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A SUM-SHUFFLE FORMULA FOR ZETA VALUES IN TATE ALGEBRAS

F. PELLARIN

Abstract. We prove a sum-shuffle formula for double zeta values in Tate algebras, introduced in [8]. This follows from an analog result for double twisted power sums, implying that an $\mathbb{F}_p$-vector space generated by multiple zeta values in Tate algebras is an $\mathbb{F}_p$-algebra.

1. Introduction

Let $A = \mathbb{F}_q[\theta]$ be the ring of polynomials in an indeterminate θ with coefficients in $\mathbb{F}_q$ the finite field with q elements and characteristic p, and let K be the fraction field of A. We consider variables t_i independent over K, for all $i \in \mathbb{N}^* := \{1, \ldots\}$ the set of positive natural numbers. For $\Sigma \subset \mathbb{N}^*$ a finite subset, we denote by $\underline{t}_\Sigma$ the collection of variables $(t_i)_{i\in\Sigma}$, so that $\mathbb{F}_q(\underline{t}_\Sigma)$, $K(\underline{t}_\Sigma)$ denote the fields $\mathbb{F}_q(t_i : i \in \Sigma)$ etc.

For $d \geq 0$ an integer, we denote by $A^+(d)$ the set of monic polynomials of A of degree d, which is a multiplicative monoid. We denote by $\chi_{t_i} : A \to \mathbb{F}_q[t_i]$ the unique $\mathbb{F}_q$-linear map which sends θ to t. More generally for $\Sigma \subset \mathbb{N}^*$ a finite subset, we denote by σ_Σ the semi-character

$$\sigma_\Sigma : A^+ \to \mathbb{F}_q[\underline{t}_\Sigma]$$

defined by $\sigma_\Sigma(a) = \prod_{i\in\Sigma} \chi_{t_i}(a)$. The associated *twisted power sum* of order k and degree d is the sum:

$$S_d(k; \sigma_\Sigma) = \sum_{a\in A^+(d)} a^{-k}\sigma_\Sigma(a) \in K[\underline{t}_\Sigma].$$

If $\Sigma = \emptyset$ we recover the power sums already studied by several authors; see Thakur's [12] and the references therein. For general Σ these sums have been the object of study, for example, in the papers [1, 6]. These twisted power sums are also the basic tools to construct certain zeta values in Tate algebras, see for example [1, 2, 3, 7]. We set:

$$\zeta_A(n; \sigma_\Sigma) := \sum_{d\geq 0} S_d(n; \sigma_\Sigma) \in \mathbb{T}_\Sigma(K_\infty), \quad n > 0$$

where, for L a complete valued field, $\mathbb{T}_\Sigma(L)$ denotes the completion of the polynomial ring $L[\underline{t}_\Sigma]$ for the Gauss valuation. In [2, 3], the following result is proved, where $|\Sigma|$ denotes the cardinality of Σ:

Theorem 1. *If $|\Sigma| \equiv 1 \pmod{q-1}$ and $|\Sigma| > 1$, there exists a polynomial $\lambda_{1,\Sigma} \in A[\underline{t}_\Sigma]$, monic of degree $r := \frac{s-q}{q-1}$ in θ, such that*

$$\zeta_A(1; \sigma_\Sigma) = (-1)^{\frac{s-1}{q-1}} \frac{\widetilde{\pi}\lambda_{1,\Sigma}}{\prod_{i\in\Sigma} \omega(t_i)},$$

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where $\widetilde{\pi}$ is a fundamental period of Carlitz module, and ω denotes the Anderson-Thakur function.

We also recall from [7] the formula

$$\zeta_A(1;\chi_t) = \frac{\widetilde{\pi}}{(\theta - t)\omega(t)} \tag{1}$$

which complements Theorem 1 in the case $\Sigma = \{1\}$ (and $t = t_1$). One captivating peculiarity of the above formulas is that they constitute a bridge to a class of quite simple objects. For example, the formula (1) can be rewritten (see [7]) as

$$\zeta_A(1,\chi_t) = \prod_{i>0} \frac{1 - \frac{t}{\theta^{q^i}}}{1 - \frac{\theta}{\theta^{q^i}}},$$

and shows that $\zeta_A(1,\chi_t)$ is the reduction modulo p of a simple formal series in $1 + \frac{1}{\theta}\mathbb{Z}[t][[\frac{1}{\theta}]]$ (a "Mahler's series"). The dependency of q, although unavoidable, is very transparent, as it is only involved in the "raising to the power q" process, while this is not necessarily visible at the first sight in the initial definition of $\zeta_A(1,\sigma_\Sigma)$, and neither is in its retranscription as an Eulerian product

$$\zeta_A(1,\sigma_\Sigma) = \prod_P \left(1 - \frac{\chi_t(P)}{P}\right)^{-1},$$

running over the irreducible polynomials P of A^+.

A similar phenomenon holds in the case $s = |\Sigma| > 1$ of Theorem 1, at least for q big enough. The polynomial $\lambda_{1,\Sigma}$ itself, for example, is a polynomial monic in θ of degree r independently of q, as we have already pointed out, and it is easy to deduce, from the elegant arguments presented in [4] (see also the table therein), that the coefficient of θ^{r-1} in $\lambda_{1,\Sigma}$ is equal, for q big enough, to the reduction modulo p of

$$-e_{s-q+1}(\underline{t}_\Sigma) - e_{s-2(q-1)}(\underline{t}_\Sigma) - \cdots - e_q(\underline{t}_\Sigma) \in \mathbb{Z}[\underline{t}_\Sigma],$$

where $e_n(\underline{t}_\Sigma)$ is the elementary symmetric polynomial of degree n in the variables $\underline{t}_\Sigma$ (this for $s \geq 2q-1$ and $s \equiv 1 \pmod{q-1}$, while in the case $s = q$, we have $\lambda_{1,\Sigma} = 1$). These phenomena ultimately arise because certain universality features of sequences of power sums hold, and more generally, the same principles govern the behavior of twisted power sums (see [12] and [6]).

In this paper, we analyze another aspect of the above principles. We shall show (Theorem 3) that any product $\zeta_A(1,\sigma_U)\zeta_A(1,\sigma_V)$ ($U, V \subset \mathbb{N}^*$, $U \cap V = \emptyset$) of such zeta values satisfies a *sum-shuffle product formula,* and this will be again deduced from properties of twisted power sums (Theorem 6). We deduce that a certain $\mathbb{F}_p$-vector space of multiple zeta values in Tate algebras also has a structure of $\mathbb{F}_p$-algebra (Theorem 5). The formula of Theorem 3 is submitted to universal rules very similar to those of the above remarks.

2. The result

Before presenting the result, we have to now recall, from [8], multiple twisted power sums and multiple zeta values in our context.

Definition 2. Let $\Sigma \subset \mathbb{N}^*$ be a finite subset. If U, V are two subsets of Σ such that $U \cap V = \emptyset$, we denote by $U \sqcup V$ their union. Now, suppose that for an integer $r > 0$, we have subsets U_i

$(i = 1, \dots, r)$ such that $\Sigma = U_1 \sqcup \cdots \sqcup U_r$. Further, let d be a non-negative integer. We have the *multiple twisted power sum* of degree d associated to these data:

$$S_d\begin{pmatrix}\sigma_{U_1} & \sigma_{U_2} & \cdots & \sigma_{U_r}\\ n_1 & n_2 & \cdots & n_r\end{pmatrix} = S_d(n_1;\sigma_{U_1}) \sum_{d>i_2>\cdots>i_r\geq 0} S_{i_2}(n_2;\sigma_{U_2})\cdots S_{i_r}(n_r;\sigma_{U_r}) \in K[\underline{t}_\Sigma].$$

The integer $\sum_i n_i$ is called the *weight* and the integer r is called its *depth*.

We can write in both ways $S_d(n;\sigma_\Sigma) = S_d\binom{\sigma_\Sigma}{n}$. Observe also that, if $\Sigma = \emptyset$, then $\sigma_\Sigma = \mathbf{1}$ the trivial semi-character.

$$S_d\begin{pmatrix}\mathbf{1} & \mathbf{1} & \cdots & \mathbf{1}\\ n_1 & n_2 & \cdots & n_r\end{pmatrix} = S_d(n_1, n_2, \dots, n_r) \in K,$$

in the notations of Thakur.

