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Abstract

The 1-2 Conjecture raised by Przybyło and Woźniak in 2010 asserts that every undirected graph admits a 2-total-weighting such that the sums of weights "incident" to the vertices yield a proper vertex-colouring. Following several recent works bringing related problems and notions (such as the well-known 1-2-3 Conjecture, and the notion of locally irregular decompositions) to digraphs, we here introduce and study several variants of the 1-2 Conjecture for digraphs. For every such variant, we raise conjectures concerning the number of weights necessary to obtain a desired total-weighting in any digraph. We verify some of these conjectures, while we obtain close results towards the ones that are still open.

1. Introduction

Posed by Karonski, Łuczak and Thomason in 2004, the **1-2-3 Conjecture** reads as follows [8]. An edge-weighting w of an undirected graph G is called *sum-colouring* if the sums of weights "incident" to the vertices yield a proper vertex-colouring of G. More precisely, for each vertex v of G one can compute

$$\sigma^e(v) := \sum_{u \in N(v)} w(vu)$$

and we require σ^e to be proper. The smallest $k \geq 1$ such that G admits a sum-colouring k-edge-weighting (if any) is denoted[1] $\chi^e_\sigma(G)$. The 1-2-3 Conjecture then states the following.

[☆]The first, third and fourth authors were supported by CNRS-PICS Project no. 6367 "GraphPar". The second author was supported by ANR grant no. ANR-13-BS02-0007 "STINT", ERC Advanced Grant GRACOL, project no. 320812, and PEPS grant POCODIS. The third author was supported by the National Science Centre, Poland, grant no. 2014/13/B/ST1/01855. The third and fourth authors were partly supported by the Polish Ministry of Science and Higher Education. The fourth author was supported by the National Science Centre, Poland, grant no. DEC- 2013/09/B/ST1/01772.

[1]This notation and its variants should be understood as follows throughout: χ means the parameter is a chromatic parameter; the superscript refers to the elements to be weighted or coloured; the subscript refers to the aggregate, computed from the weighting or colouring, to be distinguished on the adjacent vertices.

Conjecture 1.1 (1-2-3 Conjecture)**.** *For every graph G with no isolated edge, we have $\chi^e_\sigma(G) \leq 3$.*

Since its introduction, the 1-2-3 Conjecture has been attracting ingrowing attention, resulting in many research works considering either the conjecture itself or variants of it. As a best result towards it, it was proved by Kalkowski, Karoński and Pfender that $\chi^e_\sigma(G) \leq 5$ holds for every graph G with no isolated edge [7]. For more information, we refer the interested reader to [11] for a survey by Seamone on this wide topic.

In this paper, we mainly focus on the following two notions related to the 1-2-3 Conjecture. The first one is the total version of the 1-2-3 Conjecture, called the **1-2 Conjecture**, which was introduced by Przybyło and Woźniak in [10]. Quite similarly as in the context of weighting edges only, we say that a total-weighting w of G is *sum-colouring* if the vertex-colouring σ^t defined as

$$\sigma^t(v) := w(v) + \sum_{u \in N(v)} w(vu)$$

for every vertex v is proper. We then denote by $\chi^t_\sigma(G)$ the least $k \geq 1$ such that G admits a sum-colouring k-total-weighing. It is believed that being granted the possibility to "locally" modify the sums of weights incident to the vertices should, compared to the original edge version, reduce the number of needed weights.

Conjecture 1.2 (1-2 Conjecture)**.** *For every graph G, we have $\chi^t_\sigma(G) \leq 2$.*

The 1-2 Conjecture is known to hold for several families of graphs, such as 3-colourable graphs, complete graphs, and 4-regular graphs [10]. As for upper bounds on χ^t_σ, the best known one is due to Kalkowski who proved that $\chi^t_\sigma(G) \leq 3$ holds for every graph G [6].

The second notion considered in this paper is the one of *locally irregular decompositions.* We say that a graph G is *locally irregular* if every two of its adjacent vertices have distinct degrees. A *locally irregular edge-colouring* of G is then an edge-colouring whose each colour induces a locally irregular subgraph. We denote by $\chi^e_{\text{irr}}(G)$ the least number of colours giving a locally irregular edge-colouring of G (if any). Intuitively, the parameter χ^e_{irr} can be seen as a measure of how "far" from (locally) irregular is a graph. This parameter was introduced and studied by the current authors in [1] mainly because of its link with the 1-2-3 Conjecture and some of its variants. In particular, let us mention that in very particular settings, such as when dealing with regular graphs and only two colours, finding a sum-colouring edge-weighting is equivalent to finding a locally irregular edge-colouring. Since its introduction, this particular edge-colouring notion gave birth to several investigations, related mainly to the following conjecture raised in [1].

