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# Shidlovsky's multiplicity estimate and Irrationality of zeta values

Stéphane Fischler

September 30, 2016

## Abstract

In this paper we follow the approach of Bertrand-Beukers (and of later work of Bertrand), based on differential Galois theory, to prove a very general version of Shidlovsky's lemma that applies to Padé approximation problems at several points, both at functional and numerical levels (i.e., before and after evaluating at a specific point). This allows us to obtain a new proof of the Ball-Rivoal theorem on irrationality of infinitely many values of Riemann zeta function at odd integers, inspired by the proof of the Siegel-Shidlovsky theorem on values of  $E$ -functions: Shidlovsky's lemma is used to replace Nesterenko's linear independence criterion with Siegel's, so that no lower bound is needed on the linear forms in zeta values. The same strategy provides a new proof, and a refinement, of Nishimoto's theorem on values of  $L$ -functions of Dirichlet characters.

MSC 2010 : 11J72 (Primary); 11M06, 34M03 (Secondary).

## 1 Introduction

The motivation of this paper comes from the Ball-Rivoal theorem:  $\zeta(s)$  is irrational for infinitely many odd integers  $s \geq 3$ . Its proof is based on *explicit* Padé approximation to polylogarithms. In order to try to generalize this result to other functions, it would be natural to use *non-explicit* Padé approximation instead, for instance through Siegel's lemma. Several difficulties arise; the first one is the need for a lower bound on the linear forms in zeta values, in order to apply Nesterenko's linear independence criterion: such a lower bound cannot be obtained from a non-explicit construction.

In this paper we overcome this difficulty by giving a new proof of the Ball-Rivoal theorem in which no lower bound on the linear forms is used. Indeed Nesterenko's linear independence criterion is replaced with Siegel's combined with a multiplicity estimate, namely a new generalization of Shidlovsky's lemma. We combine an explicit construction of the linear forms with the strategy used to prove the Siegel-Shidlovsky theorem on values of  $E$ -functions (see for instance [21, Chapter 3]).

Let  $q$  be a positive integer, and  $A \in M_q(\mathbb{C}(z))$ . We fix  $P_1, \dots, P_q \in \mathbb{C}[z]$  and  $n \in \mathbb{N} = \{0, 1, 2, \dots\}$  such that  $\deg P_i \leq n$  for any  $i$ . Then with any solution  $Y = {}^t(y_1, \dots, y_q)$  of the differential system  $Y' = AY$  is associated a remainder  $R(Y)$  defined by

$$R(Y)(z) = \sum_{i=1}^q P_i(z)y_i(z).$$

Let  $\Sigma$  be a finite subset of  $\mathbb{C} \cup \{\infty\}$ . For each  $\sigma \in \Sigma$ , let  $(Y_j)_{j \in J_\sigma}$  be a family of solutions of  $Y' = AY$  such that the functions  $R(Y_j)$ ,  $j \in J_\sigma$ , are  $\mathbb{C}$ -linearly independent and holomorphic at  $\sigma$ ; here  $\sigma \in \mathbb{C} \cup \{\infty\}$  might be a singularity of the differential system  $Y' = AY$ . We agree that  $J_\sigma = \emptyset$  if  $\sigma \notin \Sigma$ , and let  $M(z) = [P_{k,i}(z)]_{1 \leq i, k \leq q} \in M_q(\mathbb{C}(z))$  where the rational functions  $P_{k,i} \in \mathbb{C}(z)$  are defined for  $k \geq 1$  and  $1 \leq i \leq q$  by

$$\begin{pmatrix} P_{k,1} \\ \vdots \\ P_{k,q} \end{pmatrix} = \left( \frac{d}{dz} + {}^tA \right)^{k-1} \begin{pmatrix} P_1 \\ \vdots \\ P_q \end{pmatrix}. \quad (1.1)$$

Obviously the poles of the coefficients  $P_{k,i}$  of  $M$  are among those of  $A$ .

The following multiplicity estimate appears essentially (see below) in [3, Théorème 2].

**Theorem 1.** *There exists a positive constant  $c_1$ , which depends only on  $A$  and  $\Sigma$ , such that if*

$$\sum_{\sigma \in \Sigma} \sum_{j \in J_\sigma} \text{ord}_\sigma(R(Y_j)) \geq (n+1)q - n\#J_\infty - \tau \quad (1.2)$$

with  $0 \leq \tau \leq n - c_1$ , then  $\det M(z)$  is not identically zero.

The special case where  $\Sigma = \{0\}$ ,  $\#J_0 = 1$ , and  $Y_j$  is analytic at 0 is essentially Shidlovsky's lemma (see [22, Chapter 3, Lemma 8]). When  $\Sigma \subset \mathbb{C}$ ,  $\#J_\sigma = 1$  for any  $\sigma$ , and all functions  $Y_j$  are obtained by analytic continuation from a single one, analytic at all  $\sigma \in \Sigma$ , this result was proved by Bertand-Beukers [4] with more details on the constant  $c_1$ . Then Bertrand has allowed [3, Théorème 2] an arbitrary number of solutions at each  $\sigma$ , proving Theorem 1 under the additional assumptions that  $\infty \notin \Sigma$  and the functions  $Y_j$ ,  $j \in J_\sigma$ , are analytic at  $\sigma$ .

Our proof of Theorem 1 (like that of [3, Théorème 2]) follows the strategy of [4], based on differential Galois theory. The point is that we allow  $\Sigma$  to contain  $\infty$ , and/or singularities of the differential system  $Y' = AY$ : only the remainders  $R(Y_j)$  are assumed to be holomorphic at  $\sigma$  (but not the functions  $Y_j$ , and not at points  $\sigma' \in \Sigma$  distinct from  $\sigma$ ). These features make Theorem 1 general enough to cover essentially all Padé approximation problems related to polylogarithms we have found in the literature, for instance the ones of Beukers [5, 6], Sorokin [23, 24, 25], and those of [9]. In such a setting,  $\tau$  in Eq. (1.2) appears as the difference between the number of unknowns and the number of equations.

Then we evaluate at a point  $\alpha$ , going from functional to numerical linear forms (see [22, Chapter 3, Lemma 10] for the classical setting). The point here is that we allow  $\alpha$  to be a singularity of the differential system  $Y' = AY$ , and/or an element of  $\Sigma$  (in our proof of the Ball-Rivoal theorem,  $\alpha$  is both).

**Theorem 2.** *There exists a positive constant  $c_2$ , which depends only on  $A$  and  $\Sigma$ , with the following property. Assume that, for some  $\alpha \in \mathbb{C}$ :*

- (i) *The differential system  $Y' = AY$  has a basis of local solutions at  $\alpha$  in  $\mathbb{C}[\log(z - \alpha)][[z - \alpha]]$ .*
- (ii) *All rational functions  $P_{k,i}$ , with  $1 \leq i \leq q$  and  $1 \leq k \leq \tau + c_2$ , are holomorphic at  $z = \alpha$ .*
- (iii) *Eq. (1.2) holds for some  $\tau$  with  $0 \leq \tau \leq n - c_1$ .*

*Then the matrix  $[P_{k,i}(\alpha)]_{1 \leq i \leq q, 1 \leq k \leq c_2} \in M_{q,c_2}(\mathbb{C})$  has rank at least  $q - \#J_\alpha$ .*

If  $\alpha$  is a singularity, assertion (i) means it is regular and all exponents at  $\alpha$  are integers. As far as we know, this result is the first general one in which  $\alpha$  is allowed to be a singularity. The case where  $\alpha$  is not a singularity is much easier, and assumptions (i) and (ii) are then trivially satisfied.

If  $\alpha \notin \Sigma$  then  $J_\alpha = \emptyset$  so that we obtain a matrix of maximal rank  $q$ . On the opposite, if  $\alpha \in \Sigma$  then  $\#J_\alpha$  linearly independent linear combinations of the rows of the matrix  $[P_{k,i}(z)]_{i,k}$  are holomorphic at  $\alpha$  and (probably) vanish at  $\alpha$ : the lower bound  $q - \#J_\alpha$  is best possible.

Using a zero estimate such as Theorem 2 is the key point in the classical proof of the Siegel-Shidlovsky theorem on values of  $E$ -functions. Following a different but similar strategy, Nikishin constructed explicitly [16] linearly independent linear forms in  $1, \text{Li}_1(\alpha), \dots, \text{Li}_a(\alpha)$  to prove that these numbers are linearly independent over  $\mathbb{Q}$  when  $\alpha = u/v$  is a rational number with  $v$  sufficiently large in terms of  $|u|$ . His approach was used by several authors, including Marcovecchio [12] to bound from below the dimension of the  $\mathbb{Q}$ -vector space spanned by these numbers, for any fixed algebraic number  $\alpha$  with  $|\alpha| < 1$  (thereby generalizing to non-real numbers  $\alpha$  Rivoal's result [19] based on Nesterenko's linear independence criterion). The zero estimate used by Marcovecchio is similar to Theorem 2 but deals only with a specific situation in which (essentially)  $\tau = 1$  in Eq. (1.2),  $\alpha \notin \Sigma$ , and  $\alpha$  is not a singularity. Moreover he does not define  $P_{k,i}$  for  $k \geq 2$  using Eq. (1.1) (i.e., differentiating the linear forms as in the proof of the Siegel-Shidlovsky theorem): following Nikishin he uses an additional parameter instead.

In this paper we use Theorem 2 to obtain a new proof, and a refinement, of the following result of Nishimoto [17] on  $L$ -functions  $L(\chi, s) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$  associated with Dirichlet characters  $\chi$ . He proved it with  $d$  instead of  $N$  in the lower bound (1.3); see §4.1 for this easy improvement.

**Theorem 3.** *Let  $\chi$  be a Dirichlet character modulo  $d$ , of conductor  $N$ . Let  $p \in \{0, 1\}$  and  $a \geq 2$ . Denote by  $\delta_{\chi,p,a}$  the dimension of the  $\mathbb{Q}$ -vector space spanned by 1 and the numbers  $L(\chi, s)$  with  $2 \leq s \leq a$  and  $s \equiv p \pmod{2}$ . Then*

$$\delta_{\chi,p,a} \geq \frac{1 + o(1)}{N + \log 2} \log a \quad (1.3)$$

where  $o(1)$  is a sequence that depends on  $N$  and  $a$ , and tends to 0 as  $a \rightarrow \infty$  (for any  $N$ ).

If  $p$  and  $\chi$  have the same parity then  $L(\chi, s)\pi^{-s}$  is a non-zero algebraic number for any  $s \geq 2$  such that  $s \equiv p \pmod{2}$  (see for instance [15, Chapter VII, §2]): this result is interesting when  $p$  and  $\chi$  have opposite parities.

Nishimoto's proof is similar to Ball-Rivoal's, except that obtaining the lower bound necessary to apply Nesterenko's criterion is very technical: the saddle point method has to be used because cancellations take place (see [14]). In this paper we present an alternative proof of Theorem 3, based on the zero estimate stated above. It makes it unnecessary to use the saddle point method, since Siegel's criterion is applied instead of Nesterenko's. In the special case  $d = N = 1$  (so that  $\chi(n) = 1$  for any  $n$ , and  $L(\chi, s) = \zeta(s)$ ) this is exactly the proof of the Ball-Rivoal theorem mentioned above.

We also obtain the following refinement of Theorem 3, by improving the arithmetic estimates.

**Theorem 4.** *In the setting of Theorem 3, if  $N$  is a multiple of 4 then Eq. (1.3) can be replaced with*

$$\delta_{\chi,p,a} \geq \frac{1 + o(1)}{(N/2) + \log 2} \log a.$$

When  $\chi$  is the non-principal character mod  $d = 4$ , so that  $N = 4$ , this result was proved by Rivoal-Zudilin [20] as a first step towards the (conjectural) irrationality of Catalan's constant  $L(\chi, 2) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2}$ .

The structure of this paper is as follows. We first sketch in §2 our proof of the Ball-Rivoal theorem. Then §3 is devoted to Shidlovsky's lemma: we prove Theorems 1 and 2. At last, in §4 we prove in details a general result which contains Theorem 3, Theorem 4, and the Ball-Rivoal theorem.

## 2 A new proof of the Ball-Rivoal theorem

We sketch in this section the new proof of the Ball-Rivoal theorem obtained as a special case of the proof of Theorem 6 in §4 below (namely  $N = 1$ ,  $f(n) = 1$  for any  $n$ ,  $p = 1$ ,  $z_0 = 1$ ,  $i_0 = 2$ ,  $\xi_1 = 0$ , and  $\xi_j = \zeta(j)$  for any  $j \geq 2$ ). Of course we refer to §4 for more details.

