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Generalized Luenberger observers for fault detection in switched systems using H- index

Ahmad Farhat[1] and Damien Koenig[1]

***Abstract*—In this paper, a linear matrix inequality (LMI) technique for the fault detection and state estimation of switched discrete systems is proposed. This approach is employed to generate generalized Luenberger minimal order observers giving best state estimation and sensitive fault detection. It consists on using H_- index design. Sufficient conditions of global convergence of observers are proposed using switched Lyapunov functions. Numerical example is given to illustrate this method.**

Keywords : Fault detection, reduced order observers, switched systems, LMI, H_- .

I. Introduction

Model based techniques have been widely studied and developed for fault detection (FD) problems [1]–[4].

On another hand, switched system modeling has been of considerable interest, since many physical plants show switching feature between sub-models, where multi-controllers/observer with suitable switching rules are needed [5], [6].

The classical observers for FD consist on generating the residual signal. In literature, numerous methods use LMIs techniques, such as H_∞ approach, the sensitivity of fault approach, H_∞/H_- approach [7]–[9].

Only few works have extended the H_- fault detection technique by adding weighting filters (analogue to the standard H_∞ control problem) [8], [10]. It greatly improves the fault sensitivity specially when the spectral frequency of the faults is known (zeros- frequencies for offset or Boolean faults, low frequencies for low varying faults...) .

In this paper the H_- approach is adopted, coupled with a minimal order observer strategy. The motivation for the reduced order state observer stems from the fact that only p of n states are directly available from measurement , and only the other $n-p$ states are unavailable, hence it suffices to build an observer that estimates at least the unmeasured states. Numerous technique have been studied based on algebraic equation from generalized Luenberger observers [11], [12].

The main contributions of this paper are:

- To give sufficient conditions for existence of generalized Luenberger minimal order observer for state estimation and fault detection.
- To give a general scheme to insure fault sensitivity, by solving an optimization problem using LMIs formulated by multi-objectives H_-/stability design.

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To the best of our knowledge, the problem of the minimal order observers design for state estimation and fault detection for switched systems has not been investigated yet. The H_- switched observer design uses switched Lyapunov functions (SLF) in order to avoid conservatism introduced by one common Lyapunov function.

The outline of this paper is as follows. After the introduction, general structure of the observer is given in Section II. In section III, preliminaries and sufficient conditions on the existence and parametrization of the observer are stated. The synthesis is discussed in Section IV with the use of LMI to satisfy H_- and stability objectives. A general design scheme is given in Section V, then followed by results illustrated in a numerical example in Section VI. Finally, the last section shows the concluding remarks and the possible future work.

Notations: The notation used in this paper is standard. X^T is the transposed of matrix X, the star symbol ($\star$) in a symmetric matrix denotes the transposed block in the symmetric position; The notation $P > (<, \leq)\ 0$ means P is real symmetric positive (negative, semi) definite matrix; A^+ is the generalized inverse matrix of A satisfying $AA^+A = A$; 0 and I denote zeros and identity matrix of appropriate dimensions.

II. Problem Formulation

Consider the following class of linear discrete-time switched system with faults and disturbances:

$$\left\{ \begin{array}{rcl} x_{k+1} & = & A_{\alpha(k)}x_k + B_{\alpha(k)}u_k + E_{f,\alpha(k)}f_k \\ y_k & = & C_{\alpha(k)}x_k + F_{f,\alpha(k)}f_k \end{array} \right. \quad (1)$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^p$, $u \in \mathbb{R}^m$ and $f \in \mathbb{R}^{n_f}$ denote respectively the state vector, the measurement output vector, the input vector and the vector of faults to be detected.

The switching rule takes values in the finite index set $\mathbb{S} = \{1, 2, .., N_\alpha\}$. It means that the switched signal is a piecewise constant function, where N_α is the number of subsystems. When $\alpha(k) = i$, it means that the ith subsystem is activated. Moreover, at the switching time k: $i = \alpha(k) \neq \alpha(k+1) = j$.

Then (1) can be transformed to:

$$\left\{ \begin{array}{rcl} x_{k+1} & = & A_i x_k + B_i u_k + E_{f,i} f_k \\ y_k & = & C_i x_k + F_{f,i} f_k \end{array} \right. \quad (2)$$

The aim is to design a generalized observer in the form:

$$\left\{ \begin{array}{rcl} z_{k+1} & = & F_i z_k + G_i y_k + J_i u_k \\ \hat{x}_k & = & M_i z_k + N_i y_k \end{array} \right. \quad (3)$$

where $z \in \mathbb{R}^q$ is the state vector of the observer, $\hat{x} \in \mathbb{R}^n$ is the estimation of x, Matrices F_i, G_i, J_i, M_i and N_i are unknown to be determined and of appropriate dimensions.

