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# Boundary Local Null Controllability of the Kuramoto-Sivashinsky equation

Takéo Takahashi

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## Abstract

We prove that the Kuramoto-Sivashinsky equation is locally controllable in 1D and in 2D with one boundary control. Our method consists in combining several general results in order to reduce the null-controllability of this nonlinear parabolic equation to the exact controllability of a linear beam or plate system. This improves known results on the controllability of Kuramoto-Sivashinsky equation and gives a general strategy to handle the null-controllability of nonlinear parabolic systems.

**Key words:** Kuramoto-Sivashinsky equation, null controllability

**2010 Mathematics Subject Classification:** 93B05, 35K55, 93C10

## 1 Introduction

In this article, we study the boundary controllability of the Kuramoto-Sivashinsky equation, in space dimension 1 and 2. This equation is due to Kuramoto [12] (phase disturbance in chemical reaction system) and Sivashinsky [19] (model for a laminar flame). Lots of works are devoted to this equation: let us quote for instance [15], [16], [9], [11], etc. for the long time behavior. From a control point of view, this equation have been tackled in several papers: [10], [2], [4], [5], [7], etc. In particular, in [2], the author obtains the boundary null controllability of a linear system associated with the Kuramoto-Sivashinsky equation in 1D. In [4], the authors prove the local exact controllability to the trajectories of the Kuramoto-Sivashinsky equation in 1D by using boundary controls. In that case, they need two controls at the boundary and their method is based upon a Carleman estimate. Here our aim is to use only one control, in the 1D or in the 2D case. More precisely, we consider the control problem

$$\begin{cases} y_t + y_{xxxx} + ay_{xx} + yy_x = 0, & x \in (0, 1), t \in (0, T) \\ y(t, 0) = 0, \quad y(t, 1) = 0, & t \in (0, T), \\ y_x(t, 0) = u(t), \quad y_x(t, 1) = 0, & t \in (0, T), \\ y(0, x) = y_0(x), & x \in (0, 1). \end{cases} \quad (1.1)$$

In the above system, the state  $y$  is controlled by the function  $u = u(t)$ . One important step to study the controllability of (1.1) consists in studying the controllability of the linear system associated with (1.1). This was done in [2] and the author shows the null-controllability provided that  $a \notin \mathcal{N}$ , where

$$\mathcal{N} := \{ \pi^2(k^2 + \ell^2) ; k, \ell \in \mathbb{N}^*, k \neq \ell, k \equiv \ell \pmod{2} \}.$$

Our first main results shows the local null-controllability of (1.1) with the same hypothesis on  $a$ :

**Theorem 1.1.** *Assume that  $a \notin \mathcal{N}$ . Then the system (1.1) is locally null-controllable for any  $T > 0$ : there exists  $c_0 > 0$  such that for any  $\|y_0\|_{H^{-1}(0,1)} \leq c_0$ , there exists  $u \in L^2(0, T; \mathbb{R})$  and a solution  $y \in C([0, T]; H^{-1}(0, 1))$  of (1.1) with*

$$y(T) = 0.$$

The method of [2] is based on a precise spectral analysis, but does not allow to consider the nonlinear case. Nevertheless, we will use this spectral analysis in our proof, but in order to obtain a first result on an hyperbolic problem associated with (1.1).

**Remark 1.2.** *The space for the initial conditions is different from [2] but in this paper, the author uses a control  $u \in H^1(0, T)$ .*

**Remark 1.3.** *If  $a \in \mathcal{N}$ , then the linear system is not exactly controllable and thus the method developed here can not be applied. Nevertheless, the nonlinear equation (1.1) can be null-controllable even if  $a \in \mathcal{N}$ . One can look at [1] and [3] for a similar problem in the case of the Korteweg-de Vries equation.*

Our approach for the Kuramoto-Sivashinsky equation in 1D can be extended to the Kuramoto-Sivashinsky equation in 2D, but the boundary conditions are different from the 1D case:

$$\begin{cases} y_t + \Delta^4 y + a\Delta y + \frac{1}{2} |\nabla y|^2 = 0, & x \in \Omega, t \in (0, T) \\ y(t) = 0, & x \in \partial\Omega, t \in (0, T), \\ \Delta y(t) = u1_\Gamma, & x \in \partial\Omega, t \in (0, T), \\ y(0, x) = y_0(x), & x \in \Omega. \end{cases} \quad (1.2)$$

Our condition on  $a$  is more restrictive in that case: let us consider the eigenvalues  $(\lambda_k)_{k \geq 1}$  of the operator

$$A_0 : \{\phi \in H^4(\Omega) ; \phi = \Delta\phi = 0 \text{ on } \partial\Omega\} \rightarrow L^2(\Omega), \quad \phi \mapsto \Delta^4\phi.$$

We consider that  $\lambda_1$  is the smallest eigenvalue. We know that  $\lambda_1 > 0$ . Our second main result is

**Theorem 1.4.** *Assume that  $a < \sqrt{\lambda_1}$ . Assume also one of the following assertions holds:*

1.  $\partial\Omega$  is of class  $C^5$  and  $\Gamma$  satisfies the geometric optics conditions;
2.  $\Omega$  is a rectangle  $(0, a) \times (0, b)$  and  $\Gamma$  contains both a horizontal and a vertical segment of nonzero length.

*Then the system (1.2) is locally null-controllable for any  $T > 0$ : there exists  $c_0 > 0$  such that for any  $\|y_0\|_{L^2(\Omega)} \leq c_0$ , there exists  $u \in L^2(0, T; L^2(\Gamma))$  and a solution  $y \in C([0, T]; L^2(\Omega))$  of (1.2) with*

$$y(T) = 0.$$

**Remark 1.5.** *The conditions on  $\Gamma$  and on  $a$  are sufficient conditions for the null-controllability of the linear system associated with (1.2). These conditions may not be necessary: we should expect for instance a condition on  $a$  as in Theorem 1.1 (not in a countable set) and for  $\Gamma$  we should expect a condition as for the heat equation (non empty interior). However, to obtain such a result on  $\Gamma$ , one of the only methods is based on some Carleman estimates for a plate type system. Even in 1D, such a method fails to obtain only one control (see for instance [4]). As for Theorem 1.1, one could try to obtain the null-controllability of (1.2) even if the corresponding linear system is not controllable.*

The proofs of Theorem 1.1 and Theorem 1.4 are based on several general results described in Section 3. Using a result in [14], we can reduce the study to linear systems associated with (1.1) and (1.2). The corresponding linear systems are parabolic and to prove their null-controllability, we apply a method due to Russell [18], and consider the controllability or the observability of ‘‘hyperbolic’’ systems associated with them. Finally, in the 2D case, the observability of the corresponding plate system is obtained through a perturbation method developed in [6].