Let  $a, r, r', n$  be such that  $a$  is odd and  $r, r' < a/2$ . It turns out that the best estimates come from the case where  $r$  and  $r'$  have essentially the same size, so we shall restrict in §4 to the case  $r' = r$ ; however the proof works in the same way if  $r' \neq r$ . Consider the rational function

$$F(t) = n!^{a-r-r'} \frac{(t-rn)_{rn}(t+n+1)_{r'n}}{(t)_{n+1}^a}$$

where  $(\alpha)_k = \alpha(\alpha+1)\dots(\alpha+k-1)$  is Pochhammer's symbol, and let

$$S_0(z) = \sum_{t=n+1}^{\infty} F(-t)z^t, \quad S_{\infty}(z) = \sum_{t=1}^{\infty} F(t)z^{-t}.$$

For any  $k \geq 1$  we let

$$\Lambda_k = S_0^{(k-1)}(1) - S_{\infty}^{(k-1)}(1), \quad (2.1)$$

where  $S^{(k-1)}$  is the  $(k-1)$ -th derivative of  $S$ . We shall use a symmetry phenomenon to get rid of even zeta values, but it does not appear exactly as in the original proof of Ball-Rivoal. Indeed, even if  $r' = r$ ,  $S_0^{(k-1)}(1)$  and  $S_{\infty}^{(k-1)}(1)$  involve both odd and even zeta values when  $k \geq 2$ : they are values at  $z = 1$  of hypergeometric series which are no more well-poised. The cancellation of even zeta values comes at a different stage, by considering  $\Lambda_k$  in Eq. (2.1). Indeed there exist integers  $s_{k,i}$ ,  $2 \leq i \leq a$ , and  $u_k, v_k$  such that for any  $k \leq (a-r-r')n+a-1$ , we have both

$$d_n^a S_0^{(k-1)}(1) = u_k + \sum_{i=2}^a (-1)^i s_{k,i} \zeta(i)$$

and

$$d_n^a S_{\infty}^{(k-1)}(1) = v_k + \sum_{i=2}^a s_{k,i} \zeta(i)$$

where  $d_n = \text{lcm}(1, 2, \dots, n)$ , so that  $d_n^a \Lambda_k = d_n^a S_0^{(k-1)}(1) - d_n^a S_{\infty}^{(k-1)}(1)$  is a  $\mathbb{Z}$ -linear combination of 1 and odd zeta values:

$$d_n^a \Lambda_k = s_{k,a+1} - 2 \sum_{\substack{2 \leq i \leq a \\ i \text{ odd}}} s_{k,i} \zeta(i),$$

with  $s_{k,a+1} = u_k - v_k$ . Using Theorem 2 we prove that the matrix  $[s_{k,i}]_{2 \leq i \leq a+1, 1 \leq k \leq c_2}$  has maximal rank, equal to  $a$  (see below). This enables one to apply Siegel's linear independence criterion (see §4.6) instead of Nesterenko's: no lower bound on  $|\Lambda_k|$  is needed. The upper bounds on  $|s_{k,j}|$  and  $|\Lambda_k|$  are essentially the same as in the proof of Ball-Rivoal, so that we obtain the same lower bound:

$$\dim_{\mathbb{Q}} \text{Span}_{\mathbb{Q}}(1, \zeta(3), \zeta(5), \dots, \zeta(a)) \geq \frac{1+o(1)}{1+\log 2} \log a.$$

Let us focus now on the functional aspects of this proof, which play an important role (whereas the proof of Ball-Rivoal can be written with  $z = 1$  throughout). For simplicity we restrict ourselves to the case  $r' = r$ . The functions  $S_0(z)$  and  $S_\infty(z)$  are solutions of the following Padé approximation problem: find polynomials  $P_1, \dots, P_{a+2}$  of degree at most  $n$  such that:

$$\begin{cases} S_0(z) := P_{a+1}(z) + \sum_{i=1}^a P_i(z)(-1)^i \text{Li}_i(z) = O(z^{(r+1)n+1}), & z \rightarrow 0, \\ S_\infty(z) := P_{a+2}(z) + \sum_{i=1}^a P_i(z) \text{Li}_i(1/z) = O(z^{-rn-1}), & z \rightarrow \infty, \\ \sum_{i=1}^a P_i(z)(-1)^{i-1} \frac{(\log z)^{i-1}}{(i-1)!} = O((z-1)^{(a-2r)n+a-1}), & z \rightarrow 1. \end{cases} \quad (2.2)$$

This is exactly the Padé approximation problem of [9, Théorème 1]: it has a unique solution up to proportionality,  $(n+1)(a+2)$  unknowns and  $(n+1)(a+2) - 1$  equations. Let  $A \in M_{a+2}(\mathbb{C}(z))$  denote the following matrix:

$$A = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 0 & \frac{1}{z-1} & \frac{1}{z(1-z)} \\ \frac{-1}{z} & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & \frac{-1}{z} & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{-1}{z} & \dots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \frac{-1}{z} & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \end{bmatrix}$$

and consider the following solutions of the differential system  $Y' = AY$ :

$$\begin{aligned} Y_0(z) &= {}^t(-\text{Li}_1(z), \text{Li}_2(z), \dots, (-1)^a \text{Li}_a(z), 1, 0), \\ Y_\infty(z) &= {}^t(\text{Li}_1(1/z), \text{Li}_2(1/z), \dots, \text{Li}_a(1/z), 0, 1), \\ Y_1(z) &= {}^t(1, -\log z, \frac{(\log z)^2}{2}, \dots, (-1)^{a-1} \frac{(\log z)^{a-1}}{(a-1)!}, 0, 0). \end{aligned}$$

Let  $\Sigma = \{0, 1, \infty\}$  and  $J_0 = \{0\}$ ,  $J_1 = \{1\}$ ,  $J_\infty = \{\infty\}$ . Then with the notation of the introduction, we have  $R(Y_0) = S_0(z)$ ,  $R(Y_\infty) = S_\infty(z)$ , and  $R(Y_1)$  is the left hand side of the third equation of (2.2); Eq. (1.2) stated in the introduction holds with  $\tau = 1$  as a consequence of the Padé approximation problem (2.2). In general,  $\tau$  corresponds in Eq. (1.2) to the difference between the number of unknowns and the number of equations. To apply Theorem 2 it is not useful to prove that the problem has a unique solution up to proportionality: the upper bound  $\tau \leq n/2$ , for instance, would be sufficient since  $n$  is taken arbitrarily large.

Defining  $P_{k,i}$  as in the introduction by Eq. (1.1), it is well-known (see [22, Chapter 3, §4]) that for any  $k \geq 1$ ,

$$S_0^{(k-1)}(z) = P_{k,a+1}(z) + \sum_{i=1}^a P_{k,i}(z)(-1)^i \text{Li}_i(z) \quad \text{and} \quad S_\infty^{(k-1)}(z) = P_{k,a+2}(z) + \sum_{i=1}^a P_{k,i}(z) \text{Li}_i(1/z). \quad (2.3)$$

Moreover  $P_{k,i}$  is a rational function of which 0 is the only possible pole if  $i \leq a$ . If  $i = a + 1$  or  $i = a + 2$ , both 0 and 1 may be poles of  $P_{k,i}$ ; but if  $k \leq (a - 2r)n + a - 1$ , the functions  $S_0^{(k-1)}(z)$  and  $S_\infty^{(k-1)}(z)$  have finite limits as  $z \rightarrow 1$  so that 1 is not a pole.

Finally Theorem 2 applies at  $\alpha = 1$ : the matrix  $[P_{k,i}(1)]_{1 \leq i \leq a+2, 1 \leq k \leq c_2}$  has rank at least  $a + 1$ . Actually  $P_{k,1}(1) = 0$  for any  $k \leq (a - 2r)n + a - 1$  (which can be seen by letting  $z$  tend to 1 in Eq. (2.3)) so that the first row of this matrix is zero (provided  $n$  is large enough) and its rank is exactly  $a + 1$ . Since the coefficients  $s_{k,i}$  defined above are given by  $s_{k,i} = d_n^a P_{k,i}(1)$  for  $2 \leq i \leq a$  and  $s_{k,a+1} = d_n^a (P_{k,a+1}(1) - P_{k,a+2}(1))$ , the matrix  $[s_{k,i}]_{2 \leq i \leq a+1, 1 \leq k \leq c_2}$  has rank  $a$ : Siegel's criterion (stated and proved in §4.6) applies.

### 3 Zero estimates

In this section we prove Theorems 1 and 2. We start with the functional part of the proof (§3.1), in which we follow the approach of Bertrand-Beukers [4] to generalize Shidlovsky's lemma (see Theorem 5). Then we deduce in §3.2 Theorems 1 and 2 stated in the introduction: the important point is to evaluate at  $\alpha$  which may be a singularity and/or an element of  $\Sigma$ .

#### 3.1 Functional zero estimate

Throughout this section we consider a positive integer  $q$  and a matrix  $A \in M_q(\mathbb{C}(X))$ . We let  $P_1, \dots, P_q \in \mathbb{C}[X]$  with  $\deg P_i \leq n$  for any  $i$ . We also denote by  $\Omega$  a simply connected open subset of  $\mathbb{C}$  in which  $A$  has no pole. We assume that  $\Omega$  is obtained from  $\mathbb{C}$  by removing finitely many half-lines, so that  $\Omega$  is dense in  $\mathbb{C}$ , and denote by  $\mathcal{H}$  the space of functions holomorphic on  $\Omega$ . A solution  $Y$  of the differential system  $Y' = AY$  will always be a column matrix in  $M_{q,1}(\mathcal{H})$ , identified with the corresponding element  $(y_1, \dots, y_q)$  of  $\mathcal{H}^q$ . Since  $P_1, \dots, P_q$  are fixed, to such a solution is associated a remainder  $R(Y)$  defined on  $\Omega$  by

$$R(Y)(z) = \sum_{i=1}^q P_i(z)y_i(z).$$

Let  $\Sigma$  be a finite subset of  $\mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$ . For each  $\sigma \in \Sigma$ , let  $(Y_j)_{j \in J_\sigma}$  be a family of solutions of  $Y' = AY$  such that:

- For any  $j \in J_\sigma$ , the function  $R(Y_j)$  is holomorphic at  $\sigma$ .
- The functions  $R(Y_j)$ , for  $j \in J_\sigma$ , are linearly independent over  $\mathbb{C}$ .

Here we do not assume that  $\sigma \in \Omega$ : in the case  $\sigma \notin \Omega$  (for instance if  $\sigma = \infty$ ), by  $R(Y_j)$  is holomorphic at  $\sigma$  we mean that  $R(Y_j)$  can be continued analytically to a function holomorphic at  $\sigma$ . Moreover, we denote by  $\text{ord}_\sigma(R(Y_j))$  its order of vanishing at  $z = \sigma$ .

The point is that we do not assume any relation (or lack of relation) between the families  $(Y_j)_{j \in J_\sigma}$  at distinct points  $\sigma$ , except of course that all are solutions of the same differential system.



At last, we let  $J_\sigma = \emptyset$  when  $\sigma \notin \Sigma$ .

Defining  $M(z)$  and  $P_{k,i}$  as in the introduction, our functional multiplicity estimate is the following generalization of Bertrand-Beukers' version of Shidlovsky's lemma; if  $\infty \notin \Sigma$  and the functions  $Y_j$ ,  $j \in J_\sigma$ , are analytic at  $\sigma$  it is due to Bertrand [3, Théorème 2]. The constant  $c_1$  is the same as in Theorem 1 (that we shall deduce from Theorem 5 at the beginning of §3.2).

**Theorem 5.** *Let  $\mu$  denote the order of a non-zero differential operator  $L \in \mathbb{C}(z)[\frac{d}{dz}]$  such that  $L(R(Y_j)) = 0$  for any  $\sigma$  and any  $j \in J_\sigma$ . Then*

$$\sum_{\sigma \in \Sigma} \sum_{j \in J_\sigma} \text{ord}_\sigma(R(Y_j)) \leq (n+1)(\mu - \#J_\infty) + c_1 \quad (3.1)$$

where  $c_1$  is a constant that depends only on  $A$  and  $\Sigma$ .

In the special case where  $\Sigma \subset \mathbb{C}$ ,  $J_\sigma$  consists of a single element  $j_\sigma$ , and the function  $Y_{j_\sigma}$  is the same for all  $\sigma$ , this is exactly [4, Théorème 2] except that we did not try to make the constant  $c_1$  explicit (we refer to [4], and to [1, Appendix of Chapter III] in the Fuchsian case, for discussions on effectivity which are not relevant to our purposes). Indeed we have fixed a simply connected open subset  $\Omega$  only for convenience: analytic continuation from a point of  $\Sigma$  to another could be performed along any fixed path.

Let us prove Theorem 5 now, following the strategy of [4].

