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# Boundary Local Null Controllability of the Kuramoto-Sivashinsky equation

Takéo Takahashi

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**Abstract**

We prove that the Kuramoto-Sivashinsky equation is locally controllable in 1D and in 2D with one boundary control. Our method consists in combining several general results in order to reduce the null-controllability of this nonlinear parabolic equation to the exact controllability of a linear beam or plate system. This improves known results on the controllability of Kuramoto-Sivashinsky equation and gives a general strategy to handle the null-controllability of nonlinear parabolic systems.

**Key words:** Kuramoto-Sivashinsky equation, null controllability

**2010 Mathematics Subject Classification:** 93B05, 35K55, 93C10

## 1 Introduction

In this article, we study the boundary controllability of the Kuramoto-Sivashinsky equation, in space dimension 1 and 2. This equation is due to Kuramoto [12] (phase disturbance in chemical reaction system) and Sivashinsky [19] (model for a laminar flame). Lots of works are devoted to this equation: let us quote for instance [15], [16], [9], [11], etc. for the long time behavior. From a control point of view, this equation have been tackled in several papers: [10], [2], [4], [5], [7], etc. In particular, in [2], the author obtains the boundary null controllability of a linear system associated with the Kuramoto-Sivashinsky equation in 1D. In [4], the authors prove the local exact controllability to the trajectories of the Kuramoto-Sivashinsky equation in 1D by using boundary controls. In that case, they need two controls at the boundary and their method is based upon a Carleman estimate. Here our aim is to use only one control, in the 1D or in the 2D case. More precisely, we consider the control problem

$$\begin{cases} y_t + y_{xxxx} + a y_{xx} + y y_x = 0, \quad x \in (0,1),\ t \in (0,T) \\ y(t,0) = 0, \quad y(t,1) = 0, \quad t \in (0,T), \\ y_x(t,0) = u(t), \quad y_x(t,1) = 0, \quad t \in (0,T), \\ y(0,x) = y_0(x), \quad x \in (0,1). \end{cases} \tag{1.1}$$

In the above system, the state $y$ is controlled by the function $u = u(t)$. One important step to study the controllability of (1.1) consists in studying the controllability of the linear system associated with (1.1). This was done in [2] and the author shows the null-controllability provided that $a \notin \mathcal{N}$, where

$$\mathcal{N} := \left\{ \pi^2(k^2+\ell^2)\ ;\ k,\ell \in \mathbb{N}^*,\ k \neq \ell,\ k \equiv \ell \mod 2 \right\}.$$

Our first main results shows the local null-controllability of (1.1) with the same hypothesis on $a$:

**Theorem 1.1.** *Assume that $a \notin \mathcal{N}$. Then the system (1.1) is locally null-controllable for any $T > 0$: there exists $c_0 > 0$ such that for any $\|y_0\|_{H^{-1}(0,1)} \leqslant c_0$, there exists $u \in L^2(0,T;\mathbb{R})$ and a solution $y \in C([0,T];H^{-1}(0,1))$ of (1.1) with*

$$y(T) = 0.$$

The method of [2] is based on a precise spectral analysis, but does not allow to consider the nonlinear case. Nevertheless, we will use this spectral analysis in our proof, but in order to obtain a first result on an hyperbolic problem associated with (1.1).

**Remark 1.2.** *The space for the initial conditions is different from [2] but in this paper, the author uses a control* $u \in H^1(0,T)$.

**Remark 1.3.** *If* $a \in \mathcal{N}$*, then the linear system is not exactly controllable and thus the method developed here can not be applied. Nevertheless, the nonlinear equation* (1.1) *can be null-controllable even if* $a \in \mathcal{N}$*. One can look at [1] and [3] for a similar problem in the case of the Korteweg-de Vries equation.*

Our approach for the Kuramoto-Sivashinsky equation in 1D can be extended to the Kuramoto-Sivashinsky equation in 2D, but the boundary conditions are different from the 1D case:

$$\begin{cases} y_t + \Delta^4 y + a\Delta y + \dfrac{1}{2}\left|\nabla y\right|^2 = 0, \quad x \in \Omega,\ t \in (0,T) \\ y(t) = 0, \quad x \in \partial\Omega,\ t \in (0,T), \\ \Delta y(t) = u 1_\Gamma, \quad x \in \partial\Omega,\ t \in (0,T), \\ y(0,x) = y_0(x), \quad x \in \Omega. \end{cases} \tag{1.2}$$

Our condition on $a$ is more restrictive in that case: let us consider the eigenvalues $(\lambda_k)_{k\geqslant 1}$ of the operator

$$A_0 : \left\{\phi \in H^4(\Omega)\ ;\ \phi = \Delta\phi = 0 \quad \text{on } \partial\Omega\right\} \to L^2(\Omega), \quad \phi \mapsto \Delta^4\phi.$$

We consider that $\lambda_1$ is the smallest eigenvalue. We know that $\lambda_1 > 0$. Our second main result is

**Theorem 1.4.** *Assume that* $a < \sqrt{\lambda_1}$*. Assume also one of the following assertions holds:*

1. $\partial\Omega$ *if of class* $C^5$ *and* $\Gamma$ *satisfies the geometric optics conditions;*
2. $\Omega$ *is a rectangle* $(0,a)\times(0,b)$ *and* $\Gamma$ *contains both a horizontal and a vertical segment of nonzero length.*

*Then the system* (1.2) *is locally null-controllable for any* $T > 0$*: there exists* $c_0 > 0$ *such that for any* $\|y_0\|_{L^2(\Omega)} \leqslant c_0$*, there exists* $u \in L^2(0,T;L^2(\Gamma))$ *and a solution* $y \in C([0,T];L^2(\Omega))$ *of* (1.2) *with*

$$y(T) = 0.$$

**Remark 1.5.** *The conditions on* $\Gamma$ *and on* $a$ *are sufficient conditions for the null-controllability of the linear system associated with* (1.2)*. These conditions may not be necessary: we should expect for instance a condition on* $a$ *as in Theorem 1.1 (not in a countable set) and for* $\Gamma$ *we should expect a condition as for the heat equation (non empty interior). However, to obtain such a result on* $\Gamma$*, one of the only methods is based on some Carleman estimates for a plate type system. Even in 1D, such a method fails to obtain only one control (see for instance [4]). As for Theorem 1.1, one could try to obtain the null-controllability of* (1.2) *even if the corresponding linear system is not controllable.*

The proofs of Theorem 1.1 and Theorem 1.4 are based on several general results described in Section 3. Using a result in [14], we can reduce the study to linear systems associated with (1.1) and (1.2). The corresponding linear systems are parabolic and to prove their null-controllability, we apply a method due to Russell [18], and consider the controllability or the observability of "hyperbolic" systems associated with them. Finally, in the 2D case, the observability of the corresponding plate system is obtained through a perturbation method developed in [6].

