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Chaos-assisted tunneling in the presence of Anderson localization

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Tunneling between two classically disconnected regular regions can be strongly affected by the presence of a chaotic sea in between. This phenomenon, known as chaos-assisted tunneling, gives rise to large fluctuations of the tunneling rate. Here we study chaos-assisted tunneling in the presence of Anderson localization effects in the chaotic sea. Our results show that the standard tunneling rate distribution is strongly modified by localization, going from the known Cauchy distribution in the ergodic regime to a log-normal distribution in the strongly localized case. We develop an analytical single-parameter scaling theory which accurately describes the numerical data, for both a deterministic and a disordered model. Several possible experimental implementations using cold atoms, photonic lattices or microwave billiards are discussed.

Tunneling has been known since early quantum mechanics as a striking example of a purely quantum effect that is classically forbidden. However, the simplest presentation based on tunneling through a one-dimensional barrier does not readily generalize to more generic situations. Indeed, in dimension two or higher or in time-dependent systems, the dynamics becomes more complex with various degrees of chaos, and the tunneling effect can become markedly different [1]. An especially spectacular effect of chaos in this context is known as chaos-assisted tunneling [2]: in this case, tunneling is mediated by ergodic states in a chaotic sea, and tunneling amplitudes have reproducible fluctuations by orders of magnitude over small changes of a parameter. This is reminiscent of universal conductance fluctuations which arise in condensed matter disordered systems. There, reproducible fluctuations of the conductance when e.g. a magnetic field is varied are an interferential signature of the disorder configuration [3].

A typical example of chaos-assisted tunneling arises for systems having a mixed classical dynamics where regular zones with stable trajectories coexist with chaotic regions with ergodic trajectories. In the presence of a discrete symmetry, one can have two symmetric regular zones which are classically disconnected in the sense that classical transport between these two zones is forbidden. In the quantum regime however, transport between these two structures is possible through what is called dynamical tunneling [4]. Regular eigenstates are regrouped in pairs of symmetric and antisymmetric states whose eigenenergies differ by a splitting δ inversely proportional to the characteristic tunneling time. If a chaotic region is present between the two regular structures, tunneling becomes strongly dependent on the specificities of the energy and phase-space distributions of the chaotic states. It results in strong fluctuations of the tunneling splittings which is known to be well-described by a Cauchy distribution [5]. Chaos-assisted tunneling has been extensively studied in different contexts, both theoretically [6–12] and experimentally [13–21].

However, states in a chaotic sea are not necessarily

ergodic; indeed, it is well-known that in condensed matter, quantum interference effects can stop classical diffusion and induce Anderson localization [22–24]. In chaotic systems, a similar effect known as dynamical localization can take place where states are exponentially localized with a characteristic localization length ξ [25–28]. When the size L of the chaotic sea is much smaller than ξ , chaotic states are effectively ergodic, whereas a new regime arises when $\xi \ll L$ where strong localization effects should change the standard picture of chaos-assisted tunneling. Indeed, it is well known for disordered systems that in quasi-1D universal Gaussian conductance fluctuations are replaced by much larger fluctuations in the localized regime with a specific characteristic log-normal distribution [29]. Moreover, for chaotic systems, it was shown in [30] that tunneling is drastically suppressed by dynamical localization, and that the Anderson transition between dynamical localization and diffusive transport manifests itself as a sharp enhancement of the average tunneling rate at the transition [31].

In this paper, we study the hitherto unexplored effect of Anderson localization on the statistics of chaos-assisted tunneling in quasi-1D systems. We show that the distribution of level splittings changes from the Cauchy distribution characteristic of the ergodic regime to a log-normal distribution in the strongly localized regime. We consider two models to study these effects: a deterministic model having a mixed phase space and whose quantum chaotic states display dynamical localization, and a disordered model based on the famous Anderson model [22]. This allows us to study numerically the two extreme ergodic and localized regimes as well as the full crossover between them. The numerical data are found to follow a one-parameter scaling law with ξ/L . We present a simple analytical theory which accounts for the observed behaviors throughout the full range of parameters studied for both models. This shows that the fluctuations of chaos-assisted tunneling precisely characterize the different physical regimes of transport in these systems.