Given  $\sigma \in \mathbb{P}^1(\mathbb{C})$ , we let  $\mathcal{A}_\sigma$  denote the set of all finite sums

$$\sum_{\alpha \in E} \sum_{Q \in \mathcal{P}} \sum_{j=0}^J u_{\alpha, Q, j} (z - \sigma) (z - \sigma)^\alpha (\log(z - \sigma))^j \exp(Q((z - \sigma)^{-1/q!})) \quad (3.2)$$

where  $E \subset \mathbb{C}$  and  $\mathcal{P} \subset \mathbb{C}[X]$  are finite subsets,  $J \geq 0$ , and  $u_{\alpha, Q, j}(z - \sigma) \in \mathbb{C}[[ (z - \sigma)^{1/q!} ]]$  for any  $\alpha, Q, j$ . Here and below, we agree that  $z - \sigma$  stands for  $1/z$  si  $\sigma = \infty$ . Then the differential system  $Y' = AY$  has a complete system of formal solutions in  $\mathcal{A}_\sigma^q$ . Moreover we let  $\mathcal{K}_\sigma$  denote the fraction field of  $\mathcal{A}_\sigma$ , and  $F_\sigma$  denote the differential subfield of  $\mathcal{K}_\sigma$  generated over  $\mathbb{C}(z)$  by all components of all solutions of  $Y' = AY$  in  $\mathcal{K}_\sigma^q$ . Then the differential extension  $F_\sigma/\mathbb{C}(z)$  is Picard-Vessiot, and we denote by  $G_\sigma$  its group of differential automorphisms.

To prove Theorem 5 we may assume that  $0 \in \Sigma$ , that  $\mu$  is the minimal order of a non-zero differential operator that annihilates  $R(Y_j)$  for any  $j \in J_\sigma$  and any  $\sigma \in \Sigma$ , and that the coefficient of  $(\frac{d}{dz})^\mu$  in  $L$  is 1.

Given  $\sigma \in \Sigma$  and  $j \in J_\sigma$ , all components of  $Y_j$  are holomorphic on the cut plane  $\Omega$ , and can be seen as elements of  $\mathcal{A}_0$ . Indeed, if 0 is a regular singularity (or an ordinary point) of the system  $Y' = AY$  then all components of  $Y_j$  have a generalized Taylor expansion at the origin in  $\mathcal{A}_0$  (of the form (3.2) with  $\mathcal{P} = \{0\}$ ). In the general case, we identify each

component of  $Y_j$  with its asymptotic expansion at 0 in a fixed large sector (see [18]). By definition of  $F_0$ , all components of  $Y_j$  (seen in  $\mathcal{A}_0$ ) belong to  $F_0$  so that  $R(Y_j) \in F_0$ . We consider the  $\mathbb{C}$ -vector space  $V \subset F_0$  spanned by the images  $\gamma(R(Y_j))$  of all  $R(Y_j)$ ,  $j \in J_\sigma$ ,  $\sigma \in \Sigma$ , under all  $\gamma \in G_0$ . Since the kernel of  $L : F_0 \rightarrow F_0$  is stable under  $G_0$ , we have  $V \subset \ker L$  so that  $m \leq \mu$ , where  $m = \dim_{\mathbb{C}} V$ . Let  $(R_1, \dots, R_m)$  be a basis of  $V$ , such that  $R_i = \gamma_i(R(Y_{j_i}))$  with  $\gamma_i \in G_0$  and  $j_i \in J_{\sigma_i}$  for any  $i \in \{1, \dots, m\}$ .

Arguing as in the proof of [4, Proposition 3], we have

$$Ly = \frac{1}{W(R_1, \dots, R_m)} \det \begin{bmatrix} y & y' & \dots & y^{(m)} \\ R_1 & R_1' & \dots & R_1^{(m)} \\ \vdots & \vdots & & \vdots \\ R_m & R_m' & \dots & R_m^{(m)} \end{bmatrix}$$

where  $W(R_1, \dots, R_m) = \det[R_i^{(j-1)}]_{1 \leq i, j \leq m}$  is the wronskian determinant. In particular, we have  $m = \mu$  and  $V = \ker L$ .

Now we claim that for any  $\sigma \in \mathbb{P}^1(\mathbb{C})$  there exist  $\mu$  solutions  $Y^{[\sigma, j]} = (y_1^{[\sigma, j]}, \dots, y_q^{[\sigma, j]})$  of  $Y' = AY$  in  $F_\sigma^q$ , with  $1 \leq j \leq \mu$ , such that  $R(Y^{[\sigma, 1]}), \dots, R(Y^{[\sigma, \mu]})$  span the  $\mathbb{C}$ -vector space of solutions of  $Ly = 0$  in  $F_\sigma$ . Indeed, as in [4, Corollaire], using a differential isomorphism  $F_0 \rightarrow F_\sigma$  we may assume  $\sigma = 0$ . Then for any  $i \in \{1, \dots, m\}$ ,  $\gamma_i(Y_{j_i}) \in F_0^q$  is a solution of  $Y' = AY$  and  $\pi_0(\gamma_i(Y_{j_i})) = R(\gamma_i(Y_{j_i})) = \gamma_i(R(Y_{j_i})) = R_i$  so that the claim is proved since  $(R_1, \dots, R_m)$  is a basis of  $V = \ker L$ .

Let us recall the following terminology from [4]: an element of  $\mathcal{A}_\sigma$  has rank  $\leq \kappa \in \frac{1}{q!}\mathbb{N}$  and generalized order  $\geq r$  if it is of the form (3.2) with  $\deg Q \leq q!\kappa$  for any  $Q \in \mathcal{P}$  and  $\operatorname{Re} \alpha \geq r$  for any  $\alpha \in E$ . The differential operator  $L$  has rank  $\leq \kappa$  at  $\sigma$  and  $(r_1, \dots, r_\mu) \in \mathbb{R}^\mu$  is an admissible system of exponents of  $L$  at  $\sigma$  if the differential equation  $Ly = 0$  has a complete system of solutions  $(y_1, \dots, y_\mu)$  in  $\mathcal{A}_\sigma^\mu$  such that each  $y_i$  has rank  $\leq \kappa$  and generalized order  $\geq r_i$ .

Given  $\sigma \in \mathbb{P}^1(\mathbb{C})$  all functions  $y_i^{[\sigma, j]}$  with  $1 \leq i \leq q$  and  $1 \leq j \leq \mu$  have rank  $\leq \kappa_\sigma$  and generalized order  $\geq r_\sigma$  for some  $\kappa_\sigma \in \frac{1}{q!}\mathbb{N}$  and  $r_\sigma \in \mathbb{R}$  which depend only on  $A$  and  $\sigma$  (see [4, Proposition 1]). If  $\sigma \neq \infty$ ,  $R(Y^{[\sigma, j]}) = \sum_{i=1}^q P_i(z)y_i^{[\sigma, j]}(z)$  has rank  $\leq \kappa_\sigma$  and generalized order  $\geq r_\sigma$ ; these functions make up a complete system of solutions of  $L$  (using the claim above). Moreover, if  $\sigma \in \Sigma \setminus \{\infty\}$  then for any  $j \in J_\sigma$  the function  $R(Y_j)$  is holomorphic at  $\sigma$ , so it can be seen as an element of  $\mathcal{A}_\sigma$  with rank  $\leq 0$  and generalized order  $\geq \operatorname{ord}_\sigma(R(Y_j))$ . Combining these  $\mathbb{C}$ -linearly independent solutions of  $Ly = 0$  with suitable functions  $R(Y^{[\sigma, j]})$ , we obtain that  $L$  has rank  $\leq \kappa_\sigma$  at  $\sigma$  and an admissible system of exponents of  $L$  at  $\sigma$  consists in  $r_\sigma$  repeated  $\mu - \#J_\sigma$  times, and  $\operatorname{ord}_\sigma(R(Y_j))$  for each  $j \in J_\sigma$ . In the same way, at infinity, for any  $j \in J_\infty$  the function  $R(Y_j) \in \mathcal{A}_\infty$  has rank  $\leq 0$  and generalized order  $\geq \operatorname{ord}_\infty(R(Y_j))$ . To obtain a complete system of solutions of  $Ly = 0$  in  $\mathcal{A}_\infty$  we use also  $\mu - \#J_\infty$  functions  $R(Y^{[\infty, j]})$ , which have rank  $\leq \kappa_\sigma$  and generalized order  $\geq r_\infty - n$  since  $\deg P_i(z) \leq n$  for any  $i \in \{1, \dots, q\}$ . Therefore  $L$  has rank  $\leq \kappa_\infty$  at

$\infty$  and an admissible system of exponents of  $L$  at  $\infty$  consists in  $r_\infty - n$  repeated  $\mu - \#J_\infty$  times, and  $\text{ord}_\infty(R(Y_j))$  for each  $j \in J_\infty$ .

So far we have found an upper bound on the rank of  $L$ , and an admissible system of exponents of  $L$ , at any  $\sigma \in \Sigma$ . Enlarging  $\Sigma$  if necessary, we may assume that it contains  $\infty$  and all poles of  $A$ . Then for any  $\sigma \in \mathbb{P}^1(\mathbb{C}) \setminus \Sigma$  the differential system  $Y' = AY$  has a complete system of solutions holomorphic at  $\sigma$ , and therefore the same property holds for the differential equation  $Ly = 0$  using the claim above. Accordingly  $\Sigma$  contains  $\infty$  and all non-apparent singularities of  $L$ , so that the Corollary of [4, Théorème 3] provides an inequality involving upper bounds on the ranks of  $L$  and admissible systems of exponents of  $L$  at all points of  $\Sigma$ , namely:

$$\begin{aligned} & (\mu - \#J_\infty)(r_\infty - n) + \left( \sum_{j \in J_\infty} \text{ord}_\infty(R(Y_j)) \right) - (\kappa_\infty + 1)\mu(\mu - 1)/2 \\ & + \sum_{\sigma \in \Sigma \setminus \{\infty\}} \left[ (\mu - \#J_\sigma)r_\sigma + \left( \sum_{j \in J_\sigma} \text{ord}_\sigma(R(Y_j)) \right) - (\kappa_\sigma + 1)\mu(\mu - 1)/2 \right] \leq -\mu(\mu - 1) \end{aligned}$$

so that

$$\left( \sum_{\sigma \in \Sigma} \sum_{j \in J_\sigma} \text{ord}_\sigma(R(Y_j)) \right) - (n + 1)(\mu - \#J_\infty) \leq c_1$$

where  $c_1$  is a constant that can be written down explicitly in terms of  $\Sigma$ ,  $\mu$ ,  $\kappa_\sigma$ ,  $r_\sigma$  and  $\#J_\sigma$  for  $\sigma \in \Sigma$ . This concludes the proof of Theorem 5.

## 3.2 Numerical zero estimate

In this section we prove Theorems 1 and 2 stated in the introduction. The proof falls into 3 steps; the first one is Theorem 1.

**Step 1:**  $M(z) \in M_q(\mathbb{C}(z))$  is an invertible matrix.

As in [22], if  $M$  is singular in  $M_q(\mathbb{C}(z))$  then there is a non-trivial linear relation with coefficients in  $\mathbb{C}(z)$  between the  $\text{rk}(M) + 1$  first columns of  $M$ ; this provides a differential operator  $L$  of order  $\mu = \text{rk}(M)$  to which Theorem 5 applies, in contradiction with Eq. (1.2) since  $\tau \leq n - c_1$ . Indeed, for any solution  $Y$  of the differential system  $Y' = AY$  we have

$${}^tYM = \begin{bmatrix} R(Y) & R(Y)' & \dots & R(Y)^{(q-1)} \end{bmatrix}.$$

**Step 2:** Determination of  $\det M(z)$  up to factors of bounded degree.

Let  $S$  denote the set of finite singularities of the differential system  $Y' = AY$ , i.e. poles of coefficients of  $A$ . For any  $s \in S$ , let  $N_s$  denote the maximal order of  $s$  as a pole of a coefficient of  $A$ ; let  $N_s = 0$  for  $s \in \mathbb{C} \setminus S$ . Then Eq. (1.1) shows that  $(z - s)^{(k-1)N_s} P_{k,i}(z)$  is holomorphic at  $z = s$  for any  $k \geq 1$  and any  $i \in \{1, \dots, q\}$ . Therefore  $\det M(z) \cdot \prod_{s \in S} (z - s)^{q(q-1)N_s}$  has no pole: is it a polynomial.

Now let  $\sigma \in \Sigma$ , and denote by  $T_\sigma \in M_{\#J_\sigma, q}(\mathcal{H})$  the matrix with rows  ${}^t Y_j$ ,  $j \in J_\sigma$ . The vector-valued functions  $Y_j$ ,  $j \in J_\sigma$ , are linearly independent over  $\mathbb{C}$  because the functions  $R(Y_j)$  are; therefore they are the  $\#J_\sigma$  first elements of a basis of solutions  $\mathcal{B}$  of the differential system  $Y' = AY$ . The wronskian determinant of  $\mathcal{B}$  may vanish at  $\sigma$  if  $\sigma$  is a singularity, but even in this case it cannot have generalized order  $\geq c_0(\sigma)$  at  $\sigma$  (with the terminology of §3.1) where  $c_0(\sigma)$  is a constant depending only on  $A$  and  $\sigma$  (not on  $\mathcal{B}$ ). On the other hand, all components of all elements of  $\mathcal{B}$  have generalized order  $\geq r_\sigma$  at  $\sigma$  (as in §3.1). Therefore there exists a subset  $I_\sigma$  of  $\{1, \dots, q\}$ , with  $\#I_\sigma = q - \#J_\sigma$ , such that the determinant of the submatrix of  $T_\sigma$  corresponding to the columns indexed by  $\{1, \dots, q\} \setminus I_\sigma$  cannot have generalized order  $\geq c(\sigma)$  at  $\sigma$ , where  $c(\sigma) = c_0(\sigma) - r_\sigma \#I_\sigma$  depends only on  $A$  and  $\sigma$ .