With $n_1, \dots, n_r \geq 1$ and the semi-characters $\sigma_{U_1}, \dots, \sigma_{U_r}$ as above, we introduce the associated *multiple zeta value*

$$\zeta_A\begin{pmatrix}\sigma_{U_1} & \sigma_{U_2} & \cdots & \sigma_{U_r}\\ n_1 & n_2 & \cdots & n_r\end{pmatrix} := \sum_{d\geq 0} S_d\begin{pmatrix}\sigma_{U_1} & \sigma_{U_2} & \cdots & \sigma_{U_r}\\ n_1 & n_2 & \cdots & n_r\end{pmatrix} \in \mathbb{T}_\Sigma(K_\infty).$$

The sum thus converges in the Tate algebra $\mathbb{T}_\Sigma(K_\infty)$ where K_∞ is the completion $\mathbb{F}_q((1/\theta))$ of K at the infinity place. Explicitly, we have:

$$\zeta_A\begin{pmatrix}\sigma_{U_1} & \sigma_{U_2} & \cdots & \sigma_{U_r}\\ n_1 & n_2 & \cdots & n_r\end{pmatrix} = \sum_{d\geq 0} \sum_{\substack{a_1,\dots,a_r\in A^+\\ d=\deg_\theta(a_1)>\cdots>\deg_\theta(a_r)\geq 0}} \frac{\sigma_{U_1}(a_1)\cdots\sigma_{U_r}(a_r)}{a_1^{n_1}\cdots a_r^{n_r}}. \tag{2}$$

Again, the integer $\sum_i n_i$ is called the *weight* of the above multiple zeta value and the integer r is called its *depth*. These elements of the above considered Tate algebras have been introduced and first discussed in [8]; it is easy, by using [12, Theorem 4], to show that they are non-zero. We prove, in this paper, a sum shuffle formula for products $\zeta_A(1,\sigma_U)\zeta_A(1,\sigma_V)$, with $U \sqcup V = \Sigma \subset \mathbb{N}^*$:

Theorem 3. *The following formula holds, for all $\Sigma \subset \mathbb{N}^*$ finite, and $U \sqcup V = \Sigma$:*

$$\zeta_A\begin{pmatrix}\sigma_U\\ 1\end{pmatrix}\zeta_A\begin{pmatrix}\sigma_V\\ 1\end{pmatrix} - \zeta_A\begin{pmatrix}\sigma_\Sigma\\ 2\end{pmatrix} = \zeta_A\begin{pmatrix}\sigma_U & \sigma_V\\ 1 & 1\end{pmatrix} + \zeta_A\begin{pmatrix}\sigma_V & \sigma_U\\ 1 & 1\end{pmatrix} - \sum_{\substack{I\sqcup J=\Sigma\\ |J|\equiv 1 \pmod{q-1}\\ J\subset U \text{ or } J\subset V}} \zeta_A\begin{pmatrix}\sigma_I & \sigma_J\\ 1 & 1\end{pmatrix}.$$

The reader will notice the universality phenomenon mentioned above: the coefficients of the right-hand side of the above formula are $0, 1, -1$ and they are determined upon a simple divisibility by $q-1$ condition, and the position I, J of the subsets of Σ relative to U, V. We will use techniques of Thakur in [13] and some linear algebra over $\mathbb{F}_p$ to show the existence of the coefficients. To compute them, as it is easily verified that they are uniquely determined, we will use the sum-shuffle formula of Chen [5], which also arises in several ways from our formula by specialization. This might not be the shortest way to prove our result. Nevertheless, in our proof it is possible to imagine a connection with shtuka functions which could possibly help to understand the deep reason for the existence of these sum-shuffle relations.

We now recall Chen's result.

Theorem 4 (Chen)**.** *For all $n, m > 0$,*

$$\zeta_A(m)\zeta_A(n) - \zeta_A(m+n) = \sum_{\substack{0<j<m+n\\ q-1|j}} f_j\zeta_A(m+n-j, j),$$

where

$$f_j = (-1)^{m-1}\binom{j-1}{m-1} + (-1)^{n-1}\binom{j-1}{n-1}.$$

It is further possible to deduce, by quite standard methods (applying a twisted Frobenius endomorphism a certain amount of times and specializing some variables), the following:

Theorem 5. *For all $\Sigma \subset \mathbb{N}^*$ a finite subset, the $\mathbb{F}_p$-subvector space of $\mathbb{T}_\Sigma(K_\infty)$ generated by the multiple zeta values (2) with $r \geq 1$ and $U_1, \ldots, U_r \subset \Sigma$ is an $\mathbb{F}_p$-algebra.*

The case $\Sigma = \emptyset$ is Thakur's [13, Theorem 3].

3. SUM-SHUFFLE RELATIONS

We follow the main idea of Thakur in [13], where he proves a sum-shuffle formula for the product of two power sums and he deduces from this result that the $\mathbb{F}_p$-sub-vector space of K_∞ generated by his multiple zeta values $\zeta_A(n_1, \ldots, n_k)$ is an $\mathbb{F}_p$-algebra (this is the case $\Sigma = \emptyset$). Thakur's result thus relies in universal families of sum-shuffle relations for power sums products that he proves by reducing to the case of one-degree power sums (the case of $d = 1$), a technique which is also naturally suggested by the philosophy of "solitons". We follow the principles of this proof. The main difference between this part of our proof and Thakur's is situated in the case of $d = 1$, which presents new structures.

3.1. Sum-shuffle formulas for power sums.

We shall prove:

Theorem 6. *Let U, V be subsets of Σ finite, such that $U \sqcup V = \Sigma$. Then, the following formula holds*

$$S_d\begin{pmatrix}\sigma_U\\ 1\end{pmatrix} S_d\begin{pmatrix}\sigma_V\\ 1\end{pmatrix} - S_d\begin{pmatrix}\sigma_\Sigma\\ 2\end{pmatrix} = -\sum_{\substack{I\sqcup J=\Sigma\\ |J|\equiv 1 \pmod{q-1}\\ J\subset U \text{ or } J\subset V}} S_d\begin{pmatrix}\sigma_I & \sigma_J\\ 1 & 1\end{pmatrix}, \quad d \geq 0.$$

Theorem 3 easily follows by taking the sum for $d \geq 0$. The identity is clearly satisfied if $d = 0$. We first develop some tools involving certain vector spaces generated by multiples of twisted power sums, then we show the identity for $d = 1$.

3.1.1. *$\mathbb{F}_p$-subvector spaces of twisted power sums.* We set $[1] = \theta^q - \theta$. We note that

$$\frac{[1]}{\theta-\lambda} = \frac{\theta^q-\theta}{\theta-\lambda} = \frac{\theta^q-\lambda^q+\lambda-\theta}{\theta-\lambda} = (\theta-\lambda)^{q-1}-1, \quad \forall \lambda \in \mathbb{F}_q.$$

Hence, for all $U \subset \Sigma$, using that $[1] = \prod_{\lambda\in\mathbb{F}_q}(\theta-\lambda)$,

$$P_U := [1]S_1(1,\sigma_U) = [1]\sum_{\lambda\in\mathbb{F}_q}\frac{\prod_{i\in U}(t_i-\lambda)}{\theta-\lambda} = \sum_{\lambda\in\mathbb{F}_q}((\theta-\lambda)^{q-1}-1)\prod_{i\in U}(t_i-\lambda) \in A[\underline{t}_U].$$

In fact it is, more precisely, a polynomial of $\mathbb{F}_p[\theta][\underline{t}_U]$ of degree $\leq q-1$ in θ ([1]). The claim on the degree in θ being clear, we indeed observe that P_V is invariant, by construction, under the action of $\mathrm{Gal}(\mathbb{F}_p^{ac}/\mathbb{F}_p)$. Let $\mathcal{V}_\Sigma$ be the $\mathbb{F}_p$-subvector space of $\mathbb{F}_p[\theta][\underline{t}_\Sigma](< q)$ (a shortcut for polynomials of degree $< q$ in θ) generated by the polynomials P_U with $U \subset \Sigma$. We have that $\mathcal{V}_{\Sigma'} \subset \mathcal{V}_\Sigma$ if $\Sigma' \subset \Sigma$. In particular, $\mathcal{V}_\emptyset = 1 \cdot \mathbb{F}_p$. In Lemma 12 we will show that the $\mathbb{F}_p$-vector space $\mathcal{V}_\Sigma$ has dimension $2^{|\Sigma|}$ but we do not need this information right now.

[1] The degree is exactly $q-1$ if $|\Sigma| \geq q$.