## 2 Notation and preliminaries

Let us consider a Hilbert space $\mathcal{X}$ and $A : \mathcal{D}(A) \to \mathcal{X}$ a generator of a strongly continuous semigroup $\left(e^{tA}\right)_{t\geqslant 0}$. Let us also consider a Hilbert space $U$ and $B \in \mathcal{L}(U, \mathcal{D}(A)^*)$ an operator. We consider the following control problem

$$\dot{z} = Az + Bu, \quad z(0) = z_0. \tag{2.1}$$

We fix $\lambda_\# \in \rho(A)$. We can equip $\mathcal{D}(A)$ with the norm $\|z\|_{\mathcal{D}(A)} := \|(\lambda_\# - A)z\|_{\mathcal{X}}$. Its dual $\mathcal{D}(A^*)^*$ with respect to pivot space $\mathcal{X}$ is the completion of $\mathcal{X}$ with respect to the norm $\|z\|_{\mathcal{D}(A^*)^*} := \|(\lambda_\# - A)^{-1}z\|_{\mathcal{X}}$. We keep the notation $A$ and $\left(e^{tA}\right)_{t\geqslant 0}$. for the extensions of the generator to $\mathcal{X}$ and of the semigroup to $\mathcal{D}(A^*)^*$. Assume $T > 0$. For $z_0 \in \mathcal{D}(A^*)^*$ and $u \in L^2(0, T; U)$, we can consider a solution $z$ of (2.1) as a mild solution

$$z(t) = e^{tA}z_0 + \int_0^t e^{(t-s)A}Bu(s)\, ds.$$

We say that $B$ is admissible if for any $z_0 \in \mathcal{X}$ and for any $u \in L^2(0, T; U)$, the above solution satisfies $z \in C([0, T]; \mathcal{X})$. This is the case if for instance $A$ is self-adjoint and positive (parabolic case) and if $B \in \mathcal{L}(U, \mathcal{D}(A^{1/2})^*)$. In the case where $B$ is admissible, we say that the system (2.1) (or the pair $(A, B)$) is null-controllable in time $T$ if for any $z_0 \in \mathcal{X}$, there exists $u \in L^2(0, T; U)$ such that the solution of (2.1) verifies $z(T) = 0$. We also say that the system (2.1) (or the pair $(A, B)$) is exactly controllable in time $T$ if for any $z_0, z_1 \in \mathcal{X}$, there exists $u \in L^2(0, T; U)$ such that $z(T) = z_1$. These two notions are related by duality to observability properties:

$$\dot{\phi} = A^*\phi, \quad \psi = B^*\phi, \quad \phi(0) = \phi_0. \tag{2.2}$$

For $\phi_0 \in \mathcal{D}(A^*)$, $\phi(t) = e^{tA^*}\phi_0$ and $\psi(t) = B^*e^{tA^*}\phi_0$. We say that the system (2.2) (or the pair $(A^*, B^*)$) is final-state observable in time $T$ if there exists $K(T) > 0$ such that

$$\|e^{TA^*}\phi_0\|^2_{\mathcal{X}} \leqslant K(T)^2 \int_0^T \|B^*e^{tA^*}\phi_0\|^2_U\, dt \quad (\phi_0 \in \mathcal{D}(A^*)).$$

We say that the system (2.2) (or the pair $(A^*, B^*)$) is exactly observable in time $T$ if there exists $K(T) > 0$ such that

$$\|\phi_0\|^2_{\mathcal{X}} \leqslant K(T)^2 \int_0^T \|B^*e^{tA^*}\phi_0\|^2_U\, dt \quad (\phi_0 \in \mathcal{D}(A^*)).$$

The equivalence between observability and controllability is recalled in the proposition below, due to Dolecki and Russell [8] (see also Proposition 12.1.2 and Theorem 11.2.1 in [21]).

**Proposition 2.1.** *Assume that $B$ is admissible. Then the pair $(A, B)$ is exactly controllable in time $T$ if and only if the pair $(A^*, B^*)$ is exactly observable in time $T$. The pair $(A, B)$ is null controllable in time $T$ if and only if the pair $(A^*, B^*)$ is final state observable in time $T$. Moreover, if the pair $(A^*, B^*)$ is exactly observable in time $T$, then there exists an operator $L_T \in \mathcal{L}(\mathcal{X}, U)$ such that a control for (2.1) is given by $u = L_T z_0$ and such that $\|L_T\|_{\mathcal{L}(\mathcal{X},U)} \leqslant K(T)$.*

## 3 Some general results

This section is devoted to the presentation of several important general results that are used in the proofs of Theorem 1.1 and Theorem 1.4.

### 3.1 The Russell method

We recall here a method introduced by Russell [18] and that is completely described in [21] (Section 9.2). The idea is that the final state observability of a parabolic system can be obtained by the exact observability of

an hyperbolic system associated with the parabolic system. More precisely, let us consider a Hilbert space $\mathcal{H}$ and a self-adjoint operator $A_0 : \mathcal{D}(A_0) \to \mathcal{H}$, strictly positive. We also assume that $A_0^{-1}$ is compact. Let us also consider a Hilbert space $U$ and $C_0 \in \mathcal{L}(\mathcal{D}(A_0^{1/2}), U)$ an operator. We consider the following observability problems

$$\dot{z} = -A_0 z, \quad y = C_0 z, \quad z(0) = z_0 \tag{3.1}$$

and

$$\ddot{w} = -A_0 w, \quad y = C_0 \dot{w}, \quad w(0) = w_0, \quad \dot{w}(0) = w_1. \tag{3.2}$$

We denote by $\left(e^{-tA_0}\right)_{t\geqslant 0}$ the semigroup generated by $-A_0$. We introduce an orthonormal basis $(\varphi_n)_{n\in\mathbb{N}^*}$ in $\mathcal{H}$ consisting of eigenvectors of $A_0$ and $(\lambda_n)_{n\in\mathbb{N}^*}$ the corresponding eigenvalues. Then we have the following result:

**Theorem 3.1.** *Assume that* (3.2) *is exactly observable in* $\mathcal{D}(A_0^{1/2}) \times \mathcal{H}$ *in time* $T_0 > 0$ *and that*

$$\sum_{n\in\mathbb{N}^*} \frac{1}{\lambda_n^k} < \infty \tag{3.3}$$

*for some* $k \in \mathbb{N}^*$. *Then* (3.1) *is final state observable in* $\mathcal{H}$ *for any time* $T$*:*

$$\|e^{-A_0 T}\varphi_0\|_{\mathcal{H}} \leqslant K(T) \left( \int_0^T \|C_0 e^{-A_0 t}\varphi_0\|_U^2 \, dt \right)^{1/2}, \tag{3.4}$$

*with*

$$K(T) := Ce^{M/T}. \tag{3.5}$$

This theorem is stated in this way in [21] (Theorem 9.2.2) but we give a sketch of the proof to precise the cost $K(T)$ in (3.5).

*Proof.* First, (3.2) is exactly controllable in $\mathcal{D}(A_0^{1/2}) \times \mathcal{H}$ in time $T_0 > 0$ means that the pair $(A, C)$ defined by

$$A = \begin{bmatrix} 0 & I \\ -A_0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & C_0 \end{bmatrix}$$

is exactly observable in $\mathcal{D}(A_0^{1/2}) \times \mathcal{H}$ in time $T_0 > 0$.

We define

$$\mu_k = \operatorname{sgn}(k)\sqrt{\lambda_{|k|}} \quad (k \in \mathbb{Z}^*).$$

Then we use Lemma 9.2.1 in [21] and the exact observability of $(A, C)$ to obtain the existence of a bounded sequence $(F_n)_{n\in\mathbb{Z}^*}$ in $L^2(0, T_0; U)$ that is biorthogonal in $L^2(0, T_0; U)$ to the sequence $(e^{i\mu_k t} C_0 \varphi_k)_{n\in\mathbb{Z}^*}$ in $L^2(0, T_0; U)$.

