Choquet integral calculus on a continuous support and its applications

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Abstract

In this paper we give representation results about the calculation of the Choquet integral of a monotone function on the nonnegative real line. Next, we represent the Choquet integral of nonmonotone functions, by construction of monotone functions from nonmonotone ones, by using the increasing and decreasing rearrangement of a nonmonotone function. Finally, this paper is completed with some applications of these results to the continuous agregation operator OWA, and to the representation of risk measures by Choquet integral.

Keywords: Choquet integral, distorted Lebesgue measure, risk measure, OWA operator.

1 Introduction

The notion of measure is a very important concept in mathematics, particularly for the theory of integrals. These measures are based on the property of additivity. This property has been abandoned in many areas such as in decision theory and in the theory of cooperative games. It becomes essential to define nonadditive measures, which are usually called capacities [2] or fuzzy measures [17]. A fundamental concept that uses these nonadditive measures is the Choquet integral [2], defined as an integral with respect to a capacity.

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The Choquet integral is known as a nonadditive integral of a function with respect to a capacity (or nonadditive measure, or fuzzy measure). It was characterized mathematically by Schmeidler [15], and then by Murofushi and Sugeno [8] using the concept of the capacity introduced by Choquet. Later it was used in utility theory [16], leading to the so-called Choquet Expected Utility.

So far, many studies have focused on the theory and the applications of the Choquet integral defined on a discrete set (Faigle and Grabisch [5], Grabisch and Labreuche [7]). In the discrete case, the Choquet integral of a function with respect to a capacity is easy to calculate. However, this is not the case for the Choquet integrals of functions on a continuous support. Recent developments that have been conducted on the Choquet integral of real functions [18, 19] appear to open up new horizons.

This paper is in the continuation of the seminal work of Sugeno [18, 19] and results and applications already established by Narukawa and Torra [10, 12]. In particular, we provide on the theoretical side methods for the calculation of the Choquet integral for nonmonotone functions, first by providing an analytic calculation for functions with a single maximum or minimum, and second by providing a general method based on the increasing or decreasing rearrangement of a function. In the second part of the paper, we give some possible applications of these new methods, e.g., for computing the continuous version of OWA operators, and for computing distortion risk measures

This paper is organized as follows. Section 2 is devoted to the concepts of measures and capacities, and to the presentation of the Choquet integral, and essentially focuses on definitions. In Section 3, we present the results obtained by Sugeno [19] for calculating the Choquet integral of a monotone function on the nonnegative real line with respect to a capacity, in particular with respect to distorted Lebesgue measure. Then we represent the Choquet integral of nonmonotone functions, exploring analytical calculation methods with examples, and by using the increasing and decreasing rearrangement of a nonmonotone function to turn it into a monotone function. In section 4, we focus on the application of the results obtained from the previous sections above, on the continuous aggregation operator OWA, and we make some links between the Choquet integral in the context of a distorted probability, and concepts used in finance such as the notions of risk measure. Finally, we end this paper with some concluding remarks.

2 Preliminaries

In this section, we present some basic definitions and properties of measures and Choquet integral.

2.1 Additive and nonadditive measures

We recall some definitions and properties of additive measures and capacities.

Let Ω be a set, and let $\mathcal{A}$ a collection of subsets of Ω. $\mathcal{A}$ is a *σ-algebra* if it satisfies the following conditions:
(1) $\Omega \in \mathcal{A}$,
(2) $A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}$,
(3) $\forall n \in \mathbb{N}$, $A_n \in \mathcal{A} \Rightarrow \bigcup_n A_n \in \mathcal{A}$.

A pair $(\Omega, \mathcal{A})$ is called a *measurable space.*

Let $(\Omega, \mathcal{A})$ be a measurable space. A set function μ: $\mathcal{A} \to \mathbb{R}^+$ is a *σ-additive measure* if it satisfies the following conditions:
(1) $\mu(\emptyset) = 0$,
(2) $\mu(\bigcup_{n=1}^{+\infty} A_i) = \sum_{n=1}^{+\infty} \mu(A_i)$ for every countable sequence $(A_n)_{n\in\mathbb{N}}$ of $\mathcal{A}$ of pairwise disjoint sets ($A_i \cap A_j = \emptyset$ for all $i \neq j$).

The triplet $(\Omega, \mathcal{A}, \mu)$ is called a *measure space.*

A *probability measure* P on $(\Omega, \mathcal{A})$ is an additive measure such that $\mu(\Omega) = 1$. The triplet $(\Omega, \mathcal{A}, P)$ is called a probability space.

Let $\mathcal{B}$ be the smallest σ-algebra including all the closed intervals of $\mathbb{R}$. There is a measure λ on $(\mathbb{R}, \mathcal{B})$ such that: $\lambda([a, b]) = b - a$ for every interval $[a, b]$, with $-\infty < a \leq b < \infty$. This measure is called the *Lebesgue measure.*

Consider two measurable spaces $(\Omega_1, \mathcal{A}_1)$ and $(\Omega_2, \mathcal{A}_2)$. A function $f : \Omega_1 \to \Omega_2$ is *measurable* if $\forall E \in \mathcal{A}_2$, $f^{-1}(E) \in \mathcal{A}_1$.