Conjecture 1.3. *For every graph G not among a well-identified set of graphs with maximum degree at most 3, we have $\chi^e_{\text{irr}}(G) \leq 3$.*

Conjecture 1.3 was mainly verified for several families of graphs, including regular graphs of large degree in [1] and graphs of large minimum degree [9]. No matter whether Conjecture 1.3 is true or not, it has to be known that it is difficult in general to compute the exact value of $\chi^e_{\mathrm{irr}}(G)$ for a given graph G, as shown in [2] by Baudon, Bensmail and Sopena. In a recent work [4], Bensmail, Merker and Thomassen provided the first constant upper bound on χ^e_{irr}, showing that $\chi^e_{\mathrm{irr}}(G) \leq 328$ holds for every graph G admitting locally irregular decompositions.

This paper is mainly inspired by two papers, namely [3] and [5], which brought Conjectures 1.1 and 1.3 in the context of digraphs in the particular setting where all notions of "incident weights" and "locally irregular graphs" are with respect to the *outdegree* parameter. So that we avoid any confusion, we omit the formal definitions and statements here and will rather recall them in the corresponding upcoming sections. Let us nevertheless mention that the directed version of Conjecture 1.1 considered in [3] by Baudon, Bensmail and Sopena was entirely solved, while the directed version of Conjecture 1.3 in [5] was only proved in a weaker form by Bensmail and Renault, where 6 instead of 3 is proved.

Section 2 is dedicated to sum-colouring edge-weightings and total-weightings in digraphs, while Section 3 is devoted to irregular decompositions in digraphs. The three series of results from these sections are comparable, and should hence be regarded in parallel. We start, in Section 2, by filling in the space showing that the directed version of the 1-2 Conjecture in the setting of [3] is false in a strong sense, and introduce a holding variant. In Section 3, we start by improving the main result of [5] from 6 down to 5, making one step closer to what is conjectured to be the right bound. We then investigate two total versions of the same problem inspired by the 1-2 Conjecture. Regarding these two versions, we provide bounds which are close to what we estimate to be optimal. Some conclusions are gathered in Section 4.

Notation and terminology: Throughout this paper, we focus on *simple digraphs*, i.e. loopless digraphs with no two arcs directed in the same direction between any pair of distinct vertices. Note that this definition allows our digraphs to have digons, i.e. directed cycles of length 2. Any arc (u, v) of a digraph D will be denoted $\overrightarrow{uv}$ to lighten the notation and make the arc's direction apparent. The *outdegree* (resp. *indegree*) of a vertex v of D is its number $d^+_D(v)$ (resp. $d^-_D(v)$) of outgoing (resp. incoming) incident arcs. In case no ambiguity is possible, the subscript in this notation will be freely omitted. The *maximum outdegree* (resp. *maximum indegree*) of D, denoted $\Delta^+(D)$ (resp. $\Delta^-(D)$) refers to the maximum outdegree (resp. indegree) over the vertices of D.

2. Sum-colouring arc- and total-weightings in digraphs

In this section, we extend the results from [3] to the 1-2 Conjecture. We start in Section 2.1 by recalling the investigations from [3]. Then we consider, in Sections 2.2 and 2.3, two directed analogues of the 1-2 Conjecture regarding the problem considered in that paper. The first such variant is shown to be false, even in a strong sense, while the second one is shown to hold.

2.1. Outsum-colouring arc-weightings

Let D be a digraph, and w be an arc-weighting of D. From w, one can compute, for every vertex v, the sum $\sigma_+^e(v)$ of "outgoing weights", formally defined as

$$\sigma_+^e(v) := \sum_{u \in N^+(v)} w(\overrightarrow{vu}).$$

In case σ_+^e is proper, we call w *outsum-colouring*. The least number $k \geq 1$ of weights needed to obtain an outsum-colouring k-arc-weighting of D is denoted $\chi_{\sigma+}^e(D)$. Using a very simple argument, Baudon, Bensmail and Sopena showed in [3] that the tightest upper bound on $\chi_{\sigma+}^e$ is 3, which cannot be improved as deciding whether $\chi_{\sigma+}^e(D) \leq 2$ holds for a given digraph D is NP-complete in general. Since this upper bound will be of some use in the upcoming sections, we state it here.

Theorem 2.1. *For every digraph D, we have $\chi_{\sigma+}^e(D) \leq 3$.*

2.2. Outsum-colouring total-weightings

We now consider the natural directed variant of the 1-2 Conjecture, where the terminology we use is inspired by that introduced in Section 2.1. Assume w is a total-weighting of a digraph D. To every vertex v, we associate the colour $\sigma_+^t(v)$, where

$$\sigma_+^t(v) := w(v) + \sum_{u \in N^+(v)} w(\overrightarrow{vu}).$$

We say that w is *outsum-colouring* if σ_+^t is proper. Again, the least number $k \geq 1$ of weights needed to deduce an outsum-colouring k-total-weighting of D is denoted $\chi_{\sigma+}^t(D)$.

Due to Theorem 2.1, clearly we have $\chi_{\sigma+}^t(D) \leq 3$ for every digraph D (start from an outsum-colouring $\chi_{\sigma+}^e(D)$-arc-weighting, and put weight 1 on all vertices). As a straight directed analogue of the 1-2 Conjecture, one could naturally wonder about the following question.