Let $(\mathcal{D}_n)_{n\geq 0}$ be the system of higher derivatives of $\mathbb{F}_p[\theta][\underline{t}_\Sigma]$ in θ which is $\mathbb{F}_p[\underline{t}_\Sigma]$-linear and such that $\mathcal{D}_n(\theta^m) = \binom{m}{n}\theta^{m-n}$. The main result of this subsection is the following:

Proposition 7. *For all Σ finite, the space $\mathcal{V}_\Sigma$ is stable under the higher derivations $(\mathcal{D}_n)_{n\geq 0}$.*

The proof of this proposition occupies the rest of this subsection. We note, since $\mathcal{V}_\Sigma \subset \mathbb{F}_p[\theta][\underline{t}_\Sigma](< q)$, that it suffices to show that

$$\mathcal{D}_n(\mathcal{V}_\Sigma) \subset \mathcal{V}_\Sigma, \quad n = 1, \ldots, q-1.$$

We define, for all $n \geq 0$:

$$\mathcal{V}_\Sigma^{(n)} = \mathrm{Vect}_{\mathbb{F}_p}(\mathcal{V}_U : |\Sigma \setminus U| \geq n).$$

Hence, we have

$$\mathcal{V}_\Sigma = \mathcal{V}_\Sigma^{(0)} \supset \mathcal{V}_\Sigma^{(1)} \supset \cdots \supset \mathcal{V}_\Sigma^{(|\Sigma|)} = \mathcal{V}_\emptyset = \mathbb{F}_p \cdot 1.$$

By convention, we set $\mathcal{V}_\Sigma^{(n)} = \{0\}$ if $n > |\Sigma|$. We will make use of the polynomials $B_1(\sigma_U) := \prod_{i\in U}(t_i - \theta)$, $U \subset \Sigma$. We note that for any $U \subset \Sigma$ such that $Nq < |U| < (N+1)q$ there exist polynomials $a_0^U, \ldots, a_N^U \in \mathbb{F}_p[\theta][\underline{t}_\Sigma](< q)$, uniquely determined, such that

$$B_1(\sigma_U) = a_0^U + a_1^U[1] + \cdots + a_N^U[1]^N. \tag{3}$$

Also, we need the next Lemma:

Lemma 8. *For all $n \geq 1$ and $m \geq 0$, $\mathcal{D}_n([1]^m)$ is a polynomial of $\mathbb{F}_p[\theta^q - \theta]$ of degree $\leq \min\{m - \frac{n}{q}, m-1\}$ in $[1] = \theta^q - \theta$.*

Proof. We have $\mathcal{D}_1([1]) = -1$, $\mathcal{D}_q([1]) = 1$ and $\mathcal{D}_n([1]) = 0$ for all $n \geq 1$ with $n \notin \{1, q\}$. The statement is thus clear for $m = 1$ and the proof can now be obtained by induction on $m \geq 1$. □

We note that, for all $n \geq 1$,

$$\mathcal{D}_n(B_1(\sigma_\Sigma)) = (-1)^n \sum_{\substack{W\subset\Sigma \\ |\Sigma\setminus W| = n}} B_1(\sigma_W). \tag{4}$$

This follows from Leibnitz's formula (we set, for commodity, $\Sigma = \{1, \ldots, s\}$):

$$\begin{aligned}
\mathcal{D}_n(B_1(\sigma_\Sigma)) &= \mathcal{D}_n(\prod_{i\in\Sigma}(t_i - \theta)) = \\
&= \sum_{i_1+\cdots+i_s=n} \mathcal{D}_{i_1}((t_1-\theta))\cdots\mathcal{D}_{i_s}((t_s-\theta)) \\
&= \sum_{\substack{i_1+\cdots+i_s=n \\ 0\leq i_j\leq 1, j=1,\ldots,s}} \mathcal{D}_{i_1}((t_1-\theta))\cdots\mathcal{D}_{i_s}((t_s-\theta)) \\
&= (-1)^n \sum_{\substack{W\subset\Sigma \\ |\Sigma\setminus W|=n}} \prod_{k\in W}(t_k - \theta).
\end{aligned}$$

Our Proposition 7 is a direct consequence of the next reinforced statement:

Proposition 9. *For all $U \subset \Sigma$ and $n, m \geq 0$, we have $\mathcal{D}_n(\mathcal{V}_U^{(m)}) \subset \mathcal{V}_U^{(m+n)}$. Moreover, if $N = \lfloor \frac{|\Sigma|}{q} \rfloor$, we have $a_i^\Sigma \in \mathcal{V}_\Sigma^{(qi)}$ for all $i = 0, \ldots, N$, where the polynomials a_i^Σ are those of the expansion (3).*

Proof. We proceed by induction on $s := |\Sigma|$. In fact, the proof makes use of two nested induction processes; they are not complicated, but in order to avoid confusion, we shall refer to the *first induction hypothesis* and to the *second induction hypothesis*. The statement is satisfied for $s = 0, 1, \ldots, q-1$. Indeed, in this case, we have $B_1(\sigma_\Sigma) = a_0^\Sigma$ and we know that $B_1(\sigma_\Sigma) = P_\Sigma$. The formula (4) then implies that $\mathcal{V}_\Sigma$, and hence $\mathcal{V}_U$ for all $U \subset \Sigma$, are stable under the operators $\mathcal{D}_1, \ldots, \mathcal{D}_{q-1}$. This implies that $\mathcal{D}_n(\mathcal{V}_U^{(m)}) \subset \mathcal{V}_U^{(m+n)}$ for all $U \subset \Sigma$ and $n, m \geq 0$.

We now suppose that $s \geq q$, so that $N = \lfloor \frac{s}{q} \rfloor \geq 1$. We suppose that the statement is satisfied for all $\Sigma' \subsetneq \Sigma$ (this is our first induction hypothesis). Let m be an integer between 1 and N. We easily verify, by using Leibnitz's formula and using Lemma 8, that

$$\mathcal{D}_{mq}(B_1(\sigma_\Sigma)) = a_m^\Sigma - \mathcal{D}_{q-1}(a_m^\Sigma) + c_0^{\langle m \rangle} + c_1^{\langle m \rangle}[1] + \cdots + c_{N-m}^{\langle m \rangle}[1]^{N-m}, \tag{5}$$

where, for all $j \in \{0, \ldots, N-m\}$, $c_j^{\langle m \rangle} \in \mathbb{F}_p[\theta][\underline{t}_\Sigma](< q)$ is an $\mathbb{F}_p$-linear combination of the polynomials

$$a_k^\Sigma, \mathcal{D}_1(a_k^\Sigma), \ldots, \mathcal{D}_{q-1}(a_k^\Sigma), \quad k = m+1, \ldots, N,$$

which are in $\mathbb{F}_p[\theta][\underline{t}_\Sigma](< q)$. We also note, again for $m = 1, \ldots, N$, by using (4), that

$$\mathcal{D}_{mq}(B_1(\sigma_\Sigma)) = \sum_{\substack{W \subset \Sigma \\ |\Sigma \setminus W| = mq}} B_1(\sigma_W), \tag{6}$$

and $B_1(\sigma_W) = a_0^W + a_1^W[1] + \cdots + a_{N-m}^W[1]^{N-m}$ by using (3). Since $m > 0$, by the (first) induction hypothesis we have that $a_i^W \in \mathcal{V}_W^{(qi)}$ and we can write, equating the coefficients of $[1]^j$ for all j in (5) and (6) and extracting the constant term, that

$$a_m^\Sigma - \mathcal{D}_{q-1}(a_m^\Sigma) + c_0^{\langle m \rangle} \in \mathcal{V}_\Sigma^{(mq)}. \tag{7}$$

The next step is to prove, by induction on $m = N - g$, $g = 0, \ldots, N-1$, that $a_m^\Sigma = a_{N-g}^\Sigma \in \mathcal{V}_\Sigma^{(mq)}$ (this is our second induction process).

We suppose that $m = N$, so that $g = 0$. Then, in (7), we see that $c_0^{\langle N \rangle} = 0$. Hence, $a_N^\Sigma - \mathcal{D}_{q-1}(a_N^\Sigma) \in \mathcal{V}_\Sigma^{(Nq)}$. If $\deg_\theta(a_N^\Sigma) < q-1$ we are done, as in this case, $\mathcal{D}_{q-1}(a_N^\Sigma) = 0$. Otherwise, note that $\mathcal{D}_{q-1}(\mathcal{D}_{q-1}(a_N^\Sigma)) = 0$. Since $N > 0$, the (first) induction hypothesis implies that $\mathcal{V}_\Sigma^{(Nq)}$ is $\mathcal{D}_{q-1}$-stable (observe that this space is $\mathbb{F}_p$-spanned by subspaces $\mathcal{V}_W$ with $W \subsetneq \Sigma$ which are $\mathcal{D}_n$-stable for all $n = 1, \ldots, q-1$ by the first induction hypothesis, as the various W are such that $|W| < s$). Applying $\mathcal{D}_{q-1}$ to $a_N^\Sigma - \mathcal{D}_{q-1}(a_N^\Sigma) \in \mathcal{V}_\Sigma^{(Nq)}$ we obtain that $\mathcal{D}_{q-1}(a_N^\Sigma) \in \mathcal{V}_\Sigma^{(Nq)}$ and summing we get that $a_N^\Sigma \in \mathcal{V}_\Sigma^{(Nq)}$ as desired.