Let $(\Omega, \mathcal{A}, P)$ be a probability space. A measurable function f from $(\Omega, \mathcal{A})$ to $(\mathbb{R}, \mathcal{B})$ is called a *random variable.*

We call a *distribution function* of a random variable X the function $F : \mathbb{R} \to [0, 1]$ given by:

$$\forall x \in \mathbb{R}, F(x) = P(X \leq x).$$

The notion of capacity was introduced by Choquet [2] in his theory of capacities. A similar concept was proposed by Sugeno [17] under the name "fuzzy measure", and by Denneberg [3] under the name "nonadditive measure".

Definition 1. Let $(\Omega, \mathcal{A})$ be a mesurable space. A set function μ: $\mathcal{A} \to [0, 1]$ is called a *capacity* [2] or *fuzzy measure* [17] if it satisfies the following conditions:
(1) $\mu(\emptyset) = 0$,
(2) $\mu(A) \leq \mu(B)$, if $A \subseteq B$, $A, B \in \mathcal{A}$.

For any capacity μ, the *dual capacity* $\overline{\mu}$ is defined by $\overline{\mu}(A) = 1 - \mu(A^c)$ for any $A \in \mathcal{A}$. The capacity μ is normalized if in addition $\mu(\Omega) = 1$.

Definition 2. Let μ be a capacity on $(\Omega, \mathcal{A})$. μ is called *concave* or *submodular*, if $\mu(A) + \mu(B) \geq \mu(A \cup B) + \mu(A \cap B)$, for all $A, B \in \mathcal{A}$.
A capacity μ is called *convex* or *supermodular*, if it satisfies the previous property with reverse inequality.

2.2 Distortion measures

We call a *distortion function* every nondecreasing function $m : [0,1] \to [0,1]$ with $m(0) = 0$ et $m(1) = 1$.

Let P be a probability measure on $(\Omega, \mathcal{A})$ and let m be a distortion function. The set function $m \circ P$ defined by $m \circ P(A) = m(P(A)), \forall A \in \mathcal{A}$ is called a *distorted probability.*

Definition 3. [19] Consider a Lebesgue measure λ on $(\mathbb{R}, \mathcal{B})$ and $m : \mathbb{R}_+ \to \mathbb{R}_+$ a differentiable nondecreasing function such that $m(0) = 0$. Then $\mu_m = m \circ \lambda$ is a *distorted Lebesgue measure.* We have $\mu_m([a,b]) = m(b-a)$.

2.3 Choquet integral

Definition 4. [2, 8] Let μ be a capacity on $(\Omega, \mathcal{A})$, and let $f : \Omega \to \mathbb{R}^+$ be a measurable function. The Choquet integral of f with respect to μ is defined by:

$$(C)\int f d\mu = \int_0^{+\infty} \mu_f(r) dr,$$

where $\mu_f(r) = \mu(\{\tau | f(\tau) \geq r\})$

Let $A \subset \Omega$. The Choquet integral of f with respect to μ on A is defined by [18]:

$$(C)\int_A f d\mu = \int_0^{\infty} \mu(\{\tau | f(\tau) \geq r\} \cap A) dr.$$

We give some well-known properties of the Choquet integral (see, e.g., Denneberg [3])

Proposition 1. Let μ be a capacity on $(\Omega, \mathcal{A})$. Let f and g be two measurable functions on $(\Omega, \mathcal{A})$. We have the following properties:

(1) if $f \leq g$ then $(C)\int f d\mu \leq (C)\int g d\mu$

(2) $(C)\int \lambda f d\mu = \lambda (C)\int f d\mu$, for every $\lambda \in \mathbb{R}_+$

(3) $(C)\int (f + c) d\mu = (C)\int f d\mu + c\mu(\Omega)$, for every constant $c \in \mathbb{R}$

(4) $(C)\int -f d\mu = -(C)\int f d\overline{\mu}$

Proposition 2. Let μ be a capacity on $(\Omega, \mathcal{A})$. Let f and g be two measurable functions on $(\Omega, \mathcal{A})$.
• If μ is supermodular, then the Choquet integral with respect to μ is superadditive:

$$(C)\int (f+g)d\mu \geq (C)\int f d\mu + (C)\int g d\mu.$$

• If μ is submodular, then the Choquet integral with respect to μ is subadditive:

$$(C)\int (f+g)d\mu \leq (C)\int f d\mu + (C)\int g d\mu.$$

Proposition 3. [10] Let μ be a submodular capacity on $(\Omega, \mathcal{A})$. Let f and g be two measurable functions on $(\Omega, \mathcal{A})$. The Choquet integral satisfies the following inequality:

$$\left[(C)\int (f+g)^2 d\mu\right]^{\frac{1}{2}} \leq \left[(C)\int f^2 d\mu\right]^{\frac{1}{2}} + \left[(C)\int g^2 d\mu\right]^{\frac{1}{2}}$$

3 Choquet integral calculus methods

In this section, we introduce methods for calculating the continuous Choquet integral with respect to distorted Lebesgue measures on the nonnegative real line. The calculus of continuous Choquet integrals was recently studied by Sugeno [18], [19] for nonnegative monotonic functions, and mainly nondecreasing functions. Based on the results in [18], [19], we will start by calculating the Choquet integral for nonnegative monotonic functions with respect to distorted Lebesgue measures. Next, we construct monotonic functions from nonmonotone ones for calculating the Choquet integral of nonmonotonic functions.

Let $\mathcal{F} = \{f \mid f : \mathbb{R}^+ \to \mathbb{R}^+, f\text{: measurable, derivable, monotone }\}$ be the class of measurable, differentiable, nonnegative and monotone functions.
We denote by $\mathcal{F}^+$ the nondecreasing functions of $\mathcal{F}$, and by $\mathcal{F}^-$ the nonincreasing functions of $\mathcal{F}$.