Question 2.2. *For every digraph D, do we have $\chi_{\sigma+}^t(D) \leq 2$?*

Unfortunately, easy counterexamples to Question 2.2 can be exhibited, showing that 3 is actually the best general upper bound on $\chi_{\sigma+}^t$. It can even be proved that Question 2.2 is far from being true, in the sense that there exists no constant $k \geq 3$ such that every digraph admits an outsum-colouring $(k, 2)$*-total-weighting*, i.e. an outsum-colouring total-weighting using weights among $\{1, ..., k\}$ on the vertices and among $\{1, 2\}$ on the arcs.

Proposition 2.3. *There is no $k \geq 1$ such that every digraph admits an outsum-colouring $(k, 2)$-total-weighting.*

Proof. Choose an odd integer $n \geq 5$, and let $\overrightarrow{T_n}$ be the tournament on n vertices defined as follows. Denote $0, 1, ..., n-1$ the vertices of $\overrightarrow{T_n}$, and, for every vertex i of $\overrightarrow{T_n}$, add the arcs $\overrightarrow{(i, i+1)}, \overrightarrow{(i, i+2)}, ..., \overrightarrow{(i, i+\lfloor \frac{n}{2} \rfloor)}$, where the indexes are taken modulo n.

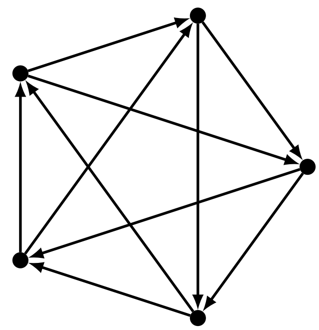

Figure 1: A digraph with no outsum-colouring 2-total-weighting.

By construction, every vertex of $\overrightarrow{T_n}$ has outdegree precisely $\lfloor \frac{n}{2} \rfloor$. For this reason, the possible values as $\sigma_+^t(v)$ by a $(k,2)$-total-weighting w of $\overrightarrow{T_n}$ are those among the set

$$\left\{ \left\lfloor \frac{n}{2} \right\rfloor + 1, \left\lfloor \frac{n}{2} \right\rfloor + 2, ..., 2 \cdot \left\lfloor \frac{n}{2} \right\rfloor + k \right\},$$

which includes $\lfloor \frac{n}{2} \rfloor + k$ values. Because $\overrightarrow{T_n}$ is a tournament, we need to have $\sigma_+^t(u) \neq \sigma_+^t(v)$ for every two vertices u and v of $\overrightarrow{T_n}$ so that w is outsum-colouring. Thus we want k to be big enough so that $\lfloor \frac{n}{2} \rfloor + k \geq n$, and hence $k \geq \lceil \frac{n}{2} \rceil$. By then making n grow, we need bigger and bigger values of k to get an outsum-colouring $(k,2)$-total-weighting of $\overrightarrow{T_n}$. This implies the claim. $\square$

Due to Proposition 2.3, digraphs may not admit outsum-colouring $(k,2)$-total-weightings with k being any fixed constant. So k should rather be expressed as some digraph invariant, as suggested in the following result, which is actually tight.

Proposition 2.4. *Every digraph D admits an outsum-colouring $(\Delta^+(D)+1, 2)$-total-weighting. Furthermore, there exist digraphs for which we cannot decrease the number of vertex weights.*

Proof. Let w be an outsum-colouring 3-arc-weighting of D. Such exists according to Theorem 2.1. Now for every vertex $v \in V(D)$, define

$$n_3(v) := \left| \{ \overrightarrow{vu} \in A(D) : w(\overrightarrow{vu}) = 3 \} \right|,$$

the number of arcs outgoing from v weighted 3 by w. Clearly, we have $n_3(v) \leq \Delta^+(D)$.

Now consider the $(\Delta^+(D)+1, 2)$-total-weighting w' of D defined as

$$\begin{cases} w'(v) &= n_3(v) + 1 \text{ for every } v \in V(D), \text{ and} \\ w'(\overrightarrow{uv}) &= \min\{2, w(\overrightarrow{uv})\} \text{ for every } \overrightarrow{uv} \in A(D). \end{cases}$$

By the way w' is defined, the value $\sigma_+^t(v)$ by w' is exactly 1 plus $\sigma_+^e(v)$ by w. Since w is outsum-colouring, then w' is also outsum-colouring.

To conclude the proof, we just note that the construction from the proof of Proposition 2.3 confirms the last part of the statement, as every considered tournament $\overrightarrow{T_n}$ verifies $\Delta^+(\overrightarrow{T_n}) + 1 = \lfloor \frac{n}{2} \rfloor + 1 = \lceil \frac{n}{2} \rceil$. $\square$

We end up this section by mentioning that Proposition 2.3 remains true even if one requires the adjacent "incident outmultisets" (rather than the "incident outsums") to be different. This will justify our upcoming investigations in the next section.