## 2 Notation and preliminaries

Let us consider a Hilbert space  $\mathcal{X}$  and  $A : \mathcal{D}(A) \rightarrow \mathcal{X}$  a generator of a strongly continuous semigroup  $(e^{tA})_{t \geq 0}$ . Let us also consider a Hilbert space  $U$  and  $B \in \mathcal{L}(U, \mathcal{D}(A)^*)$  an operator. We consider the following control problem

$$\dot{z} = Az + Bu, \quad z(0) = z_0. \quad (2.1)$$

We fix  $\lambda_{\#} \in \rho(A)$ . We can equip  $\mathcal{D}(A)$  with the norm  $\|z\|_{\mathcal{D}(A)} := \|(\lambda_{\#} - A)z\|_{\mathcal{X}}$ . Its dual  $\mathcal{D}(A^*)^*$  with respect to pivot space  $\mathcal{X}$  is the completion of  $\mathcal{X}$  with respect to the norm  $\|z\|_{\mathcal{D}(A^*)^*} := \|(\lambda_{\#} - A)^{-1}z\|_{\mathcal{X}}$ . We keep the notation  $A$  and  $(e^{tA})_{t \geq 0}$  for the extensions of the generator to  $\mathcal{X}$  and of the semigroup to  $\mathcal{D}(A^*)^*$ . Assume  $T > 0$ . For  $z_0 \in \mathcal{D}(A^*)^*$  and  $u \in L^2(0, T; U)$ , we can consider a solution  $z$  of (2.1) as a mild solution

$$z(t) = e^{tA}z_0 + \int_0^t e^{(t-s)A}Bu(s) ds.$$

We say that  $B$  is admissible if for any  $z_0 \in \mathcal{X}$  and for any  $u \in L^2(0, T; U)$ , the above solution satisfies  $z \in C([0, T]; \mathcal{X})$ . This is the case if for instance  $A$  is self-adjoint and positive (parabolic case) and if  $B \in \mathcal{L}(U, \mathcal{D}(A^{1/2})^*)$ . In the case where  $B$  is admissible, we say that the system (2.1) (or the pair  $(A, B)$ ) is null-controllable in time  $T$  if for any  $z_0 \in \mathcal{X}$ , there exists  $u \in L^2(0, T; U)$  such that the solution of (2.1) verifies  $z(T) = 0$ . We also say that the system (2.1) (or the pair  $(A, B)$ ) is exactly controllable in time  $T$  if for any  $z_0, z_1 \in \mathcal{X}$ , there exists  $u \in L^2(0, T; U)$  such that  $z(T) = z_1$ . These two notions are related by duality to observability properties:

$$\dot{\phi} = A^*\phi, \quad \psi = B^*\phi, \quad \phi(0) = \phi_0. \quad (2.2)$$

For  $\phi_0 \in \mathcal{D}(A^*)$ ,  $\phi(t) = e^{tA^*}\phi_0$  and  $\psi(t) = B^*e^{tA^*}\phi_0$ . We say that the system (2.2) (or the pair  $(A^*, B^*)$ ) is final-state observable in time  $T$  if there exists  $K(T) > 0$  such that

$$\|e^{TA^*}\phi_0\|_{\mathcal{X}}^2 \leq K(T)^2 \int_0^T \|B^*e^{tA^*}\phi_0\|_U^2 dt \quad (\phi_0 \in \mathcal{D}(A^*)).$$

We say that the system (2.2) (or the pair  $(A^*, B^*)$ ) is exactly observable in time  $T$  if there exists  $K(T) > 0$  such that

$$\|\phi_0\|_{\mathcal{X}}^2 \leq K(T)^2 \int_0^T \|B^*e^{tA^*}\phi_0\|_U^2 dt \quad (\phi_0 \in \mathcal{D}(A^*)).$$

The equivalence between observability and controllability is recalled in the proposition below, due to Dolecki and Russell [8] (see also Proposition 12.1.2 and Theorem 11.2.1 in [21]).

**Proposition 2.1.** *Assume that  $B$  is admissible. Then the pair  $(A, B)$  is exactly controllable in time  $T$  if and only if the pair  $(A^*, B^*)$  is exactly observable in time  $T$ . The pair  $(A, B)$  is null controllable in time  $T$  if and only if the pair  $(A^*, B^*)$  is final state observable in time  $T$ . Moreover, if the pair  $(A^*, B^*)$  is exactly observable in time  $T$ , then there exists an operator  $L_T \in \mathcal{L}(\mathcal{X}, U)$  such that a control for (2.1) is given by  $u = L_T z_0$  and such that  $\|L_T\|_{\mathcal{L}(\mathcal{X}, U)} \leq K(T)$ .*

## 3 Some general results

This section is devoted to the presentation of several important general results that are used in the proofs of Theorem 1.1 and Theorem 1.4.

### 3.1 The Russell method

We recall here a method introduced by Russell [18] and that is completely described in [21] (Section 9.2). The idea is that the final state observability of a parabolic system can be obtained by the exact observability of

an hyperbolic system associated with the parabolic system. More precisely, let us consider a Hilbert space  $\mathcal{H}$  and a self-adjoint operator  $A_0 : \mathcal{D}(A_0) \rightarrow \mathcal{H}$ , strictly positive. We also assume that  $A_0^{-1}$  is compact. Let us also consider a Hilbert space  $U$  and  $C_0 \in \mathcal{L}(\mathcal{D}(A_0^{1/2}), U)$  an operator. We consider the following observability problems

$$\dot{z} = -A_0 z, \quad y = C_0 z, \quad z(0) = z_0 \quad (3.1)$$

and

$$\ddot{w} = -A_0 w, \quad y = C_0 \dot{w}, \quad w(0) = w_0, \quad \dot{w}(0) = w_1. \quad (3.2)$$

We denote by  $(e^{-tA_0})_{t \geq 0}$  the semigroup generated by  $-A_0$ . We introduce an orthonormal basis  $(\varphi_n)_{n \in \mathbb{N}^*}$  in  $\mathcal{H}$  consisting of eigenvectors of  $A_0$  and  $(\lambda_n)_{n \in \mathbb{N}^*}$  the corresponding eigenvalues. Then we have the following result:

**Theorem 3.1.** *Assume that (3.2) is exactly observable in  $\mathcal{D}(A_0^{1/2}) \times \mathcal{H}$  in time  $T_0 > 0$  and that*

$$\sum_{n \in \mathbb{N}^*} \frac{1}{\lambda_n^k} < \infty \quad (3.3)$$

for some  $k \in \mathbb{N}^*$ . Then (3.1) is final state observable in  $\mathcal{H}$  for any time  $T$ :

$$\|e^{-A_0 T} \varphi_0\|_{\mathcal{H}} \leq K(T) \left( \int_0^T \|C_0 e^{-A_0 t} \varphi_0\|_U^2 dt \right)^{1/2}, \quad (3.4)$$

with

$$K(T) := C e^{M/T}. \quad (3.5)$$

This theorem is stated in this way in [21] (Theorem 9.2.2) but we give a sketch of the proof to precise the cost  $K(T)$  in (3.5).