In [18], [19], Sugeno used the Laplace transform for establishing the basis of Choquet integral calculus of nondecreasing functions.

Let f be a function defined for all real numbers $t \geq 0$. The *Laplace transform* of f is the function $F(s) = \mathcal{L}[f(t)]$ defined by

$$F(s) = \int_0^\infty f(t)e^{-st}dt,$$

for those $s \in \mathbb{C}$ for which the integral is defined.

The inverse Laplace transform is given by the following complex integral:

$$f(t) = \mathcal{L}^{-1}[F(s)] = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} F(s)e^{st}ds,$$

where γ is a real number so that the contour path of integration is in the region of convergence of $F(s)$.

Without any assumption on f, the integral $\int_0^\infty f(t)e^{-st}dt$ does not necessarily exist. Sufficient condition for the existence of Laplace transform are:
(i) f is a piecewise continuous function.
(ii) $|f(t)| \leq Me^{\alpha t}$, $M > 0$ and $\alpha \in \mathbb{R}$.

We list properties of the Laplace transform which will be useful in the sequel:
(i) $\mathcal{L}\left[\int_0^t f dt\right] = F(s)/s$,
(ii) $\mathcal{L}\left[df/dt\right] = sF(s) - f(0)$,
(iii) $\mathcal{L}(f * g) = F(s)G(s)$, where $*$ is the convolution product.

3.1 Choquet integral of monotonic functions

Let μ be a capacity, and consider that $\mu([\tau, t])$ is differentiable with respect to τ on $[0, t]$, for every $t > 0$.

Theorem 1. [18] Let $g \in \mathcal{F}^+$, then the Choquet integral of g with respect to μ on $[0, t]$ is represented as:

$$(C)\int_{[0,t]} g d\mu = -\int_0^t \mu'([\tau, t])g(\tau)d\tau$$

In particular, if $\mu = \mu_m$, we have:

$$(C)\int_{[0,t]} g d\mu_m = \int_0^t m'(t - \tau)g(\tau)d\tau$$

Remark 1. $(C)\int_{[0,t]} g d\mu_m = m' * g(t)$, where $*$ is the convolution product.

The next proposition shows how to calculate the Choquet integral by using the Laplace transformation.

Theorem 2. [18] Let $g \in \mathcal{F}^+$, and let μ_m be a distorted Lebesgue measure. The Choquet integral of g with respect to μ_m on $[0, t]$ is given by:

$$(C)\int_{[0,t]} g d\mu_m = \mathcal{L}^{-1}[sM(s)G(s)]$$

where $M(s)$ is the Laplace transform of m, $G(s)$ is the Laplace transform of g, and $\mathcal{L}^{-1}$ is the inverse Laplace transform.

Example 1. Let $m(t) = t$, and $g(t) = e^t - 1$, then:

$$(C)\int_{[0,t]} gd\mu_m(\tau) = (C)\int_{[0,t]} (e^\tau - 1)d\mu_m(\tau) = \mathcal{L}^{-1}[sM(s)G(s)]$$

with $M(s) = \mathcal{L}[m(t)] = \dfrac{1}{s^2}$, et $G(s) = \mathcal{L}[g(t)] = \dfrac{1}{s-1} - \dfrac{1}{s}$.
Then $sM(s)G(s) = \dfrac{1}{s(s-1)} - \dfrac{1}{s^2}$, therefore:

$$(C)\int_{[0,t]} gd\mu_m(\tau) = e^t - t - 1.$$

Theorem 3. [18] Let μ_m be a distorted Lebesgue measure, let f be a nondecreasing and continuous function with $f(0) = 0$, then the solution of the Choquet integral equation $f(t) = (C)\int_{[0,t]} g(\tau)d\mu_m(\tau)$ is given by :

$$g(t) = \mathcal{L}^{-1}\left[\frac{F(s)}{sM(s)}\right],$$

with $g \in \mathcal{F}^+$.

Theorem 4. [18] Let $g \in \mathcal{F}^-$, then the Choquet integral of g with respect to μ on $[0,t]$ is represented by:

$$(C)\int_{[0,t]} gd\mu = \int_0^t \mu'([0,\tau])g(\tau)d\tau$$

In particular, if $\mu = \mu_m$, we have:

$$(C)\int_{[0,t]} gd\mu_m = \int_0^t m'(\tau)g(\tau)d\tau$$

3.2 Choquet integral of nonmonotonic functions

In the previous section, we have shown methods for calculating the continuous Choquet integral of monotonic functions on the nonnegative real line. In this section, we will use the nondecreasing and nonincreasing rearrangements of a nonmonotonic function, for the transformation into a monotonic function. This issue was addressed by Ralescu and Sugeno [14], Ralescu [13], and recently studied by Sugeno [19] and Narukawa et al. [12].

Let $\mathcal{G}$ be the class of nonnegative and continuous functions on $[0,t]$, for some fixed $t \in \mathbb{R}^+$: $\mathcal{G} = \{g \mid g : [0,t] \to \mathbb{R}^+, g \text{ continuous}\}$.

3.2.1 Analytic calculation

In this section, we provide an explicit expression of the Choquet integral for functions with one maximum or minimum, that is, which are increasing then decreasing on an interval (or the converse).