Models.— The deterministic model that we use is a variant introduced in [30] of the quantum kicked rotor.

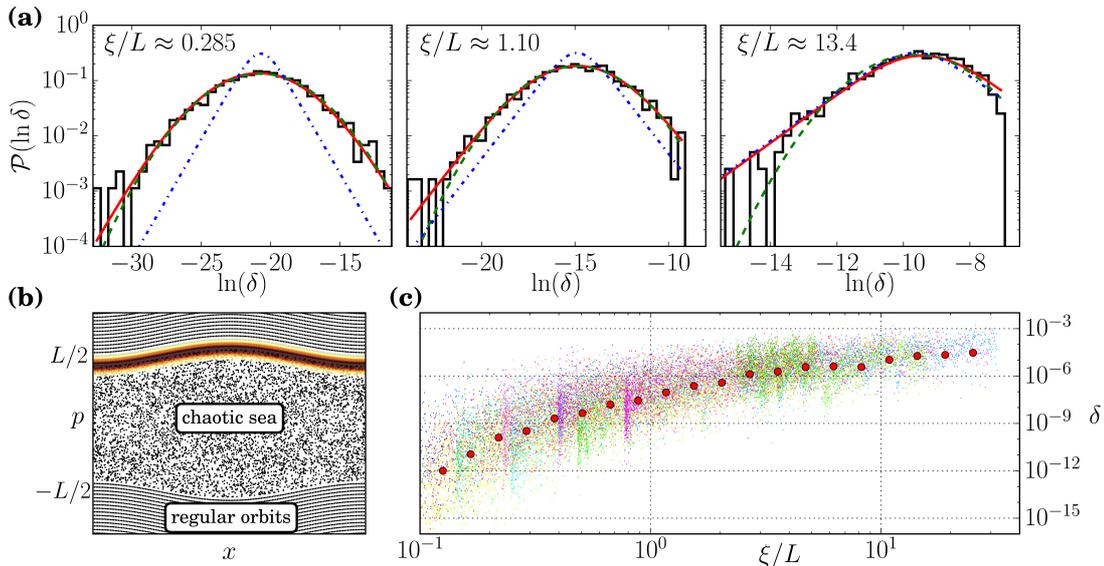


FIG. 1. (Color online). (a) Splitting distribution $\mathcal{P}(\ln \delta)$ for the deterministic model (1) in the localized (left), intermediate (middle) and ergodic (right) regimes. Green dashed line: fit with a log-normal distribution. Blue dashed-dotted line: fit by the Cauchy distribution. Red solid line: analytical theory Eq. (6) with Γ and δ_{typ} fitting parameters (see Fig. 2b). In the localized (ergodic) regime the log-normal (Cauchy) distributions overlap almost perfectly with Eq. (6). The data correspond to $K = 30$, $p_r = 0.4n$ ($n = 2048$) and 6400 values of \hbar in a small range around $\hbar = 0.85, 0.45, 0.25$ respectively. (b) Phase-space (Husimi) distribution of the initial wave function (color shades) superimposed on the classical phase space of the deterministic model (1), with $\hbar = 0.25$, $K = 30$, $p_c = 10$ and $p_r = 1024$. (c) Splittings δ as a function of ξ/L showing single-parameter scaling behavior. There are 16 sets (different colors) corresponding to fixed values of $K = 20, 30, 40, 50$ and p_r varies from $0.1n$ to $0.7n$ ($n = 2048$ is the system size) for 1600 values of $\hbar \in [0.1, 1.5]$. Big red dots: typical value of δ averaged over all data sets.