The idea is to construct from $(F_n)_{n\in\mathbb{Z}^*}$ a sequence $(G_n)_{n\in\mathbb{N}^*}$ in $L^2(0, T; U)$ biorthogonal to $(e^{-\lambda_k t} C_0 \varphi_k)_{n\in\mathbb{N}^*}$. This is done in the proof of Theorem 9.2.2 in [21] (see formula (9.2.24), p.294). Using Lemma 9.2.3 in [21], we have

$$\sum_{n\in\mathbb{N}^*} e^{-2T\lambda_n} \|G_n\|^2_{L^2(0,T;U)} \leqslant C \sum_{n\in\mathbb{N}^*} e^{-T\lambda_n} (\nu+1)^2 e^{3\nu/2} e^{-\frac{\beta\lambda_n}{2\sqrt{\nu+1}}},$$

with $\delta > \max(0, 2(T_0 - \pi))$, $\beta \in (0, \frac{T}{2})$ and $\nu = (\pi+\delta)^2/\beta$. The constant $C$ appearing in the above estimate only depends on $\delta$. Using that $e^{-x} < k!/x^k$ for all $x > 0$, we deduce that for all $k \in \mathbb{N}^*$, there exists $C = C(\delta, k)$ such that

$$\sum_{n\in\mathbb{N}^*} e^{-2T\lambda_n} \|G_n\|^2_{L^2(0,T;U)} \leqslant C (\nu+1)^{2+3k/2} e^{3\nu/2} \sum_{n\in\mathbb{N}^*} \frac{1}{\lambda_n^k}.$$

Gathering the above estimate, (3.3) and the above conditions on $\beta, \nu, \delta$, we deduce

$$\sum_{n\in\mathbb{N}^*} e^{-2T\lambda_n} \|G_n\|^2_{L^2(0,T;U)} \leqslant C^2 e^{2M/T}.$$

Applying Lemma 9.1.4 in [21], we deduce (3.4) and (3.5). □

## 3.2 The source term method

We recall a method introduced in [14] and that allows us to deal with the controllability of nonlinear parabolic systems. Assume $T > 0$. In what follows, we consider the subset $\mathcal{R}_T$ of the functions $\rho$ satisfying the following properties

$$\rho : [0,T] \to [0,\infty) \text{ continuous, non increasing, positive on } [0,T),\ \rho(T) = 0. \tag{3.6}$$

We denote by $L^2_\rho(0,T;Y)$ the $L^2$ space with values in $Y$ and with measure $\lambda_1/\rho^2$ where $\lambda_1$ denotes the Lebesgue measure on $(0,T)$. We consider the same hypotheses as in the previous subsection: $\mathcal{H}$ is a Hilbert space, $A_0 : \mathcal{D}(A_0) \to \mathcal{H}$ is a self-adjoint operator strictly positive such that $A_0^{-1}$ is compact. We also consider a Hilbert space $U$ and $B_0 \in \mathcal{L}(U, \mathcal{D}(A_0^{1/2})^*)$ an operator. To shorten the notation in this subsection, we write

$$\mathcal{H}_{1/2} := \mathcal{D}(A_0^{1/2}), \quad \mathcal{H}_{-1/2} := \mathcal{D}(A_0^{1/2})^*.$$

In this subsection, we consider the following control problem

$$\dot{z} = -A_0 z + B_0 u + f, \quad z(0) = z_0. \tag{3.7}$$

**Theorem 3.2.** *Assume that for all $t > 0$ we have the observability inequality*

$$\|e^{-A_0 t}\varphi_0\|_{\mathcal{H}} \leqslant K(t) \left( \int_0^t \|B_0^* e^{-A_0 t}\varphi_0\|_U^2 \, dt \right)^{1/2}, \tag{3.8}$$

*with $K : (0,\infty) \to (0,\infty)$ continuous and non increasing. There exist $\rho_0, \rho_1 \in \mathcal{R}_T$ such that for any $z_0 \in \mathcal{H}$ and $f \in L^2_{\rho_1}(0,T;\mathcal{H}_{-1/2})$, there exists $u \in L^2_{\rho_0}(0,T;U)$ such that the solutions of (3.7) satisfies*

$$\|z\|^2_{L^2_{\rho_0}(0,T;\mathcal{H}_{1/2})} + \left\| \frac{z}{\rho_0} \right\|^2_{C^0([0,T];\mathcal{H})} + \|\dot{z}\|^2_{L^2_{\rho_0}(0,T;\mathcal{H}_{-1/2})} + \|u\|^2_{L^2_{\rho_0}(0,T;U)} \leqslant C \left( \|z_0\|_{\mathcal{H}} + \|f\|^2_{L^2_{\rho_1}(0,T;\mathcal{H}_{-1/2})} \right). \tag{3.9}$$

*In particular, (3.7) is null-controllable.*

In the proof of this theorem, $\rho_0$ and $\rho_1$ are constructed in the following way: we consider $q > 1$ and

$$\rho_0(t) := \rho_1(q^2(t-T) + T)K((q-1)(T-t)) \quad \left( t \in \left[ T\left(1 - \frac{1}{q^2}\right), T \right] \right), \tag{3.10}$$

and $\rho_0$ extended on $\left[0, T\left(1 - \frac{1}{q^2}\right)\right]$ so that $\rho_0 \in \mathcal{R}_T$.

The above statement is slightly different from the result in [14] since in our case $B_0^*$ is not a bounded operator of $\mathcal{H}$. The proof is nevertheless completely similar, we only remind the main steps of proof for sake of completeness.

*Sketch of the proof of Theorem 3.2.* We set

$$T_k := T - \frac{T}{q^k} \quad (k \in \mathbb{N}).$$

From (3.10), we have the relation

$$\rho_0(T_{k+2}) = \rho_1(T_k)K(T_{k+2} - T_{k+1}). \tag{3.11}$$

Then we define the following sequence: $a_0 := z_0$ and for any $k \geqslant 0$, $a_{k+1} := z_1(T_{k+1}^-)$ where $z_1$ is the solution of the system

$$\begin{cases} \dot{z}_1 = -Az_1 + f & \text{in } (T_k, T_{k+1}), \\ z_1(T_k^+) = 0. \end{cases} \tag{3.12}$$

Using the classical results for parabolic system, we have for all $k \geqslant 0$,

$$\|a_{k+1}\|^2_{\mathcal{H}} \leqslant C\|f\|^2_{L^2(T_k,T_{k+1};\mathcal{H}_{-1/2})}. \tag{3.13}$$

where the constant $C$ is independent of $k$.