*Proof.* First, (3.2) is exactly controllable in  $\mathcal{D}(A_0^{1/2}) \times \mathcal{H}$  in time  $T_0 > 0$  means that the pair  $(A, C)$  defined by

$$A = \begin{bmatrix} 0 & I \\ -A_0 & 0 \end{bmatrix}, \quad C = [0 \quad C_0]$$

is exactly observable in  $\mathcal{D}(A_0^{1/2}) \times \mathcal{H}$  in time  $T_0 > 0$ .

We define

$$\mu_k = \operatorname{sgn}(k) \sqrt{\lambda_{|k|}} \quad (k \in \mathbb{Z}^*).$$

Then we use Lemma 9.2.1 in [21] and the exact observability of  $(A, C)$  to obtain the existence of a bounded sequence  $(F_n)_{n \in \mathbb{Z}^*}$  in  $L^2(0, T_0; U)$  that is biorthogonal in  $L^2(0, T_0; U)$  to the sequence  $(e^{i\mu_k t} C_0 \varphi_k)_{n \in \mathbb{Z}^*}$  in  $L^2(0, T_0; U)$ .

The idea is to construct from  $(F_n)_{n \in \mathbb{Z}^*}$  a sequence  $(G_n)_{n \in \mathbb{N}^*}$  in  $L^2(0, T; U)$  biorthogonal to  $(e^{-\lambda_k t} C_0 \varphi_k)_{n \in \mathbb{N}^*}$ . This is done in the proof of Theorem 9.2.2 in [21] (see formula (9.2.24), p.294). Using Lemma 9.2.3 in [21], we have

$$\sum_{n \in \mathbb{N}^*} e^{-2T\lambda_n} \|G_n\|_{L^2(0, T; U)}^2 \leq C \sum_{n \in \mathbb{N}^*} e^{-T\lambda_n} (\nu + 1)^2 e^{3\nu/2} e^{-\frac{\beta\lambda_n}{2\sqrt{\nu+1}}},$$

with  $\delta > \max(0, 2(T_0 - \pi))$ ,  $\beta \in (0, \frac{T}{2})$  and  $\nu = (\pi + \delta)^2 / \beta$ . The constant  $C$  appearing in the above estimate only depends on  $\delta$ . Using that  $e^{-x} < k! / x^k$  for all  $x > 0$ , we deduce that for all  $k \in \mathbb{N}^*$ , there exists  $C = C(\delta, k)$  such that

$$\sum_{n \in \mathbb{N}^*} e^{-2T\lambda_n} \|G_n\|_{L^2(0, T; U)}^2 \leq C (\nu + 1)^{2+3k/2} e^{3\nu/2} \sum_{n \in \mathbb{N}^*} \frac{1}{\lambda_n^k}.$$

Gathering the above estimate, (3.3) and the above conditions on  $\beta, \nu, \delta$ , we deduce

$$\sum_{n \in \mathbb{N}^*} e^{-2T\lambda_n} \|G_n\|_{L^2(0, T; U)}^2 \leq C^2 e^{2M/T}.$$

Applying Lemma 9.1.4 in [21], we deduce (3.4) and (3.5).  $\square$

### 3.2 The source term method

We recall a method introduced in [14] and that allows us to deal with the controllability of nonlinear parabolic systems. Assume  $T > 0$ . In what follows, we consider the subset  $\mathcal{R}_T$  of the functions  $\rho$  satisfying the following properties

$$\rho : [0, T] \rightarrow [0, \infty) \text{ continuous, non increasing, positive on } [0, T), \rho(T) = 0. \quad (3.6)$$

We denote by  $L^2_\rho(0, T; Y)$  the  $L^2$  space with values in  $Y$  and with measure  $\lambda_1/\rho^2$  where  $\lambda_1$  denotes the Lebesgue measure on  $(0, T)$ . We consider the same hypotheses as in the previous subsection:  $\mathcal{H}$  is a Hilbert space,  $A_0 : \mathcal{D}(A_0) \rightarrow \mathcal{H}$  is a self-adjoint operator strictly positive such that  $A_0^{-1}$  is compact. We also consider a Hilbert space  $U$  and  $B_0 \in \mathcal{L}(U, \mathcal{D}(A_0^{1/2})^*)$  an operator. To shorten the notation in this subsection, we write

$$\mathcal{H}_{1/2} := \mathcal{D}(A_0^{1/2}), \quad \mathcal{H}_{-1/2} := \mathcal{D}(A_0^{1/2})^*.$$

In this subsection, we consider the following control problem

$$\dot{z} = -A_0 z + B_0 u + f, \quad z(0) = z_0. \quad (3.7)$$

**Theorem 3.2.** *Assume that for all  $t > 0$  we have the observability inequality*

$$\|e^{-A_0 t} \varphi_0\|_{\mathcal{H}} \leq K(t) \left( \int_0^t \|B_0^* e^{-A_0 t} \varphi_0\|_U^2 dt \right)^{1/2}, \quad (3.8)$$

with  $K : (0, \infty) \rightarrow (0, \infty)$  continuous and non increasing. There exist  $\rho_0, \rho_1 \in \mathcal{R}_T$  such that for any  $z_0 \in \mathcal{H}$  and  $f \in L^2_{\rho_1}(0, T; \mathcal{H}_{-1/2})$ , there exists  $u \in L^2_{\rho_0}(0, T; U)$  such that the solutions of (3.7) satisfies

$$\|z\|_{L^2_{\rho_0}(0, T; \mathcal{H}_{1/2})}^2 + \left\| \frac{z}{\rho_0} \right\|_{C^0([0, T]; \mathcal{H})}^2 + \|\dot{z}\|_{L^2_{\rho_0}(0, T; \mathcal{H}_{-1/2})}^2 + \|u\|_{L^2_{\rho_0}(0, T; U)}^2 \leq C \left( \|z_0\|_{\mathcal{H}} + \|f\|_{L^2_{\rho_1}(0, T; \mathcal{H}_{-1/2})} \right). \quad (3.9)$$

In particular, (3.7) is null-controllable.

In the proof of this theorem,  $\rho_0$  and  $\rho_1$  are constructed in the following way: we consider  $q > 1$  and

$$\rho_0(t) := \rho_1(q^2(t - T) + T)K((q - 1)(T - t)) \quad \left( t \in \left[ T \left( 1 - \frac{1}{q^2} \right), T \right] \right), \quad (3.10)$$

and  $\rho_0$  extended on  $\left[ 0, T \left( 1 - \frac{1}{q^2} \right) \right]$  so that  $\rho_0 \in \mathcal{R}_T$ .

The above statement is slightly different from the result in [14] since in our case  $B_0^*$  is not a bounded operator of  $\mathcal{H}$ . The proof is nevertheless completely similar, we only remind the main steps of proof for sake of completeness.