The Hamiltonian is given by:

$$\hat{H} = T(p) + K \cos(x) \sum_{n=-\infty}^{\infty} \delta(t - n), \quad (1)$$

where $p\hbar$ is momentum, x is a dimensionless position (or phase) with period 2π , t is time and K is the strength of the kicks. The dispersion relation $T(p)$ is chosen such that the phase space is divided into well-separated chaotic and regular (integrable) regions (cf. Ref. [30]):

$$T(p) = \frac{(\hbar p)^2}{2} \quad \text{for } |p| \leq p_r/2, \quad (2a)$$

$$T(p) = \omega|p| + \omega_0 \quad \text{for } |p| > p_r/2. \quad (2b)$$

The value of ω should be irrational, throughout this work we use $\omega = 2\sqrt{5} \approx 4.4721\dots$. ω_0 is chosen such that $T(p)$ is continuous. A discrete symmetry $p \rightarrow -p$ exists; the region of phase space $|p| \leq p_r/2$ is strongly chaotic for $K \gg 1$. The characteristic size of the chaotic sea is $L = p_r + K/[\hbar \sin(\omega/2)]$ (see Fig. 1b). The two regions $|p| > L/2$ correspond to two momentum symmetric regular zones. In the following, we will consider an initial state located on a classical torus in the regular part of phase space as obtained using Einstein-Brillouin-Keller quantization (see [30]), with initial mean momentum $\langle p \rangle(t=0) = p_c + L/2$. The distance to the chaotic sea p_c is set to $p_c = 10$ such that the coupling to the

chaotic sea is constant. In Fig. 1b we portray this initial state superimposed on the classical phase space representation of (2).

The second model we use is a disordered system based on the Anderson model:

$$\hat{H}_A = \sum_{i \neq 1, L} w_i a_i^\dagger a_i + \sum_{\langle i, j \rangle} a_i^\dagger a_j + \text{H.c.} \\ + t_c (a_1^\dagger a_2 + a_L^\dagger a_{L-1} + \text{H.c.}) \quad (3)$$

where the summation $\langle i, j \rangle$ is over indices $1 \leq |i - j| \leq 3$ ($i, j \neq 1, L$), L is the lattice size and $a^{(\dagger)}$ are annihilation (creation) operators. The on-site energies w_i are independent random numbers, uniformly distributed $\in [-W/2, W/2]$, with the important constraint that the w_i 's are *symmetric* with respect to the center of the lattice (creating the analog of the discrete symmetry $p \rightarrow -p$ for model (1)). The coefficient $t_c \ll 1$ represents a weak coupling of edge sites to the disordered chain. Tunneling beyond the nearest neighbor $|i - j| > 1$ is required to reach the effective quasi-one-dimensional regime, as the Anderson model with only nearest-neighbor hopping permits only localized and ballistic behavior.

One can identify two extreme regimes in these systems, characterized by the ratio of the localization length ξ to the size of the chaotic/disordered sea L : i) $\xi \gg L$: the *ergodic* regime and ii) $\xi \ll L$: the *localized* regime. We do not consider the ballistic regime where L is much smaller