Finally, we consider the control system

$$\begin{cases} \dot{z}_2 = -Az_2 + Bu \quad \text{in } (T_k, T_{k+1}), \\ z_2(T_k^+) = a_k. \end{cases} \tag{3.14}$$

From (3.8) and Proposition 2.1, there exists a control $u \in L^2(T_k, T_{k+1}; U)$ such that

$$z_2(T_{k+1}^-) = 0$$

and

$$\|u\|^2_{L^2(T_k,T_{k+1};U)} \leqslant K^2(T_{k+1} - T_k)\|a_k\|^2_{\mathcal{H}}. \tag{3.15}$$

We then define $z = z_1 + z_2$ and using the relations of $z_1$ and $z_2$ at $T_k^-$ and $T_k^+$, we can check that $z$ is solution of (3.7). Moreover, from (3.11), (3.13) and (3.15), we deduce that for all $k \geqslant 0$,

$$\|u\|^2_{L^2_{\rho_0}(T_{k+1},T_{k+2};U)} \leqslant C\|f\|^2_{L^2_{\rho_1}(T_k,T_{k+1};\mathcal{H}_{-1/2})}. \tag{3.16}$$

From the classical results for parabolic system, we have for any $k \geqslant 0$,

$$\begin{aligned} &\|z\|^2_{L^2(T_{k+1},T_{k+2};\mathcal{H}_{1/2})} + \|z\|^2_{C^0([T_{k+1},T_{k+2}];\mathcal{H})} + \|\dot{z}\|^2_{L^2(T_{k+1},T_{k+2};\mathcal{H}_{-1/2})} \\ &\qquad \leqslant C\left(\|a_{k+1}\|^2_{\mathcal{H}} + \|u\|^2_{L^2(T_{k+1},T_{k+2};U)} + \|f\|^2_{L^2(T_{k+1},T_{k+2};\mathcal{H}_{-1/2})}\right) \end{aligned} \tag{3.17}$$

Thus, from the above relation, from (3.13) and from (3.15) , we deduce that for all $k \geqslant 0$,

$$\|z\|^2_{L^2(T_{k+1},T_{k+2};\mathcal{H}_{1/2})} + \|z\|^2_{C^0([T_{k+1},T_{k+2}];\mathcal{H})} + \|\dot{z}\|^2_{L^2(T_{k+1},T_{k+2};\mathcal{H}_{-1/2})} \leqslant C\rho_0(T_{k+2})^2\|f\|^2_{L^2_{\rho_1}(T_k,T_{k+2};\mathcal{H}_{-1/2})}. \tag{3.18}$$

Since $\rho_0$ is non increasing we deduce

$$\|z\|^2_{L^2_{\rho_0}(T_{k+1},T_{k+2};\mathcal{H}_{1/2})} + \left\|\frac{z}{\rho_0}\right\|^2_{C^0([T_{k+1},T_{k+2}];\mathcal{H})} + \|\dot{z}\|^2_{L^2_{\rho_0}(T_{k+1},T_{k+2};\mathcal{H}_{-1/2})} \leqslant C\|f\|^2_{L^2_{\rho_1}(T_k,T_{k+2};\mathcal{H}_{-1/2})}. \tag{3.19}$$

A similar estimate is valid for $(T_0, T_1)$ and thus we deduce (3.9).

□

## 3.3 The Perturbation method

In this subsection, we recall some results obtained in [6] in order to deduce the exact observability of

$$\ddot{w} = -A_0 w + aA_0^{1/2}w, \quad y = C_0 v, \quad w(0) = w_0, \quad \dot{w}(0) = w_1 \tag{3.20}$$

from the exact observability of

$$\ddot{w} = -A_0 w, \quad y = C_0 v, \quad w(0) = w_0, \quad \dot{w}(0) = w_1, \tag{3.21}$$

where as the previous subsections, $\mathcal{H}$ is a Hilbert space, $A_0 : \mathcal{D}(A_0) \to \mathcal{H}$ is a strictly positive self-adjoint operator such that $A_0^{-1}$ is compact. We have denoted by $A_0^{1/2}$ the square root of $A_0$ (see for instance [21, p.81]). We also consider a Hilbert space $U$ and $B_0 \in \mathcal{L}(U, \mathcal{D}(A_1^{1/2})^*)$ an admissible operator (see Section 2).

We denote by $\lambda_n$, $n \in \mathbb{N}^*$, the eigenvalues of $A_0$. They are real positive and we denote by $\lambda_1$ the smallest one.

We have the following result.

**Theorem 3.3.** *Assume that* (3.21) *is exactly observable and* $a < \sqrt{\lambda_1}$. *Then* (3.20) *is exactly observable.*

This result is stated [6, Proposition 3.1]. The proof is done in the case where $C_0$ is a bounded operator (i.e. $C_0 \in \mathcal{L}(\mathcal{H}, U)$) but this assumption is not used. Note that the time of exact observability is not obtained in this theorem, and this is due to the method of proof based on a spectral method (see [17]).

# 4 Controllability of the Kuramoto-Sivashinsky equation in 1D

We are now using the result of the previous section in order to obtain the null-controllability of the Kuramoto-Sivashinsky equation in 1D.

## 4.1 Abstract framework

First we introduce operators in order to rewrite (1.1).

We define

$$\mathcal{D}(A_0) = \left\{ y \in H^4(0,1) \ ; \ y = y_x = 0 \quad \text{in } \{0,1\} \right\}, \tag{4.1}$$

$$A_0 : \mathcal{D}(A_0) \to L^2(0,1), \quad y \mapsto y_{xxxx} + a y_{xx}. \tag{4.2}$$

The operator $A_0$ is self-adjoint and $-A_0$ is the infinitesimal generator of a parabolic semigroup $(e^{-tA_0})_{t \geqslant 0}$.

In what follows, we consider $\lambda_\# \in \rho(A_0)$ such that $A_\# := \lambda_\# + A_0$ is positive and we set

$$\mathcal{H}^s := \mathcal{D}(A_\#^{s/4}) \quad (s \geqslant 0), \quad \mathcal{H}^{-s} := \mathcal{D}(A_\#^{s/4})' \quad (s \geqslant 0), \tag{4.3}$$

where $\mathcal{X}'$ denotes the dual of $\mathcal{X}$ with respect to the space $L^2(0,1)$.

We can consider $A_0$ as an operator

$$A_0 : \mathcal{H}^{s+4} \to \mathcal{H}^s, \quad (s \in \mathbb{R}).$$

In particular,

$$A_\# : \mathcal{H}^{s+4} \to \mathcal{H}^s, \quad (s \in \mathbb{R})$$

is a unitary transformation.

In order to obtain a weak formulation for system (1.1), we multiply formally the first equation of (1.1) by $\varphi \in \mathcal{H}^4$. After some integrations by parts, we obtain

$$\frac{d}{dt} \int_0^1 y\varphi \, dx + \int_0^1 y A_0 \varphi \, dx - \int_0^1 \left( \frac{y^2}{2} \right) \varphi_x \, dx = u(t) \varphi_{xx}(t,0). \tag{4.4}$$

In order to write the corresponding control operator, we use a classical approach and define the operator $D_0 : U := \mathbb{R} \to \mathcal{H}^1 = H_0^1(0,1)$ as follows: $D_0 u = y$ is the solution of

$$\begin{cases} \lambda_\# y + (y_{xxxx} + a y_{xx}) = 0, \quad x \in (0,1), \\ y(0) = y(1) = 0, \\ y_x(0) = u, \quad y_x(1) = 0. \end{cases} \tag{4.5}$$

More precisely, $D_0 u$ is the unique element of $\mathcal{H}^1 = H_0^1(0,1)$ such that

$$\langle D_0 u, f \rangle_{\mathcal{H}^1, \mathcal{H}^{-1}} = u \left( A_\#^{-1} f \right)_{xx} (0). \tag{4.6}$$

Since $A_\#^{-1}(\mathcal{H}^{-1}) = \mathcal{H}^3 \subset H^3(0,1)$, $D_0 u$ is well-defined and $D_0 \in \mathcal{L}(U, \mathcal{H}^1)$.