*Sketch of the proof of Theorem 3.2.* We set

$$T_k := T - \frac{T}{q^k} \quad (k \in \mathbb{N}).$$

From (3.10), we have the relation

$$\rho_0(T_{k+2}) = \rho_1(T_k)K(T_{k+2} - T_{k+1}). \quad (3.11)$$

Then we define the following sequence:  $a_0 := z_0$  and for any  $k \geq 0$ ,  $a_{k+1} := z_1(T_{k+1}^-)$  where  $z_1$  is the solution of the system

$$\begin{cases} \dot{z}_1 = -Az_1 + f & \text{in } (T_k, T_{k+1}), \\ z_1(T_k^+) = 0. \end{cases} \quad (3.12)$$

Using the classical results for parabolic system, we have for all  $k \geq 0$ ,

$$\|a_{k+1}\|_{\mathcal{H}}^2 \leq C \|f\|_{L^2(T_k, T_{k+1}; \mathcal{H}_{-1/2})}^2. \quad (3.13)$$

where the constant  $C$  is independent of  $k$ .

Finally, we consider the control system

$$\begin{cases} \dot{z}_2 = -Az_2 + Bu & \text{in } (T_k, T_{k+1}), \\ z_2(T_k^+) = a_k. \end{cases} \quad (3.14)$$

From (3.8) and Proposition 2.1, there exists a control  $u \in L^2(T_k, T_{k+1}; U)$  such that

$$z_2(T_{k+1}^-) = 0$$

and

$$\|u\|_{L^2(T_k, T_{k+1}; U)}^2 \leq K^2(T_{k+1} - T_k) \|a_k\|_{\mathcal{H}}^2. \quad (3.15)$$

We then define  $z = z_1 + z_2$  and using the relations of  $z_1$  and  $z_2$  at  $T_k^-$  and  $T_k^+$ , we can check that  $z$  is solution of (3.7). Moreover, from (3.11), (3.13) and (3.15), we deduce that for all  $k \geq 0$ ,

$$\|u\|_{L_{\rho_0}^2(T_{k+1}, T_{k+2}; U)}^2 \leq C \|f\|_{L_{\rho_1}^2(T_k, T_{k+1}; \mathcal{H}_{-1/2})}^2. \quad (3.16)$$

From the classical results for parabolic system, we have for any  $k \geq 0$ ,

$$\begin{aligned} & \|z\|_{L^2(T_{k+1}, T_{k+2}; \mathcal{H}_{1/2})}^2 + \|z\|_{C^0([T_{k+1}, T_{k+2}]; \mathcal{H})}^2 + \|\dot{z}\|_{L^2(T_{k+1}, T_{k+2}; \mathcal{H}_{-1/2})}^2 \\ & \leq C \left( \|a_{k+1}\|_{\mathcal{H}}^2 + \|u\|_{L^2(T_{k+1}, T_{k+2}; U)}^2 + \|f\|_{L^2(T_{k+1}, T_{k+2}; \mathcal{H}_{-1/2})}^2 \right) \end{aligned} \quad (3.17)$$

Thus, from the above relation, from (3.13) and from (3.15), we deduce that for all  $k \geq 0$ ,

$$\|z\|_{L^2(T_{k+1}, T_{k+2}; \mathcal{H}_{1/2})}^2 + \|z\|_{C^0([T_{k+1}, T_{k+2}]; \mathcal{H})}^2 + \|\dot{z}\|_{L^2(T_{k+1}, T_{k+2}; \mathcal{H}_{-1/2})}^2 \leq C \rho_0(T_{k+2})^2 \|f\|_{L_{\rho_1}^2(T_k, T_{k+2}; \mathcal{H}_{-1/2})}^2. \quad (3.18)$$

Since  $\rho_0$  is non increasing we deduce

$$\|z\|_{L_{\rho_0}^2(T_{k+1}, T_{k+2}; \mathcal{H}_{1/2})}^2 + \left\| \frac{z}{\rho_0} \right\|_{C^0([T_{k+1}, T_{k+2}]; \mathcal{H})}^2 + \|\dot{z}\|_{L_{\rho_0}^2(T_{k+1}, T_{k+2}; \mathcal{H}_{-1/2})}^2 \leq C \|f\|_{L_{\rho_1}^2(T_k, T_{k+2}; \mathcal{H}_{-1/2})}^2. \quad (3.19)$$

A similar estimate is valid for  $(T_0, T_1)$  and thus we deduce (3.9). □

### 3.3 The Perturbation method

In this subsection, we recall some results obtained in [6] in order to deduce the exact observability of

$$\ddot{w} = -A_0 w + a A_0^{1/2} w, \quad y = C_0 v, \quad w(0) = w_0, \quad \dot{w}(0) = w_1 \quad (3.20)$$

from the exact observability of

$$\ddot{w} = -A_0 w, \quad y = C_0 v, \quad w(0) = w_0, \quad \dot{w}(0) = w_1, \quad (3.21)$$

where as the previous subsections,  $\mathcal{H}$  is a Hilbert space,  $A_0 : \mathcal{D}(A_0) \rightarrow \mathcal{H}$  is a strictly positive self-adjoint operator such that  $A_0^{-1}$  is compact. We have denoted by  $A_0^{1/2}$  the square root of  $A_0$  (see for instance [21, p.81]). We also consider a Hilbert space  $U$  and  $B_0 \in \mathcal{L}(U, \mathcal{D}(A_0^{1/2})^*)$  an admissible operator (see Section 2).

We denote by  $\lambda_n$ ,  $n \in \mathbb{N}^*$ , the eigenvalues of  $A_0$ . They are real positive and we denote by  $\lambda_1$  the smallest one.

We have the following result.

**Theorem 3.3.** *Assume that (3.21) is exactly observable and  $a < \sqrt{\lambda_1}$ . Then (3.20) is exactly observable.*

This result is stated [6, Proposition 3.1]. The proof is done in the case where  $C_0$  is a bounded operator (i.e.  $C_0 \in \mathcal{L}(\mathcal{H}, U)$ ) but this assumption is not used. Note that the time of exact observability is not obtained in this theorem, and this is due to the method of proof based on a spectral method (see [17]).

## 4 Controllability of the Kuramoto-Sivashinsky equation in 1D

We are now using the result of the previous section in order to obtain the null-controllability of the Kuramoto-Sivashinsky equation in 1D.

### 4.1 Abstract framework

First we introduce operators in order to rewrite (1.1).

We define

$$\mathcal{D}(A_0) = \{y \in H^4(0, 1) ; y = y_x = 0 \text{ in } \{0, 1\}\}, \quad (4.1)$$

$$A_0 : \mathcal{D}(A_0) \rightarrow L^2(0, 1), \quad y \mapsto y_{xxxx} + ay_{xx}. \quad (4.2)$$

The operator  $A_0$  is self-adjoint and  $-A_0$  is the infinitesimal generator of a parabolic semigroup  $(e^{-tA_0})_{t \geq 0}$ .