Remark 2.5. *Even the directed multiset analogue of the 1-2 Conjecture is false.*

2.3. Pair-colouring total-weightings

As pointed out in the previous section, the directed analogue of the 1-2 Conjecture in the setting of [3] is false in a strong sense (recall Proposition 2.3 and Remark 2.5). We herein show that by modifying the aggregate to be distinguished on the adjacent vertices, we get another directed variant of the 1-2 Conjecture, which, here, is verified.

Let w be a total-weighting of a digraph D. From w, we compute at every vertex v the value

$$\rho_+^t(v) := \left(w(v), \sum_{u \in N^+(v)} w(\overrightarrow{vu}) \right).$$

In case the vertex-colouring ρ_+^t is proper, we call w *pair-colouring*. The least number $k \geq 1$ of weights needed to obtain a pair-colouring k-total-weighting of D is denoted $\chi_{\rho+}^t(D)$.

We now prove the analogue of the 1-2 Conjecture for pair-colouring total-weighting.

Theorem 2.6. *For every digraph D, we have $\chi_{\rho+}^t(D) \leq 2$.*

Proof. Set $n := |V(D)|$. Consider the following ordering $v_1, ..., v_n$ over the vertices of D, defined from last (v_n) to first (v_1). Let v_n be a vertex of D satisfying $d_D^-(v_n) \leq d_D^+(v_n)$. Such a vertex exists since

$$\sum_{v \in V(D)} d_D^-(v) = \sum_{v \in V(D)} d_D^+(v).$$

Now consider the digraph $D - \{v_n\}$, and denote v_{n-1} one vertex of $V(D) \setminus \{v_n\}$ satisfying $d_{D-\{v_n\}}^-(v_{n-1}) \leq d_{D-\{v_n\}}^+(v_{n-1})$. Repeat the same procedure until all vertices of D are labelled. Namely, assuming that the vertices $v_{n-i+1}, ..., v_n$ have been defined, choose v_{n-i} as a vertex of $D - \{v_{n-i+1}, ..., v_n\}$ satisfying

$$d_{D-\{v_{n-i+1},...,v_n\}}^-(v_{n-i}) \leq d_{D-\{v_{n-i+1},...,v_n\}}^+(v_{n-i}),$$

which again exists according to the same argument as above.

We construct a pair-colouring 2-total-weighting w of D by considering the vertices $v_1, ..., v_n$ from "left" to "right", i.e. in increasing order of their indexes. Assume $v_1, ..., v_{i-1}$ have already been correctly treated, i.e. $\rho_+^t(v_1)$, ..., $\rho_+^t(v_{i-1})$ are defined (these vertices and their outgoing arcs have been each assigned a weight) and $\rho_+^t(v_j) \neq \rho_+^t(v_{j'})$ for every $j, j' \in \{1, ..., i-1\}$ such that v_j and $v_{j'}$ are adjacent. Let $D_i := D - \{v_{i+1}, ..., v_n\}$. We now assign a weight to v_i and its outgoing arcs by w in such a way that no conflict arises. An important thing to keep in mind is that when weighting an arc $\overrightarrow{v_i v_j}$, the couple $\rho_+^t(v_j)$ is not altered. Note further that $\rho_+^t(v_i) \neq \rho_+^t(v_j)$ whenever $w(v_i) \neq w(v_j)$.

For every $\alpha \in \{1, 2\}$, let

$$n_\alpha := \left| \left\{ v_j \in N_{D_i}^-(v_i) \cup N_{D_i}^+(v_i) : j < i \text{ and } w(v_j) = \alpha \right\} \right|$$

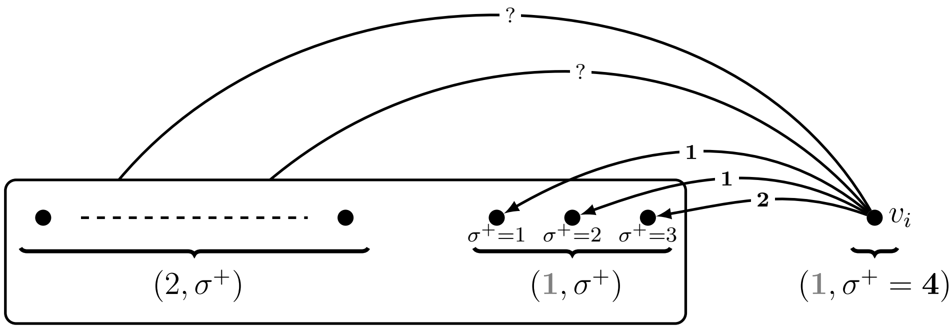

Figure 2: Illustration of the proof of Theorem 2.6.

be the number of already treated adjacent vertices which have been assigned weight α. There has to be a value of $\alpha \in \{1, 2\}$ for which

$$n_\alpha \leq \left\lfloor \frac{d^+_{D_i}(v_i) + d^-_{D_i}(v_i)}{2} \right\rfloor.$$

Let us assume $\alpha = 1$ in what follows.