Let  $P_\sigma \in M_q(\mathcal{H})$  denote the matrix of which the  $\#J_\sigma$  first rows are that of  $T_\sigma$ , and the other rows are the  ${}^t e_i$ ,  $i \in I_\sigma$ , where  $(e_1, \dots, e_q)$  is the canonical basis of  $M_{q,1}(\mathbb{C})$ . Then  $P_\sigma M$  has its first rows equal to  $[ R(Y_j) \ R(Y_j)' \ \dots \ R(Y_j)^{(q-1)} ]$  with  $j \in J_\sigma$ , and its last rows equal to  $[ P_{1,i} \ \dots \ P_{q,i} ]$  with  $i \in I_\sigma$ . Therefore all coefficients in the row corresponding to  $j \in J_\sigma$  vanish at  $\sigma$  with order at least  $\text{ord}_\sigma R(Y_j) - q + 1$ , and (if  $\sigma \neq \infty$ ) all coefficients in the row corresponding to  $i \in I_\sigma$  are either holomorphic at  $\sigma$ , or have a pole of order at most  $(q-1)N_\sigma$  if  $\sigma \in S$ . Since  $N_\sigma = 0$  if  $\sigma \notin S$ , we have for any  $\sigma \in \Sigma \setminus \{\infty\}$ :

$$\text{ord}_\sigma \det(P_\sigma M) \geq \left( \sum_{j \in J_\sigma} \text{ord}_\sigma R(Y_j) \right) - (q-1)\#J_\sigma - (q-1)N_\sigma(q - \#J_\sigma).$$

Since  $\det P_\sigma$  cannot have generalized order  $\geq c(\sigma)$  at  $\sigma$ , we obtain

$$\text{ord}_\sigma \det(M) \geq \left( \sum_{j \in J_\sigma} \text{ord}_\sigma R(Y_j) \right) - c(\sigma).$$

Now let

$$Q_2(z) = \left( \prod_{s \in S} (z - s)^{q(q-1)N_s} \right) \cdot \left( \prod_{\sigma \in \Sigma \setminus \{\infty\}} (z - \sigma)^{c(\sigma)} \right)$$

so that  $Q_2(z) \det M(z)$  is a polynomial and vanishes at any  $\sigma \in \Sigma \setminus \{\infty\}$  with order at least  $\sum_{j \in J_\sigma} \text{ord}_\sigma R(Y_j)$ . To bound from above the degree of this polynomial, we define  $P_\infty$  as above if  $\infty \in \Sigma$ , and let  $P_\infty$  denote the identity matrix (and  $J_\infty = \emptyset$ ) otherwise. Then we have  $R(Y_j)^{(k-1)} = O(z^{-\text{ord}_\infty R(Y_j)})$  as  $|z| \rightarrow \infty$  for any  $j \in J_\infty$  and any  $k \geq 1$ , and  $P_{k,i}(z) = O(z^{n+(q-1)d})$  for any  $i \in I_\infty$  and any  $k \in \{1, \dots, q\}$  (where  $d$  is greater than or equal to the degree of all coefficients of  $A$ ). Therefore we have  $\det M(z) = O(z^u)$  as  $|z| \rightarrow \infty$ , with

$$u = (q - \#J_\infty)(n + (q-1)d) - \sum_{j \in J_\infty} \text{ord}_\infty R(Y_j),$$

so that

$$\deg(Q_2(z) \det M(z)) \leq u + \deg Q_2 \leq \sum_{\sigma \in \Sigma \setminus \{\infty\}} \sum_{j \in J_\sigma} \text{ord}_\sigma R(Y_j) + \tau + c_1$$

using Eq. (1.2), where  $c_1$  depends only on  $A$  and  $\Sigma$  (since  $0 \leq \#J_\sigma \leq q$  for any  $\sigma$ ). To sum up, we have found a polynomial  $Q_1$  of degree at most  $\tau + c_1$  such that

$$\det M(z) = \frac{Q_1(z)}{Q_2(z)} \prod_{\sigma \in \Sigma \setminus \{\infty\}} (z - \sigma)^{\sum_{j \in J_\sigma} \text{ord}_\sigma R(Y_j)}.$$

**Step 3:** Evaluation at  $\alpha$ .

Let  $q_\alpha = \#J_\alpha$  and  $q'_\alpha = q - q_\alpha$ , where  $J_\alpha = \emptyset$  if  $\alpha \notin \Sigma$ ; for simplicity we assume that  $J_\alpha = \{1, \dots, q_\alpha\}$ . Since the solutions  $Y_1, \dots, Y_{q_\alpha}$  of the differential system  $Y' = AY$  are linearly independent over  $\mathbb{C}$ , there exist solutions  $Y_{q_\alpha+1}, \dots, Y_q$  such that  $(Y_1, \dots, Y_q)$  is a local basis of solutions at  $\alpha$ . Let  $\mathcal{Y} \in M_q(\mathcal{L}_\alpha)$  be the matrix with columns  $Y_1, \dots, Y_q$ , where  $\mathcal{L}_\alpha = \mathbb{C}[\log(z - \alpha)][[z - \alpha]]$ . Then  ${}^t\mathcal{Y}M$  is the matrix  $[R(Y_i)^{(k-1)}]_{1 \leq i, k \leq q}$ .

For any subset  $E$  of  $\{1, \dots, q\}$  of cardinality  $q'_\alpha = q - q_\alpha$ , we denote by  $\Delta_E$  the determinant of the submatrix of  $[R(Y_i)^{(k-1)}]$  obtained by considering only the rows with index  $i \geq q_\alpha + 1$  and the columns with index  $k \in E$ , and by  $\tilde{\Delta}_E$  the one obtained by removing these rows and columns. Then Laplace expansion by complementary minors yields

$$\det \mathcal{Y}(z) \cdot \det M(z) = \sum_{\substack{E \subset \{1, \dots, q\} \\ \#E = q'_\alpha}} \varepsilon_E \Delta_E(z) \tilde{\Delta}_E(z) \quad (3.3)$$

with  $\varepsilon_E \in \{-1, 1\}$ . Now  $\det \mathcal{Y}$  is the wronskian of  $Y_1, \dots, Y_q$ : it is a solution of the first order differential equation

$$w'(z) = w(z) \text{trace}(A(z)). \quad (3.4)$$

Moreover it is non-zero, and belongs to  $\mathcal{L}_\alpha$ . Therefore we have  $\det \mathcal{Y}(z) \sim \gamma_1(z - \alpha)^{\varpi_1}$  as  $z \rightarrow \alpha$ , for some  $\gamma_1 \in \mathbb{C}^*$  and  $\varpi_1 \in \mathbb{N}$ . On the other hand we have  $\det M(z) \sim \gamma_2(z - \alpha)^{\varpi_2}$  with  $\gamma_2 \in \mathbb{C}^*$  and  $\varpi_2 \in \mathbb{N}$  using Step 1 and the assumption that all entries of  $M(z)$  are holomorphic at  $\alpha$ . Now for  $f \in \mathcal{L}_\alpha \setminus \{0\}$  let  $\text{ord}_\alpha f$  denote its generalized order at  $\alpha$ , namely the maximal integer  $N$  such that  $f \in (z - \alpha)^N \mathcal{L}_\alpha$ . Then Eq. (3.3) shows that, for some subset  $E$ ,

$$\varpi_3 := \text{ord}_\alpha \Delta_E(z) \leq \varpi_1 + \varpi_2 - \text{ord}_\alpha \tilde{\Delta}_E(z) \text{ and } \Delta_E(z) \sim \gamma_3(z - \alpha)^{\varpi_3} \text{ with } \gamma_3 \in \mathbb{C}^*. \quad (3.5)$$

Now letting  $\omega_\alpha = \sum_{j \in J_\alpha} \text{ord}_\alpha R(Y_j)$  if  $\alpha \in \Sigma$  and  $\omega_\alpha = 0$  otherwise, Step 2 shows that  $\varpi_2 \leq \omega_\alpha + \tau + c_1$ . Moreover the order of vanishing at  $\alpha$  of any non-zero solution of Eq. (3.4), and in particular  $\varpi_1$ , can be bounded from above in terms of  $A$  only. At last, for any  $i \in J_\alpha = \{1, \dots, q_\alpha\}$  and any  $k \in \{1, \dots, q\}$  the function  $R(Y_i)^{(k-1)}$  vanishes at  $\alpha$  with order at least  $\text{ord}_\alpha R(Y_i) - (q - 1)$  so that  $\text{ord}_\alpha \tilde{\Delta}_E(z) \geq \omega_\alpha - q_\alpha(q - 1)$ . Therefore Eq. (3.5) yields  $\varpi_3 \leq \tau + c_3$  for some constant  $c_3$  depending only on  $A$  and  $\Sigma$ .

Now let us consider the  $\varpi_3$ -th derivative  $\Delta_E^{(\varpi_3)}(z)$ : it has a finite non-zero limit (equal to  $\gamma_3 \varpi_3!$ ) as  $z \rightarrow \alpha$ . Moreover this derivative is a  $\mathbb{Z}$ -linear combination of determinants of matrices of the form

$$N_{k_1, \dots, k_{q'_\alpha}} = [R(Y_{q_\alpha+i})^{(k_j-1)}]_{1 \leq i, j \leq q'_\alpha}$$

with  $1 \leq k_1 < \dots < k_{q'_\alpha} \leq q + \varpi_3 \leq \tau + c_2$ ; this constant  $c_2$  (which depends only on  $A$  and  $\Sigma$ ) is the one in the statement of assertion (ii) of Theorem 2. Now  $P_{k,i}$  is assumed to be holomorphic at  $\alpha$  for any  $i$  and any  $k \leq \tau + c_2$ , so that  $R(Y_i)^{(k-1)} \in \mathcal{L}_\alpha$  since  $Y_i \in M_{q,1}(\mathcal{L}_\alpha)$ . Therefore  $\det N_{k_1, \dots, k_{q'_\alpha}} \in \mathcal{L}_\alpha$ ; since  $\Delta_A^{(\varpi_3)}(z)$  has a finite non-zero limit as  $z \rightarrow \alpha$ , there exists at least one term  $\det N_{k_1, \dots, k_{q'_\alpha}}$  in the above-mentioned  $\mathbb{Z}$ -linear combination which also has a finite non-zero limit as  $z \rightarrow \alpha$ . For this tuple we consider the equality  ${}^t\tilde{\mathcal{Y}}\tilde{M} = N_{k_1, \dots, k_{q'_\alpha}}$ , where  $\tilde{\mathcal{Y}} \in M_{q, q'_\alpha}(\mathcal{L}_\alpha)$  is the matrix with columns  $Y_{q_\alpha+1}, \dots, Y_q$ , and  $\tilde{M} = [P_{k_j, i}]_{1 \leq i \leq q, 1 \leq j \leq q'_\alpha}$ . The Cauchy-Binet formula yields

$$\det N_{k_1, \dots, k_{q'_\alpha}} = \sum_{\substack{B \subset \{1, \dots, q\} \\ \#B = q'_\alpha}} \det {}^t\tilde{\mathcal{Y}}_B \cdot \det \tilde{M}_B \quad (3.6)$$

where  $\tilde{\mathcal{Y}}_B$  (resp.  $\tilde{M}_B$ ) is the square matrix consisting in the rows of  $\tilde{\mathcal{Y}}$  (resp. of  $\tilde{M}$ ) corresponding to indices in  $B$ .