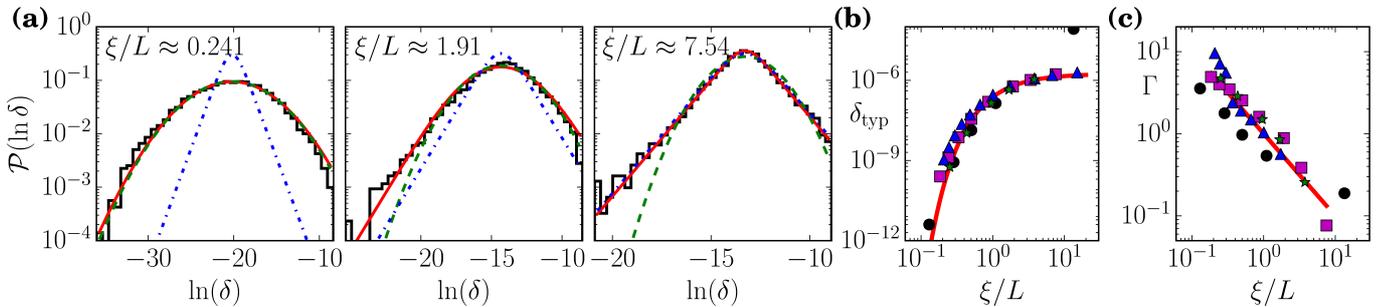


FIG. 2. (Color online). **(a)** Splitting distribution $\mathcal{P}(\ln \delta)$ for the disordered model (3), for $W = 3, 1, 1/2$ from left to right. Note the strong similarity with the distributions for the deterministic model (1) of Fig. 1a. Colored lines: as in Fig. 1. $L = 256$, $t_c = 0.001$, $N = 5$ channels, and 10000 realizations (see text). **(b)** Scaling behavior of δ_{typ} as a function of ξ/L . Disordered model (3): $L = 128$ (blue triangles), $L = 256$ (magenta squares) and $L = 512$ (green stars), parameters as in **(a)**. Black circles: deterministic model (1), $K = 30$, 6400 values of \hbar in a small range around 0.25, 0.45, 0.65, 0.85, 1.45, $p_r = 0.4n$ ($n = 2048$). Red solid line: theoretical prediction $\delta_{\text{typ}} = \delta_{\text{ch,typ}} \exp(-2L/\xi)$, with $\delta_{\text{ch,typ}}$ fitted to $L = 512$ data. **(c)** Scaling behavior of Γ (see Eq. (6)) as a function of ξ/L (symbols defined as in the left panel). Red solid line: theoretical prediction $\Gamma = L/\xi$.

than the mean free path.

Splitting statistics for the deterministic model.— The dynamics of the model (1) can be integrated over one period to give the evolution operator $\hat{U} = e^{-\frac{i}{\hbar}T(p)}e^{-\frac{i}{\hbar}K \cos(x)}e^{-\frac{i}{\hbar}T(p)}$, where we have chosen a symmetric version of the map. Tunneling may be studied through the properties of the Floquet eigenstates $|\Psi_\alpha\rangle$ of \hat{U} with associated quasi-energies ε_α obeying $\hat{U}|\Psi_\alpha\rangle = e^{i\varepsilon_\alpha}|\Psi_\alpha\rangle$. Classically, transport to the chaotic sea or the other regular island is forbidden by the presence of invariant curves. However, in the quantum regime, the initial state, having a certain expectation value of momentum $\langle p \rangle(t=0) = p_c + L/2$, will tunnel through the chaotic sea to the other side characterized by an oscillation period T_{osc} , so that $\langle p \rangle(T_{\text{osc}}/2) = -\langle p \rangle(t=0)$. T_{osc} can be identified with $2\pi/\delta$, where $\delta = \varepsilon^S - \varepsilon^A$ is the splitting between the symmetric and anti-symmetric Floquet eigenstates having the largest overlap with the initial state.

In Fig. 1a we show the distributions of $\ln \delta$ in the different regimes of the chaotic sea: localized $\xi \ll L$, ergodic $\xi \gg L$ and intermediate $\xi \approx L$. One can see that the distributions are markedly different: we recover the known Cauchy distribution in the ergodic regime whereas a log-normal distribution is observed in the localized regime. In the crossover regime, an intermediate distribution is obtained with a non-trivial shape to be discussed later. In order to reveal which scales control these behaviors, we considered the splittings δ for many different parameters (see caption of Fig. 1). In Fig. 1c we show that, strikingly, the data follow a single parameter scaling law as a function of ξ/L , where the localization length obeys the known expression [32] $\xi = (K^2/4\hbar^2)[1 - 2J_2(\tilde{K})(1 - J_2(\tilde{K}))]$ (J_2 denotes the Bessel function) with $\tilde{K} = (2K/\hbar) \sin \hbar/2$.