Set $w(v_i) = 1$. Then v_i is already distinguished from all its already treated adjacent vertices which received weight 2 by w. Now what remains to do is to weight the arcs outgoing from v_i in D so that v_i is distinguished by its outsum from all its already treated adjacent vertices which received weight 1 by w (refer to Figure 2 for an illustration). The possible outsums for v_i in D by w are those among

$$S := \left\{ d^+_D(v_i), d^+_D(v_i) + 1, ..., 2d^+_D(v_i) \right\},$$

forming a set with cardinality $d^+_D(v_i) + 1$. But the outsum of v_i has to be different from the outsums of its n_1 previously treated adjacent vertices which also received weight 1 by w. By the ordering of the vertices of D, we have

$$d^+_{D_i}(v_i) + d^-_{D_i}(v_i) \leq 2d^+_{D_i}(v_i),$$

yielding

$$n_1 \leq d^+_{D_i}(v_i) < |S|.$$

There is thus at least one value s among S which does not appear as the outsum by w of any vertex v_j with $j < i$ neighbouring v_i which received weight 1. Then just weight the arcs outgoing from v_i so that the outsum of v_i is s. Now v_i gets also distinguished from its previously considered neighbours weighted 1.

By repeating the above procedure until we reach v_1, we eventually get the claimed pair-colouring 2-total-weighting w, concluding the proof. $\square$

3. Irregular arc- and total-decompositions in digraphs

We now focus on irregular decompositions of digraphs, where the notion of irregularity is with respect to the one introduced in [5] by Bensmail and Renault. We start by recalling the needed terminology and notation in Section 3.1. In the same section, we then improve the main result from [5] by showing that every digraph is decomposable into at most five locally irregular digraphs (while six is proved there). Total counterparts for irregular decompositions of the notions we have introduced in Sections 2.2 and 2.3, are then studied in Sections 3.2 and 3.3.

3.1. Locally irregular arc-colourings

A digraph D is called *locally irregular* if its adjacent vertices have distinct outdegrees. An arc-colouring of D is said *locally irregular* if its every colour class induces a locally irregular subdigraph. The smallest number of colours in a locally irregular arc-colouring of D is denoted by $\chi^e_{\text{irr+}}(D)$.

The main conjecture stated in [5] is the following.

Conjecture 3.1. *For every digraph D, we have $\chi^e_{\text{irr+}}(D) \leq 3$.*

The originators of Conjecture 3.1 proved its following weakening.

Theorem 3.2. *For every digraph D, we have $\chi^e_{\text{irr+}}(D) \leq 6$.*

The proof of Theorem 3.2 consists in first arc-decomposing any D into two acyclic digraphs, i.e. two digraphs with no directed cycles. The claimed upper bound then follows by showing that Conjecture 3.1 holds for acyclic digraphs (as first proved in [5]). We formally state this result as it will be used in some of our upcoming proofs.

Lemma 3.3. *For every acyclic digraph D, we have $\chi^e_{\text{irr+}}(D) \leq 3$.*

Our improvement on the bound in Theorem 3.2 from 6 down to 5, consists in showing that every digraph admits an arc-decomposition into one acyclic digraph and one *degree-decreasing acyclic digraph*, which we define as an acyclic digraph admitting an ordering $v_1, ..., v_n$ of its vertices such that

1. all arcs are directed "to the right", and
2. $d^+(v_i) \geq d^+(v_j)$ whenever $i < j$.

Since acyclic digraphs D verify $\chi^e_{\text{irr+}}(D) \leq 3$ (according to Lemma 3.3) and degree-decreasing acyclic digraphs D verify $\chi^e_{\text{irr+}}(D) \leq 2$ (which we show in Lemma 3.6 below), our result follows.

Theorem 3.4. *For every digraph D, we have $\chi^e_{\text{irr+}}(D) \leq 5$.*

Towards Theorem 3.4, as a first step we start by pointing out that every digraph indeed admits an arc-decomposition into one acyclic digraph and one degree-decreasing acyclic digraph.

Lemma 3.5. *Every digraph D admits an arc-decomposition into one acyclic digraph and one degree-decreasing acyclic digraph.*

Proof. Consider the following ordering $v_1, ..., v_n$ over the vertices of D. Start with v_1 being one vertex of D with largest outdegree (if there are several choices as v_1, pick any of them). Now remove v_1 from D and choose v_2 to be one vertex of $D - \{v_1\}$ with the largest outdegree. Then remove v_2 from $D - \{v_1\}$ and continue the procedure until all vertices are labelled. Basically, if we just read the vertices from "left" (i.e. v_1) to "right" (i.e. v_n) we get that for every two vertices v_i and v_j with $i < j$, vertex v_i has more outneighbours than v_j towards the right.