Let  $\text{ev}_\alpha : \mathcal{L}_\alpha \rightarrow \mathbb{C}$  denote regularized evaluation at  $\alpha$ , defined by  $\text{ev}_\alpha(f) = c_{0,0}$  for any  $f = \sum_{i,j} c_{i,j}(z - \alpha)^i (\log(z - \alpha))^j$ . The important point is that  $\text{ev}_\alpha$  is a  $\mathbb{C}$ -algebra homomorphism, and that  $\text{ev}_\alpha(f)$  is equal to the limit of  $f(z)$  as  $z \rightarrow \alpha$  whenever this limit exists. Extending  $\text{ev}_\alpha$  coefficientwise to matrices, Eq. (3.6) yields

$$\text{ev}_\alpha \left( \det N_{k_1, \dots, k_{q'_\alpha}} \right) = \sum_{\substack{B \subset \{1, \dots, q\} \\ \#B = q'_\alpha}} \text{ev}_\alpha \left( \det {}^t\tilde{\mathcal{Y}}_B \right) \cdot \text{ev}_\alpha \left( \det \tilde{M}_B \right).$$

Now the left hand side is non-zero, so that  $\text{ev}_\alpha(\det \tilde{M}_B) \neq 0$  for some  $B$ . Since all coefficients  $P_{k,i}$  are holomorphic at  $\alpha$ , so is  $\det \tilde{M}_B$  and therefore  $\det(\tilde{M}_B(\alpha)) = \text{ev}_\alpha(\det \tilde{M}_B) \neq 0$ . We have found an invertible submatrix of  $M(\alpha)$  of size  $q'_\alpha$ , so that  $\text{rk} M(\alpha) \geq q'_\alpha$ : this concludes the proof of Theorem 2.

## 4 Diophantine part of the proof

In this section we prove Theorem 4 stated in the introduction, and give in details new proofs of the Ball-Rivoal theorem and Nishimoto's Theorem 3. To provide a unified treatment, we state a general result (namely Theorem 6) and deduce these results from it in §4.1. In order to help the reader, we first sketch the proof of Theorem 6 in §4.2, then construct the linear forms (§4.3), apply the zero estimate (namely Theorem 2) to obtain an invertible matrix (§4.4), and study the arithmetic and asymptotic properties (§4.5). At last we state and prove Siegel's linear independence criterion in §4.6.

## 4.1 Statement of the main theorem and consequences

**Theorem 6.** *Let  $N \geq 1$ , and  $f : \mathbb{N} \rightarrow \mathbb{C}$  be such that  $f(n + N) = f(n)$  for any  $n$ . Let  $p \in \{0, 1\}$ ,  $a \geq 2$ , and  $z_0 \in \{1, e^{i\pi/N}\}$ ; put*

$$\xi_j = \sum_{n=1}^{\infty} \frac{f(n)z_0^n}{n^j} \text{ for any } j \in \{1, \dots, a\},$$

except that  $\xi_1 = 0$  if  $z_0 = 1$ . Then as  $a \rightarrow \infty$ ,

$$\dim_{\mathbb{Q}} \text{Span}_{\mathbb{Q}}(\{\xi_j, 1 \leq j \leq a, j \equiv p \pmod{2}\}) \geq \frac{1 + o(1)}{N + \log 2} \log a.$$

We refer to §2 for the special case of the Ball-Rivoal theorem.

Let us deduce Theorems 3 and 4 stated in the introduction from this result. Let  $\chi$  be a Dirichlet character mod  $d$ . Its conductor is the smallest divisor  $e$  of  $d$  for which there exists a character  $\chi'$  mod  $e$  such that  $\chi(n) = \chi'(n)$  for any  $n$  coprime to  $d$ . Comparing the  $L$ -functions of  $\chi$  and  $\chi'$  (see for instance [11, §§3.2 and 3.3]) yields

$$L(\chi, s) = L(\chi', s) \prod_{\substack{p|d \\ p \nmid e}} (1 - \chi'(p)p^{-s})$$

so that  $\delta_{\chi,p,a} = \delta_{\chi',p,a}$  for any  $p, a$  (with the notation of Theorem 3). Therefore we may assume that  $e = d$ , i.e.  $\chi$  is primitive. Then Theorem 3 follows from Theorem 6 by letting  $z_0 = 1$  and  $f = \chi$ .

To prove Theorem 4, we first prove that for any primitive Dirichlet character  $\chi$  modulo a multiple  $e$  of 4,

$$\chi\left(n + \frac{e}{2}\right) = -\chi(n) \text{ for any } n \in \mathbb{Z}. \quad (4.1)$$

Indeed we have  $n(\frac{e}{2} + 1) \equiv n + \frac{e}{2} \pmod{e}$  if  $n$  is odd, so that  $\chi(n + \frac{e}{2}) = \chi(n)\chi(\frac{e}{2} + 1)$  for any  $n \in \mathbb{Z}$  (since both sides vanish if  $n$  is even). Moreover  $(\chi(\frac{e}{2} + 1))^2 = 1$  since  $(\frac{e}{2} + 1)^2 \equiv 1 \pmod{e}$ , and  $\chi(\frac{e}{2} + 1) \neq 1$  because  $\chi$  is primitive (so that  $\chi(n + \frac{e}{2}) \neq \chi(n)$  for some  $n$ ). Therefore  $\chi(\frac{e}{2} + 1) = -1$ : this concludes the proof of (4.1).

Now let  $N = e/2$  and define  $f : \mathbb{N} \rightarrow \mathbb{C}$  by  $f(r) = \chi(r)z_0^{-r}$  for any  $r \in \{1, \dots, N\}$ , where  $z_0 = e^{i\pi/N}$ . Then Eq. (4.1) yields

$$\sum_{n=1}^{\infty} \frac{f(n)z_0^n}{n^j} = \sum_{r=1}^N f(r)z_0^r \sum_{\substack{n \geq 1 \\ n \equiv r \pmod{2N}}} \left( \frac{1}{n^j} - \frac{1}{(n+N)^j} \right) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^j} = L(\chi, j)$$

so that Theorem 6 implies Theorem 4.

## 4.2 Sketch of the proof

To prove Theorem 6, we let  $r, n \geq 1$  be such that  $r < \frac{a}{2N}$  and  $N$  divides  $n$ . We define  $\xi'_1, \dots, \xi'_{a+N}$  as follows:

$$\begin{cases} \xi'_j &= 2(-1)^p \xi_j & \text{for } j \in \{1, \dots, a\} \text{ such that } j \equiv p \pmod{2} \\ \xi'_j &= 0 & \text{for } j \in \{1, \dots, a\} \text{ such that } j \not\equiv p \pmod{2} \\ \xi'_{a+1+\lambda} &= z_0^\lambda f(\lambda) & \text{for any } \lambda \in \{0, \dots, N-1\}. \end{cases} \quad (4.2)$$

We also let

$$\delta_n = (Nd_n)^a N^{an/N},$$

and define  $i_0$  to be equal to 1 if  $z_0 = e^{i\pi/N}$ , and equal to 2 otherwise (i.e., if  $z_0 = 1$ ).

In §4.3 (see (4.16)) we shall construct integers  $s_{k,i}$ ,  $i_0 \leq i \leq a+N$ , such that as  $n \rightarrow +\infty$ :

$$\max_{i_0 \leq i \leq a+N} |s_{k,i}| \leq \beta^{n(1+o(1))} \quad \text{and} \quad \left| \sum_{i=i_0}^{a+N} s_{k,i} \xi'_i \right| \leq \alpha^{n(1+o(1))} \quad (4.3)$$

where

$$\alpha = e^a 4^{a/N-r} (N+1)^{2r+2} r^{-a/N+4r+2} \quad \text{and} \quad \beta = (2e^N)^{a/N} (rN+1)^{2r+2}.$$

Then Lemma 1 (that will be stated and proved in §4.4 using Theorem 2) provides a positive constant  $c_2$  (which depends only on  $a$  and  $N$ ) and integers  $1 \leq k_{i_0} < k_{i_0+1} < \dots < k_{a+N} \leq c_2$  (which depend on  $a, N, r$ , and  $n$ ) such that the matrix  $[s_{k_j,i}]_{i_0 \leq i, j \leq a+N}$  is invertible. Since  $k_j \leq c_2$  for any  $j$ , the symbols  $o(1)$  in (4.3) with  $k = k_j$  can be made uniform with respect to  $k$ . Therefore Siegel's linear independence criterion applies (see §4.6). Taking  $a$  very large,  $N$  fixed, and  $r$  equal to the integer part of  $\frac{a}{(\log(a))^2}$  concludes the proof of Theorem 6 since  $\xi'_i = 0$  if  $i \leq a$  and  $i \not\equiv p \pmod{2}$ , and

$$1 - \frac{\log \alpha}{\log \beta} = \frac{1 + \varepsilon_a}{N + \log 2} \log a \quad \text{where} \quad \lim_{a \rightarrow +\infty} \varepsilon_a = 0.$$

## 4.3 Construction of the linear forms

Let  $a, r, N$  be positive integers such that  $1 \leq r < \frac{a}{2N}$ . For any integer multiple  $n$  of  $N$  we let

$$F(t) = (n/N)!^{a-2rn} \frac{(t-rn)_{rn} (t+n+1)_{rn}}{\prod_{h=0}^{n/N} (t+Nh)^a}.$$

Then  $F$  is a rational function, and its degree  $-d_0$  satisfies

$$d_0 := a\left(\frac{n}{N} + 1\right) - 2rn = -\deg F \geq n + a \geq 2. \quad (4.4)$$

Its partial fraction expansion reads

$$F(t) = \sum_{h=0}^{n/N} \sum_{j=1}^a \frac{p_{j,h}}{(t+Nh)^j}$$



with rational coefficients  $p_{j,h}$ . Let

$$P_j(z) = \sum_{h=0}^{n/N} p_{j,h} z^{Nh} \in \mathbb{Q}[z]_{\leq n} \text{ for any } j \in \{1, \dots, a\},$$

and also

$$S_0(z) = \sum_{t=n+1}^{\infty} F(-t) z^t, \quad S_{\infty}(z) = \sum_{t=1}^{\infty} F(t) z^{-t}. \quad (4.5)$$

As in [2] we have

$$S_{\infty}(z) = V(z) + \sum_{j=1}^a P_j(z) \text{Li}_j(1/z)$$

where

$$V(z) = - \sum_{t=0}^{n-1} z^t \sum_{j=1}^a \sum_{h=\lceil (t+1)/N \rceil}^{n/N} \frac{p_{j,h}}{(Nh-t)^j} \in \mathbb{Q}[z]_{\leq n}.$$

In the same way (see [9]) we have

$$S_0(z) = U(z) + \sum_{j=1}^a P_j(z) (-1)^j \text{Li}_j(z)$$

with the same polynomials  $P_1, \dots, P_a$ , and

$$U(z) = - \sum_{t=1}^n z^t \sum_{j=1}^a \sum_{h=0}^{\lfloor (t-1)/N \rfloor} \frac{p_{j,h}}{(t-Nh)^j} \in \mathbb{Q}[z]_{\leq n}.$$

Now let  $P_{1,j} = P_j$  for any  $j \in \{1, \dots, a\}$ , and define inductively  $P_{k,j} \in \mathbb{Q}(z)$  by

$$P_{k,j}(z) = P'_{k-1,j}(z) - \frac{1}{z} P_{k-1,j+1}(z) \text{ for any } k \geq 2 \text{ and any } j \in \{1, \dots, a\}, \quad (4.6)$$

where  $P_{k-1,a+1} = 0$  for any  $k$ ; we shall check in §4.4 below that this notation  $P_{k,j}$  is consistent with the one used in the introduction. We let also  $U_1 = U$ ,  $V_1 = V$ , and define  $U_k, V_k$  for any  $k \geq 2$  by the recurrence relations

$$U_k(z) = U'_{k-1}(z) - \frac{1}{1-z} P_{k-1,1}(z), \quad (4.7)$$

$$V_k(z) = V'_{k-1}(z) + \frac{1}{z(1-z)} P_{k-1,1}(z). \quad (4.8)$$

Then for any  $k \geq 1$  we have

$$S_0^{(k-1)}(z) = U_k(z) + \sum_{j=1}^a P_{k,j}(z) (-1)^j \text{Li}_j(z) \quad (4.9)$$

$$\text{and } S_\infty^{(k-1)}(z) = V_k(z) + \sum_{j=1}^a P_{k,j}(z) \text{Li}_j(1/z). \quad (4.10)$$

Moreover Eqns. (4.6), (4.7) and (4.8) show that the rational functions  $P_{k,j}$  with  $1 \leq j \leq a$  (resp.  $U_k$  and  $V_k$ ) have only 0 (resp. only 0 and 1) as possible finite poles. Now we have

$$S_0^{(k-1)}(z) = \sum_{t=n+1}^{\infty} F(-t)(t-k+2)_{k-1} z^{t-k+1} \text{ for } |z| < 1$$

$$\text{and } S_\infty^{(k-1)}(z) = \sum_{t=1}^{\infty} F(t)(-1)^{k-1}(t)_{k-1} z^{-t-k+1} \text{ for } |z| > 1.$$

Let us assume that  $k-1 \leq d_0-2$ , where  $d_0 = -\deg F$  is defined by Eq. (4.4); then these formulas hold also when  $|z| = 1$  and we may let  $z$  tend to 1 in Eqns. (4.9) and (4.10). Since  $P_{k,j}$  is holomorphic at  $z = 1$  for any  $k \geq 1$  and any  $j$ , a possible divergence may come only from poles of  $U_k$  or  $V_k$  at  $z = 1$ , or from the logarithmic term involving  $\text{Li}_1(z)$  or  $\text{Li}_1(1/z)$ . Since a pole and a logarithmic term cannot cancel each other out, and  $S_0^{(k-1)}(z)$  and  $S_\infty^{(k-1)}(z)$  have finite limits as  $z \rightarrow 1$ , we obtain:

$$\text{For any } k \leq d_0 - 1, \quad P_{k,1}(1) = 0 \quad \text{and} \quad U_k, V_k \text{ do not have a pole at } z = 1. \quad (4.11)$$