In what follows, we consider  $\lambda_{\#} \in \rho(A_0)$  such that  $A_{\#} := \lambda_{\#} + A_0$  is positive and we set

$$\mathcal{H}^s := \mathcal{D}(A_{\#}^{s/4}) \quad (s \geq 0), \quad \mathcal{H}^{-s} := \mathcal{D}(A_{\#}^{s/4})' \quad (s \geq 0), \quad (4.3)$$

where  $\mathcal{X}'$  denotes the dual of  $\mathcal{X}$  with respect to the space  $L^2(0, 1)$ .

We can consider  $A_0$  as an operator

$$A_0 : \mathcal{H}^{s+4} \rightarrow \mathcal{H}^s, \quad (s \in \mathbb{R}).$$

In particular,

$$A_{\#} : \mathcal{H}^{s+4} \rightarrow \mathcal{H}^s, \quad (s \in \mathbb{R})$$

is a unitary transformation.

In order to obtain a weak formulation for system (1.1), we multiply formally the first equation of (1.1) by  $\varphi \in \mathcal{H}^4$ . After some integrations by parts, we obtain

$$\frac{d}{dt} \int_0^1 y \varphi \, dx + \int_0^1 y A_0 \varphi \, dx - \int_0^1 \left( \frac{y^2}{2} \right) \varphi_x \, dx = u(t) \varphi_{xx}(t, 0). \quad (4.4)$$

In order to write the corresponding control operator, we use a classical approach and define the operator  $D_0 : U := \mathbb{R} \rightarrow \mathcal{H}^1 = H_0^1(0, 1)$  as follows:  $D_0 u = y$  is the solution of

$$\begin{cases} \lambda_{\#} y + (y_{xxxx} + ay_{xx}) = 0, & x \in (0, 1), \\ y(0) = y(1) = 0, \\ y_x(0) = u, \quad y_x(1) = 0. \end{cases} \quad (4.5)$$

More precisely,  $D_0 u$  is the unique element of  $\mathcal{H}^1 = H_0^1(0, 1)$  such that

$$\langle D_0 u, f \rangle_{\mathcal{H}^1, \mathcal{H}^{-1}} = u \left( A_{\#}^{-1} f \right)_{xx} (0). \quad (4.6)$$

Since  $A_{\#}^{-1}(\mathcal{H}^{-1}) = \mathcal{H}^3 \subset H^3(0, 1)$ ,  $D_0 u$  is well-defined and  $D_0 \in \mathcal{L}(U, \mathcal{H}^1)$ .

Then, we define the operator  $B_0$  by

$$B_0 := A_{\#} D_0 : U = \mathbb{R} \rightarrow \mathcal{H}^{-3}. \quad (4.7)$$

One can check that for all  $\varphi \in \mathcal{H}^3$  and  $u \in \mathbb{R}$ ,

$$\langle B_0 u, \varphi \rangle_{\mathcal{H}^{-3}, \mathcal{H}^3} = u \varphi_{xx}(0).$$

Finally, we define  $F : L^2(0, 1) \rightarrow \mathcal{H}^{-2}$  by

$$\langle F(y), \varphi \rangle_{\mathcal{H}^{-2}, \mathcal{H}^2} := - \int_0^1 \left( \frac{y^2}{2} \right) \varphi_x \, dx \quad (\varphi \in \mathcal{H}^2). \quad (4.8)$$



The function  $F$  is well-defined since  $H^1(0, 1) \subset L^\infty(0, 1)$ .

The above definitions allow us to write (4.4) as

$$y_t + A_0 y + F(y) = B_0 u, \quad (4.9)$$

and if  $u \in L^2(0, T)$ , we consider solutions satisfying the following regularity

$$y \in H^1(0, T; \mathcal{H}^{-3}) \cap L^2(0, T; \mathcal{H}^1) \cap C([0, T]; \mathcal{H}^{-1}). \quad (4.10)$$

Note that if  $y$  satisfies (4.10), then

$$y \in L^4(0, T; L^2(0, 1))$$

and in particular  $F(y) \in L^2(0, T; \mathcal{H}^{-2})$ .

## 4.2 Controllability of the linear problem

Using the previous section, we can write

$$\begin{cases} y_t + y_{xxxx} + ay_{xx} = 0, & x \in (0, 1), t \in (0, T) \\ y(t, 0) = 0, \quad y(t, 1) = 0, & t \in (0, T), \\ y_x(t, 0) = u(t), \quad y_x(t, 1) = 0, & t \in (0, T), \\ y(0, x) = y_0(x), & x \in (0, 1). \end{cases} \quad (4.11)$$

as

$$y_t + A_0 y = B_0 u, \quad y(0) = y_0.$$

The main result of this section is the following result.

**Theorem 4.1.** *Assume  $a \notin \mathcal{N}$ . For  $y_0 \in \mathcal{H}^{-1}$ , there exists  $u \in L^2(0, T)$  such that  $y(T) = 0$ . Moreover, there exists an operator  $L_T \in \mathcal{L}(\mathcal{H}^{-1}, \mathbb{R})$  such that  $u$  can be given by  $u = L_T(y_0)$ . We have*

$$\|L_T\|_{\mathcal{L}(\mathcal{H}^{-1}, \mathbb{R})} \leq K(T)$$

with

$$K(T) := Ce^{M/T}, \quad (4.12)$$

for some positive constants  $C, M$ .

*Proof of Theorem 4.1.* First, we show that the adjoint of  $B_0$  defined by (4.7) is

$$C_0 := B_0^* : \mathcal{H}^1 \rightarrow \mathbb{R}, \quad \varphi \mapsto \left( A_{\#}^{-1/2} \varphi \right)_{xx}(0). \quad (4.13)$$

Indeed, considering  $A_0$  as an operator of  $\mathcal{H}^{-1}$  with domain  $D(A_0) = \mathcal{H}^3$ , we have  $D(A_0^{1/2}) = \mathcal{H}^1$  and the dual  $D(A_0^{1/2})^*$  of  $D(A_0^{1/2})$  with respect to  $\mathcal{H}^{-1}$  is  $D(A_0^{1/2})^* = \mathcal{H}^{-3}$  and

$$\langle B_0 u, \varphi \rangle_{D(A_0^{1/2})^*, D(A_0^{1/2})} = \langle A_{\#}^{1/2} D_0 u, \varphi \rangle_{\mathcal{H}^{-1}, \mathcal{H}^1} = u(A_{\#}^{-1/2} \varphi)_{xx}(0).$$

where we have used (4.6).