Now let A_2 be the subset of arcs of D containing all arcs of the form $\overrightarrow{v_i v_j}$ with $i < j$ (i.e. the arcs going to the right). Clearly $D[A_2]$ cannot have a directed cycle. Besides, due to the ordering of the vertices, $D[A_2]$ is degree-decreasing. Now let A_1 be the subset of the remaining arcs, i.e. those going to the left. For the same reason as previously, $D[A_1]$ is acyclic (but clearly it may be not degree-decreasing). Then A_1 and A_2 yield the desired arc-decomposition. $\square$

We now prove the second ingredient of our proof of Theorem 3.4, namely that degree-decreasing acyclic digraphs D verify $\chi^e_{\mathrm{irr}+}(D) \leq 2$. The proof is algorithmic and the result is also of interest as, as noted in [5], there are acyclic digraphs D verifying $\chi^e_{\mathrm{irr}+}(D) = 3$. So our result provides a new class of acyclic digraphs for which two colours are sufficient.

Lemma 3.6. *For every degree-decreasing acyclic digraph D, we have $\chi'_{\mathrm{irr}}(D) \leq 2$.*

Proof. Let $v_1, ..., v_n$ be an ordering of $V(D)$ such that all arcs are directed to the right, and verifying $d^+(v_1) \geq ... \geq d^+(v_n)$. To prove the claim, we start from v_n, iteratively put back the vertices $v_{n-1}, v_{n-2}, ..., v_1$ to D following this order, and, at each iteration, extend a locally irregular 2-arc-colouring to the added arcs. Assume we consider one vertex v of the v_i's, and all of its outneighbours $u_1, ..., u_k$, which all have smaller outdegree, have already been considered. We then just need to show that the arcs outgoing from v can be coloured without creating any conflict with the previously coloured arcs. Note that colouring any such arc does not affect the outdegrees of the u_i's in the subdigraphs induced by colours 1 and 2.

In the following, we say that one of the u_i's, say u_1, is a *(d_1, d_2)-vertex* if u_1 has d_1 outgoing arcs coloured 1 by the partial colouring, and d_2 outgoing arcs coloured 2. Now consider the following procedure for colouring the arcs outgoing from v. First start by colouring 1 all arcs outgoing from v. If this extension of the arc-colouring is not locally irregular (otherwise we are done), it means that at least one neighbour of v, say u_1, has k outgoing arcs coloured 1. Actually, by the ordering of the vertices of D, we even get that u_1 is a $(k, 0)$-vertex. This being known, for the next steps we know that the arc $\overrightarrow{vu_1}$ can be safely coloured 1 unless all arcs outgoing from v are couloured 1. Now colour all arcs outgoing from v with colour 2. If a conflict arises (otherwise, again we are done), by the previous arguments we know it originates from at least one vertex, say u_2, which is a $(0, k)$-vertex different from u_1. Once again, we note that u_2 cannot cause more trouble in the next iterations unless all arcs outgoing from v are coloured 2.

Now colour 2 the arc $\overrightarrow{vu_2}$ and colour 1 all other arcs outgoing from v. By the previous arguments, if the obtained arc-colouring is still not locally irregular, it means that at least one vertex different from u_1 and u_2, say u_3, is a $(k-1, \leq 1)$-vertex. This time, for the

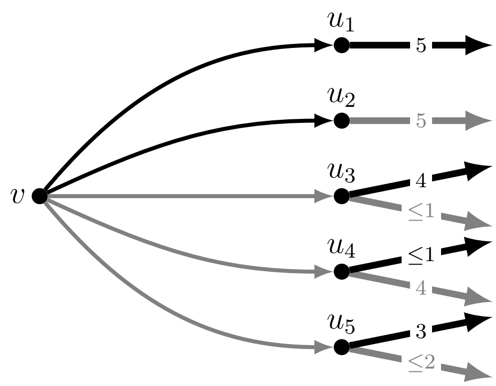

Figure 3: Illustration of the proof of Lemma 3.6 for $k = 5$.

next iterations, note that u_3 will not cause more problem unless $k-1$ arcs including $\overrightarrow{vu_3}$ are coloured 1, or maybe only the arc $\overrightarrow{vu_3}$ is coloured 2 (in case u_3 is a $(k-1,1)$-vertex). Now colouring 1 the arc $\overrightarrow{vu_1}$ and 2 all the other arcs outgoing from v, if again we run into a conflict, it involves at least one vertex different from u_1, u_2, u_3, say u_4, which is actually a $(\leq 1, k-1)$-vertex. Similar deductions as for u_3 can be made for the next iterations.