Let $g \in \mathcal{G}$. We calculate the Choquet integral of g on $[0, t]$, with $\tau_m \in [0, t]$ such that $\overline{g} = g(\tau_m) = \max_{0 \leq \tau \leq t} g(\tau)$ (or $\underline{g} = g(\tau_m) = \min_{0 \leq \tau \leq t} g(\tau)$).

We assume first that function g is nondecreasing on $[0, \tau_m]$, and nonincreasing on $[\tau_m, t]$, with $g(\tau_m) = \overline{g}$ (see Fig. 1).

Let $g_1(\tau) = g(\tau)$ on $[0, \tau_m]$, and $g_2(\tau) = g(\tau)$ on $[\tau_m, t]$. If $g(t) \geq g(0)$ (respectively, $g(t) \leq g(0)$), for every $r \in [g(t), \overline{g}]$ (respectively, $r \in [g(0), \overline{g}]$), there exists a unique pair $(\underline{\tau_r}, \overline{\tau_r})$, such that: $\underline{\tau_r} = g_1^{-1}(r)$, and $\overline{\tau_r} = g_2^{-1}(r)$ (see Fig. 1).

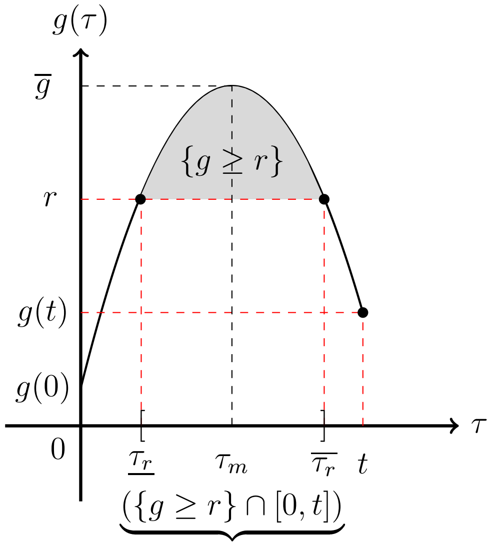

Figure 1: case of a nonmonotonic function

• If $g(t) \geq g(0)$, the Choquet integral of g with respect to measure μ is:

$$\begin{aligned}(C)\int_{[0,t]} g d\mu &= \int_0^\infty \mu(\{\tau | g(\tau) \geq r\} \cap [0,t]) dr \\ &= \int_0^{g(0)} \mu([0,t]) dr + \int_{g(0)}^{g(t)} \mu([g_1^{-1}(r), t]) dr + \int_{g(t)}^{\overline{g}} \mu([g_1^{-1}(r), g_2^{-1}(r)]) dr \\ &= \mu([0,t]) g(0) + \int_{g(0)}^{g(t)} \mu([g_1^{-1}(r), t]) dr + \int_{g(t)}^{\overline{g}} \mu([g_1^{-1}(r), g_2^{-1}(r)]) dr\end{aligned}$$

In particular, if $\mu = \mu_m$, we have:

$$(C)\int_{[0,t]} g d\mu_m = m(t) g(0) + \int_{g(0)}^{g(t)} m(t - g_1^{-1}(r)) dr + \int_{g(t)}^{\overline{g}} m(g_2^{-1}(r) - g_1^{-1}(r)) dr \qquad (1)$$

• If $g(t) \leq g(0)$, the Choquet integral of g with respect to measure μ is represented by:

$$(C)\int_{[0,t]} g d\mu = \mu([0,t]) g(t) + \int_{g(t)}^{g(0)} \mu([0, g_2^{-1}(r)]) dr + \int_{g(0)}^{\overline{g}} \mu([g_1^{-1}(r), g_2^{-1}(r)]) dr$$

In particular, if $\mu = \mu_m$, we have:

$$(C)\int_{[0,t]} g d\mu_m = m(t) g(t) + \int_{g(t)}^{g(0)} m(g_2^{-1}(r)) dr + \int_{g(0)}^{\overline{g}} m(g_2^{-1}(r) - g_1^{-1}(r)) dr \qquad (2)$$

We assume now that g is nonincreasing on $[0, \tau_m]$, and nondecreasing on $[\tau_m, t]$, with $\underline{g} = g(\tau_m) = min_{0 \leq \tau \leq t} g(\tau)$.

Proceeding similary, we find:

• If $g(t) \geq g(0)$, the Choquet integral of g with respect with respect to measure μ is:

$$(C)\int_{[0,t]} g d\mu = \mu([0,t]) \underline{g} + \int_{\underline{g}}^{g(0)} \mu([0, g_1^{-1}(r)] \cup [g_2^{-1}(r), t]) dr + \int_{g(0)}^{g(t)} \mu([g_2^{-1}(r), t]) dr$$

In particular, if $\mu = \mu_m$, we have:

$$(C)\int_{[0,t]} g d\mu_m = m(t) \underline{g} + \int_{\underline{g}}^{g(0)} m(t + g_1^{-1}(r) - g_2^{-1}(r)] dr + \int_{g(0)}^{g(t)} m(t - g_2^{-1}(r)] dr \qquad (3)$$

• If $g(t) \leq g(0)$, the Choquet integral of g with respect with respect to measure μ is represented by:

$$(C)\int_{[0,t]} g d\mu = \mu([0,t]) \underline{g} + \int_{\underline{g}}^{g(t)} \mu([0, g_1^{-1}(r)] \cup [g_2^{-1}(r), t]) dr + \int_{g(t)}^{g(0)} \mu([0, g_1^{-1}(r)]) dr$$