The second inductive process is similar (we use the same trick of applying $\mathcal{D}_{q-1}$ as above). We suppose by (second) induction hypothesis that $a_{m+1}^\Sigma \in \mathcal{V}_\Sigma^{((m+1)q)}, \ldots, a_N^\Sigma \in \mathcal{V}_\Sigma^{(Nq)}$, so that $a_{m+1}^\Sigma, \ldots, a_N^\Sigma \in \mathcal{V}_\Sigma^{((m+1)q)}$ (and we have, by the first induction hypothesis, that $\mathcal{V}_\Sigma^{((m+1)q)}$ is $(\mathcal{D}_1, \ldots, \mathcal{D}_{q-1})$-stable). In (7), we observe that $c_0^{\langle m \rangle} \in \mathcal{V}_\Sigma^{((m+1)q)}$. Therefore $a_m^\Sigma - \mathcal{D}_{q-1}(a_m^\Sigma) \in \mathcal{V}_\Sigma^{(mq)}$ so that, the same trick as above yields that $a_m^\Sigma \in \mathcal{V}_\Sigma^{(mq)}$ and this, for all $m = 1, \ldots, N$. For $m = 0$ we have observed that $a_0^\Sigma = P_\Sigma$ so our property that $a_m^\Sigma \in \mathcal{V}_\Sigma^{(mq)}$ for $m = 0, \ldots, N$ is completely checked.

The last step of the proof is to show that $\mathcal{V}_\Sigma$ is $(\mathcal{D}_1, \ldots, \mathcal{D}_{q-1})$-stable and that $\mathcal{D}_n(\mathcal{V}_\Sigma^{(m)}) \subset \mathcal{V}_\Sigma^{(m+n)}$. All we need to show, thanks to the first induction hypothesis, is that $\mathcal{D}_n(a_0^\Sigma) \in \mathcal{V}_\Sigma^{(n)}$ for

$n = 1, \dots, q-1$. Let k be an integer between 1 and $q-1$. We have, by using (3) and Lemma 8:

$$\begin{aligned}\mathcal{D}_k(B_1(\sigma_\Sigma)) &= \mathcal{D}_k(a_0^\Sigma) + \mathcal{D}_k(a_1^\Sigma[1] + \cdots + a_N^\Sigma[1]^N) \\ &= \mathcal{D}_k(a_0^\Sigma) + e_0^{\langle k\rangle} + e_1^{\langle k\rangle}[1] + \cdots + e_{N-1}^{\langle k\rangle}[1]^{N-1},\end{aligned}$$

where the elements $e_j^{\langle k\rangle}$ are polynomials of $\mathbb{F}_p[\theta][\underline{t}_\Sigma]$ which are $\mathbb{F}_p$-linear combinations of elements $\mathcal{D}_j(a_l^\Sigma)$ with $j = 0, \dots, q-1$ and $l = 1, \dots, N$. Since these elements are in $\mathcal{V}_\Sigma^{(q)}$ by what we have seen above, and since this latter space is $\mathcal{D}_j$-stable for $j = 1, \dots, q-1$ by the first induction hypothesis, we obtain in particular that $e_0^{\langle k\rangle} \in \mathcal{V}_\Sigma^{(q)}$. Combining with (4) and comparing the coefficients of $[1]^l$ for all l, we deduce that $\mathcal{D}_k(a_0^\Sigma) \in \mathcal{V}_\Sigma^{(k)} \subset \mathcal{V}_\Sigma$. Any element x of $\mathcal{V}_\Sigma$ is a combination $\sum_i x_i$ with $x_i \in \mathcal{V}_{U_i}$. Hence, $\mathcal{D}_n(\mathcal{V}_\Sigma) \subset \mathcal{V}_\Sigma^{(n)}$ for all n. Let now x be an element of $\mathcal{V}_\Sigma^{(r)}$. Then, $x = \sum_i x_i$ with $x_i \in \mathcal{V}_{U_i}$ and $|\Sigma \setminus U_i| \geq r$, so that $\mathcal{D}_n(x_i) \in \mathcal{V}_{U_i}^{(n)} \subset \mathcal{V}_\Sigma^{(n+r)}$. We deduce that $\mathcal{D}_n(\mathcal{V}_\Sigma^{(r)}) \subset \mathcal{V}_\Sigma^{(n+r)}$. $\square$

We deduce, from the above proof, that for all $U \subset \Sigma$,

$$D_1(P_U) = -\sum_{i \in U} P_{U \setminus \{i\}} + Q, \quad \exists Q \in \mathcal{V}_U^{(2)}.$$

3.1.2. *The case of $d = 1$ in the Theorem 6: existence of certain coefficients $f_{I,J}$.* We have that (Δ designates the diagonal subset):

$$\begin{aligned}P_{U,V} &:= [1](S_1(1, \sigma_U)S_1(1, \sigma_V) - S_1(2, \sigma_\Sigma)) = \\ &= [1]\left(\sum_{\lambda,\mu \in \mathbb{F}_q} \frac{\prod_{i \in U}(t_i - \lambda)}{\theta - \lambda}\frac{\prod_{j \in V}(t_j - \mu)}{\theta - \mu} - \sum_{\nu \in \mathbb{F}_q} \frac{\prod_{k \in \Sigma}(t_k - \nu)}{\theta - \nu}\right) \\ &= [1] \sum_{(\lambda,\mu) \in \mathbb{F}_q^2 \setminus \Delta} \frac{\prod_{i \in U}(t_i - \lambda)\prod_{j \in V}(t_j - \mu)}{(\theta - \lambda)(\theta - \mu)} \\ &= \sum_{(\lambda,\mu) \in \mathbb{F}_q^2 \setminus \Delta} \left(\prod_{i \in U}(t_i - \lambda)\prod_{j \in V}(t_j - \mu)\right) \prod_{\nu \notin \{\mu,\nu\}} (\theta - \nu) \in \mathbb{F}_p[\theta][\underline{t}_\Sigma],\end{aligned}$$

which is, in particular, a polynomial of degree $\leq q-2$ in θ (again, we use $\mathrm{Gal}(\mathbb{F}_p^{ac}/\mathbb{F}_p)$-invariance, which is easily checked, to prove that the coefficients are in $\mathbb{F}_p$). We now compute:

$$\begin{aligned}P_{U,V} &= [1] \sum_{(\lambda,\mu) \in \mathbb{F}_q^2 \setminus \Delta} \frac{\prod_{i \in U}(t_i - \lambda)\prod_{j \in V}(t_j - \mu)}{(\theta - \lambda)(\theta - \mu)} = \\ &= [1] \sum_{\lambda \in \mathbb{F}_q} \frac{\prod_{i \in U}(t_i - \lambda)}{\theta - \lambda} \sum_{\mu \in \mathbb{F}_q \setminus \{\lambda\}} \frac{\prod_{j \in V}(t_j - \lambda + \lambda - \mu)}{(\theta - \lambda + \underbrace{\lambda - \mu}_{=:\eta})} \\ &= [1] \sum_{\lambda \in \mathbb{F}_q} \frac{\prod_{i \in U}(t_i - \lambda)}{\theta - \lambda} \sum_{\eta \in \mathbb{F}_q^\times} \frac{\prod_{j \in V}(t_j - \lambda + \eta)}{(\theta - \lambda + \eta)} \\ &= [1] \sum_{\lambda \in \mathbb{F}_q} \frac{\prod_{i \in U}(t_i - \lambda)}{\theta - \lambda} \sum_{\eta \in \mathbb{F}_q^\times} \sum_{M \sqcup W = V} \frac{\prod_{j \in W}(t_j - \lambda)\eta^{|M|}}{(\theta - \lambda)(1 + \frac{\eta}{\theta - \lambda})}.\end{aligned}$$

Hence, by developing $\frac{1}{1+\frac{\eta}{\theta-\lambda}}$ in $\mathbb{F}_p((\frac{1}{\theta-\lambda}))$:

$$\begin{aligned} P_{U,V} &= [1]\sum_{\lambda\in\mathbb{F}_q}\frac{\prod_{i\in U}(t_i-\lambda)}{(\theta-\lambda)^2}\sum_{\eta\in\mathbb{F}_q^\times}\sum_{M\sqcup W=V}\prod_{j\in W}(t_j-\lambda)\eta^{|M|}\sum_{n\geq 0}(-1)^n\frac{\eta^n}{(\theta-\lambda)^n} \\ &= [1]\sum_{\lambda\in\mathbb{F}_q}\prod_{i\in U}(t_i-\lambda)\sum_{M\sqcup W=V}\prod_{j\in W}(t_j-\lambda)\sum_{n\geq 0}\frac{(-1)^n}{(\theta-\lambda)^{n+2}}\sum_{\eta\in\mathbb{F}_q^\times}\eta^{|M|+n} \\ &= -[1]\sum_{\substack{M\sqcup W=V\\ |M|+n>0\\ |M|+n\equiv 0 \pmod{q-1}}}(-1)^n S_1(n+2,\sigma_{U\sqcup W}). \end{aligned}$$

We observe that, if $n+2\geq q+1$, then $\|[1]S_1(n+2,\sigma_U)\|\leq q^{-1}$. Since $P_{U,V}\in\mathbb{F}_p[\theta][\underline{t}_\Sigma](<q-1)$, we thus see that the part of the sum on the right for which $n\geq q-1$ is in the maximal ideal $\mathfrak{M}$ of $\mathbb{F}_p[\underline{t}_\Sigma][[1/\theta]]$. More precisely:

$$P_{U,V}\equiv -[1]\sum_{\substack{M\sqcup W=V\\ |M|+n>0, n<q-1\\ |M|+n\equiv 0 \pmod{q-1}}}(-1)^n S_1(n+2,\sigma_{U\sqcup W}) \pmod{\mathfrak{M}}. \tag{8}$$

We set, for $U\subset\Sigma$:

$$P_U^{(k)}:=[1]S_1(k,\sigma_U),\quad k\geq 1,$$

so that $P_U^{(1)}=P_U$ in the previous notations.