By repeatedly applying this colouring scheme, i.e. colouring 1 or 2 some "safe" arcs outgoing from v whose outdegrees in the two subdigraphs induced by the partial arc-colouring have been revealed in the previous steps, we either get a locally irregular extension at some point, or reveal that v has at least one $(k-i, \leq i)$-outneighbour and at least one $(\leq i, k-i)$-outneighbour (unless k is odd) for every value of i within some range. Actually, since we reveal the status of two new outneighbours at each step (unless for the last step if k is odd), the procedure ends within $\lceil k/2 \rceil$ steps. Now the conclusion is that if k is even, then the procedure ends with none of the u_i's being a $(k/2, k/2)$-vertex; so to get a locally irregular extension, we can just arbitrarily colour $k/2$ arcs outgoing from v with colour 1, and colour the remaining $k/2$ arcs with colour 2. In case k is odd, it just has to be noted that if, in the procedure above, at each step we try to reveal some outdegree in the subdigraph induced by colour 1 first, then the procedure ends with none of the u_i's having outdegree $\lfloor k/2 \rfloor$ in the subdigraph induced by colour 1, and none of the u_i's having outdegree $\lceil k/2 \rceil$ in the subdigraph induced by colour 2. Then we can just arbitrarily colour $\lfloor k/2 \rfloor$ arcs outgoing from v with colour 1, and colour 2 the $\lceil k/2 \rceil$ remaining arcs (see Figure 3). $\square$

3.2. Locally irregular total-colourings

Let us now discuss a total counterpart of the problem above. Within this, arcs and vertices of a given digraph D receive colours, and each colour class is expected to induce a locally irregular subdigraph in D with respect to outdegrees. This time however, the outdegree of a vertex v in the subdigraph induced by colour i is being increased by 1 if v is coloured with colour i (and is not modified otherwise).

In other words, the colour classes define a decomposition of D into *locally irregular total subdigraphs*, where a *total digraph* might be regarded as a triplet $D^t = (V_0, V_1; A)$ with $V := V_0 \cup V_1$ ($V_0 \cap V_1 = \emptyset$) constituting its vertex set and A corresponding to the set of arcs (defined as in a usual digraph). The vertices in V_0 are called *hollow*, while those in

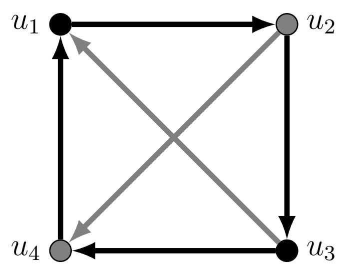

Figure 4: A locally irregular 2-total-colouring (with colours black and gray) of a digraph.

V_1 are said *solid*. The (total) outdegree (resp. indegree) of a vertex v, denoted $d_t^+(v)$ (resp. $d_t^-(v)$), is then understood as only the number of arcs outgoing from (resp. incoming to) v if v is a hollow vertex, or this quantity plus 1 if v is solid. Such a total digraph is called *locally irregular* if $d_t^+(u) \neq d_t^+(v)$ for every arc $\overrightarrow{uv} \in A$. A *locally irregular k-total-colouring* of a digraph D is then a total-colouring c of D with k colours such that each colour class induces a locally irregular total subdigraph (where for any given colour i, the vertices of D coloured with i define V_1 for the corresponding total digraph, and the rest of vertices of D yield V_0). The least number of colours needed to colour D is then denoted by $\chi^t_{\mathrm{irr}+}(D)$.

See Figure 4 for an illustration of a locally irregular 2-total-colouring of a given digraph. We note in particular that the gray subdigraph is locally irregular since its only pairs of adjacent vertices are (u_1, u_3) and (u_2, u_4), which verify $(d_t^+(u_1), d_t^+(u_3)) = (0, 1)$ and $(d_t^+(u_2), d_t^+(u_4)) = (2, 1)$. It is worth mentioning that all these notions were introduced and studied in the context of undirected graphs by the authors [1].

Concerning a general upper bound on $\chi^t_{\mathrm{irr}+}$, it is worth observing that if $\chi^t_{\mathrm{irr}+}(D) \leq 2$ held for every digraph D, then every digraph would admit a 2-total-weighting distinguishing the adjacent vertices by their "incident outmultisets", contradicting Remark 2.5. We however believe that the following conjecture should be the right direction.

Conjecture 3.7. *For every digraph D, we have $\chi^t_{\mathrm{irr}+}(D) \leq 3$.*

As already mentioned, the upper bound in Conjecture 3.7, if verified, would be best possible. We below get quite close to Conjecture 3.7 by proving that 4 bounds $\chi^t_{\mathrm{irr}+}$ above.

Theorem 3.8. *For every digraph D, we have $\chi^t_{\mathrm{irr}+}(D) \leq 4$.*

Proof. By Lemma 3.5, digraph D admits an arc-decomposition into a degree-decreasing acyclic digraph D_1 and an acyclic digraph D_2. We will regard these as total subdigraphs where all vertices of D_1 are hollow and all vertices of D_2 are solid. By Lemma 3.6, we immediately obtain that D_1 can be further decomposed into two locally irregular total subdigraphs (as all vertices of D_1 are hollow, and thus do not influence their corresponding outdegrees). It is thus left to show that D_2 (with all vertices solid) can be decomposed into two such total subdigraphs as well. For this aim, a slightly increased range of potential outdegrees of all vertices is exactly as much as is necessary.