Now let  $k \leq d_0 - 1$ , and  $z \in \mathbb{C}$  be such that  $|z| = 1$ . Then Eqns. (4.9) and (4.10) hold, upon agreeing that the sums start at  $j = 2$  if  $z = 1$ ; the same remark applies in what follows. Since  $P_j(z) \in \mathbb{Q}[z^N]$  for any  $j \in \{1, \dots, a\}$ , Eq. (4.6) yields  $P_{k,j} \in z^{1-k} \mathbb{Q}[z^N]$  (see the proof of Proposition 1 in §4.5 for details). On the other hand, since  $U_k, V_k \in \mathbb{Q}[z, z^{-1}]$  for any  $k \leq d_0 - 1$ , we can write

$$z^{k-1}U_k(z) = \sum_{\lambda=0}^{N-1} z^\lambda U_{k,\lambda}(z) \quad \text{and} \quad z^{k-1}V_k(z) = \sum_{\lambda=0}^{N-1} z^\lambda V_{k,\lambda}(z) \quad (4.12)$$

with  $U_{k,\lambda}, V_{k,\lambda} \in \mathbb{Q}[z^N, z^{-N}]$ . Then Eqns. (4.9) and (4.10) yield

$$z^{k-1}S_0^{(k-1)}(z) = \sum_{\lambda=0}^{N-1} z^\lambda U_{k,\lambda}(z) + \sum_{j=1}^a z^{k-1}P_{k,j}(z)(-1)^j \text{Li}_j(z) \quad (4.13)$$

$$\text{and } z^{k-1}S_\infty^{(k-1)}(z) = \sum_{\lambda=0}^{N-1} z^\lambda V_{k,\lambda}(z) + \sum_{j=1}^a z^{k-1}P_{k,j}(z) \text{Li}_j(1/z). \quad (4.14)$$

The point now is that  $U_{k,\lambda}(z)$ ,  $V_{k,\lambda}(z)$ , and  $z^{k-1}P_{k,j}(z)$  depend only on  $z^N$ . For any  $\ell \in \{1, \dots, N\}$  we consider

$$\mu_\ell = \frac{1}{N} \sum_{\lambda=1}^N f(\lambda) \omega^{-\ell\lambda}. \quad (4.15)$$

Let  $z_0 \in \{1, e^{i\pi/N}\}$  and  $p \in \{0, 1\}$  be as in Theorem 6, and recall that  $\omega = e^{2i\pi/N}$ . For any  $k \leq d_0 - 1$  we let

$$\Lambda_k = \sum_{\ell=1}^N \mu_\ell \left[ (\omega^\ell z_0)^{k-1} S_0^{(k-1)}(\omega^\ell z_0) + (-1)^p (\omega^\ell z_0)^{1-k} S_\infty^{(k-1)}\left(\frac{1}{\omega^\ell z_0}\right) \right].$$

Then Eqns. (4.13) and (4.14) yield, since  $U_{k,\lambda}(z)$ ,  $V_{k,\lambda}(z)$ , and  $z^{k-1}P_{k,j}(z)$  depend only on  $z^N$  and  $(\omega^\ell z_0)^N = (\omega^\ell z_0)^{-N} = z_0^N$ :

$$\begin{aligned} \Lambda_k &= \sum_{\lambda=0}^{N-1} \left[ \left( \sum_{\ell=1}^N \mu_\ell (\omega^\ell z_0)^\lambda \right) U_{k,\lambda}(z_0) + (-1)^p \left( \sum_{\ell=1}^N \mu_\ell (\omega^\ell z_0)^{-\lambda} \right) V_{k,\lambda}(z_0) \right] \\ &\quad + \sum_{j=1}^a z_0^{k-1} P_{k,j}(z_0) \sum_{\ell=1}^N \mu_\ell \text{Li}_j(\omega^\ell z_0) ((-1)^j + (-1)^p). \end{aligned}$$

Now Eq. (4.15) yields

$$\sum_{\ell=1}^N \mu_\ell \omega^{n\ell} = f(n) \text{ for any } n \in \mathbb{Z}, \text{ so that } \sum_{\ell=1}^N \mu_\ell \text{Li}_j(\omega^\ell z_0) = \sum_{n=1}^{\infty} \frac{f(n) z_0^n}{n^j} = \xi_j \text{ for any } j \leq a.$$

Letting  $V_{k,N} = V_{k,0}$  we obtain:

$$\Lambda_k = 2(-1)^p \sum_{\substack{1 \leq j \leq a \\ j \equiv p \pmod{2}}} z_0^{k-1} P_{k,j}(z_0) \xi_j + \sum_{\lambda=0}^{N-1} (U_{k,\lambda}(z_0) + (-1)^p V_{k,N-\lambda}(z_0)) z_0^\lambda f(\lambda).$$

As announced in §4.2 we now define the coefficients  $s_{k,i}$ :

$$\begin{cases} s_{k,i} = \delta_n z_0^{k-1} P_{k,i}(z_0) \text{ for } 1 \leq i \leq a, \\ s_{k,a+1+\lambda} = \delta_n (U_{k,\lambda}(z_0) + (-1)^p V_{k,N-\lambda}(z_0)) \text{ for } 0 \leq \lambda \leq N-1, \end{cases} \quad (4.16)$$

where  $\delta_n = (Nd_n)^a N^{an/N}$ , so that

$$\delta_n \Lambda_k = \sum_{i=i_0}^{a+N} s_{k,i} \xi'_i$$

since  $\xi'_1 = 0$  if  $z_0 = 1$  (recall from §4.2 that  $i_0 = 2$  in this case, and  $i_0 = 1$  otherwise, i.e. if  $z_0 = e^{i\pi/N}$ ;  $\xi'_i$  is defined in Eq. (4.2)).

Since  $z_0^N \in \{-1, 1\}$  and  $z^{k-1}P_{k,j}(z)$ ,  $U_{k,\lambda}(z)$  and  $V_{k,N-\lambda}(z)$  are polynomials in  $z^N$  with rational coefficients, the numbers  $s_{k,1}, \dots, s_{k,a+N}$  are rational. We shall prove in Proposition 1 (§4.5) that they are integers, thanks to the factor  $\delta_n$ .

## 4.4 Application of the zero estimate

In this section we deduce from Theorem 2 the following lemma, used at the end of §4.2. It provides an invertible matrix which enables us to apply Siegel's linear independence criterion (see §4.6).

**Lemma 1.** *In the setting of §4.2, let  $i_0 = 1$  if  $z_0 = e^{i\pi/N}$  and  $i_0 = 2$  if  $z_0 = 1$ ; let  $s_{k,i}$  be defined by Eq. (4.16). Then there exist a positive constant  $c_2$  (which depends only on  $a$  and  $N$ ) and integers  $1 \leq k_{i_0} < k_{i_0+1} < \dots < k_{a+N} \leq c_2$  (which depend on  $a$ ,  $N$ ,  $r$ , and  $n$ ) such that the matrix  $[s_{k_j,i}]_{i_0 \leq i, j \leq a+N}$  is invertible.*

To begin with, let us recall from §4.2 that  $\omega = e^{2i\pi/N}$ ,  $a$ ,  $r$ ,  $N$ ,  $n$  are positive integers such that  $1 \leq r < \frac{a}{2N}$ ,  $n$  is a multiple of  $N$ , and

$$F(t) = (n/N)!^{a-2rN} \frac{(t-rn)_{rn}(t+n+1)_{rn}}{\prod_{h=0}^{n/N} (t+Nh)^a}.$$

We have

$$S_0(z) = \sum_{t=n+1}^{\infty} F(-t)z^t = U(z) + \sum_{j=1}^a P_j(z)(-1)^j \text{Li}_j(z)$$

$$\text{and } S_{\infty}(z) = \sum_{t=1}^{\infty} F(t)z^{-t} = V(z) + \sum_{j=1}^a P_j(z) \text{Li}_j(1/z).$$

Since  $P_j \in \mathbb{C}[z^N]$  for any  $j \in \{1, \dots, a\}$ , we have  $P_j(\omega^\ell z) = P_j(z)$  for any  $\ell \in \mathbb{Z}$ . Therefore letting

$$R_{0,\ell}(z) = S_0(\omega^\ell z), \quad R_{\infty,\ell}(z) = S_{\infty}(\omega^\ell z), \quad \bar{P}_{0,\ell}(z) = U(\omega^\ell z), \quad \bar{P}_{\infty,\ell}(z) = V(\omega^\ell z) \quad (4.17)$$

for any  $\ell \in \{1, \dots, N\}$ , we have

$$R_{0,\ell}(z) = \bar{P}_{0,\ell}(z) + \sum_{j=1}^a P_j(z)(-1)^j \text{Li}_j(\omega^\ell z) = O(z^{(r+1)n+1}), \quad z \rightarrow 0, \quad (4.18)$$

$$\text{and } R_{\infty,\ell}(z) = \bar{P}_{\infty,\ell}(z) + \sum_{j=1}^a P_j(z) \text{Li}_j\left(\frac{1}{\omega^\ell z}\right) = O(z^{-rn-1}), \quad z \rightarrow \infty. \quad (4.19)$$

Moreover, recall that  $d_0 = -\deg F = a(\frac{n}{N} + 1) - 2rn$ ; Lemma 3 of [9] shows that

$$\sum_{j=1}^a P_j(z)(-1)^{j-1} \frac{(\log z)^{j-1}}{(j-1)!} = O((z-1)^{d_0-1}), \quad z \rightarrow 1.$$

Using again the fact that  $P_j(\omega^{-\ell} z) = P_j(z)$ , we obtain for any  $\ell \in \{1, \dots, N\}$ :

$$R_{\omega^\ell}(z) := \sum_{j=1}^a P_j(z)(-1)^{j-1} \frac{(\log(\omega^{-\ell} z))^{j-1}}{(j-1)!} = O((z - \omega^\ell)^{d_0-1}), \quad z \rightarrow \omega^\ell. \quad (4.20)$$

Combining Eqns. (4.18), (4.19), and (4.20) with  $1 \leq \ell \leq N$ , we have solved a simultaneous Padé approximation problem. The  $(n+1)(a+2N)$  unknowns are the coefficients of  $P_1, \dots, P_a, \overline{P}_{0,1}, \dots, \overline{P}_{0,N}, \overline{P}_{\infty,1}, \dots, \overline{P}_{\infty,N}$ , which are polynomials of degree less than or equal to  $n$ . There are

$$2N((r+1)n+1) + N(d_0 - 1) = n(a+2N) + (a+1)N$$

linear equations, since a priori we have  $R_{\infty,\ell}(z) = O(z^n)$  as  $z \rightarrow \infty$ . The difference between the number of unknowns and the number of equations is equal to  $N - a(N-1)$ . If  $N = 1$  this is equal to 1: the Padé approximation problem is exactly (2.2), i.e. the one of [9, Théorème 1], which has a unique solution up to proportionality. Whenever  $N \geq 2$  we have  $N - a(N-1) < 0$ : the problem we have solved has more equations than unknowns. This is due to the fact that we always assume  $n$  to be an integer multiple of  $N$ . Anyway to complete the proof, it is sufficient to bound from above the difference between the number of unknowns and the number of equations by a constant independent from  $n$ ; we do not need to study whether the Padé approximation problem has a unique solution or not.

Let  $q = a + 2N$ , and  $A \in M_q(\mathbb{C}(z))$  be the matrix of which the coefficients  $A_{i,j}$  are given by:

$$\begin{cases} A_{i,i-1}(z) = \frac{-1}{z} \text{ for any } i \in \{2, \dots, a\} \\ A_{1,a+\ell}(z) = \frac{\omega^\ell}{\omega^\ell z - 1} \text{ for any } \ell \in \{1, \dots, N\} \\ A_{1,a+N+\ell}(z) = \frac{1}{z(1-\omega^\ell z)} \text{ for any } \ell \in \{1, \dots, N\} \end{cases}$$

and all other coefficients are zero. We consider the following solutions of the differential system  $Y' = AY$ , with  $1 \leq \ell \leq N$ :

$$Y_{0,\ell}(z) = {}^t \left( -\text{Li}_1(\omega^\ell z), \text{Li}_2(\omega^\ell z), \dots, (-1)^a \text{Li}_a(\omega^\ell z), 0, \dots, 0, 1, 0, \dots, 0 \right),$$

$$Y_{\infty,\ell}(z) = {}^t \left( \text{Li}_1\left(\frac{1}{\omega^\ell z}\right), \text{Li}_2\left(\frac{1}{\omega^\ell z}\right), \dots, \text{Li}_a\left(\frac{1}{\omega^\ell z}\right), 0, \dots, 0, 1, 0, \dots, 0 \right),$$

$$Y_{\omega^\ell}(z) = {}^t \left( 1, -\log(\omega^{-\ell} z), \frac{(\log(\omega^{-\ell} z))^2}{2!}, \dots, (-1)^{a-1} \frac{(\log(\omega^{-\ell} z))^{a-1}}{(a-1)!}, 0, \dots, 0 \right)$$

where the coefficient 1 in  $Y_{0,\ell}(z)$  (resp.  $Y_{\infty,\ell}(z)$ ) is in position  $a + \ell$  (resp.  $a + N + \ell$ ).