Lemma 10. *We have the congruence*

$$P_U^{(1+n)}\equiv(-1)^n\mathcal{D}_n(P_U)\pmod{\mathfrak{M}},\quad n=0,\dots,q-1.$$

Proof. If $n=0$, the statement is clear. Assume that $q>n>0$. By Leibnitz's formula we see that

$$\begin{aligned} \mathcal{D}_n(P_U) &= \mathcal{D}_1([1])\mathcal{D}_{n-1}(S_1(1,\sigma_U))+[1]\mathcal{D}_n(S_1(1,\sigma_U)) \\ &= -\mathcal{D}_{n-1}(S_1(1,\sigma_U))+(-1)^n P_U^{(n+1)}. \end{aligned}$$

Now, note that $\|\mathcal{D}_{n-1}(S_1(1,\sigma_U))\|<1$. □

Hence, combining (8), Lemma 10 and Proposition 7, we see that

$$P_{U,V}\in\mathcal{V}_\Sigma.$$

We have proved (multiply the above by $[1]^{-1}$) that, given $U,V\subset\Sigma$ such that $\Sigma=U\sqcup V$ there exist, for any decomposition $\Sigma=I\sqcup J$, an element $f_{I,J}\in\mathbb{F}_p$, so that

$$S_1(1,\sigma_U)S_1(1,\sigma_V)-S_1(2,\sigma_\Sigma)=\sum_{I\sqcup J=\Sigma}f_{I,J}S_1(1,\sigma_I) \tag{9}$$

(the title of this subsection refers to these coefficients). We now claim that, if $f_{I,J}\neq 0$, then $|I|\equiv|\Sigma|-1 \pmod{q-1}$ (that is, $|J|\equiv 1 \pmod{q-1}$). For this, we consider, for all $\mu\in\mathbb{F}_q^\times$, the map

$$\psi_\mu:\mathbb{F}_q(\underline{t}_\Sigma,\theta)\to\mathbb{F}_q(\underline{t}_\Sigma,\theta)$$

which sends t_i to μt_i and θ to $\mu\theta$. Observe that, for all $n \geq 1$,

$$\begin{aligned}\psi_\mu(S_1(n,\sigma_\Sigma)) &= \sum_{\lambda\in\mathbb{F}_q} \frac{\prod_{i\in\Sigma}(\mu t_i-\lambda)}{(\mu\theta-\lambda)^n} \\ &= \sum_{\lambda\in\mathbb{F}_q} \frac{\mu^{|\Sigma|}\prod_{i\in\Sigma}\left(t_i-\frac{\lambda}{\mu}\right)}{\mu^n\left(\theta-\frac{\lambda}{\mu}\right)^n} \\ &= \mu^{|\Sigma|-n}\sum_{\lambda'\in\mathbb{F}_q}\frac{\prod_{i\in\Sigma}(t_i-\lambda')}{(\theta-\lambda')^n}.\end{aligned}$$

Hence, for all $\mu\in\mathbb{F}_q^\times$ and $I\subset\Sigma$,

$$\psi_\mu(S_1(1,\sigma_I)) = \mu^{|I|-1}S_1(1,\sigma_I).$$

In particular, if L is the left-hand side of the identity (9), we have

$$\psi_\mu(L) = \mu^{|\Sigma|-2}L, \quad \mu\in\mathbb{F}_q^\times,$$

and this proves our claim. For later use, we write the result that we have reached, in the case $d=1$:

Proposition 11. *If $\Sigma = U\sqcup V$ there exists, for any decomposition $\Sigma = I\sqcup J$, an element $f_{I,J}\in\mathbb{F}_p$, so that*

$$S_1(1,\sigma_U)S_1(1,\sigma_V) - S_1(2,\sigma_\Sigma) = \sum_{\substack{I\sqcup J=\Sigma\\ |I|\equiv|\Sigma|-1 \pmod{q-1}}} f_{I,J}S_1(1,\sigma_I) = \sum_{\substack{I\sqcup J=\Sigma\\ |J|\equiv 1 \pmod{q-1}}} f_{I,J}S_1(1,\sigma_I).$$

Note that the congruence conditions on $|J|$ for the $f_{I,J}$ above could have been encoded directly in the proof of Proposition 7 but we have preferred to handle them separately to avoid too technical discussions.

3.1.3. *Computation of the coefficients $f_{I,J}$ in Proposition 11.* We use the explicit computation of the $\mathbb{F}_p$-coefficients of the sum-shuffle product formula for double zeta values of Chen in [5] to compute the coefficients $f_{I,J}$.

Lemma 12. *The $\mathbb{F}_p$-vector space $\mathcal{V}_\Sigma$ has dimension $2^{|\Sigma|}$ and the polynomials P_U, $U\subset\Sigma$ form a basis of it.*

Proof. It suffices to show that the elements $S_1(1,\sigma_U)$, $U\subset\Sigma$, are $\mathbb{F}_p$-linearly independent. First of all, the fractions $S_1(n) = \sum_{\lambda\in\mathbb{F}_q}\frac{1}{(\theta-\lambda)^n}\in\mathbb{F}_p(\theta)$, $n=1,2,\ldots$ are linearly independent over $\mathbb{F}_p$. Indeed, for each n, $S_1(n)$ has poles of order n at each $\theta=\lambda\in\mathbb{F}_q$. Observe that the map $\{U : U\subset\Sigma\}\to\{0,\ldots,q^s-1\}$ ($\Sigma=\{1,\ldots,s\}$) defined by $U\mapsto\sum_{i\in U}q^{i-1}$ is injective (it is bijective if and only if $q=2$). Thus, the map $\{U : U\subset\Sigma\}\to\{1,\ldots,q^s\}$ which sends U to $q^s-\sum_{i\in U}q^{i-1}$ is also injective; let G be its image. Then, the map

$$\{S_1(1,\sigma_U) : U\subset\Sigma\}\xrightarrow{\psi}\mathcal{W} := \{S_1(n) : n\in G\}$$

defined by

$$\tau^s(S_1(1,\sigma_U))_{\substack{t_i=\theta q^{i-1}\\ i=1,\ldots,s}} = S_1\left(q^s-\sum_{i\in U}q^{i-1}\right)$$

is injective, and induces an injective $\mathbb{F}_p$-linear map $\mathcal{V}_\Sigma\to\mathrm{Vect}_{\mathbb{F}_p}(\mathcal{W})$. Since the latter space has dimension $2^s=2^{|\Sigma|}$, the lemma follows. □

In fact, the map ψ defines an $\mathbb{F}_p$-algebra homomorphism

$$K[\underline{t}_\Sigma] \xrightarrow{\psi} K.$$

In Lemma 12, we have seen that ψ induces an isomorphism of $\mathbb{F}_p$-vector spaces

$$\mathcal{U}_\Sigma := \mathrm{Vect}_{\mathbb{F}_p}(S_1(1,\sigma_U) : U \subset \Sigma) \xrightarrow{\psi} \mathcal{W}_\Sigma := \mathrm{Vect}_{\mathbb{F}_p}\left(S_1(n) : n = q^n - \sum_{i=1}^{s} c_i q^{i-1}, c_i \in \{0,1\}\right).$$

If $U \subset \Sigma$, we write $n_U = q^s - \sum_{i\in U} q^{i-1}$. The map ψ thus sends $S_1(1,\sigma_U)$ to $S_1(n_U)$. We have seen that $S_1(1,\sigma_U)S_1(1,\sigma_V) - S_1(2,\sigma_\Sigma) \in \mathcal{U}_\Sigma$, and we have that

$$\psi(S_1(1,\sigma_U)S_1(1,\sigma_V) - S_1(2,\sigma_\Sigma)) = S_1(n_U)S_1(n_V) - S_1(n_U+n_V),$$

because $\psi(S_1(2,\sigma_\Sigma)) = S_1(2q^s - \sum_{i\in\Sigma} q^{i-1}) = S_1(q^s - \sum_{i\in U} q^{i-1} + q^s - \sum_{j\in V} q^{j-1})$. Now, we invoke Chen's explicit formula in [5] which, we recall, says that

$$S_1(n_U)S_1(n_V) - S_1(n_U+n_V) = \sum_{\substack{0<n<n_U+n_V \\ q-1|n}} f_n S_1(n_U+n_V-n), \tag{10}$$

where

$$f_n = (-1)^{n_U-1}\binom{n-1}{n_U-1} + (-1)^{n_V-1}\binom{n-1}{n_V-1}.$$

Assuming that the right-hand side of (10) belongs to $\mathcal{W}_\Sigma$, it is then equal to a combination

$$\psi\left(\sum_n f_n S_1(1,\sigma_{I_n})\right), \quad I_n \subset \Sigma,$$

and we obtain our result by pulling back this relation, thanks to Lemma 12. So, everything we need to show is that, if $f_n \neq 0$, then $n = n_J$ for some $J \subset \Sigma$, and then, compute f_n.