Splitting statistics for the disordered model.— The disordered model (3) can be used to describe chaos-

assisted tunneling, with the localization length in position space, dependent on W , analogous to the localization length in momentum space of the deterministic model (1), dependent on K and \hbar . The splitting δ is determined by computing the eigenfunctions most strongly overlapping with the first site (due to symmetry, this is equivalent to the last site). The corresponding splitting distributions for this model are shown in Fig. 2a. Strikingly, very similar behavior is observed in this disordered model compared to the deterministic model. In particular, we recover the Cauchy distribution at small values of W where the disordered sea has delocalized ergodic states, whereas for large W (localized states) the distribution has a log-normal shape, with again an intermediate behavior at the crossover. In Fig. 2b we show, for both the deterministic and disordered models, the scaling behaviors of $\delta_{\text{typ}} = \exp(\overline{\ln \delta})$, where $\overline{\ln \delta}$ denotes the ensemble average of $\ln \delta$, and the fitted parameter Γ related to the width of the distribution (see Eq. (6) below). Here we approximate ξ in the disordered model (3) according to $\xi \approx N\xi_{\text{1D}}$, where $N = 5$ is the number of channels (related to the number of neighbors to which particles can hop), and the localization length for a 1D Anderson chain obeys $\xi_{\text{1D}}^{-1} = \ln(1+W^2/16)/2 + 4 \arctan(W/4)/W - 1$ [33].

Analytical theory.— We can derive an expression for the splitting distribution valid in all regimes, including the crossover regime, for both the disordered model (3) and the deterministic model (1). Following [5], the splitting between the symmetric and antisymmetric regular states can be obtained from the displacements $\delta^{S,A}$ of their quasi-energies due to the coupling to the chaotic/disordered sea. Because this coupling v is classically forbidden, it is exponentially small and a perturbation theory can be used to obtain: $\delta^{S,A} \approx v^2/(E - E_{\text{ch}}^{S,A})$, with $E_{\text{ch}}^{S,A}$ the energy of the near resonant chaotic state of the same symmetry of the regular state of energy E [34]. The splitting δ is then given by $\delta = \delta^S - \delta^A$. In the

delocalized ergodic regime, the overlap between chaotic states is large which excludes that symmetric states are resonant concomitantly with antisymmetric states, thus δ^S and δ^A are uncorrelated. The splitting distribution is then given by the distribution of $\delta^{S,A}$, and follows a Cauchy law [5].

In the localized regime, symmetric and antisymmetric chaotic states have exponentially small overlap, therefore δ^S and δ^A are strongly correlated. This correlation is at the origin of the departure of the splitting distribution from the Cauchy law (see [2, 5] for similar correlations arising due to partial classical barriers in the chaotic sea). Defining the auxiliary quantities $E_{\text{ch}}^S = E_{\text{loc}} - \delta_{\text{loc}}/2$ and $E_{\text{ch}}^A = E_{\text{loc}} + \delta_{\text{loc}}/2$, with δ_{loc} the splitting of the localized chaotic states, $|\delta_{\text{loc}}| \ll |E - E_{\text{loc}}|$ in the localized regime, which yields:

$$\delta = |\delta^S - \delta^A| \approx \left| \frac{v^2}{E - E_{\text{loc}}} \frac{\delta_{\text{loc}}}{E - E_{\text{loc}}} \right|. \quad (4)$$

In the first factor, $v = v_{\text{ch}} \exp(-2L_{\text{reg}}/\xi_{\text{loc}})$ where v_{ch} is the usual coupling strength to the chaotic sea in the case where it has ergodic states. $\exp(-2L_{\text{reg}}/\xi_{\text{loc}})$ describes the effect of localization of the chaotic state, with localization length ξ_{loc} , located at a distance L_{reg} from the regular border. In the second factor of (4), $\delta_{\text{loc}} = \Delta \exp(-2L_{\text{loc}}/\xi_{\text{loc}})$ where L_{loc} is the distance between the two peaks of the localized (anti)symmetric chaotic state whereas $E - E_{\text{loc}} \approx \Delta$ the mean level spacing. As $L = 2L_{\text{reg}} + L_{\text{loc}}$, the splitting can be approximated by the simple expression:

$$\delta = \delta_{\text{ch}} \exp(-2L/\xi_{\text{loc}}) \quad (5)$$

with δ_{ch} the ‘‘standard’’ chaos-assisted tunneling splitting. The first term of (5) has the known Cauchy distribution $P_{\text{ch}}(\ln \delta_{\text{ch}}) = 2e^{\ln \hat{\delta}_{\text{ch}}} / [\pi(1 + e^{2 \ln \hat{\delta}_{\text{ch}}})]$, where $\hat{\delta} \equiv \delta/\delta_{\text{typ}}$. In our case $\delta_{\text{ch,typ}} = \exp(\overline{\ln \delta_{\text{ch}}})$ depends only on t_c or p_c (see also [11]). Because the inverse localization length $1/\xi_{\text{loc}}$ in quasi-1D has a normal distribution with width $\propto 1/L$, the second term $\tilde{\delta}_{\text{loc}} \equiv \exp(-2L/\xi_{\text{loc}})$ of (5) has a log-normal distribution, analogous to that of conductance in the strongly localized limit [29, 35]: $P_{\text{loc}}(\ln \tilde{\delta}_{\text{loc}}) = (1/4\Gamma\sqrt{2\pi}) \exp[-(\ln 1/\tilde{\delta}_{\text{loc}} - 2\Gamma)^2/8\Gamma]$, where $\Gamma = L/\xi$, $\xi = \exp(\overline{\ln \xi_{\text{loc}}})$ denoting the typical localization length. The total distribution $P(\ln \delta)$ can be obtained by convolution and is given by:

$$P(\ln \delta) = \frac{1}{4} \exp\left(-\ln \hat{\delta} + 2\Gamma\right) \left[1 + \operatorname{erf}\left(\frac{-4\Gamma + \ln \hat{\delta}}{\sqrt{8\Gamma}}\right) + \exp(2 \ln \hat{\delta}) \operatorname{erfc}\left(\frac{4\Gamma + \ln \hat{\delta}}{\sqrt{8\Gamma}}\right) \right]. \quad (6)$$

The expression (6) can be fitted to the distributions obtained numerically (see Fig. 1a and Fig. 2a). A very good agreement is found with the numerical data, both in the

two extreme localized and ergodic regimes where (6) describes a log-normal or Cauchy distribution, respectively. The intermediate regime ($\xi/L \approx 1$) is characterized by log-normal behavior in the center of the distribution, around $\delta = \delta_{\text{typ}}$, with Cauchy-type behavior in the tails. The fitted values of δ_{typ} and Γ are represented in Fig. 2b for both models. A good agreement is found with the expected behavior $\delta_{\text{typ}} = \delta_{\text{ch,typ}} \exp(-2L/\xi)$ and $\Gamma = L/\xi$ in the localized regime.

Experimental implementation.— We have shown that the effects we have presented are observable for both deterministic chaotic or disordered systems, provided they are in the localized regime. It is possible to implement a realistic version of the model (1) using a variant of the well-known cold-atom implementation of the quantum kicked rotor [27, 28, 36]. The dispersion relation (2) cannot be implemented directly. However, in the atomic kicked rotor, the kicking potential is usually realized through a sequence of short impulses of a stationary wave periodic in time. By truncating the Fourier series of this sequence at some specified harmonic p_r , one realizes a mixed system with a classically ergodic chaotic sea between $-p_r$ and p_r [37] which realizes an experimentally accessible version of (1) (see [38] for experimental details on an analogous system). The disordered model (3) can be implemented readily using photonic lattices [39–41]. A spatially symmetric disorder can be implemented in the direction transverse to propagation, and tunneling will result in oscillations in this transverse direction which can be easily measured. Lastly, both chaos-assisted tunneling [13–17] and Anderson localization [42, 43] have been observed in microwave chaotic billiards. In this context, both disordered and deterministic models could be implemented and can yield very precise distributions of the tunneling rate.

Conclusion.— We have shown that the presence of localization in the chaotic sea brings a new regime to chaos-assisted tunneling, manifested by a distinctive distribution of tunneling rates. Remarkably, the crossover from the ergodic to the localized regimes is governed by a single-parameter scaling law. The simple analytical theory presented describes accurately the different behaviors for both disordered and deterministic models, both of which could be implemented experimentally.

Our results show that chaos assisted tunneling could be used as a probe of the non-ergodic character of the chaotic sea. It would be interesting to generalize these ideas in higher dimensional systems where localization properties can be markedly different, with e.g. multifractal states at the Anderson transition.

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