In order to find the unknown matrices, an error variable is introduced: $\epsilon_k = z_k - T_i x_k$, where $T_i \in \mathbb{R}^{q\times n}$ is an arbitrary matrix.

Remark 1: Assume that $rank(C_i) = p$, if $q = n - p$, a minimal order observer is obtained; if $q = n$, a reduced order observer is obtained, the full-order observer is obtained if $n - p < q < n$. The minimal order observer problem is investigated in this study.

Assumptions: In the sequel it is assumed that:

A. *1:* The pairs (A_i, C_i) are observable, or without loss of generality are detectable. This is a standard for all fault detection problems in the free unknown inputs (IU) case. This could be written as:

$$rank \begin{pmatrix} zI_n - A_i \\ C_i \end{pmatrix} = n \tag{4}$$

A. *2: rank* $\begin{pmatrix} T_i \\ C_i \end{pmatrix} = n$, and that there exist matrices $E_i \in \mathbb{R}^{q\times n}$ such that:

$$rank \begin{pmatrix} E_i \\ C_i \end{pmatrix} = rank \begin{pmatrix} T_i \\ C_i \end{pmatrix} = n \tag{5}$$

A. *3:* The switching signal $\alpha(t)$ is not known *a priori*, but it is measured and its value is real time available.

A. *4:* Faults occur in low frequencies, it is the region where most faults are spectraly located (offsets...).

III. Observer Parametrization

Lemma 1: For $f_k = 0$, (3) is a generalized observer for system (1) if there exists arbitrary matrices T_i and T_j such that (s.t.) the following constraints are satisfied:

$$F_i T_i + G_i C_i - T_j A_i = 0 \tag{6a}$$
$$J_i - T_j B_i = 0 \tag{6b}$$
$$M_i T_i + N_i C_i = I_n \tag{6c}$$

and the matrices F_i are Hurwitz.

Proof 1: By definition, the state estimate is:

$$\hat{x}_k = M_i z_k + N_i y_k = M_i \epsilon_k + (M_i T_i + N_i C_i) x_k \tag{7}$$

If $\epsilon_k \to 0$ then $\hat{x}_k \to x_k$; it follows

$$\hat{x}_k = M_i \epsilon_k + x_k \tag{8}$$

which is true if (6c) holds.

Secondly, from the definition of the error ϵ:

$$\begin{aligned} \epsilon_{k+1} &= z_{k+1} - T_j x_{k+1} \\ &= F_i z_k + G_i y_k + J_i u_k - T_j A_i x_k - T_j B_i u_k \\ &= F_i \epsilon_k + (F_i T_i + G_i C_i - T_j A_i) x_k + (J_i - T_j B_i) u_k \end{aligned} \tag{9}$$

The dynamics of the error is independent of x and u if the constraints (6a)-(6b) are satisfied.

Now, if the constraints in (6a)-(6c) are fulfilled, and defining the estimation error $e = x - \hat{x}$, the following autonomous system is deduced:

$$\begin{cases} \epsilon_{k+1} &= F_i \epsilon_k \\ e_k &= M_i \epsilon_k \end{cases} \tag{10}$$

It is then obvious that the estimation error $e_k \to 0$ if $\epsilon_k \to 0$, and $\epsilon_k \to 0$ if and only if (iff) F_i is Hurwitz. □

In this part, the observer design problem in the fault/disturbance free case is reduced to the stabilization problem in (10).

The equation (6c) can be put on the form:

$$\begin{bmatrix} M_i & N_i \end{bmatrix} \begin{bmatrix} T_i \\ C_i \end{bmatrix} = I_n \tag{11}$$

which has a solution according to [13] iff :

$$\text{rank} \begin{pmatrix} T_i \\ C_i \\ I_n \end{pmatrix} = \text{rank} \begin{pmatrix} T_i \\ C_i \end{pmatrix} = n. \tag{12}$$

Since T_i are unknown, there always exists matrices T_i and K_i such that:

$$\begin{bmatrix} T_i \\ C_i \end{bmatrix} = \begin{bmatrix} I_q & -K_i \\ 0 & I_p \end{bmatrix} \begin{bmatrix} E_i \\ C_i \end{bmatrix} \tag{13a}$$
$$\iff \quad T_i = E_i - K_i C_i \tag{13b}$$
$$\iff \begin{bmatrix} T_i & K_i \end{bmatrix} \begin{bmatrix} I_n \\ C_i \end{bmatrix} = E_i \tag{13c}$$

