frac{x-b}{x-a}-2}}}{2(b-a)} & -\frac{\sqrt{\frac{x-b}{x-a}-\sqrt{\frac{x-a}{x-b}}}}{2(b-a)} \\ -\frac{\sqrt{\frac{x-b}{x-a}-\sqrt{\frac{x-a}{x-b}}}}{2(b-a)} & \frac{\sqrt{\frac{x-a}{x-b}+\sqrt{\frac{x-b}{x-a}+2}}}{2(b-a)} \end{pmatrix} \end{aligned} \quad (\text{B.30})$$

Computing  $\mathcal{V}(x)S(x)\mathcal{V}(x)^TC$  leads to :

$$\begin{aligned} \mathcal{V}(x)S(x)\mathcal{V}(x)^TC &= \begin{pmatrix} \frac{\theta_\infty(q_0-t)}{2t(t-1)} + \frac{\theta_\infty(q_0-t)(t-\frac{a+b}{2})}{2(x-t)} & -\frac{\theta_\infty(q_0-t)(b-a)}{4t(t-1)(x-t)} \\ \frac{\theta_\infty(q_0-t)(b-a)}{4t(t-1)(x-t)} & -\frac{\theta_\infty(q_0-t)}{2t(t-1)} - \frac{\theta_\infty(q_0-t)(t-\frac{a+b}{2})}{2(x-t)} \end{pmatrix} \\ &= \pm \begin{pmatrix} \frac{\theta_t}{\sqrt{(t-a)(t-b)}} + \frac{(t-\frac{a+b}{2})t(t-1)\theta_t}{2\sqrt{(t-a)(t-b)}(x-t)} & -\frac{\theta_t(b-a)}{4\sqrt{(t-a)(t-b)}(x-t)} \\ \frac{\theta_t(b-a)}{4\sqrt{(t-a)(t-b)}(x-t)} & -\frac{\theta_t}{\sqrt{(t-a)(t-b)}} - \frac{(t-\frac{a+b}{2})t(t-1)\theta_t}{2\sqrt{(t-a)(t-b)}(x-t)} \end{pmatrix} \end{aligned}$$

(B.31)

where we used (B.22) to replace  $q_0$ . Eventually a direct computation from (B.28) and (B.23) shows that :

$$\begin{aligned} V(x, t) &= v(t)\mathcal{V}(x) \text{ with } v(t) = \begin{pmatrix} 0 & \frac{4}{\theta_\infty(b-a)} \\ 1 & 0 \end{pmatrix} \\ L^{(0)}(x, t) &= v(t)\mathcal{V}(x)Y(x)\mathcal{V}(x)^T C v(t)^T \\ R^{(0)}(x, t) &= v(t)\mathcal{V}(x)S(x)\mathcal{V}(x)^T C v(t)^T \end{aligned} \quad (\text{B.32})$$

so that assumption 4 is verified.

Eventually since

$$L_{\text{VI}}(x, t, \hbar) = \frac{A_0(t, \hbar)}{x} + \frac{A_1(t, \hbar)}{x-1} + \frac{A_t(t, \hbar)}{x-t}, \quad R_{\text{VI}}(x, t, \hbar) = -\frac{A_t(t, \hbar)}{x-t} - \frac{(q-t)(\theta_\infty - \hbar)}{2t(t-1)}\sigma_3 \quad (\text{B.33})$$

we see that there is no mixing between the  $x$ -dependence and the  $\hbar$ -expansion. In particular,  $L^{(k)}$  has poles only at  $x \in \{0, 1, t\}$  and assumption 5 is trivially verified. Finally, the symmetry condition is answered in [18] where it is proved that

$$\Gamma_{\text{VI}}(t) = \begin{pmatrix} -\frac{t^2 z_0(z_0 + \theta_0)}{q} + \frac{(t-1)^2 z_1(z_1 + \theta_1)}{q-1} & 0 \\ 0 & 1 \end{pmatrix} \quad (\text{B.34})$$

satisfies assumption 6. Note that at order  $\hbar^0$  computations from [18] gives :

$$\Gamma_{\text{VI}}^{(0)}(t) = \begin{pmatrix} -\frac{\theta_\infty^2 (b-a)^2}{16} & 0 \\ 0 & 1 \end{pmatrix} \quad (\text{B.35})$$

Since  $\Gamma$  is only determined up to a global multiplication by a constant, we can easily match it with the direct computation of :

$$(v^T)^{-1} C v^{-1} = \frac{b-a}{2} \begin{pmatrix} -\frac{\theta_\infty^2 (b-a)^2}{16} & 0 \\ 0 & 1 \end{pmatrix} \quad (\text{B.36})$$

and thus assumption 6 is satisfied.