We let  $J_0 = \{(0, 1), (0, 2), \dots, (0, N)\}$ ,  $J_\infty = \{(\infty, 1), (\infty, 2), \dots, (\infty, N)\}$ ,  $J_{\omega^\ell} = \{\omega^\ell\}$  for  $1 \leq \ell \leq N$ , and  $\Sigma = \{0, \infty\} \cup \{\omega^\ell, 1 \leq \ell \leq N\}$ . We also let  $P_{a+\ell}(z) = \overline{P}_{0,\ell}(z) = U(\omega^\ell z)$  and  $P_{a+N+\ell} = \overline{P}_{\infty,\ell}(z) = V(\omega^\ell z)$  for any  $\ell \in \{1, \dots, N\}$ . Then with the notation of the introduction we have  $R(Y_{0,\ell}) = R_{0,\ell}(z)$ ,  $R(Y_{\infty,\ell}) = R_{\infty,\ell}(z)$ , and  $R(Y_{\omega^\ell}) = R_{\omega^\ell}(z)$  for any  $\ell \in \{1, \dots, N\}$ .

Since  $P_a$  is not the zero polynomial, we have  $R_{\omega^\ell}(z) \neq 0$  for any  $\ell$ ; the  $\mathbb{C}$ -linear independence of  $R_{0,1}(z), \dots, R_{0,N}(z)$  (resp. of  $R_{\infty,1}(z), \dots, R_{\infty,N}(z)$ ) follows directly (resp. up to changing  $z$  to  $1/z$ ) from the following lemma, which is not difficult to prove using monodromy (see [26]).

**Lemma 2.** *The functions 1 and  $\text{Li}_j(\omega^\ell z)$ , for  $j \geq 1$  and  $1 \leq \ell \leq N$ , are linearly independent over  $\mathbb{C}(z)$ .*

Eqns. (4.18), (4.19), and (4.20) yield  $\text{ord}_0(R_{0,\ell}(z)) \geq (r+1)n+1$ ,  $\text{ord}_\infty(R_{\infty,\ell}(z)) \geq rn+1$ , and  $\text{ord}_{\omega^\ell}(R_{\omega^\ell}(z)) \geq d_0-1$  for any  $\ell \in \{1, \dots, N\}$ , so that

$$\sum_{\sigma \in \Sigma} \sum_{j \in J_\sigma} \text{ord}_\sigma R_j(z) \geq (2r+1)Nn + N(d_0+1) = (n+1)q - nN - \tau \text{ with } \tau = N - a(N-1);$$

here  $q = a + 2N$ , and we recall that  $d_0 = -\deg F = a(\frac{n}{N} + 1) - 2rn$ . This number  $\tau$  is exactly the difference between the number of unknowns and the number of equations computed after Eq. (4.20).

Now for any  $k \geq 1$  and any  $\ell \in \{1, \dots, N\}$  we let

$$\bar{P}_{k,0,\ell} = \omega^{\ell(k-1)} U_k(\omega^\ell z) \text{ and } \bar{P}_{k,\infty,\ell} = \omega^{\ell(k-1)} V_k(\omega^\ell z), \quad (4.21)$$

and

$$\mathbf{P}_k = {}^t(P_{k,1}, P_{k,2}, \dots, P_{k,a}, \bar{P}_{k,0,1}, \dots, \bar{P}_{k,0,N}, \bar{P}_{k,\infty,1}, \dots, \bar{P}_{k,\infty,N}) \in M_{q,1}(\mathbb{C}(z)),$$

so that  $\mathbf{P}_1 = {}^t(P_1, \dots, P_{a+2N})$ . Then it is not difficult to check that

$$\mathbf{P}_k = \left( \frac{d}{dz} + {}^t A \right)^{k-1} \mathbf{P}_1.$$

To illustrate this equality, we notice that Eq. (4.17) yields

$$R_{0,\ell}^{(k-1)} = \bar{P}_{k,0,\ell}(z) + \sum_{j=1}^a P_{k,j}(z) (-1)^j \text{Li}_j(\omega^\ell z)$$

$$\text{and } R_{\infty,\ell}^{(k-1)} = \bar{P}_{k,\infty,\ell}(z) + \sum_{j=1}^a P_{k,j}(z) \text{Li}_j\left(\frac{1}{\omega^\ell z}\right)$$

since (as in [22, Chapter 3, §4])

$$S_0^{(k-1)} = U_k(z) + \sum_{j=1}^a P_{k,j}(z) (-1)^j \text{Li}_j(z)$$

$$\text{and } S_\infty^{(k-1)} = V_k(z) + \sum_{j=1}^a P_{k,j}(z) \text{Li}_j(1/z).$$

Provided  $n$  is large enough, we have checked all assumptions of Theorem 2 (using, among others, Eq. (4.11)). We apply this result with  $\alpha = z_0$ ; recall that  $z_0 \in \{1, e^{i\pi/N}\}$ . In the case  $z_0 = 1$ , we obtain positive integers  $k_2 < \dots < k_q \leq c_2$  such that the matrix

with columns  $\mathbf{P}_{k_2}(1), \dots, \mathbf{P}_{k_q}(1)$  has rank  $q - 1$ . Now  $P_{k,1}(1) = 0$  for any  $k \leq c_2$  (using Eq. (4.11) since  $n$  is large enough) so that the first row of this matrix is identically zero. Removing this row yields the following invertible matrix (with  $z_0 = 1$  and  $i_0 = 2$ ):

$$\begin{bmatrix} [z_0^{k_j-1} P_{k_j,i}(z_0)]_{i_0 \leq i \leq a, i_0 \leq j \leq q} \\ [z_0^{k_j-1} \bar{P}_{k_j,0,i}(z_0)]_{1 \leq i \leq N, i_0 \leq j \leq q} \\ [z_0^{k_j-1} \bar{P}_{k_j,\infty,i}(z_0)]_{1 \leq i \leq N, i_0 \leq j \leq q} \end{bmatrix}. \quad (4.22)$$

If  $z_0 = e^{i\pi/N} \notin \Sigma$  then Theorem 2 provides directly  $k_1 < \dots < k_q \leq c_2$  such that the matrix (4.22) with  $i_0 = 1$  is invertible.

Now Eq. (4.12) with  $z = \omega^\ell z_0$  yields, since  $U_{k,\lambda} \in \mathbb{Q}[z^N, z^{-N}]$ :

$$\omega^{(k-1)\ell} z_0^{k-1} U_k(\omega^\ell z_0) = \sum_{\lambda=0}^{N-1} \omega^{\ell\lambda} z_0^\lambda U_{k,\lambda}(z_0) \text{ for any } \ell \in \{1, \dots, N\}.$$

Therefore we have for any  $\lambda \in \{0, \dots, N-1\}$ :

$$U_{k,\lambda}(z_0) = \frac{z_0^{k-1-\lambda}}{N} \sum_{\ell=1}^N \omega^{(k-1-\lambda)\ell} U_k(\omega^\ell z_0) = \frac{z_0^{k-1-\lambda}}{N} \sum_{\ell=1}^N \omega^{-\lambda\ell} \bar{P}_{k,0,\ell}(z_0) \quad (4.23)$$

using Eq. (4.21). Moreover the same relation holds with  $V_{k,\lambda}$  and  $\bar{P}_{k,\infty,\ell}$  for  $\lambda \in \{0, \dots, N-1\}$ . We recall that  $s_{k,i}$  was defined in Eq. (4.16) (§4.3) by

$$\begin{aligned} s_{k,i} &= \delta_n z_0^{k-1} P_{k,i}(z_0) \text{ for } 1 \leq i \leq a, \\ \text{and } s_{k,a+1+\lambda} &= \delta_n (U_{k,\lambda}(z_0) + (-1)^p V_{k,N-\lambda}(z_0)) \text{ for } 0 \leq \lambda \leq N-1. \end{aligned}$$

For any  $\lambda \in \{0, \dots, N-1\}$  we deduce that

$$s_{k,a+1+\lambda} = \frac{\delta_n}{N} z_0^{-\lambda} \sum_{\ell=1}^N \omega^{-\lambda\ell} z_0^{k-1} \bar{P}_{k,0,\ell}(z_0) \pm (-1)^p \frac{\delta_n}{N} z_0^{-N+\lambda} \sum_{\ell=1}^N \omega^{\lambda\ell} z_0^{k-1} \bar{P}_{k,\infty,\ell}(z_0)$$

where  $\pm$  is  $+$  if  $1 \leq \lambda \leq N-1$ , and  $z_0^N$  if  $\lambda = 0$ ; indeed  $V_{k,N} = V_{k,0}$  satisfies the equation analogous to Eq. (4.23) with  $\lambda = 0$ , but not with  $\lambda = N$  if  $z_0 = e^{i\pi/N}$ .

Let  $M = [m_{i,j}]_{i_0 \leq i \leq a+N, i_0 \leq j \leq a+2N}$  be the matrix defined by:

$$\begin{aligned} m_{i,i} &= \delta_n \text{ for any } i \in \{i_0, \dots, a\}, \\ m_{a+1+\lambda, a+\ell} &= \frac{\delta_n}{N} z_0^{-\lambda} \omega^{-\lambda\ell} \text{ for any } \lambda \in \{0, \dots, N-1\} \text{ and any } \ell \in \{1, \dots, N\}, \\ m_{a+1+\lambda, a+N+\ell} &= \pm (-1)^p \frac{\delta_n}{N} z_0^{-N+\lambda} \omega^{\lambda\ell} \text{ for any } \lambda \in \{0, \dots, N-1\} \text{ and any } \ell \in \{1, \dots, N\}, \end{aligned}$$

and all other coefficients are zero. Then  $M$  has rank  $a + N + 1 - i_0$ ; denoting by  $P \in \text{GL}_{a+2N+1-i_0}(\mathbb{C})$  the matrix (4.22), the matrix  $MP$  has rank  $a + N + 1 - i_0$ . Now  $MP$  is exactly the matrix  $[s_{k_j,i}]_{i_0 \leq i, j \leq a+N}$ . This concludes the proof of Lemma 1.

## 4.5 Arithmetic and Asymptotic Properties

In this section we prove the following result, used in the proof of Theorem 6; see §4.2 for the notation.

**Proposition 1.** *Let*

$$\alpha = e^a 4^{a/N-r} (N+1)^{2r+2} r^{-a/N+4r+2} \text{ and } \beta = (2e^N)^{a/N} (rN+1)^{2r+2}. \quad (4.24)$$

*Then we have  $s_{k,i} \in \mathbb{Z}$  for any  $i \in \{1, \dots, a+N\}$  and any  $k \leq d_0 - 1$ , and as  $n \rightarrow \infty$ :*

$$\left| \sum_{i=i_0}^{a+N} s_{k,i} \xi_i^k \right| \leq \alpha^{n(1+o(1))}, \quad \max_{1 \leq i \leq a+N} |s_{k,i}| \leq \beta^{n(1+o(1))}.$$

In this proposition and throughout this section, we denote by  $o(1)$  any sequence that tends to 0 as  $n \rightarrow \infty$ ; it usually depends also on  $a$ ,  $r$ ,  $N$ , and  $k$ . When Proposition 1 is applied in the proof of Theorem 6 (see §4.2), this dependence is not a problem since  $a$ ,  $r$ ,  $N$  are fixed parameters and  $k$  is bounded from above by  $c_2$ . At last we recall that  $d_n$  is the least common multiple of  $1, 2, \dots, n$ , and that

$$\delta_n = (Nd_n)^a N^{an/N}.$$

Let us start with a lemma, in which (as in §4.3)

$$F(t) = (n/N)!^{a-2rN} \frac{(t-rn)_{rn} (t+n+1)_{rn}}{\prod_{h=0}^{n/N} (t+Nh)^a} = \sum_{h=0}^{n/N} \sum_{j=1}^a \frac{p_{j,h}}{(t+Nh)^j}.$$

**Lemma 3.** *For any  $j \in \{1, \dots, a\}$  and any  $h \in \{0, \dots, n/N\}$  we have*

$$(Nd_{n/N})^{a-j} N^{an/N} p_{j,h} \in \mathbb{Z} \quad (4.25)$$

$$\text{and } |p_{j,h}| \leq \left( (2/N)^{a/N} (rN+1)^{2r+2} \right)^{n(1+o(1))} \quad (4.26)$$

where  $o(1)$  is a sequence that tends to 0 as  $n \rightarrow \infty$  and may depend also on  $N$ ,  $a$ , and  $r$ .