We recall that $q = p^e$, $e > 0$. We now write

$$n_U - 1 = \sum_{k=0}^{es-1} c_k^U p^k, \quad c_k^U \in \{1,\dots,p-1\},$$

and similarly for n_V etc. so that $c_k^U = p-1$ unless $k = ei$ for $i+1 \in U$, case in which $c_k^U = p-2$. Now, let n be in $\{1,\dots,n_U+n_V-1\}$, $q-1|n$. Since $n_U+n_V-1 = 2q^s - \sum_{i\in\Sigma} q^{i-1} - 2$, we have that $n < q^s$ for all q. We write

$$n - 1 = \sum_{k=0}^{es-1} c_k p^k, \quad c_k \in \{1,\dots,p-1\}.$$

Then $\binom{n-1}{n_U-1} = \prod_{k=0}^{es-1}\binom{c_k}{c_k^U}$ by Lucas' formula, and $\binom{n-1}{n_U-1} \neq 0$ if and only if, for all $k \geq 0$, $c_k \geq c_k^U$. Hence, the latter non-vanishing condition is equivalent to $c_k = p-1$ if $e \nmid k$ and if $e \mid k$, then $c_k \in \{p-2, p-1\}$. This means that $\binom{n-1}{n_U-1} \neq 0$ if and only if $n = n_J$ for some $J \subset \Sigma$. In this case, we easily check that $\binom{n-1}{n_U-I} = \binom{n_J-1}{n_U-I} = (-1)^{|U\setminus J|}$. Hence,

$$(-1)^{n_U-1}\binom{n_J-1}{n_U-I} = (-1)^{q^s-\sum_{i\in U} q^{i-1}-1}(-1)^{|U\setminus J|} = (-1)^{|U|}(-1)^{|U\setminus J|} = (-1)^{|J|}.$$

Now, observe that $f_n \neq 0$ if and only if either $\binom{n-1}{n_U-1} \neq 0$ or $\binom{n-1}{n_V-1} \neq 0$ and, for what seen above, these two terms cannot be simultaneously non-zero. We conclude that $f_n \neq 0$ if and only if $n = n_J$

with either $J \subset U$ or $J \subset V$, and if this is the case, then $f_n = (-1)^{|J|}$. Observe also that if $U \sqcup V = I \sqcup J = \Sigma$, then $n_U + n_V = n_I + n_J$. Hence, $n_U + n_V - n = n_U + n_V - n_J = n_I$ with $I \sqcup J = \Sigma$, and therefore, the right-hand side of (10) is:

$$\sum_{\substack{0<n<n_U+n_V \\ q-1|n}} f_n S_1(n_U + n_V - n) = - \sum_{\substack{I \sqcup J = \Sigma \\ J \subset U \text{ or } J \subset V \\ |J| \equiv 1 \pmod{q-1}}} \psi(S_1(1, \sigma_I)),$$

and we are done because ψ is injective on $\mathcal{U}_\Sigma$. We have proved:

Proposition 13. *If $\Sigma = U \sqcup V$, then*

$$S_1(1, \sigma_U) S_1(1, \sigma_V) - S_1(2, \sigma_\Sigma) = \sum_{\substack{I \sqcup J = \Sigma \\ |I| \equiv |\Sigma| - 1 \pmod{q-1}}} f_{I,J} S_1(1, \sigma_I) = - \sum_{\substack{I \sqcup J = \Sigma \\ |J| \equiv 1 \pmod{q-1} \\ J \subset U \text{ or } J \subset V}} S_1(1, \sigma_I).$$

3.1.4. *The case of $d \geq 1$ in the Theorem 6.* In this subsection, we prove Theorem 6. This part follows closely the principles introduced by Thakur in [13]. For the sake of completeness, we give full details. We denote by $A^+(d)$ the set of monic polynomials of degree d in A and by $A^+(< d)$ the set of monic polynomials of A which have degree $< d$. For $n \in A^+(d)$ and $m \in A^+(< d)$, we write

$$S_{n,m} = \{(n + \mu m, n + \nu); \mu, \nu \in \mathbb{F}_q, \mu \neq \nu\}.$$

We have that

$$S_{n,m} \subset A^+(d) \times A^+(d) \setminus \Delta,$$

where Δ is the diagonal of $A^+(d) \times A^+(d)$. Further, we recall from [13], the next:

Lemma 14. *The following properties hold, for $(n, m), (n', m') \in A^+(d) \times A^+(< d)$.*

(1) $S_{n,m} \cap S_{n',m'} \neq \emptyset$ if and only if $m = m'$ and $n = n' + \lambda m'$ for some $\lambda \in \mathbb{F}_q$.
(2) If $S_{n,m} \cap S_{n',m'} \neq \emptyset$ then $S_{n,m} = S_{n',m'}$.
(3) For all $(a, b) \in A^+(d) \times A^+(d) \setminus \Delta$ there exists $(n, m) \in A^+(d) \times A^+(< d)$ with $(a, b) \in S_{n,m}$.

Therefore, the sets $S_{n,m}$ determine a partition of $A^+(d) \times A^+(d) \setminus \Delta$.

Proof. See Thakur's [13]. □

We also have:

Lemma 15. *The following properties hold:*

(1) $S'_{n,m} \cap S'_{n',m'} \neq \emptyset$ if and only if $m = m'$ and there exists $\mu, \mu' \in \mathbb{F}_q$ such that $n + \mu m = n' + \mu' m'$.
(2) If $S'_{n,m} \cap S'_{n',m'} \neq \emptyset$ then $S'_{n,m} = S'_{n',m'}$.
(3) If $(a, b) \in A^+(d) \times A^+(< d)$, there exists $(n, m) \in A^+(d) \times A^+(< d)$ such that $(a, b) \in S'_{n,m}$.

Therefore, the sets $S'_{n,m}$ determine a partition of $A^+(d) \times A^+(< d)$ And $S'_{n,m} = S'_{n',m'}$ if and only if $S_{n,m} = S_{n',m'}$.

Proof. Immediate. □

We thus have two partitions:

$$\begin{aligned} \mathfrak{S} &= \{S; S = S_{n,m}, (n, m) \in A^+(d) \times A^+(< d)\}, \\ \mathfrak{S}' &= \{S'; S' = S'_{n,m}, (n, m) \in A^+(d) \times A^+(< d)\}. \end{aligned}$$

3.1.5. *End of proof of Theorem 6.* We recall that $\Sigma = U \sqcup V$. We have, for all $d \geq 1$:

$$\begin{aligned} S_d(1;\sigma_U)S_d(1;\sigma_V) - S_d(2;\sigma_\Sigma) &= \sum_{(a,b)\in A^+(d)\times A^+(d)\setminus\Delta} \frac{\sigma_U(a)\sigma_V(b)}{ab} \\ &= \sum_{S\in\mathfrak{S}} \sum_{(a,b)\in S} \frac{\sigma_U(a)\sigma_V(b)}{ab}. \end{aligned}$$