Second, we show that  $(-A_0, C_0)$  is final state observable in any time  $T > 0$  with

$$A_0 : \mathcal{H}^4 \rightarrow L^2(0, 1) \quad \text{and} \quad C_0 : \mathcal{H}^2 \rightarrow \mathbb{R}$$

the last operator being the restriction of  $C_0$  defined by (4.13) to  $\mathcal{H}^2$ . In order to do that, we are going to use Theorem 3.1 and thus to consider the exact observability in  $\mathcal{H}^2 \times \mathcal{H}$  of

$$\begin{cases} \ddot{w} = -A_0 w, \\ y = C_0 \dot{w}. \end{cases} \quad (4.14)$$

Using Lemma 2.1 in [2], we have that if  $a \notin \mathcal{N}$ , then the eigenvalues  $(\lambda_k)$  of  $A_0$  are simple and using Lemma 2.2 in [2], we also have that

$$\lambda_k \sim D_1 k^4, \quad D_1 > 0. \quad (4.15)$$

The two properties imply that the eigenvalues admit a gap: there exists  $\gamma > 0$  such that

$$\lambda_{k+1} - \lambda_k \geq \gamma \quad (k \in \mathbb{N}).$$

Let us denote by  $(\phi_k)$  a family of eigenvectors of  $A_0$  associated to  $(\lambda_k)$  and forming an orthonormal basis of  $\mathcal{H}$ . We show the exact observability of (4.14) by applying a spectral method: using Proposition 4.5 in [17], we have that (4.14) is exactly observable in some time  $T_0$  if and only if there exists  $\delta > 0$  such that

$$|C_0 \phi_k| \geq \delta \quad (k \in \mathbb{N}). \quad (4.16)$$

From (4.13),

$$C_0 \phi_k = \frac{(\phi_k)''(0)}{\sqrt{\lambda_{\#} + \lambda_k}}$$

and using Lemma 2.1 and Lemma 2.2 in [2], we deduce that  $C_0 \phi_k \neq 0$  and

$$\lim_{k \rightarrow \infty} |C_0 \phi_k| = d_1 > 0.$$

This implies (4.16) and thus the exact observability of (4.14) for some time  $T_0 > 0$ .

Now, from (4.15), we deduce (3.3) and thus applying Theorem 3.1, we obtain

$$\|e^{-A_0 T} \varphi_0\|_{\mathcal{H}} \leq K(T) \left( \int_0^T \|B_0^* e^{-A_0 t} \varphi_0\|_{\mathcal{U}}^2 dt \right)^{1/2} \quad (\varphi_0 \in \mathcal{H}^1) \quad (4.17)$$

and there exist  $C$  and  $M$  such that

$$K(T) := C e^{M/T}.$$

This implies in particular

$$\|e^{-A_0 T} \varphi_0\|_{\mathcal{H}^{-1}} \leq K(T) \left( \int_0^T \|B_0^* e^{-A_0 t} \varphi_0\|_{\mathcal{U}}^2 dt \right)^{1/2} \quad (\varphi_0 \in \mathcal{H}^1). \quad (4.18)$$

From Proposition 2.1, we deduce that (4.11) is null-controllable for any time  $T > 0$  in  $\mathcal{H}^{-1}$ : for any  $y_0 \in \mathcal{H}^{-1}$ , there exists  $u = L_T(y_0) \in L^2(0, T; U)$  with

$$\|u\|_{L^2(0, T; \mathbb{R})} \leq K(T) \|y_0\|_{\mathcal{H}^{-1}}$$

such that  $y(T) = 0$ . □

### 4.3 Proof of Theorem 1.1

We are now in position to prove Theorem 1.1.

We recall that  $K$  satisfies (4.12). We take

$$q \in (1, 2^{1/4}), \quad \alpha > \frac{MTq^4}{2(q-1)}, \quad (4.19)$$

$$\rho_1(t) = \exp\left(-\frac{\alpha}{(T-t)^2}\right). \quad (4.20)$$

We can define  $\rho_0$  as

$$\rho_0(t) = C \exp\left(-\frac{\alpha}{q^4(T-t)^2} + \frac{M}{(q-1)(T-t)}\right) \quad (t \in [0, T]). \quad (4.21)$$

In particular, from (4.19),  $\rho_0$  is continuous and decreasing and satisfies (3.10). We can also check that

$$\left|\frac{(\rho_0)^2}{\rho_1}\right| \leq C \quad ((t \in [0, T])). \quad (4.22)$$

*Proof.* Applying Theorem 3.2 and Theorem 4.1, we deduce that for any  $y_0 \in \mathcal{H}^{-1}$  and  $f \in L_{\rho_1}^2(0, T; \mathcal{H}^{-3})$ , there exists  $u \in L_{\rho_0}^2(0, T; U)$  such that the solutions of

$$\begin{cases} y_t + y_{xxxx} + ay_{xx} = f, & x \in (0, 1), t \in (0, T) \\ y(t, 0) = 0, \quad y(t, 1) = 0, & t \in (0, T), \\ y_x(t, 0) = u(t), \quad y_x(t, 1) = 0, & t \in (0, T), \\ y(0, x) = y_0(x), & x \in (0, 1). \end{cases} \quad (4.23)$$

satisfies

$$\|y\|_{L_{\rho_0}^2(0, T; \mathcal{H}^1)}^2 + \left\|\frac{y}{\rho_0}\right\|_{C^0([0, T]; \mathcal{H}^{-1})}^2 + \|y_t\|_{L_{\rho_0}^2(0, T; \mathcal{H}^{-3})}^2 + \|u\|_{L_{\rho_0}^2(0, T)}^2 \leq C \left(\|y_0\|_{\mathcal{H}^{-1}}^2 + \|f\|_{L_{\rho_1}^2(0, T; \mathcal{H}^{-3})}^2\right). \quad (4.24)$$

Now, let us consider the map

$$\mathcal{Z} : f \in L_{\rho_1}^2(0, T; \mathcal{H}^{-3}) \mapsto -F(y) \in L_{\rho_1}^2(0, T; \mathcal{H}^{-3})$$

where  $(y, u)$  is the solution of (4.23) and where  $F$  is defined by (4.8). We show that  $\mathcal{Z}$  is well-defined and that there exists  $R$  such that

$$\mathcal{Z}(B(0, R)) \subset B(0, R)$$

where  $B(0, R)$  is the closed ball of  $L_{\rho_1}^2(0, T; \mathcal{H}^{-3})$  of radius  $R > 0$ .