Then, we define the operator $B_0$ by

$$B_0 := A_\# D_0 : U = \mathbb{R} \to \mathcal{H}^{-3}. \tag{4.7}$$

One can check that for all $\varphi \in \mathcal{H}^3$ and $u \in \mathbb{R}$,

$$\langle B_0 u, \varphi \rangle_{\mathcal{H}^{-3}, \mathcal{H}^3} = u \varphi_{xx}(0).$$

Finally, we define $F : L^2(0,1) \to \mathcal{H}^{-2}$ by

$$\langle F(y), \varphi \rangle_{\mathcal{H}^{-2}, \mathcal{H}^2} := -\int_0^1 \left( \frac{y^2}{2} \right) \varphi_x \, dx \quad (\varphi \in \mathcal{H}^2). \tag{4.8}$$

The function $F$ is well-defined since $H^1(0,1) \subset L^\infty(0,1)$.

The above definitions allow us to write (4.4) as

$$y_t + A_0 y + F(y) = B_0 u, \tag{4.9}$$

and if $u \in L^2(0,T)$, we consider solutions satisfying the following regularity

$$y \in H^1(0,T;\mathcal{H}^{-3}) \cap L^2(0,T;\mathcal{H}^1) \cap C([0,T];\mathcal{H}^{-1}). \tag{4.10}$$

Note that if $y$ satisfies (4.10), then

$$y \in L^4(0,T;L^2(0,1))$$

and in particular $F(y) \in L^2(0,T;\mathcal{H}^{-2})$.

## 4.2 Controllability of the linear problem

Using the previous section, we can write

$$\begin{cases} y_t + y_{xxxx} + a y_{xx} = 0, \quad x \in (0,1),\ t \in (0,T) \\ y(t,0) = 0, \quad y(t,1) = 0, \quad t \in (0,T), \\ y_x(t,0) = u(t), \quad y_x(t,1) = 0, \quad t \in (0,T), \\ y(0,x) = y_0(x), \quad x \in (0,1). \end{cases} \tag{4.11}$$

as

$$y_t + A_0 y = B_0 u, \quad y(0) = y_0.$$

The main result of this section is the following result.

**Theorem 4.1.** *Assume $a \notin \mathcal{N}$. For $y_0 \in \mathcal{H}^{-1}$, there exists $u \in L^2(0,T)$ such that $y(T) = 0$. Moreover, there exists an operator $L_T \in \mathcal{L}(\mathcal{H}^{-1},\mathbb{R})$ such that $u$ can be given by $u = L_T(y_0)$. We have*

$$\|L_T\|_{\mathcal{L}(\mathcal{H}^{-1},\mathbb{R})} \leqslant K(T)$$

*with*

$$K(T) := Ce^{M/T}, \tag{4.12}$$

*for some positive constants $C, M$.*

*Proof of Theorem 4.1.* First, we show that the adjoint of $B_0$ defined by (4.7) is

$$C_0 := B_0^* : \mathcal{H}^1 \to \mathbb{R},\ \varphi \mapsto \left(A_\#^{-1/2}\varphi\right)_{xx}(0). \tag{4.13}$$

Indeed, considering $A_0$ as an operator of $\mathcal{H}^{-1}$ with domain $D(A_0) = \mathcal{H}^3$, we have $D(A_0^{1/2}) = \mathcal{H}^1$ and the dual $D(A_0^{1/2})^*$ of $D(A_0^{1/2})$ with respect to $\mathcal{H}^{-1}$ is $D(A_0^{1/2})^* = \mathcal{H}^{-3}$ and

$$\langle B_0 u, \varphi\rangle_{D(A_0^{1/2})^*, D(A_0^{1/2})} = \langle A_\#^{1/2} D_0 u, \varphi\rangle_{\mathcal{H}^{-1},\mathcal{H}^1} = u(A_\#^{-1/2}\varphi)_{xx}(0).$$

where we have used (4.6).

Second, we show that $(-A_0, C_0)$ is final state observable in any time $T > 0$ with

$$A_0 : \mathcal{H}^4 \to L^2(0,1) \quad \text{and} \quad C_0 : \mathcal{H}^2 \to \mathbb{R}$$

the last operator being the restriction of $C_0$ defined by (4.13) to $\mathcal{H}^2$. In order to do that, we are going to use Theorem 3.1 and thus to consider the exact observability in $\mathcal{H}^2 \times \mathcal{H}$ of

$$\begin{cases} \ddot{w} = -A_0 w, \\ y = C_0 \dot{w}. \end{cases} \tag{4.14}$$

Using Lemma 2.1 in [2], we have that if $a \notin \mathcal{N}$, then the eigenvalues $(\lambda_k)$ of $A_0$ are simple and using Lemma 2.2 in [2], we also have that

$$\lambda_k \sim D_1 k^4, \quad D_1 > 0. \tag{4.15}$$

The two properties imply that the eigenvalues admit a gap: there exists $\gamma > 0$ such that

$$\lambda_{k+1} - \lambda_k \geqslant \gamma \quad (k \in \mathbb{N}).$$

Let us denote by $(\phi_k)$ a family of eigenvectors of $A_0$ associated to $(\lambda_k)$ and forming an orthonormal basis of $\mathcal{H}$. We show the exact observability of (4.14) by applying a spectral method: using Proposition 4.5 in [17], we have that (4.14) is exactly observable in some time $T_0$ if and only if there exists $\delta > 0$ such that

$$|C_0 \phi_k| \geqslant \delta \quad (k \in \mathbb{N}). \tag{4.16}$$

From (4.13),

$$C_0 \phi_k = \frac{(\phi_k)''(0)}{\sqrt{\lambda_\# + \lambda_k}}$$

and using Lemma 2.1 and Lemma 2.2 in [2], we deduce that $C_0 \phi_k \neq 0$ and

$$\lim_{k \to \infty} |C_0 \phi_k| = d_1 > 0.$$

This implies (4.16) and thus the exact observability of (4.14) for some time $T_0 > 0$.

Now, from (4.15), we deduce (3.3) and thus applying Theorem 3.1, we obtain

$$\|e^{-A_0 T} \varphi_0\|_{\mathcal{H}} \leqslant K(T) \left( \int_0^T \|B_0^* e^{-A_0 t} \varphi_0\|_U^2 \, dt \right)^{1/2} \quad (\varphi_0 \in \mathcal{H}^1) \tag{4.17}$$

and there exist $C$ and $M$ such that

$$K(T) := C e^{M/T}.$$

This implies in particular

$$\|e^{-A_0 T} \varphi_0\|_{\mathcal{H}^{-1}} \leqslant K(T) \left( \int_0^T \|B_0^* e^{-A_0 t} \varphi_0\|_U^2 \, dt \right)^{1/2} \quad (\varphi_0 \in \mathcal{H}^1). \tag{4.18}$$

From Proposition 2.1, we deduce that (4.11) is null-controllable for any time $T > 0$ in $\mathcal{H}^{-1}$: for any $y_0 \in \mathcal{H}^{-1}$, there exists $u = L_T(y_0) \in L^2(0, T; U)$ with

$$\|u\|_{L^2(0,T;\mathbb{R})} \leqslant K(T) \|y_0\|_{\mathcal{H}^{-1}}$$

such that $y(T) = 0$. □

## 4.3 Proof of Theorem 1.1

We are now in position to prove Theorem 1.1.