We compute, for any choice of $S \in \mathfrak{S}$, the sum

$$\sum_{(a,b)\in S} \frac{\sigma_U(a)\sigma_V(b)}{ab}.$$

We note, for all $U \subset \Sigma$, as σ_U is a product of injective $\mathbb{F}_q$-algebra homomorphisms, that it extends in an unique way to a group homomorphism $\sigma_U : K^\times \to \mathbb{F}_q(\underline{t}_\Sigma)^\times$. In particular, for all $U \subset \Sigma$, we have the identities: $\sigma_U(n+\mu m) = \sigma_U(m)\sigma_U\left(\frac{n}{m}+\mu\right)$ for all $\mu \in \mathbb{F}_q$. We thus have, with $S = S_{n,m}$ and with $\psi_{n,m}$ the substitution $\theta \mapsto \frac{n}{m}$, $t_i \mapsto \chi_{t_i}(\frac{n}{m})$, $i \in \Sigma$:

$$\begin{aligned} \sum_{(a,b)\in S} \frac{\sigma_U(a)\sigma_V(b)}{ab} &= \sum_{\substack{\mu,\nu\in\mathbb{F}_q \\ \mu\neq\nu}} \frac{\sigma_U(n+\mu m)\sigma_V(n+\mu m)}{(n+\mu m)(n+\nu m)} \\ &= \frac{\sigma_U(m)\sigma_V(m)}{m^2} \sum_{\substack{\mu,\nu\in\mathbb{F}_q \\ \mu\neq\nu}} \frac{\sigma_U(\frac{n}{m}+\mu)\sigma_V(\frac{n}{m}+\nu)}{(\frac{n}{m}+\mu)(\frac{n}{m}+\nu)} \\ &= \frac{\sigma_U(m)\sigma_V(m)}{m^2}[S_1(1;\sigma_U)S_1(1;\sigma_V) - S_1(2;\sigma_\Sigma)]_{\psi_{n,m}} \\ &= -\frac{\sigma_U(m)\sigma_V(m)}{m^2}\left[\sum_{\substack{I\sqcup J=\Sigma \\ |I|\equiv|\Sigma|-1 \pmod{q-1} \\ J\subset U \text{ or } J\subset V}} S_1(1;\sigma_I)\right]_{\psi_{n,m}}, \end{aligned}$$

by Proposition 13. Hence,

$$\begin{aligned} \sum_{(a,b)\in S} \frac{\sigma_U(a)\sigma_V(b)}{ab} &= \frac{\sigma_\Sigma(m)}{m^2} \sum_{\substack{I\sqcup J=\Sigma \\ |I|\equiv|\Sigma|-1 \pmod{q-1} \\ J\subset U \text{ or } J\subset V}} \sum_{\mu\in\mathbb{F}_q} \frac{\sigma_I\left(\frac{n}{m}+\mu\right)}{\frac{n}{m}+\mu} \\ &= \sum_{\substack{I\sqcup J=\Sigma \\ |I|\equiv|\Sigma|-1 \pmod{q-1} \\ J\subset U \text{ or } J\subset V}} \sum_{\mu\in\mathbb{F}_q} \frac{\sigma_I(n+\mu m)\sigma_J(m)}{(n+\mu m)m} \\ &= \sum_{\substack{I\sqcup J=\Sigma \\ |J|\equiv 1 \pmod{q-1} \\ J\subset U \text{ or } J\subset V}} \sum_{(a,b)\in S'} \frac{\sigma_I(a)\sigma_J(b)}{ab}, \end{aligned}$$

where $S' = S'_{n,m}$. Summing over all $S \in \mathfrak{S}$ induces a sum over all $S' \in \mathfrak{S}'$. Since

$$\sum_{S'\in\mathfrak{S}'} \sum_{(a,b)\in S'} \frac{\sigma_I(a)\sigma_J(b)}{ab} = \sum_{(a,b)\in A^+(d)\times A^+(<d)} \frac{\sigma_I(a)\sigma_J(b)}{ab} = S_d\begin{pmatrix} \sigma_I & \sigma_J \\ 1 & 1 \end{pmatrix},$$

and the Theorem follows.

4. Further properties of the fractions $S_1(1,\sigma_\Sigma)$

There are completely explicit formulas for $S_1(1,\sigma_\Sigma)$ that have not been used yet. One easily proves:

$$S_1(0,\sigma_\Sigma) = \sum_{\lambda\in\mathbb{F}_q}\prod_{i\in\Sigma}(t_i-\lambda) = \sum_{j=1}^{\lfloor\frac{|\Sigma|}{q-1}\rfloor} e_{|\Sigma|-j(q-1)}(\underline{t}_\Sigma), \tag{11}$$

where $e_n(\underline{t}_\Sigma)$ is the n-th elementary polynomial in the variables $\underline{t}_\Sigma$. Then, we have the following formula, where $W_i=\Sigma\cup V_i$ with V_i a set with exactly i elements, such that $\Sigma\cap V_i=\emptyset$, and where ξ_i is the map "substitution of t_i with 0", for all $i\in V_i$:

$$S_1(1,\sigma_\Sigma) = \sum_{j=1}^{q-1}(-\theta)^j\xi_{q-1-j}(S_1(0,\sigma_{W_{q-1-j}})).$$

In particular, if $|U|\geq q$ the coefficient of θ^{q-1} in $S_1(1,\sigma_U)$ (as a polynomial in θ) is proportional to $S_1(0,\sigma_U)$. We have seen, in Lemma 12, that the fractions $S_1(1,\sigma_U)$, $U\subset\Sigma$ are linearly independent over $\mathbb{F}_p$. In the opposite direction, there are non-trivial linear forms between the polynomials $S_1(0,\sigma_U)$.

Proposition 16. *For all $N\geq 1$ and $b\geq q$, The elements $S_1(0,\sigma_V)\in\mathbb{F}_p[\underline{t}_\Sigma]$ for $V\subset\Sigma$ with $|V|=b$ and $|\Sigma|=b+p^N-1$ are non-zero and linearly dependent over $\mathbb{F}_p$.*

Proof. The non-vanishing is clear because $b\geq q$. Observe, for $0\leq a\leq b\leq c=|\Sigma|$ (recall $e_a(\underline{t}_V)=\sum_{\substack{U\subset V\\|U|=a}}\prod_{i\in U}t_i$) the relation:

$$\sum_{\substack{V\subset\Sigma\\|V|=b}} e_a(\underline{t}_V) = \sum_{\substack{V\subset\Sigma\\|V|=b}} = \sum_{\substack{U\subset V\\|U|=a}}\left(\prod_{i\in U}t_i\right)\sum_{\substack{U\subset V\subset\Sigma\\|V|=b}}1 = \binom{c-a}{b-a}\sum_{\substack{U\subset V\\|U|=a}}\left(\prod_{i\in U}t_i\right),$$

which holds in any commutative ring $R[\underline{t}_\Sigma]$. If $v_p(\binom{c-a}{b-a})>0$ (with v_p the p-adic valuation of $\mathbb{Q}$), then this yields a non-trivial $\mathbb{F}_p$-linear relation among the elementary symmetric polynomials $e_a(\underline{t}_V)\in\mathbb{F}_p[\underline{t}_\Sigma]$ for $V\subset\Sigma$. Looking at the identity (11), if

$$v_p\left(\binom{c-b+j(q-1)}{j(q-1)}\right)>0,\quad \forall j=1,\dots,\left\lfloor\frac{b}{q-1}\right\rfloor,$$

then

$$\sum_{\substack{V\subset\Sigma\\|V|=b}} S_1(0,\sigma_V)=0. \tag{12}$$

Since $p-1\mid q-1$, if

$$v_p\left(\binom{c-b+j(p-1)}{j(p-1)}\right)>0,\quad \forall j=1,\dots,\left\lfloor\frac{b}{p-1}\right\rfloor, \tag{13}$$

then (12) holds. We recall that, for p a prime number and n,k integers with $n\geq k\geq 0$, $v_p(\binom{n}{k})$ is the sum of the carry over in the base-p sum of k and $n-k$. Hence, $v_p(\binom{n}{k})>0$ if there is at least one carry over. In (13) for fixed j, we take $n=c-b+j(p-1)$ and $k=j(p-1)$. We choose $N\geq 1$ an integer, and we set $c-b=p^N-1=(p-1)\sum_{i=0}^{N-1}p^i$. We have, at once, $k,n-k<p^N$. Hence,

for all j, there is carry over in the sum of $k+(n-k)$. Finally, if we choose $|\Sigma| = c = p^N - 1 + b$ and $|V| = b$, we are done. $\square$

5. Linear relations over $K(\underline{t}_\Sigma)$ for few variables

We denote by l_d the product $(\theta-\theta^{q^d})(\theta-\theta^{q^{d-1}})\cdots(\theta-\theta^q)$ and by $b_d(t)$ the product $(t-\theta)(t-\theta^q)\cdots(t-\theta^{q^{d-1}})$. In particular, with the usual conventions on empty products, $b_0 = l_0 = 1$.

We collect, in this subsection, some explicit linear dependence relation for our double zeta values in Tate algebras with $|\Sigma|$ small. First of all, we recall the formulas

$$S_d(1,\sigma_I) = \frac{\prod_{i\in I} b_d(t_i)}{l_d}, \quad I \subset \Sigma = \{1,\dots,q\}, \quad |I| < q \tag{14}$$

(first observed by Rudolph Perkins) and the formula

$$F_d(1,\chi_t) := \sum_{j=0}^{d-1} S_j(1,\chi_t) = \sum_{j=0}^{d-1} \frac{b_j(t_i)}{l_j} = \frac{b_d(t_i)}{(t_i-\theta)l_{d-1}} = \frac{\tau(b_{d-1})}{l_{d-1}}, \quad d>0, \tag{15}$$

where $\tau : K[t] \to K[t]$ is the unique $\mathbb{F}_q[t]$-linear extension of the map $x \mapsto x^q$ for $x \in K$.