First using (4.8), we deduce that

$$\|F(y)\|_{\mathcal{H}^{-2}} \leq C \|y\|_{\mathcal{H}^1} \|y\|_{\mathcal{H}^{-1}}$$

Therefore, from (4.22) and (4.24)

$$\|F(y)\|_{L_{\rho_0}^2(0, T; \mathcal{H}^{-2})} \leq C \left(\|y_0\|_{\mathcal{H}^{-1}}^2 + \|f\|_{L_{\rho_1}^2(0, T; \mathcal{H}^{-3})}^2\right).$$

Assume that

$$R > \|y_0\|_{\mathcal{H}^{-1}}$$

and take

$$\|f\|_{L_{\rho_1}^2(0, T; \mathcal{H}^{-3})} \leq R.$$

Then,

$$\|F(y)\|_{L_{\rho_0}^2(0, T; \mathcal{H}^{-2})} \leq 2CR^2 \leq R$$

if we assume

$$R = \frac{1}{4C}. \quad (4.25)$$

We now show that  $\mathcal{N}$  is a contraction on  $B(0, R)$ : let us take  $f^{(1)}, f^{(2)} \in B(0, R)$  and let us consider the corresponding solutions and controls  $(y^{(1)}, u^{(1)})$ ,  $(y^{(2)}, u^{(2)})$ . We write

$$f := f^{(1)} - f^{(2)}, \quad y := y^{(1)} - y^{(2)}, \quad u := u^{(1)} - u^{(2)}.$$

We have

$$\|y\|_{L^2_{\rho_0}(0,T;\mathcal{H}^1)}^2 + \left\| \frac{y}{\rho_0} \right\|_{C^0([0,T];\mathcal{H}^{-1})}^2 + \|y_t\|_{L^2_{\rho_0}(0,T;\mathcal{H}^{-3})}^2 + \|u\|_{L^2_{\rho_0}(0,T)}^2 \leq C \|f\|_{L^2_{\rho_1}(0,T;\mathcal{H}^{-3})}^2. \quad (4.26)$$

From (4.8), we obtain

$$\|F(y^{(1)}) - F(y^{(2)})\|_{\mathcal{H}^{-2}} \leq C(\|y^{(1)}\|_{\mathcal{H}^1} + \|y^{(2)}\|_{\mathcal{H}^1})\|y\|_{\mathcal{H}^{-1}}$$

Combining (4.24) and (4.26) yields

$$\|F(y^{(1)}) - F(y^{(2)})\|_{L^2_{\rho_0}(0,T;\mathcal{H}^{-2})} \leq 2CR \|f\|_{L^2_{\rho_1}(0,T;\mathcal{H}^{-3})}$$

and we deduce from (4.25) that  $\mathcal{N}$  is a contraction.  $\square$

## 5 Controllability of the Kuramoto-Sivashinsky equation in 2D

We show in this section the null-controllability of the Kuramoto-Sivashinsky equation in 2D (Theorem 1.4).

### 5.1 The linear system

We start by considering the null-controllability of a linear system associated with (1.2):

$$\begin{cases} y_t + \Delta^4 y + a\Delta y = 0, & x \in \Omega, t \in (0, T) \\ y(t) = 0, & x \in \partial\Omega, t \in (0, T), \\ \Delta y(t) = u\mathbf{1}_\Gamma, & x \in \partial\Omega, t \in (0, T), \\ y(0, x) = y_0(x), & x \in \Omega. \end{cases} \quad (5.1)$$

We define

$$\mathcal{D}(A_0) = \{y \in H^4(\Omega) ; y = \Delta y = 0 \text{ on } \partial\Omega\}, \quad (5.2)$$

$$A_0 : \mathcal{D}(A_0) \rightarrow L^2(\Omega), \quad y \mapsto \Delta^4 y. \quad (5.3)$$

The operator  $A_0$  is self-adjoint, positive and with compact resolvent. We denote by  $\lambda_n, n \in \mathbb{N}^*$ , the eigenvalues of  $A_0$ . They are real positive and we denote by  $\lambda_1$  the smallest one. Assume  $a < \sqrt{\lambda_1}$ . We also consider the operator

$$\mathcal{D}(A_a) = \mathcal{D}(A_0), \quad (5.4)$$

$$A_a : \mathcal{D}(A_0) \rightarrow L^2(\Omega), \quad y \mapsto \Delta^4 y + a\Delta y. \quad (5.5)$$

The operator  $A_a$  is self-adjoint and  $-A_a$  is the generator of a semi-group  $(e^{-tA_a})_{t \geq 0}$ .

In what follows, we consider  $\lambda_\# \in \rho(A_a)$  such that  $A_\# := \lambda_\# + A_a$  is positive and we set

$$\mathcal{H}^s := \mathcal{D}(A_\#^{s/4}) \quad (s \geq 0), \quad \mathcal{H}^{-s} := \mathcal{D}(A_\#^{s/4})' \quad (s \geq 0), \quad (5.6)$$

where  $\mathcal{X}'$  denotes the dual of  $\mathcal{X}$  with respect to the space  $L^2(\Omega)$ .

We can consider  $A_a$  as an operator

$$A_a : \mathcal{H}^{s+4} \rightarrow \mathcal{H}^s, \quad (s \in \mathbb{R}).$$

In particular,

$$A_0 : \mathcal{H}^{s+4} \rightarrow \mathcal{H}^s, \quad (s \in \mathbb{R})$$

is an isomorphism.

In order to obtain a weak formulation for system (5.1), we multiply formally the first equation of (5.1) by  $\varphi \in \mathcal{H}^4$ . After some integrations by parts, we obtain

$$\frac{d}{dt} \int_{\Omega} y \varphi \, dx + \int_{\Omega} y A_a \varphi \, dx = \int_{\Gamma} u \frac{\partial \varphi}{\partial \nu} \, d\gamma. \quad (5.7)$$

In order to write the corresponding control operator, we use a classical approach and define the operator  $D_0 : U = L^2(\Gamma) \rightarrow \mathcal{H}^2$  as follows:  $D_0 u = y$  is the solution of

$$\begin{cases} \Delta^4 y = 0 & \text{in } \Omega, \\ y = 0 & \text{on } \partial\Omega, \\ \Delta y = u 1_{\Gamma} & \text{on } \partial\Omega. \end{cases} \quad (5.8)$$

More precisely,  $D_0 u$  is the unique element of  $\mathcal{H}^2$  such that

$$\langle D_0 u, f \rangle_{\mathcal{H}^2, \mathcal{H}^{-2}} = \int_{\Gamma} u \frac{\partial}{\partial \nu} (A_0^{-1} f) \, d\gamma. \quad (5.9)$$

Since  $A_0^{-1}(\mathcal{H}^{-2}) = \mathcal{H}^2$ ,  $D_0 u$  is well-defined.