Again, (13) has a solution iff [13] :

$$\text{rank} \begin{pmatrix} I_n \\ C_i \\ E_i \end{pmatrix} = \text{rank} \begin{pmatrix} I_n \\ C_i \end{pmatrix} = n \tag{14}$$

Under (**A.**2), (12) and (14) are satisfied, then one solution for T_i and K_i is:

$$T_i = E_i \begin{bmatrix} I_n \\ C_i \end{bmatrix}^+ \begin{bmatrix} I_n \\ 0 \end{bmatrix} \text{ and } K_i = E_i \begin{bmatrix} I_n \\ C_i \end{bmatrix}^+ \begin{bmatrix} 0 \\ I_p \end{bmatrix} \tag{15}$$

And (11) becomes:

$$\begin{bmatrix} M_i & N_i \end{bmatrix} \begin{bmatrix} I_q & -K_i \\ 0 & I_p \end{bmatrix} \begin{bmatrix} E_i \\ C_i \end{bmatrix} = I_n \tag{16}$$

Denote:

$$\begin{cases} \Gamma_i = \begin{bmatrix} E_i \\ C_i \end{bmatrix}, \quad \Gamma_i^{\perp} = [I_{p+q} - \Gamma_i \Gamma_i^+], \\ \Pi_{1,i} = \begin{bmatrix} I_q \\ 0 \end{bmatrix}, \quad \Pi_{2,i} = \begin{bmatrix} K_i \\ I_p \end{bmatrix}, \quad \Pi_{3,i} = \begin{bmatrix} 0 \\ I_p \end{bmatrix} \end{cases} \tag{17}$$

Then (16) has the generalized solution using the pseudo-inverse [13]:

$$\begin{bmatrix} M_i & N_i \end{bmatrix} = \left(\Gamma_i^+ - L_{1,i}\Gamma_i^{\perp}\right) \begin{bmatrix} I_q & K_i \\ 0 & I_p \end{bmatrix} \tag{18}$$

or equivalently:

$$M_i = \Gamma_i^+ \Pi_{1,i} - L_{1,i} \Gamma_i^{\perp} \Pi_{1,i}, \tag{19a}$$
$$N_i = \Gamma_i^+ \Pi_{2,i} - L_{1,i} \Gamma_i^{\perp} \Pi_{2,i} \tag{19b}$$

where $L_{1,i}$ is arbitrary matrix.

Next, in order to compute F_i and G_i, equation (6a) can be put in the form using (13b):

$$F_i(E_i - K_iC_i) + G_iC_i = T_jA_i \tag{20a}$$

$$\iff \quad \begin{bmatrix} F_i & G_i - F_iK_i \end{bmatrix} \Gamma_i = T_jA_i \tag{20b}$$

This has a solution if the rank condition is satisfied [13]:

$$\text{rank}\begin{pmatrix} \Gamma_i \\ T_jA_i \end{pmatrix} = \text{rank}(\Gamma_i) = \text{rank}\begin{pmatrix} E_i \\ C_i \end{pmatrix} = n \tag{21}$$

Which is true under (**A.2**). Then a generalized solution for the above equation is:

$$\begin{bmatrix} F_i & G_i - F_iK_i \end{bmatrix} = T_jA_i\Gamma_i^+ - L_{2,i}\Gamma_i^\perp \tag{22}$$

Or equivalently:

$$F_i = T_jA_i\Gamma_i^+\Pi_{1,i} - L_{2,i}\Gamma_i^\perp\Pi_{1,i}, \tag{23a}$$

$$\begin{aligned} G_i &= T_jA_i\Gamma_i^+\Pi_{3,i} - L_{2,i}\Gamma_i^\perp\Pi_{3,i} + F_iK_i \\ &= T_jA_i\Gamma_i^+(\Pi_{3,i} + \Pi_{1,i}K_i) - L_{2,i}\Gamma_i^\perp(\Pi_{3,i} + \Pi_{1,i}K_i) \\ &= T_jA_i\Gamma_i^+\Pi_{2,i} - L_{2,i}\Gamma_i^\perp\Pi_{2,i} \end{aligned} \tag{23b}$$

Denote:

$$\mathbb{A}_{1,i} = T_jA_i\Gamma_i^+\Pi_{1,i}, \quad \mathbb{A}_{2,i} = \Gamma_i^\perp\Pi_{1,i} \tag{24a}$$

$$\mathbb{G}_{1,i} = T_jA_i\Gamma_i^+\Pi_{2,i}, \quad \mathbb{G}_{2,i} = \Gamma_i^\perp\Pi_{2,i} \tag{24b}$$

Then the autonomous system (10) becomes:

$$\begin{cases} \epsilon_{k+1} &= (\mathbb{A}_{1,i} - L_{2,i}\mathbb{A}_{2,i})\epsilon_k \\ e_k &= M_i\epsilon_k \end{cases} \tag{25}$$

Lemma 2: The pair (A_i, C_i) is detectable is equivalent to the detectability of the pair $(\mathbb{A}_{1,i}, \mathbb{A}_{2,i})$.