The simplest example of linear relation is the so-called Euler-Thakur relation, holding for $\Sigma = \emptyset$. Thakur proved the formula

$$\zeta_A(1,q-1) = \zeta_A\begin{pmatrix}\mathbf{1} & \mathbf{1}\\ 1 & q-1\end{pmatrix} = \frac{1}{\theta-\theta^q}\zeta_A(q). \tag{16}$$

This can be viewed as an analogue of the famous formula $\zeta(2,1) = \zeta(3)$ by Euler. We recall the proof. We apply τ to both left- and right-hand sides of (15). We get the identity

$$\sum_{j=1}^{d-1} \frac{\tau(b_j)}{l_j^q} = \frac{\tau(b_d)}{(t-\theta^q)l_{d-1}^q}, \quad d \geq 1.$$

Since $\tau(b_i)(t)_{t=\theta} = l_i$ for all i, we get the identity

$$\sum_{j=0}^{d-1} \frac{1}{l_j^{q-1}} = \frac{l_d}{(\theta-\theta^q)l_{d-1}^q} = \frac{1}{l_{d-1}^{q-1}}.$$

Therefore, with $F_d(q-1) = \sum_{j=0}^{d-1} S_j(q-1)$ and for all $d \geq 1$:

$$\begin{aligned} S_d(1,q-1) &= l_d^{-1}F_d(q-1)\\ &= = l_{d-1}^{-1}l_{d-1}^{-q+1}\\ &= \frac{l_{d-1}^{-q}}{\theta-\theta^q}\\ &= \frac{S_{d-1}(q)}{\theta-\theta^q}. \end{aligned}$$

Summing over $d \geq 1$ we get the identity.

We also deduce, from (15), for $\Sigma = \{1,\dots,s\}$ with $s \leq q$:

$$S_d\begin{pmatrix}\sigma_{\Sigma\setminus\{i\}} & \chi_{t_i}\\ 1 & 1\end{pmatrix} = S_d(1,\sigma_{\Sigma\setminus\{i\}})\sum_{j=0}^{d-1} S_j(1,\chi_{t_i}) = \frac{1}{t_i-\theta}\sum_{d\geq 1}\frac{\prod_{j\in\Sigma} b_d(t_j)}{l_d l_{d-1}}, \quad i \in \Sigma. \tag{17}$$

We immediately obtain:

Lemma 17. *If $|\Sigma| < q$ and if i, j are distinct elements of Σ, then*

$$\zeta_A\begin{pmatrix}\sigma_{\Sigma\setminus\{i\}} & \sigma_{\{i\}}\\ 1 & 1\end{pmatrix} = \frac{t_j-\theta}{t_i-\theta}\zeta_A\begin{pmatrix}\sigma_{\Sigma\setminus\{j\}} & \sigma_{\{j\}}\\ 1 & 1\end{pmatrix}.$$

For the next example of linear relation, we have a similar lemma.

Lemma 18. *We suppose that $s = |\Sigma| \leq 2q-1$. We write $\Sigma = U \sqcup V = U' \sqcup V'$ with $|U| = |U'| = q$ and $|V| = |V'| = r$. Then, we have*

$$\zeta_A\begin{pmatrix}\sigma_V & \sigma_U\\ 1 & q\end{pmatrix} = \frac{\prod_{i'\in U'}(t_{i'}-\theta)}{\prod_{i\in U}(t_i-\theta)}\zeta_A\begin{pmatrix}\sigma_{V'} & \sigma_{U'}\\ 1 & q\end{pmatrix}.$$

Proof. We know the following formula (see the preprint [9]):

$$F_d(1,\sigma_U) := \sum_{i=0}^{d-1} S_i(1,\sigma_U) = l_{d-1}^{-1}\prod_{i\in U} b_{d-1}(t_i), \quad d \geq 1,$$

and a similar formula holds for U'. Therefore

$$F_d(q,\sigma_U) := \tau(F_d(1,\sigma_U)) = \sum_{i=0}^{d-1} S_i(q,\sigma_U) = l_{d-1}^{-q}\frac{\prod_{i\in U} b_d(t_i)}{\prod_{i\in U}(t_i-\theta)}$$

and analogously for U'. Since by the formula (14)

$$S_d(1,\sigma_V) = \frac{\prod_{i\in V} b_d(t_i)}{l_d}$$

and similarly for V', we get the identity

$$S_d\begin{pmatrix}\sigma_V & \sigma_U\\ 1 & q\end{pmatrix}\prod_{i\in U}(t_i-\theta) = \frac{\prod_{i\in\Sigma} b_d(t_i)}{l_d l_{d-1}^q} = S_d\begin{pmatrix}\sigma_{V'} & \sigma_{U'}\\ 1 & q\end{pmatrix}\prod_{i\in U'}(t_i-\theta).$$

Summing over $d \geq 1$ proves the lemma. □

Note that, for $\Sigma = \{1,\ldots,q\}$:

$$\tau\left(\zeta_A\begin{pmatrix}\sigma_{\Sigma\setminus\{i\}} & \sigma_{\{i\}}\\ 1 & 1\end{pmatrix}\right)_{t_1=\cdots=t_q=\theta} = \zeta_A(1,q-1), \quad i=1,\ldots,q, \quad \tau\left(\zeta_A\begin{pmatrix}\sigma_\Sigma\\ 2\end{pmatrix}\right)_{t_1=\cdots=t_q=\theta} = \zeta_A(q).$$

It would be certainly nice to see Euler-Thakur's identity arising as specialization of a linear relation between the above multiple zeta values, but this does not correspond to the correct intuition. We have already seen that the values $\zeta_A\left(\begin{smallmatrix}\sigma_{\Sigma\setminus\{i\}} & \sigma_{\{i\}}\\ 1 & 1\end{smallmatrix}\right)$ for $i = 1,\ldots,q$ generate a $K(\underline{t})$-subvector space of dimension 1. In the opposite direction, we show:

Lemma 19. *Assuming that $\Sigma = \{1,\ldots,s\} \subset \{1,\ldots,q\}$ and that $s > 0$, the values*

$$\zeta_A\begin{pmatrix}\sigma_{\Sigma\setminus\{s\}} & \chi_{t_s}\\ 1 & 1\end{pmatrix}, \quad \zeta_A\begin{pmatrix}\sigma_\Sigma\\ 2\end{pmatrix}$$

are linearly independent over $K(\underline{t})$.

Proof. We suppose by contradiction that there exist two polynomials $U, V \in A[\underline{t}]$, not both zero, such that

$$U\zeta_A\begin{pmatrix}\sigma_{\Sigma\setminus\{s\}} & \chi_{t_s}\\ 1 & 1\end{pmatrix} = V\zeta_A\begin{pmatrix}\sigma_\Sigma\\ 2\end{pmatrix}.$$

We can suppose that either $t_s - \theta \nmid U$, or $t_s - \theta \nmid V$. Evaluating the above identity at $t_s = \theta$ we thus get

$$U_{t_s=\theta}\zeta_A(1, \sigma_{\Sigma\setminus\{s\}}) = V_{t_s=\theta}\zeta_A(1, \sigma_{\Sigma\setminus\{s\}} - 1).$$

This implies that $1, \zeta_A(1, \sigma_{\Sigma\setminus\{s\}})$ are linearly dependent over $K(t_1, \ldots, t_{s-1})$ which is false. □

Remark 20. The Lemma 19 can be proved in a more elegant way after having noticed that $\zeta_A\left(\begin{smallmatrix} \sigma_{\Sigma\setminus\{s\}} & \chi_{t_s} \\ 1 & 1 \end{smallmatrix}\right)$ and $\zeta_A\left(\begin{smallmatrix} \sigma_\Sigma \\ 2 \end{smallmatrix}\right)$ are coefficients of points which generate a submodule of finite index of a Taelman unit module as in [11, 3] and in the Ph. D. Thesis of Demeslay [6]. The details will appear elsewhere, in a more general setting. Finally, we mention an arithmetical application of Theorem 3. Suppose indeed that $|U| \equiv 1 \pmod{q-1}$, $|U| > 1$. Then, by [1] or [3], $\zeta_A(1, \sigma_U)$ has a trivial zero at $t_i = \theta^{q^{k_i}}$ for all $i \in U$ and $k_i \in \mathbb{N}$; assume further that $|V| \not\equiv 1 \pmod{q-1}$ (so that $q > 2$). Then, evaluating in an appropriate way, the identity of the sum-shuffle formula returns us identities for Bernoulli-Goss polynomials $\zeta_A(n - q^k) \in A$ for n not divisible by $q-1$ and all k big enough. This phenomenon has been already observed in the special case $n = 2, \ldots, q-2$ in [8] and can be used to obtain certain congruence properties of these Bernoulli-Goss polynomials modulo irreducible polynomials of A^+. The details of these investigations will appear in another work.

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