We recall that $K$ satisfies (4.12). We take

$$q \in (1, 2^{1/4}), \quad \alpha > \frac{MTq^4}{2(q-1)}, \tag{4.19}$$

$$\rho_1(t) = \exp\left( -\frac{\alpha}{(T-t)^2} \right). \tag{4.20}$$

We can define $\rho_0$ as

$$\rho_0(t) = C\exp\left(-\frac{\alpha}{q^4(T-t)^2} + \frac{M}{(q-1)(T-t)}\right) \quad (t \in [0,T]). \tag{4.21}$$

In particular, from (4.19), $\rho_0$ is continuous and decreasing and satisfies (3.10). We can also check that

$$\left|\frac{(\rho_0)^2}{\rho_1}\right| \leqslant C \quad ((t \in [0,T]). \tag{4.22}$$

*Proof.* Applying Theorem 3.2 and Theorem 4.1, we deduce that for any $y_0 \in \mathcal{H}^{-1}$ and $f \in L^2_{\rho_1}(0,T;\mathcal{H}^{-3})$, there exists $u \in L^2_{\rho_0}(0,T;U)$ such that the solutions of

$$\begin{cases} y_t + y_{xxxx} + ay_{xx} = f, \quad x \in (0,1),\ t \in (0,T) \\ y(t,0) = 0, \quad y(t,1) = 0, \quad t \in (0,T), \\ y_x(t,0) = u(t), \quad y_x(t,1) = 0, \quad t \in (0,T), \\ y(0,x) = y_0(x), \quad x \in (0,1). \end{cases} \tag{4.23}$$

satisfies

$$\|y\|^2_{L^2_{\rho_0}(0,T;\mathcal{H}^1)} + \left\|\frac{y}{\rho_0}\right\|^2_{C^0([0,T];\mathcal{H}^{-1})} + \|y_t\|^2_{L^2_{\rho_0}(0,T;\mathcal{H}^{-3})} + \|u\|^2_{L^2_{\rho_0}(0,T)} \leqslant C\left(\|y_0\|^2_{\mathcal{H}^{-1}} + \|f\|^2_{L^2_{\rho_1}(0,T;\mathcal{H}^{-3})}\right). \tag{4.24}$$

Now, let us consider the map

$$\mathcal{Z} : f \in L^2_{\rho_1}(0,T;\mathcal{H}^{-3}) \mapsto -F(y) \in L^2_{\rho_1}(0,T;\mathcal{H}^{-3})$$

where $(y,u)$ is the solution of (4.23) and where $F$ is defined by (4.8). We show that $\mathcal{Z}$ is well-defined and that there exists $R$ such that

$$\mathcal{Z}(B(0,R)) \subset B(0,R)$$

where $B(0,R)$ is the closed ball of $L^2_{\rho_1}(0,T;\mathcal{H}^{-3})$ of radius $R > 0$.

First using (4.8), we deduce that

$$\|F(y)\|_{\mathcal{H}^{-2}} \leqslant C\|y\|_{\mathcal{H}^1}\|y\|_{\mathcal{H}^{-1}}$$

Therefore, from (4.22) and (4.24)

$$\|F(y)\|_{L^2_{\rho_0}(0,T;\mathcal{H}^{-2})} \leqslant C\left(\|y_0\|^2_{\mathcal{H}^{-1}} + \|f\|^2_{L^2_{\rho_1}(0,T;\mathcal{H}^{-3})}\right).$$

Assume that

$$R > \|y_0\|_{\mathcal{H}^{-1}}$$

and take

$$\|f\|_{L^2_{\rho_1}(0,T;\mathcal{H}^{-3})} \leqslant R.$$

Then,

$$\|F(y)\|_{L^2_{\rho_0}(0,T;\mathcal{H}^{-2})} \leqslant 2CR^2 \leqslant R$$

if we assume

$$R = \frac{1}{4C}. \tag{4.25}$$

We now show that $\mathcal{N}$ is a contraction on $B(0,R)$: let us take $f^{(1)}, f^{(2)} \in B(0,R)$ and let us consider the corresponding solutions and controls $(y^{(1)}, u^{(1)})$, $(y^{(2)}, u^{(2)})$. We write

$$f := f^{(1)} - f^{(2)}, \quad y := y^{(1)} - y^{(2)}, \quad u := u^{(1)} - u^{(2)}.$$

We have

$$\|y\|^2_{L^2_{\rho_0}(0,T;\mathcal{H}^1)} + \left\|\frac{y}{\rho_0}\right\|^2_{C^0([0,T];\mathcal{H}^{-1})} + \|y_t\|^2_{L^2_{\rho_0}(0,T;\mathcal{H}^{-3})} + \|u\|^2_{L^2_{\rho_0}(0,T)} \leqslant C\|f\|^2_{L^2_{\rho_1}(0,T;\mathcal{H}^{-3})}. \tag{4.26}$$

From (4.8), we obtain

$$\|F(y^{(1)}) - F(y^{(2)})\|_{\mathcal{H}^{-2}} \leqslant C(\|y^{(1)}\|_{\mathcal{H}^1} + \|y^{(2)}\|_{\mathcal{H}^1})\|y\|_{\mathcal{H}^{-1}}$$

Combining (4.24) and (4.26) yields

$$\|F(y^{(1)}) - F(y^{(2)})\|_{L^2_{\rho_0}(0,T;\mathcal{H}^{-2})} \leqslant 2CR\|f\|_{L^2_{\rho_1}(0,T;\mathcal{H}^{-3})}.$$

and we deduce from (4.25) that $\mathcal{N}$ is a contraction. □

# 5 Controllability of the Kuramoto-Sivashinsky equation in 2D

We show in this section the null-controllability of the Kuramoto-Sivashinsky equation in 2D (Theorem 1.4).

## 5.1 The linear system

We start by considering the null-controllability of a linear system associated with (1.2):

$$\begin{cases} y_t + \Delta^4 y + a\Delta y = 0, \quad x \in \Omega,\ t \in (0,T) \\ y(t) = 0, \quad x \in \partial\Omega,\ t \in (0,T), \\ \Delta y(t) = u1_\Gamma, \quad x \in \partial\Omega,\ t \in (0,T), \\ y(0,x) = y_0(x), \quad x \in \Omega. \end{cases} \tag{5.1}$$

We define

$$\mathcal{D}(A_0) = \left\{y \in H^4(\Omega)\ ;\ y = \Delta y = 0 \quad \text{on } \partial\Omega\right\}, \tag{5.2}$$

$$A_0 : \mathcal{D}(A_0) \to L^2(\Omega), \quad y \mapsto \Delta^4 y. \tag{5.3}$$

The operator $A_0$ is self-adjoint, positive and with compact resolvent. We denote by $\lambda_n$, $n \in \mathbb{N}^*$, the eigenvalues of $A_0$. They are real positive and we denote by $\lambda_1$ the smallest one. Assume $a < \sqrt{\lambda_1}$. We also consider the operator

$$\mathcal{D}(A_a) = \mathcal{D}(A_0), \tag{5.4}$$

$$A_a : \mathcal{D}(A_0) \to L^2(\Omega), \quad y \mapsto \Delta^4 y + a\Delta y. \tag{5.5}$$

The operator $A_a$ is self-adjoint and $-A_a$ is the generator of a semi-group $(e^{-tA_a})_{t\geqslant 0}$.