In particular, if $\mu = \mu_m$, we have:

$$(C)\int_{[0,t]} g d\mu = m(t)\underline{g} + \int_{\underline{g}}^{g(t)} m(t + g_1^{-1}(r) - g_2^{-1}(r))dr + \int_{g(t)}^{g(0)} m(g_1^{-1}(r))dr \quad (4)$$

Example 2. Let $m(r) = r^2$, and $g(r) = r^2 - 2r + 2, \forall r \in [0,2]$.
We have $g_1^{-1}(r) = 1 + \sqrt{r-1}, g_2^{-1}(r) = 1 - \sqrt{r-1}$, and for all $t \in [1,2]$, $g(t) \le 0$. From (4) we get

$$\begin{aligned}(C)\int_{[0,t]} g d\mu_m &= t^2 + \int_1^{t^2-2t+2} (t - 2\sqrt{r-1})^2 dr + \int_{t^2-2t+2}^{2} (1 - \sqrt{r-1})^2 dr \\ &= \frac{4}{3}t^4 - \frac{28}{3}t^3 + 14t^2 - 8t + 2.\end{aligned}$$

3.2.2 Choquet integral of nonmonotonic function and rearrangement

The principle of the theory of monotone equimeasurable rearrangements of functions [6] is that, for every function f defined on the real line, there exists an nondecreasing (respectively, nonincreasing) function that has the same distribution function that the function f with respect to Lebesgue measure. This function is called the nondecreasing (respectively, nonincreasing) rearrangement of the function f.

Based on the results obtained in [19] and [12], we shall further explore the calculation of the Choquet integral of a nonmonotonic function on the nonnegative real line.

Let $g \in \mathcal{G}$ be a continuous and nonnegative function on $[0,t]$, with $\tau_m \in [0,t]$ such that g is nondecreasing (respectively, nonincreasing) on $[0,\tau_m]$, and nonincreasing (respectively, nondecreasing) on $[\tau_m, t]$, and that $g(t) \ge g(0)$, with $\overline{g} = g(\tau_m) = \max_{0\le\tau\le t} g(\tau)$ (respectively, $\underline{g} = g(\tau_m) = \min_{0\le\tau\le t} g(\tau)$).

Let $\lambda_g : [g(0), \overline{g}] \to [0,t]$ (respectively, $\lambda_g : [\underline{g}, g(t)] \to [0,t]$) be a function defined by:

$$\lambda_g(r) = \lambda(\{\tau | g(\tau) \ge r\})$$

The function λ_g is continuous, and nonincreasing on $[g(0), \overline{g}]$ (respectively, $[\underline{g}, g(t)]$).

We define a function $g^* : [0,t] \to [g(0), \overline{g}]$ (respectively, $g^* : [0,t] \to [\underline{g}, g(t)]$) by:

$$g^*(\tau) = \lambda_g^{-1}(t - \tau), \quad (5)$$

where g^* is called the *nondecreasing rearrangement* of g on $[0,t]$. The function g^* is nondecreasing and continuous on $[0,t]$.

Remark 2. For all $\tau \in [0, \tau_t]$ (respectively, $\tau \in [\tau_0, t]$), where $g(\tau_t) = g(t)$ (respectively, $g(\tau_0) = g(0)$), $g^*(\tau) = g(\tau)$.

If $g(t) \leq g(0)$, we define a function $g^* : [0, t] \to [g(t), \overline{g}]$ (respectively, $g^* : [0, t] \to [g(0), \underline{g}]$) by:

$$g^*(\tau) = \lambda_g^{-1}(\tau) \tag{6}$$

The function g^* is the nonincreasing rearrangement of g on $[0, t]$, it is continuous and nonincreasing on $[0, t]$.

Remark 3. For all $\tau \in [\tau_0, t]$ (respectively, $\tau \in [\tau_t, t]$), where $g(\tau_0) = g(0)$ (respectively, $g(\tau_t) = g(t)$), $g^*(\tau) = g(\tau)$.

Proposition 4. Let $g \in \mathcal{G}$, and let g^* be a rearrangement of g on $[0, t]$, then:

$$\lambda(\{\tau | g^*(\tau) \geq r\}) = \lambda(\{\tau | g(\tau) \geq r\})$$

Corollary 1. Let $g \in \mathcal{G}$, and let g^* be a rearrangement of g on $[0, t]$, then the Choquet integral of g with respect to measure μ_m on $[0, t]$ can be written as:

$$(C) \int_{[0,t]} g d\mu_m = (C) \int_{[0,t]} g^* d\mu_m$$

Hence by Theorems 1 and 4 we find:

- If $g(t) \geq g(0)$, the Choquet integral of g with respect to measure μ_m on $[0, t]$ becomes:

$$(C) \int_{[0,t]} g d\mu_m = \int_0^t m'(t - \tau) g^*(\tau) d\tau,$$

with g^* is given by (5).

- If $g(t) \leq g(0)$, the Choquet integral of g with respect to measure μ_m on $[0, t]$ becomes:

$$(C) \int_{[0,t]} g d\mu_m = \int_0^t m'(\tau) g^*(\tau) d\tau,$$

with g^* is given by (6).