Let $v_1, ..., v_n$ be any ordering of $V(D_2)$ such that all arcs are directed to the right. To construct a desired decomposition, or equivalently the corresponding total-colouring of D_2 with colours 1 and 2, we start from v_n and iteratively put back the vertices $v_{n-1}, v_{n-2}, ..., v_1$

to D_2 following this order, and, at each iteration, show that a locally irregular 2-total-colouring exists. Assume we consider v, and all of its outneighbours $u_1, ..., u_k$ have already been considered. We then just need to show that the arcs outgoing from v and v itself can be coloured without creating any conflict with the previously considered vertices. Note that colouring any element close to v (i.e. an arc incident to v, or v itself) does not affect the total outdegrees of the u_i's in the subdigraphs induced by colours 1 and 2.

For every u_i with $i = 1, \dots, k$, we denote by d_1^i and d_2^i the total outdegrees of u_i in the total subdigraphs induced (thus far) by colours 1 and 2, respectively. By the pigeonhole principle, for at least one of the k ordered pairs $(1, k)$, $(2, k-1)$, …, $(k, 1)$, say $(r, k+1-r)$ $(r \in \{1, \dots, k\})$, we must have

$$\left|\{i : d_1^i = r\}\right| + \left|\{i : d_2^i = k+1-r\}\right| \leq 1,$$

as otherwise (i.e. when the sums of cardinalities of such pairs of sets equal at least 2, in fact exactly 2, for $r = 1, \dots, k$), none of the u_i's can have $d_1^i = d+1$ nor $d_2^i = d+1$, and we could just colour all the arcs outgoing from v and v itself with colour 1 (or all of these with 2). Suppose then first that v has at most one outneighbour, say u_i (if any) with $d_1^i = r$, and no outneighbour u_j with $d_2^j = k+1-r$ for an $r \in \{1, \dots, k\}$. Then we can assign to exactly $k+1-r$ arcs outgoing from v, including vu_i (if it exists), the colour 2, and the colour 1 to the remaining outgoing arcs of v and to v itself, thus creating no new conflicts. The construction follows the same pattern in the symmetrical case, i.e. when v has at most one outneighbour, say u_i (if any) with $d_2^i = k+1-r$ and no outneighbour u_j with $d_1^j = r$.

By repeating this colouring procedure until all vertices and arcs are coloured, we eventually get a total-colouring of D which is as desired. □

3.3. Locally pair-irregular total-colourings

As mentioned in previous Section 3.2, it is not true that every digraph admits a locally irregular 2-total-colouring. Similarly as in Section 2.3, we here introduce slightly modified notions and terminology leading to a decomposition conjecture involving two colours only which corresponds to Theorem 2.6.

As in previous Section 3.2, we consider total digraphs D with vertex set $V(D)$ partitioned into hollow and solid vertices. Once more, we consider decompositions into locally irregular total subdigraphs, in the sense that, in the total-colouring of D, every i-subdigraph (i.e. the total subdigraph induced by colour i) is locally irregular. Recall that, in every i-subdigraph, the vertices coloured with colour i are solid, while the other vertices are hollow.

Because the definition of decomposition must be in accordance with the relevant definition of colouring, we have to define the *pair-degree* of a vertex $v \in V(D)$ as the pair $(1, d^+(v))$ if v is solid, and $(0, d^+(v))$ if v is hollow. A total digraph D is called *locally pair-irregular* if the pair-degrees of u and v are distinct for every arc $\overrightarrow{uv} \in A(D)$.

The least number of colours in a locally pair-irregular total-colouring of D is denoted $\chi^t_{\rho,\mathrm{irr}+}(D)$. For this variant, we wonder, in the same spirit as our investigations in Section 2.3, whether every digraph can be coloured with at most two colours only. We believe that this is so.

Conjecture 3.9. *For every digraph D, we have $\chi^t_{\rho,\mathrm{irr}+}(D) \leq 2$.*

As a first step towards Conjecture 3.9, using Theorem 3.8 and the fact that if the sums (in the definition of locally irregular total graphs) are distinct, then the pairs (in the definition of locally pair-irregular total graphs) are distinct, we get the following.

Remark 3.10. *For every digraph D, we have $\chi^t_{\rho,\mathrm{irr}+}(D) \leq 4$.*

4. Conclusions

In this paper, we have considered several problems related to directed versions of the 1-2 Conjecture and locally irregular decompositions. Although some of our results are best possible, there is still a gap to fill in concerning some of the variants. In particular, though we improved the upper bound on $\chi^e_{\mathrm{irr}+}$ from [5] from 6 down to 5, recall Theorem 3.4, the conjectured upper bound 3 is still open. Unfortunately, we do not believe that our approach, which is already an improvement of the one used in [5] (consisting in independently colouring two arc-disjoint subdigraphs), could be improved to decrease the upper bound to 3 or even only 4. Concerning locally irregular total-colourings, our upper bound of 4 on $\chi^t_{\mathrm{irr}+}$ given in Theorem 3.8 is close to what we believe to be the optimal value, namely 3 (recall Conjecture 3.7). Here again, we doubt our proof scheme could be improved to lower the bound further. The situation is similar in the case of Conjecture 3.9. One should hence design new tools and techniques to tackle these three holding conjectures.

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