In what follows, we consider $\lambda_\# \in \rho(A_a)$ such that $A_\# := \lambda_\# + A_a$ is positive and we set

$$\mathcal{H}^s := \mathcal{D}(A_\#^{s/4}) \quad (s \geqslant 0), \quad \mathcal{H}^{-s} := \mathcal{D}(A_\#^{s/4})' \quad (s \geqslant 0), \tag{5.6}$$

where $\mathcal{X}'$ denotes the dual of $\mathcal{X}$ with respect to the space $L^2(\Omega)$.

We can consider $A_a$ as an operator

$$A_a : \mathcal{H}^{s+4} \to \mathcal{H}^s, \quad (s \in \mathbb{R}).$$

In particular,

$$A_0 : \mathcal{H}^{s+4} \to \mathcal{H}^s, \quad (s \in \mathbb{R})$$

is an isomorphism.

In order to obtain a weak formulation for system (5.1), we multiply formally the first equation of (5.1) by $\varphi \in \mathcal{H}^4$. After some integrations by parts, we obtain

$$\frac{d}{dt}\int_\Omega y\varphi\, dx + \int_\Omega yA_a\varphi\, dx = \int_\Gamma u\frac{\partial\varphi}{\partial\nu}\, d\gamma. \tag{5.7}$$

In order to write the corresponding control operator, we use a classical approach and define the operator $D_0 : U = L^2(\Gamma) \to \mathcal{H}^2$ as follows: $D_0u = y$ is the solution of

$$\begin{cases} \Delta^4 y = 0 \quad \text{in } \Omega, \\ y = 0 \quad \text{on } \partial\Omega, \\ \Delta y = u1_\Gamma \quad \text{on } \partial\Omega. \end{cases} \tag{5.8}$$

More precisely, $D_0u$ is the unique element of $\mathcal{H}^2$ such that

$$\langle D_0u, f\rangle_{\mathcal{H}^2,\mathcal{H}^{-2}} = \int_\Gamma u\frac{\partial}{\partial\nu}\left(A_0^{-1}f\right) d\gamma. \tag{5.9}$$

Since $A_0^{-1}(\mathcal{H}^{-2}) = \mathcal{H}^2$, $D_0u$ is well-defined.

Then, we define the operator $B_0$ by

$$B_0 := -A_0D_0 : U \to \mathcal{H}^{-2}. \tag{5.10}$$

One can check that for all $\varphi \in \mathcal{H}^2$ and $u \in U$,

$$\langle B_0u, \varphi\rangle_{\mathcal{H}^{-2},\mathcal{H}^2} = \int_\Gamma u\frac{\partial\varphi}{\partial\nu}\, d\gamma. \tag{5.11}$$

As a consequence, (5.1) writes as

$$\dot{y} = -A_0y - aA_0^{1/2}y + B_0u, \quad y(0) = y_0. \tag{5.12}$$

Considering $A_\#$ as an operator from $\mathcal{H}^4$ onto $\mathcal{H}$, we deduce that

$$B_0 : U \to D(A_\#^{1/2})^*$$

and from (5.11), its adjoint satisfies

$$C_0 := B_0^* : \mathcal{H}^2 \to U, \varphi \mapsto \frac{\partial\varphi}{\partial\nu}_{|\Gamma}$$

We are now in position to prove the null-controllability of (5.1)

**Theorem 5.1.** *Assume $a < \sqrt{\lambda_1}$ and assume conditions 1) or 2) of Theorem 1.4. For $y_0 \in \mathcal{H}$, there exists $u \in L^2(0,T;U)$ such that the solution of (5.1) satisfies $y(T) = 0$. Moreover, there exists an operator $L_T \in \mathcal{L}(\mathcal{H}, U)$ such that $u$ can be given by $u = L_T(y_0)$. We have*

$$\|L_T\|_{\mathcal{L}(\mathcal{H}^{-1},\mathbb{R})} \leqslant K(T)$$

*with*

$$K(T) := Ce^{M/T}, \tag{5.13}$$

*for some positive constants $C, M$.*

*Proof.* In order to obtain the controllability of (5.12), we use two results on the plate equations obtained by [13] and [20] (see also [17] and Remark 7.5.6 in [21]): assume one of the conditions of Theorem 1.4 then the system

$$\begin{cases} \ddot{w} = -A_0 w, \\ y = C_0 \dot{w}. \end{cases} \tag{5.14}$$

is exactly observable in $\mathcal{H}^3 \times \mathcal{H}^1$.

Then using Theorem 3.3, we deduce that

$$\begin{cases} \ddot{w} = -A_0 w + a A_0^{1/2} w, \\ y = C_0 \dot{w}. \end{cases} \tag{5.15}$$

is exactly observable in $\mathcal{H}^3 \times \mathcal{H}^1$ for $a < \sqrt{\lambda_1}$.

Moreover, one has (see, for instance [21, Proposition 3.6.9]) that the eigenvalues $(\lambda_n(a))_{n\in\mathbb{N}^*}$ of $A_a$ satisfies

$$c_1 n^2 \leqslant \lambda_n(a) \leqslant c_2 n^2 \quad (n \geqslant n_0).$$

for some positive constants $c_1$ and $c_2$ and some $n_0 \geqslant 1$. We can apply Theorem 3.1:

$$\begin{cases} \dot{z} = -A_0 z + a A_0^{1/2} z, \\ y = C_0 \dot{z}. \end{cases} \tag{5.16}$$

is final state state observable in $\mathcal{H}^1$ for any time $T > 0$ with a cost $K$ satisfying (3.5). This means that

$$\|e^{-A_a T}\varphi_0\|_{\mathcal{H}^1} \leqslant K(T) \left( \int_0^T \|B_0^* e^{-A_a t}\varphi_0\|_U^2 \, dt \right)^{1/2} \quad (\varphi_0 \in \mathcal{H}^2) \tag{5.17}$$

and there exist $C$ and $M$ such that

$$K(T) := C e^{M/T}.$$

This implies in particular

$$\|e^{-A_a T}\varphi_0\|_{\mathcal{H}} \leqslant K(T) \left( \int_0^T \|B_0^* e^{-A_a t}\varphi_0\|_U^2 \, dt \right)^{1/2} \quad (\varphi_0 \in \mathcal{H}^2). \tag{5.18}$$

From standard duality arguments, we deduce that (5.1) is null-controllable for any time $T > 0$ in $\mathcal{H}$: for any $y_0 \in \mathcal{H}$, there exists $u \in L^2(0,T;U)$ with

$$\|u\|_{L^2(0,T;U)} \leqslant K(T)\|y_0\|_{\mathcal{H}}$$

such that $y(T) = 0$. □

## 5.2 Proof of Theorem 1.4

We are now in position to prove Theorem 1.4. The proof is similar to the proof of Theorem 1.1: we take $q$, $\alpha$, $\rho_1$ and $\rho_0$ as in (4.19), (4.20), (4.21). In particular, (4.22) holds.