Example 3. Let $m(\tau) = \tau^2$, and $g(\tau) = 4 - (\tau - 2)^2$. The function g is nonnegative, continuous, and nonmonotonic on $[0, t]$, where $t \in]2, 4]$. A function g is nondecreasing on $[0, 2]$, and nonincreasing on $[2, t]$. We have: $g(t) \geq g(0)$, then the nondecreasing rearrangement g^* of the function g on $[0, t]$ is defined by:

$$g^*(\tau) = \begin{cases} g(\tau), & \text{if } 0 \leq \tau \leq 4 - t \\ 4 - (t - \tau)^2/4, & \text{if } 4 - t \leq \tau \leq t \end{cases}$$

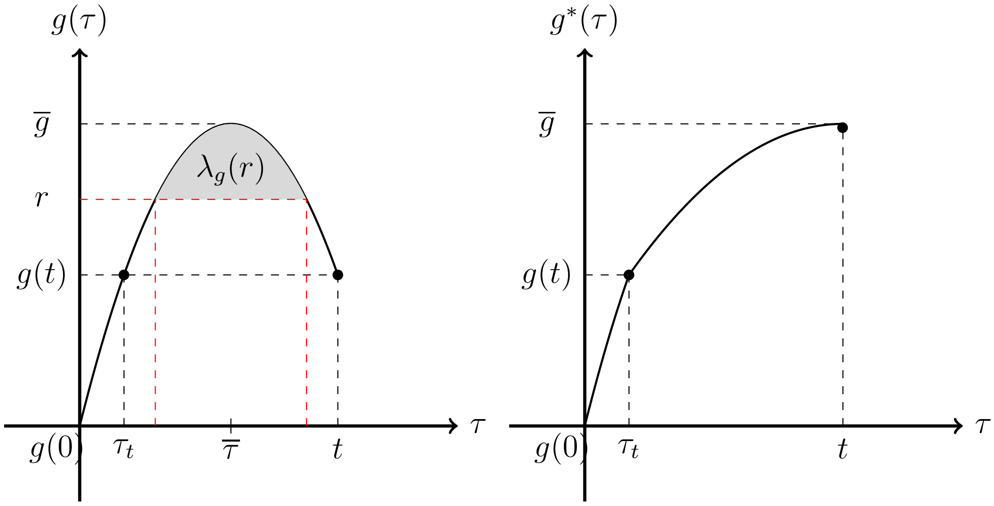

Figure 2: The function g and its nondecreasing rearrangement g^* on $[0, t]$

The Choquet integral of g with respect to measure μ_m on $[0, t]$ is represented by:

$$
\begin{aligned}
(C)\int_{[0,t]} g d\mu_m &= \int_0^t m'(t-\tau)g^*(\tau)d\tau \\
&= 2\int_0^t (\tau - t)g^*(\tau)d\tau \\
&= 2\int_0^{4-t} (\tau - t)g(\tau)d\tau + 2\int_{4-t}^t (\tau - t)(4-(t-\tau)^2/4)d\tau \\
&= 2(t^4 - 8t^3 + 16t^2 - 16) - 1/6(t-4)^2(7t^2 - 16)
\end{aligned}
$$

4 Some Applications

In this section, we review some applications of the Choquet integral on the nonnegative real line.

4.1 OWA operator on the real line

The ordered weighted averaging (OWA) operator was introduced by Yager [24].

4.1.1 OWA operator

In this section, we consider that $\Omega = \{1, \ldots, n\}$, and $\mathcal{A} = 2^\Omega$.

Definition 5. [24] Let $w = (w_1, \ldots, w_n)$, such that $w_i \in [0,1]$, and $\sum_{i=1}^{n} w_i = 1$. The *ordered weighted averaging operator (OWA)* with respect to w is defined for any $a = (a_1, \ldots, a_n) \in \mathbb{R}_+^n$ by:

$$\mathrm{OWA}_w(a) = \sum_{i=1}^{n} w_i a_{\sigma(i)},$$

where σ is a permutation on $\{1, \ldots, n\}$ such that $a_{\sigma(1)} \geqslant \ldots \geqslant a_{\sigma(n)}$.

A capacity μ is said to be *symmetric* if $\mu(A) = \mu(B)$ whenever $|A| = |B|$, $\forall A, B \in \mathcal{A}$.

Proposition 5. [9] For every OWA_w operator, there existe a symmetric capacity μ given by $\mu(\{1\}) = w_1$, and $\mu(\{1, \ldots, i\}) = w_i, i = 1, \ldots, n$, such that:

$$\mathrm{OWA}_w(a) = (C) \int a d\mu,$$

for any $a \in \mathbb{R}_+^n$.

4.1.2 Continuous OWA operator

In this section, we will use the Choquet integral for defining the continuous OWA operator (COWA) [11].

Consider a Lebesgue measure λ, and μ a capacity on $([0,1], \mathcal{B})$. μ is said to be *symmetric* if $\lambda(A) = \lambda(B)$ implies $\mu(A) = \mu(B)$.

Definition 6. [11] Let μ be a symmetric capacity, and $f : \mathbb{R}^+ \to \mathbb{R}^+$ be a measurable function. The *continuous OWA* operator is defined by:

$$\mathrm{COWA}_\mu(f) = (C) \int f d\mu.$$

Let $m : [0,1] \to [0,1]$ be a distortion function. Then a distorted Lebesgue measure μ_m is a symmetric capacity. It follows that we can consider the Choquet integral with respect to μ_m as a COWA operator.