Then, we define the operator  $B_0$  by

$$B_0 := -A_0 D_0 : U \rightarrow \mathcal{H}^{-2}. \quad (5.10)$$

One can check that for all  $\varphi \in \mathcal{H}^2$  and  $u \in U$ ,

$$\langle B_0 u, \varphi \rangle_{\mathcal{H}^{-2}, \mathcal{H}^2} = \int_{\Gamma} u \frac{\partial \varphi}{\partial \nu} \, d\gamma. \quad (5.11)$$

As a consequence, (5.1) writes as

$$\dot{y} = -A_0 y - a A_0^{1/2} y + B_0 u, \quad y(0) = y_0. \quad (5.12)$$

Considering  $A_{\#}$  as an operator from  $\mathcal{H}^4$  onto  $\mathcal{H}$ , we deduce that

$$B_0 : U \rightarrow D(A_{\#}^{1/2})^*$$

and from (5.11), its adjoint satisfies

$$C_0 := B_0^* : \mathcal{H}^2 \rightarrow U, \varphi \mapsto \frac{\partial \varphi}{\partial \nu} \Big|_{\Gamma}$$

We are now in position to prove the null-controllability of (5.1)

**Theorem 5.1.** *Assume  $a < \sqrt{\lambda_1}$  and assume conditions 1) or 2) of Theorem 1.4. For  $y_0 \in \mathcal{H}$ , there exists  $u \in L^2(0, T; U)$  such that the solution of (5.1) satisfies  $y(T) = 0$ . Moreover, there exists an operator  $L_T \in \mathcal{L}(\mathcal{H}, U)$  such that  $u$  can be given by  $u = L_T(y_0)$ . We have*

$$\|L_T\|_{\mathcal{L}(\mathcal{H}^{-1}, \mathbb{R})} \leq K(T)$$

with

$$K(T) := C e^{M/T}, \quad (5.13)$$

for some positive constants  $C, M$ .

*Proof.* In order to obtain the controllability of (5.12), we use two results on the plate equations obtained by [13] and [20] (see also [17] and Remark 7.5.6 in [21]): assume one of the conditions of Theorem 1.4 then the system

$$\begin{cases} \ddot{w} = -A_0 w, \\ y = C_0 \dot{w}. \end{cases} \quad (5.14)$$

is exactly observable in  $\mathcal{H}^3 \times \mathcal{H}^1$ .

Then using Theorem 3.3, we deduce that

$$\begin{cases} \ddot{w} = -A_0 w + aA_0^{1/2} w, \\ y = C_0 \dot{w}. \end{cases} \quad (5.15)$$

is exactly observable in  $\mathcal{H}^3 \times \mathcal{H}^1$  for  $a < \sqrt{\lambda_1}$ .

Moreover, one has (see, for instance [21, Proposition 3.6.9]) that the eigenvalues  $(\lambda_n(a))_{n \in \mathbb{N}^*}$  of  $A_a$  satisfies

$$c_1 n^2 \leq \lambda_n(a) \leq c_2 n^2 \quad (n \geq n_0).$$

for some positive constants  $c_1$  and  $c_2$  and some  $n_0 \geq 1$ . We can apply Theorem 3.1:

$$\begin{cases} \dot{z} = -A_0 z + aA_0^{1/2} z, \\ y = C_0 \dot{z}. \end{cases} \quad (5.16)$$

is final state state observable in  $\mathcal{H}^1$  for any time  $T > 0$  with a cost  $K$  satisfying (3.5). This means that

$$\|e^{-A_a T} \varphi_0\|_{\mathcal{H}^1} \leq K(T) \left( \int_0^T \|B_0^* e^{-A_a t} \varphi_0\|_U^2 dt \right)^{1/2} \quad (\varphi_0 \in \mathcal{H}^2) \quad (5.17)$$

and there exist  $C$  and  $M$  such that

$$K(T) := C e^{M/T}.$$

This implies in particular

$$\|e^{-A_a T} \varphi_0\|_{\mathcal{H}} \leq K(T) \left( \int_0^T \|B_0^* e^{-A_a t} \varphi_0\|_U^2 dt \right)^{1/2} \quad (\varphi_0 \in \mathcal{H}^2). \quad (5.18)$$

From standard duality arguments, we deduce that (5.1) is null-controllable for any time  $T > 0$  in  $\mathcal{H}$ : for any  $y_0 \in \mathcal{H}$ , there exists  $u \in L^2(0, T; U)$  with

$$\|u\|_{L^2(0, T; U)} \leq K(T) \|y_0\|_{\mathcal{H}}$$

such that  $y(T) = 0$ . □

## 5.2 Proof of Theorem 1.4

We are now in position to prove Theorem 1.4. The proof is similar to the proof of Theorem 1.1: we take  $q$ ,  $\alpha$ ,  $\rho_1$  and  $\rho_0$  as in (4.19), (4.20), (4.21). In particular, (4.22) holds.

*Proof.* Applying Theorem 3.2 and Theorem 5.1, we deduce that for any  $y_0 \in \mathcal{H}$  and  $f \in L_{\rho_1}^2(0, T; \mathcal{H}^{-2})$ , there exists  $u \in L_{\rho_0}^2(0, T; U)$  such that the solutions of

$$\begin{cases} y_t + \Delta^4 y + a \Delta y = f, & x \in \Omega, t \in (0, T) \\ y(t) = 0, & x \in \partial\Omega, t \in (0, T), \\ \Delta y(t) = u \mathbf{1}_\Gamma, & x \in \partial\Omega, t \in (0, T), \\ y(0, x) = y_0(x), & x \in \Omega \end{cases} \quad (5.19)$$

satisfies

$$\|y\|_{L^2_{\rho_0}(0,T;\mathcal{H}^2)}^2 + \left\| \frac{y}{\rho_0} \right\|_{C^0([0,T];\mathcal{H})}^2 + \|y_t\|_{L^2_{\rho_0}(0,T;\mathcal{H}^{-2})}^2 + \|u\|_{L^2_{\rho_0}(0,T;U)}^2 \leq C \left( \|y_0\|_{\mathcal{H}}^2 + \|f\|_{L^2_{\rho_1}(0,T;\mathcal{H}^{-2})}^2 \right). \quad (5.20)$$

Now, let us consider the map

$$\mathcal{Z} : f \in L^2_{\rho_1}(0,T;\mathcal{H}^{-2}) \mapsto -F(y) \in L^2_{\rho_1}(0,T;\mathcal{H}^{-2})$$

where  $(y, u)$  is the solution of (5.19)-(5.20) and where  $F$  is defined by:

$$F(y) = -\frac{1}{2}|\nabla y|^2$$

We show that  $\mathcal{Z}$  is well-defined and that there exists  $R$  such that

$$\mathcal{Z}(B(0, R)) \subset B(0, R)$$

where  $B(0, R)$  is the closed ball of  $L^2_{\rho_1}(0,T;\mathcal{H}^{-2})$  of radius  $R > 0$ .

First using the Sobolev embedding  $\mathcal{H}^2 \subset L^\infty(\Omega)$  and an interpolation argument, we deduce that

$$\|F(y)\|_{\mathcal{H}^{-2}} \leq C\|y\|_{\mathcal{H}^1}^2 \leq C\|y\|_{\mathcal{H}^2}\|y\|_{\mathcal{H}}.$$

Therefore, from (4.22) and (5.20)

$$\|F(y)\|_{L^2_{\rho_1}(0,T;\mathcal{H}^{-2})} \leq C \left( \|y_0\|_{\mathcal{H}}^2 + \|f\|_{L^2_{\rho_1}(0,T;\mathcal{H}^{-2})}^2 \right).$$

Using this estimate, we can apply the Banach fixed point theorem as in the proof of Theorem 1.1.  $\square$

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