*Proof.* Applying Theorem 3.2 and Theorem 5.1, we deduce that for any $y_0 \in \mathcal{H}$ and $f \in L^2_{\rho_1}(0,T;\mathcal{H}^{-2})$, there exists $u \in L^2_{\rho_0}(0,T;U)$ such that the solutions of

$$\begin{cases} y_t + \Delta^4 y + a\Delta y = f, \quad x \in \Omega,\ t \in (0,T) \\ \\ y(t) = 0, \quad x \in \partial\Omega,\ t \in (0,T), \\ \Delta y(t) = u 1_\Gamma, \quad x \in \partial\Omega,\ t \in (0,T), \\ y(0,x) = y_0(x), \quad x \in \Omega \end{cases} \tag{5.19}$$

satisfies

$$\|y\|^2_{L^2_{\rho_0}(0,T;\mathcal{H}^2)} + \left\|\frac{y}{\rho_0}\right\|^2_{C^0([0,T];\mathcal{H})} + \|y_t\|^2_{L^2_{\rho_0}(0,T;\mathcal{H}^{-2})} + \|u\|^2_{L^2_{\rho_0}(0,T;U)} \leqslant C\left(\|y_0\|^2_{\mathcal{H}} + \|f\|^2_{L^2_{\rho_1}(0,T;\mathcal{H}^{-2})}\right). \quad (5.20)$$

Now, let us consider the map

$$\mathcal{Z} : f \in L^2_{\rho_1}(0,T;\mathcal{H}^{-2}) \mapsto -F(y) \in L^2_{\rho_1}(0,T;\mathcal{H}^{-2})$$

where $(y,u)$ is the solution of (5.19)-(5.20) and where $F$ is defined by:

$$F(y) = -\frac{1}{2}|\nabla y|^2$$

We show that $\mathcal{Z}$ is well-defined and that there exists $R$ such that

$$\mathcal{Z}(B(0,R)) \subset B(0,R)$$

where $B(0,R)$ is the closed ball of $L^2_{\rho_1}(0,T;\mathcal{H}^{-2})$ of radius $R > 0$.

First using the Sobolev embedding $\mathcal{H}^2 \subset L^\infty(\Omega)$ and an interpolation argument, we deduce that

$$\|F(y)\|_{\mathcal{H}^{-2}} \leqslant C\|y\|^2_{\mathcal{H}^1} \leqslant C\|y\|_{\mathcal{H}^2}\|y\|_{\mathcal{H}}.$$

Therefore, from (4.22) and (5.20)

$$\|F(y)\|_{L^2_{\rho_1}(0,T;\mathcal{H}^{-2})} \leqslant C\left(\|y_0\|^2_{\mathcal{H}} + \|f\|^2_{L^2_{\rho_1}(0,T;\mathcal{H}^{-2})}\right).$$

Using this estimate, we can apply the Banach fixed point theorem as in the proof of Theorem 1.1. □

# References


[1] Eduardo Cerpa. Exact controllability of a nonlinear Korteweg-de Vries equation on a critical spatial domain. *SIAM J. Control Optim.*, 46(3):877–899 (electronic), 2007.

[2] Eduardo Cerpa. Null controllability and stabilization of the linear Kuramoto-Sivashinsky equation. *Commun. Pure Appl. Anal.*, 9(1):91–102, 2010.

[3] Eduardo Cerpa and Emmanuelle Crépeau. Boundary controllability for the nonlinear Korteweg-de Vries equation on any critical domain. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 26(2):457–475, 2009.

[4] Eduardo Cerpa and Alberto Mercado. Local exact controllability to the trajectories of the 1-D Kuramoto-Sivashinsky equation. *J. Differential Equations*, 250(4):2024–2044, 2011.

[5] Panagiotis D. Christofides and Antonios Armaou. Global stabilization of the Kuramoto-Sivashinsky equation via distributed output feedback control. *Systems Control Lett.*, 39(4):283–294, 2000.

[6] Nicolae Cîndea and Marius Tucsnak. Local exact controllability for Berger plate equation. *Math. Control Signals Systems*, 21(2):93–110, 2009.

[7] Jean-Michel Coron and Qi Lü. Fredholm transform and local rapid stabilization for a Kuramoto-Sivashinsky equation. *J. Differential Equations*, 259(8):3683–3729, 2015.

[8] Szymon Dolecki and David L. Russell. A general theory of observation and control. *SIAM J. Control Optimization*, 15(2):185–220, 1977.

[9] C. Foias, B. Nicolaenko, G. R. Sell, and R. Temam. Inertial manifolds for the Kuramoto-Sivashinsky equation and an estimate of their lowest dimension. *J. Math. Pures Appl. (9)*, 67(3):197–226, 1988.

[10] Changbing Hu and Roger Temam. Robust control of the Kuramoto-Sivashinsky equation. *Dyn. Contin. Discrete Impuls. Syst. Ser. B Appl. Algorithms*, 8(3):315–338, 2001.

[11] M. S. Jolly, R. Rosa, and R. Temam. Evaluating the dimension of an inertial manifold for the Kuramoto-Sivashinsky equation. *Adv. Differential Equations*, 5(1-3):31–66, 2000.

[12] Yoshiki Kuramoto. Diffusion-induced chaos in reaction systems. *Progress of Theoretical Physics Supplement*, 64:346–367, 1978.

[13] G. Lebeau. Contrôle de l'équation de Schrödinger. *J. Math. Pures Appl. (9)*, 71(3):267–291, 1992.

[14] Yuning Liu, Takéo Takahashi, and Tucsnak Marius. Single input controllability of a simplified fluid-structure interaction model. *ESAIM Control Optim. Calc. Var.*, 2013.

[15] B. Nicolaenko, B. Scheurer, and R. Temam. Some global dynamical properties of the Kuramoto-Sivashinsky equations: nonlinear stability and attractors. *Phys. D*, 16(2):155–183, 1985.

[16] B. Nicolaenko, B. Scheurer, and R. Temam. Some global dynamical properties of a class of pattern formation equations. *Comm. Partial Differential Equations*, 14(2):245–297, 1989.

[17] K. Ramdani, T. Takahashi, G. Tenenbaum, and M. Tucsnak. A spectral approach for the exact observability of infinite-dimensional systems with skew-adjoint generator. *J. Funct. Anal.*, 226(1):193–229, 2005.

[18] David L. Russell. A unified boundary controllability theory for hyperbolic and parabolic partial differential equations. *Studies in Appl. Math.*, 52:189–211, 1973.

[19] GI Sivashinsky. Nonlinear analysis of hydrodynamic instability in laminar flamesi. derivation of basic equations. *Acta astronautica*, 4(11):1177–1206, 1977.

[20] G. Tenenbaum and M. Tucsnak. Fast and strongly localized observation for the Schrödinger equation. *Trans. Amer. Math. Soc.*, 361(2):951–977, 2009.

[21] Marius Tucsnak and George Weiss. *Observation and control for operator semigroups*. Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks]. Birkhäuser Verlag, Basel, 2009.