Corollary 2. Let $f : [0,t] \to \mathbb{R}^+$ be a differentiable and nondecreasing function. Then we have:

$$\mathrm{COWA}_{\mu_m}(f) = \mathcal{L}^{-1}[sM(s)F(s)],$$

with $F(s) = \mathcal{L}[f(t)]$, and $M(s) = \mathcal{L}[m(t)]$

Corollary 3. Let $f : [0,t] \to \mathbb{R}^+$ be a differentiable and nonincreasing function. Then we have:

$$\mathrm{COWA}_{\mu_m}(f) = \int_0^t m'(t) f(\tau) d\tau.$$

Example 4. [12] Let $f:[0,1]\to\mathbb{R}^+$ be a differentiable and nondecreasing function. We define the sequence of functions f_k by: $f_1(t)=\int_0^t f(\tau)d\tau$, $f_{k+1}(t)=\int_0^t f_k(\tau)d\tau$, for $t\in[0,1]$, $k\in\{1,2,\ldots,n\}$. Let us show that

$$\mathrm{COWA}_{\mu_m}(f)=n!f_n(t),$$

where $m(t)=t^n$.

f being nondecreasing, the continuous OWA operator is given by:

$$(C)\int_{[0,t]} f d\mu_m=\mathcal{L}^{-1}[sM(s)F(s)],$$

where $F(s)=\mathcal{L}[f(t)]$, et $M(s)=\mathcal{L}[m(t)]=\frac{n!}{s^{n+1}}$. Then

$$(C)\int_{[0,t]} f d\mu_m=n!\mathcal{L}^{-1}\left[\frac{F(s)}{s^n}\right].$$

Note that $\frac{F(s)}{s}=\mathcal{L}\left[\int_0^t f(\tau)d\tau\right]=\mathcal{L}[f_1(t)]=F_1(s)$, hence $\frac{F(s)}{s^n}=\mathcal{L}[f_n(t)]$, therefore:

$$\begin{aligned}\mathrm{COWA}_{\mu_m}(f)&=n!\mathcal{L}^{-1}[\mathcal{L}[f_n(t)]]\\&=n!f_n(t).\end{aligned}$$

To complete the example, let us take the function $f(x)=e^x-1$ of Example 1, with $m(t)=t^n$. We have $f_n(x)=e^x-\frac{t^n}{n!}-\frac{t^{n-1}}{(n-1)!}\ldots\frac{t^2}{2}-t-1$, then

$$\mathrm{COWA}_{\mu_m}(f)=n!(e^t-t-1)-t^n-nt^{n-1}\ldots(n(n-1)\ldots3)t^2, x\in[0,1].$$

4.2 Risk Measures and Choquet integral

Risk management is a subject of concern in finance and insurance. One of the most significant problems in managing risk is the determination of a measure that can take into account the different characteristics of the distribution of losses. For this, there are tools to quantify and predict the risk, which are called *risk measures*. They allow risk assessment and to compare differents risks between them. To manage these risks, several risk measurements have been proposed, each having its advantages and disadvantages.

Artzner et al. [1] sought to characterize what would determine whether a risk measurement is "effective". For this they introduced the notion of a coherent risk measurement.

4.2.1 Risk Measures

Let $(\Omega, \mathcal{A})$ be a measurable space. Denote by $\mathcal{X}$ the space of the measurable functions X such that $\|X\|_\infty = \sup_{w\in\Omega} |X(w)|$ is bounded:

$$\mathcal{X} := \{X : \Omega \to \mathbb{R}, \mathcal{A}\text{-measurable}, \ \|X\|_\infty < \infty\}.$$

Definition 7. [4] A mapping $\rho : \mathcal{X} \to \mathbb{R}$ is called *coherent risk measure* if it satisfies the following conditions for all $X, Y \in \mathcal{X}$:
(1) *monotonicity*: $X \le Y \Rightarrow \rho(X) \le \rho(Y)$
(2) *translation invariance*: $\rho(X + c) = \rho(X) + c, \forall c \in \mathbb{R}$.
(3) *positive homogeneity*: $\rho(\lambda X) = \lambda\rho(X)$, $\forall\lambda \ge 0$
(4) *subadditivity*: $\rho(X + Y) \le \rho(X) + \rho(Y)$

We adopt the definition that is used in the case where the random variables are interpreted as losses.

The risk measure definition given by Artzner and al. [1] coincides up to the sign with the definition referred to above. However, in the definition of Artzner et al. [1], the sign " $+$ " of property (2) is changed to the sign " $-$ " , and property (1) becomes: $X \le Y \Rightarrow \rho(X) \ge \rho(Y)$.

Definition 8. [4] A mapping $\rho : X \in \mathcal{X} \to \mathbb{R}$ is called *monetary risk measure* if it satisfies the axioms of monotonicity and translation invariance.

Definition 9. [4] A monetary risk measure ρ is called a *convex risk measure* if it satisfies the following property, $\forall X, Y \in \mathcal{X}, \forall\lambda \in [0, 1]$:

$$\rho(\lambda X + (1 - \lambda)Y) \le \lambda\rho(X) + (1 - \lambda)\rho(Y).$$

Let $(\Omega, \mathcal{A})$ be a measurable space, and let μ be a capacity. We define the function $\rho_\mu : \mathcal{X} \to \mathbb{R}$ by:

$$\rho_\mu(X) = (C) \int X d\mu, X \in \mathcal{X}.$$

In the case where the random variables are interpreted as gains, we define the function ρ_μ as follows:

$$\rho_\mu(X) = (C) \int -X d\mu, X \in \mathcal{X}.$$

By the properties of Choquet integral, we know that the function ρ_μ is a positive homogeneous monetary risk measure, and if μ is a submodular capacity, the function ρ_μ is a convex risk measure.