**Proof** of Lemma 3: We follow the approach of [10] and [7] by letting

$$\begin{aligned} F_0(t) &= \frac{(n/N)!}{\prod_{h=0}^{n/N} (t+Nh)} = \sum_{h=0}^{n/N} \frac{(-1)^h N^{-n/N} \binom{n/N}{h}}{t+Nh}, \\ G_i(t) &= \frac{(t-in/N)_{n/N}}{\prod_{h=0}^{n/N} (t+Nh)} = \sum_{h=0}^{n/N} \frac{(-1)^{h+n/N} N^{-n/N} \binom{n/N}{h} \binom{Nh+in/N}{n/N}}{t+Nh} \text{ for } 1 \leq i \leq rN, \\ H_i(t) &= \frac{(t+1+in/N)_{n/N}}{\prod_{h=0}^{n/N} (t+Nh)} = \sum_{h=0}^{n/N} \frac{(-1)^h N^{-n/N} \binom{n/N}{h} \binom{-Nh+(i+1)n/N}{n/N}}{t+Nh} \text{ for } N \leq i \leq (r+1)N-1. \end{aligned}$$



Then the partial fraction expansion of  $F = F_0^{a-2rN} G_1 \dots G_{rN} H_N \dots H_{(r+1)N-1}$  can be obtained by multiplying those of  $F_0$ ,  $G_i$  and  $H_i$  using repeatedly the formula

$$\frac{1}{(t + Nh)(t + Nh')^\ell} = \frac{1}{N^\ell (h' - h)^\ell (t + Nh)} - \sum_{i=1}^{\ell} \frac{1}{N^{\ell+1-i} (h' - h)^{\ell+1-i} (t + Nh')^i} \quad (4.27)$$

with  $h \neq h'$ . The denominator of  $p_{j,h}$  comes both from this formula (and this contribution divides  $(Nd_{n/N})^{a-j}$ ) and from the denominators of the coefficients in the partial fraction expansions of  $F_0$ ,  $G_i$ ,  $H_i$  (which belong to  $N^{-n/N}\mathbb{Z}$ , so that  $N^{an/N}$  accounts for this contribution). This concludes the proof of (4.25).

On the other hand, bounding from above the coefficients of the partial fraction expansions of  $F_0$ ,  $G_i$ ,  $H_i$  yields

$$|p_{j,h}| \leq n^{O(1)} N^{-an/N} 2^{an/N} \prod_{i=1}^{rN} \frac{(n + in/N)!}{(n/N)!(n + (i-1)n/N)!} \prod_{i=N}^{(r+1)N-1} \frac{((i+1)n/N)!}{(n/N)!(in/N)!}$$

where  $O(1)$  is a constant depending only on  $a$ ,  $r$ ,  $N$  which can be made explicit (see [7] for details). Simplifying the products and using the bound  $\frac{m!}{m_1! \dots m_c!} \leq c^m$  valid when  $m_1 + \dots + m_c = m$ , one obtains

$$|p_{j,h}| \leq n^{O(1)} (2/N)^{an/N} \left( \frac{((r+1)n)!}{n!(n/N)!^{rN}} \right)^2 \leq n^{O(1)} (2/N)^{an/N} (rN + 1)^{(2r+1)n}.$$

This concludes the proof of Lemma 3.

**Proof** of Proposition 1: Let  $H(P)$  denote the exponential height of a polynomial  $P \in \mathbb{C}[X]$ , that is the maximum modulus of a coefficient of  $P$ . Recall that  $P_j(z) = \sum_{h=0}^{n/N} p_{j,h} z^{Nh}$ ,  $U(z) = -\sum_{t=1}^n z^t \sum_{j=1}^a \sum_{h=0}^{\lfloor (t-1)/N \rfloor} \frac{p_{j,h}}{(t-Nh)^j}$  and  $V(z) = -\sum_{t=0}^{n-1} z^t \sum_{j=1}^a \sum_{h=\lceil (t+1)/N \rceil}^{n/N} \frac{p_{j,h}}{(Nh-t)^j}$ . Using Lemma 3 we see that these polynomials have coefficients in  $\delta_n^{-1}\mathbb{Z}$  and height less than  $H_n$  for some  $H_n \leq \left( (2/N)^{a/N} (rN + 1)^{2r+2} \right)^{n(1+o(1))}$ . Now let  $\tilde{P}_{k,j} = z^{k-1} P_{k,j}$  for any  $k, j$ . Then the recurrence relation (4.6) yields

$$\tilde{P}_{k,j} = z \tilde{P}'_{k-1,j} - (k-2) \tilde{P}_{k-1,j} - \tilde{P}_{k-1,j+1}$$

where  $\tilde{P}_{k-1,j+1} = 0$  if  $j = a$ , so that  $\tilde{P}_{k,j}$  is a polynomial of degree at most  $n$ , with coefficients in  $\delta_n^{-1}\mathbb{Z}$  and height  $H(\tilde{P}_{k,j}) \leq (n+1)_{k-1} H_n$ , by induction on  $k$ .

In the same way, letting  $\tilde{U}_k = z^{k-1} U_k$ , Eq. (4.7) yields

$$\tilde{U}_k = z \tilde{U}'_{k-1} - (k-2) \tilde{U}_{k-1} - z Q_{k-1}$$

where  $Q_{k-1} = \frac{1}{1-z} \tilde{P}_{k-1,1}$ . Provided  $k \leq d_0 - 1$ , Eq. (4.11) asserts that  $P_{k-1,1}(1) = 0$  so that  $Q_{k-1}$  is a polynomial and  $H(Q_{k-1}) \leq n H(P_{k-1,1}) \leq (n)_{k-1} H_n$ . By induction on

$k \leq d_0 - 1$ , we deduce that  $\tilde{U}_k$  is a polynomial of degree at most  $n$ , with coefficients in  $\delta_n^{-1}\mathbb{Z}$  and height  $H(\tilde{U}_k) \leq k(n)_{k-1}H_n$ . Now Eq. (4.12) reads  $\tilde{U}_k(z) = \sum_{\lambda=1}^N z^{\lambda-1}U_{k,\lambda}(z)$  with  $U_{k,\lambda} \in \mathbb{Q}[z^N, z^{-N}]$ . If  $k \leq d_0 - 1$  then  $U_{k,\lambda}$  belongs to  $\mathbb{Q}[z^N]$ , has degree at most  $n$  (as a polynomial in  $z$ ), coefficients in  $\delta_n^{-1}\mathbb{Z}$  and height  $H(U_{k,\lambda}) \leq k(n)_{k-1}H_n$ .

Proceeding in the same way, it is not difficult to prove that the same properties hold for  $V_{k,\lambda}$ . Assertion (i) follows at once, since  $d_n = e^{n(1+o(1))}$ .

To prove (ii), we recall that  $d_0 = -\deg F$  and write, as  $|t| \rightarrow \infty$ :

$$F(t) = \sum_{d=d_0} \frac{\mathfrak{A}_d}{t^d} \text{ where } \mathfrak{A}_d = \sum_{j=1}^a \sum_{h=0}^{n/N} (-Nh)^{d-j} \binom{d-1}{d-j} p_{j,h}$$

since  $(t + Nh)^{-j} = \sum_{\ell=0}^{\infty} \binom{\ell+j-1}{\ell} (-Nh)^{\ell} t^{-j-\ell}$  (see [9, p. 1378]). Lemma 3 provides a positive real number  $A_n \leq \left( (2/N)^{a/N} (rN+1)^{2r+2} \right)^{n(1+o(1))}$  such that  $|\mathfrak{A}_d| \leq (2n)^d A_n$  for any  $d \geq d_0$ . Then we have for any  $t \in \mathbb{Z}$  such that  $|t| \geq 2n+1$ :

$$|F(t)| \leq A_n \sum_{d=d_0}^{\infty} (2n/t)^d \leq (2n+1)A_n(2n/t)^{d_0}. \quad (4.28)$$

For any  $z \in \mathbb{C}$  such that  $|z| \leq 1$ , and any  $k \leq d_0 - 1$ , we obtain

$$\begin{aligned} |S_0^{(k-1)}(z)| &= \left| \sum_{t=(r+1)n+1}^{\infty} F(-t)(t-k+2)_{k-1} z^{t-k+1} \right| \leq (2n+1)A_n(2n)^{d_0} \sum_{t=(r+1)n+1}^{\infty} t^{k-1-d_0} \\ &\leq (2n+1)A_n(2n)^{d_0} \int_{(r+1)n}^{\infty} t^{k-1-d_0} dt \leq (2n+1)A_n 2^{d_0} n^k r^{k-d_0}. \end{aligned}$$

Moreover the same upper bound holds for  $S_{\infty}(z) = \sum_{t=rn+1}^{\infty} F(t)z^{-t}$  provided  $|z| \geq 1$ . Since

$$S(z) = \sum_{\ell=1}^N \omega^{\ell} \mu_{\ell} S_0(\omega^{\ell} z) + \omega^{\ell} \nu_{N-\ell} S_{\infty}(\omega^{\ell} z).$$

and  $d_0 = a(n/N + 1) - 2rn$ , we obtain  $|\delta_n S^{(k-1)}(1)| \leq \alpha^{n(1+o(1))}$  for any  $z \in \mathbb{C}$  such that  $|z| = 1$ , and any  $k \leq d_0 - 1$ ; here the constant implied in  $o(1)$  may depend on  $k$  (but not on  $n$ ). This concludes the proof of Proposition 1.

## 4.6 Siegel's linear independence criterion

The proofs of all linear independence results in this paper rely on the following criterion, which is based on Siegel's ideas (see for instance [8, p. 81–82 and 215–216], [13, §3] or [12, Proposition 4.1]).

**Proposition 2.** *Let  $\theta_1, \dots, \theta_p$  be real numbers, not all zero. Let  $\tau > 0$ , and  $(Q_n)$  be a sequence of real numbers with limit  $+\infty$ . Let  $\mathcal{N}$  be an infinite subset of  $\mathbb{N}$ , and for any  $n \in \mathcal{N}$  let  $L^{(n)} = [\ell_{i,j}^{(n)}]_{1 \leq i,j \leq p}$  be a matrix with integer coefficients and non-zero determinant, such that as  $n \rightarrow \infty$  with  $n \in \mathcal{N}$ :*

$$\max_{1 \leq i,j \leq p} |\ell_{i,j}^{(n)}| \leq Q_n^{1+o(1)}$$

$$\text{and } \max_{1 \leq j \leq p} |\ell_{1,j}^{(n)}\theta_1 + \dots + \ell_{p,j}^{(n)}\theta_p| \leq Q_n^{-\tau+o(1)}.$$

Then we have

$$\dim_{\mathbb{Q}} \text{Span}_{\mathbb{Q}}(\theta_1, \dots, \theta_p) \geq \tau + 1.$$

In the proof of Theorem 6 we apply this proposition with  $Q_n = \beta^n$  and  $\tau = -\frac{\log \alpha}{\log \beta}$  (so that  $Q_n^{-\tau} = \alpha^n$ ), where  $\alpha$  and  $\beta$  are defined in §4.2;  $\mathcal{N}$  is the set of integer multiples of  $N$ .

Eventhough it is a classical result, let us recall the proof of Proposition 2. Let  $d = \dim_{\mathbb{Q}} \text{Span}_{\mathbb{Q}}(\theta_1, \dots, \theta_p)$ , and  $F$  be a subspace of  $\mathbb{R}^p$  defined over  $\mathbb{Q}$ , of dimension  $d$ , which contains the point  $(\theta_1, \dots, \theta_p)$ . Let  $n \in \mathcal{N}$  be sufficiently large, and denote by  $L_j^{(n)}$  the linear form  $\ell_{1,j}^{(n)}X_1 + \dots + \ell_{p,j}^{(n)}X_p$  on  $\mathbb{R}^p$ . Up to reordering  $L_1^{(n)}, \dots, L_p^{(n)}$ , we may assume the restrictions of  $L_1^{(n)}, \dots, L_d^{(n)}$  to  $F$  to be linearly independent linear forms on  $F$ . Denoting by  $(u_1, \dots, u_d)$  an  $\mathbb{R}$ -basis of  $F$  consisting in vectors of  $\mathbb{Z}^p$ , the matrix  $[L_j^{(n)}(u_t)]_{1 \leq j,t \leq d}$  has a non-zero integer determinant. Now  $(\theta_1, \dots, \theta_p)$  is a linear combination of  $u_1, \dots, u_d$ ; the same linear combination of the columns has coefficients less than  $Q_n^{-\tau+o(1)}$  in absolute value. Therefore  $Q_n^{d-1-\tau+o(1)}$  is an upper bound on this non-zero integer determinant: this concludes the proof of Proposition 2.

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