4.2.2 Distortion risk measures

Distortion risk measures originated in Yaari's paper [23], they consist in measuring the risks from the distortion of probabilities. Wang et al. [21] developed the concept of distortion risk measure by calculating expected losses from a nonlinear transformation of the cumulative probability of the risk factor. Distortion risk measures allow the production of better risk measures from the distortion of the originals.

Definition 10. [21] Let $g : [0, 1] \to [0, 1]$ be a distortion function, and $X \in \mathcal{X}$ a random variable. The *distortion risk measure* associated with distortion function g for X is defined by:

$$\rho_g(X) = \int_0^{+\infty} g(G_X(x))dx + \int_{-\infty}^{0} [g(G_X(x)) - 1]dx,$$

where $G_X = P(X > x) = 1 - P(X \leq x) = 1 - F_X(x)$, and $F_X(x)$ is the distribution function of X.

Several popular risk measures belong to the family of distortion risk measures. For example, the value-at-risk (VaR), the tail value-at-risk (TVaR) and the Wang distortion measure.

Remark 4. When the distortion function is concave, the distortion risk measure is also subadditive (Wang and Dhaene [20], Wirch and Hardy [22]).

If X is nonnegative, then the distortion risk measure associated with distortion function g is defined by:

$$\rho_g(X) = \int_0^{+\infty} g(G_X(x))dx.$$

Remark 5. $\rho_g(X)$ is the Choquet integral of X with respect to the distorted probability $\mu_g = g \circ P$.

Remark 6. If the distortion function g is differentiable, and the distribution function F is continuous (and strictly increasing), then the distortion risk measure can be written as follows:

$$\rho_g(X) = \int_0^{1} F_X^{-1}(1 - x)g'(x)dx. \tag{7}$$

Indeed, by the change of variable $u = F_X(x)$, and integration by parts :

$$
\begin{aligned}
\rho_g(X) &= \int_0^{+\infty} g(G_X(x))dx + \int_{-\infty}^{0} [g(G_X(x)) - 1]dx \\
&= \int_0^{+\infty} g(1 - F_X(x))dx + \int_{-\infty}^{0} [g(1 - F_X(x)) - 1]dx \\
&= \int_{F_X(0)}^{1} g(1-u)(F_X^{-1})'(u)du + \int_0^{F_X(0)} [g(1-u) - 1](F_X^{-1})'(u)du \\
&= \int_{F_X(0)}^{1} g'(1-u)F_X^{-1}(u)du + \int_0^{F_X(0)} g'(1-u)F_X^{-1}(u)du \\
&= \int_0^1 F_X^{-1}(1-u)g'(u)du.
\end{aligned}
$$

Proposition 6. Let F_X be a continuous and strictly increasing distribution function of X. If X is nonnegative, then $\rho_g(X) = (C)\displaystyle\int_{[0,1]} F_X^{-1} d\mu_g$, avec $\mu_g = g \circ P$.

Proof. Let X be a nonnegative random variable,then we have:

$$
\begin{aligned}
\rho_g(X) &= \int_0^1 F_X^{-1}(1-u)g'(u)du \\
&= \int_0^1 g'(1-u)F_X^{-1}(u)du \\
&= (C)\int_{[0,1]} F_X^{-1} d\mu_g, \text{by theorem 1.}
\end{aligned}
$$

□

Remark 7. The Value at Risk (VaR) is one of the most popular risk measures due to its simplicity and intuitiveness. However it is known that the distortion function associated to this risk measure is not differentiable. Therefore our results do not apply in this case. One way to obviate this problem may be to consider a differentiable approximation to the discontinuous function that yields the VaR. This approach, which may be very interesting to study, is left for future work.

Proposition 7. For every $X \in \mathcal{X}$, $\rho_g(X) = -\rho_{\overline{g}}(-X)$, with $\overline{g}(x) = 1 - g(1-x)$ the dual function of g.

Proof. For every $X \in \mathcal{X}$, we have:

$$\begin{aligned}
\rho_g(X) &= (C)\int -(-X)d\mu_g \\
&= -(C)\int -X d\overline{\mu_g} \\
&= -(C)\int -X d\mu_{\overline{g}},\ (\overline{\mu_g} = \mu_{\overline{g}}) \\
&= -\rho_{\overline{g}}(-X).
\end{aligned}$$

□

5 Conclusion

The work presented in this paper revolves around two axes. The first axis focused on the methods of calculating the Choquet integral with respect to a Lebesgue distortion measure. The second purpose is to apply the calculation of the Choquet integral.

After introducing basic concepts of Choquet integrals, we presented calculation methods for monotone and nonmonotone functions on the positive real line.

In the case of nonmonotone functions, we used the nondecreasing and nonincreasing rearrangements of a nonmonotone function to turn it into a monotone function, in order to apply the calculation results of the Choquet integral for monotone functions.

In this paper, we used the results of calculation of Choquet integral for calculating the continuous aggregation operator OWA, and we represented the risk measures by Choquet integrals to facilitate verification of the convexity risk measures.

Many aspects of the work remain obviously to deepen. An interesting subject of study would be to try to obtain results about the calculation of the Choquet integral with respect to a general capacity, not limited to distorted Lebesgue measures.

It is also important to find a general representation for the calculation of the Choquet integral of a monotone or nonmonotone function on the real line. Using these calculations in practice would allow addressing many areas.

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