Proof 2: See appendix

Theorem 1: Considering the system (25), necessary and sufficient condition to the stability of (25) is to find matrices U_i and positive symmetric matrices P_i and P_j, $\forall i, j \in \mathbb{S}$ satisfying:

$$\begin{bmatrix} -P_i & \mathbb{A}_{1,i}^TP_j + \mathbb{A}_{2,i}^TU_j^T \\ P_j\mathbb{A}_{1,i} + U_i\mathbb{A}_{2,i} & -P_j \end{bmatrix} < 0 \tag{26}$$

where $L_{2,i} = -P_j^{-1}U_j$ is a stabilizing gain.

Proof 3: Consider the Switched Lyapunov Functions (SLF) $V_{\alpha(k)} = \epsilon_k^TP_{\alpha(k)}\epsilon_k$, where $P_{\alpha(k)} > 0$ is a positive definite matrix. If such a Lyapunov function exists, and its difference $\Delta V = V_{\alpha(k+1)} - V_{\alpha(k)}$ is negative definite along system trajectories of (25), then the origin of the system (25) is globally asymptotically stable. By computing the difference ΔV along the solution of (25), ΔV is given by:

$$\Delta V = \epsilon_{k+1}^TP_{\alpha(k+1)}\epsilon_{k+1} - \epsilon_k^TP_{\alpha(k)}\epsilon_k < 0 \tag{27}$$

$$\begin{aligned} \Leftrightarrow \epsilon_k^T(\mathbb{A}_{1,\alpha(k)} - L_{2,\alpha(k)}\mathbb{A}_{2,\alpha(k)})^TP_{\alpha(k+1)}(\mathbb{A}_{1,\alpha(k)} \\ - L_{2,\alpha(k)}\mathbb{A}_{2,\alpha(k)})\epsilon_k - \epsilon_k^TP_{\alpha(k)}\epsilon_k < 0 \end{aligned} \tag{28}$$

That holds $\forall \epsilon_k \neq 0$, and by Schur complement with $U_{\alpha(k)} = L_{2,\alpha(k)}P_\alpha(k)$:

$$\begin{bmatrix} -P_{\alpha(k)} & \mathbb{A}_{1,\alpha(k)}^TP_{\alpha(k+1)} - \mathbb{A}_{2,\alpha(k)}^TU_{\alpha(k+1)}^T \\ \star & -P_{\alpha(k+1)} \end{bmatrix} < 0 \tag{29}$$

As this inequality has to be satisfied under the arbitrary switching law, it follows that it should hold for special configuration $\alpha(k+1) = j$ and $\alpha(k) = i$, $\forall i, j \in \mathbb{S}$. □

Remark 2: Switched Lyapunov Functions (SLF) are used to avoid the conservatism generated when using a unique Lyapunov function for all sub-models. For each state i, different values of P_i and $L_{2,i}$ are calculated. The condition on the edges between consecutive switches is guaranteed by the inclusion of P_j in the LMIs.

To summarize, the algorithm for the proposed observer in fault/disturbances free case is:

1) Choose free matrices E_i according to (5);
2) Compute T_i and K_i from (15) ;
3) Compute Γ_i^+, $\Gamma_i^\perp$, $\Pi_{1,i}$, $\Pi_{2,i}$, $\Pi_{3,i}$, from (17), then N_i, M_i and J_i from (19) and (6b);
4) Compute the gain $L_{2,i}$ solution of (26);
5) Deduce the matrices F_i and G_i according to (23a) and (23b).

In the following section, the step 4) is detailed and a procedure to compute the gain $L_{2,i}$ is given such that the designed observer meets the H_-/stability performances.

Remark 3: Since $L_{1,i}$ in (19) have no affect the system's dynamics, they could be chosen equal to 0. This simplifies the calculations without altering the optimization problem in the following sections.

IV. $\mathcal{H}_-$ FAULT DETECTION OBSERVER (FDO) DESIGN

In this section, the observer design problem is studied as a fault detection problem. The aim of the observer design is to find a residual signal r_k that is sensitive to faults at some rate β_i. This is expressed by:

$$\|r_i\|_2 > \beta_i \|f\|_2 \tag{30a}$$

The objective of the $\mathcal{H}_-$ switched FD observer is resumed by the following condition:

$$\|T_{rf_i}\|_- > \beta_i \tag{31a}$$

where $T_{rf_i}(z)$ denotes the transfer function of the residual sensitivities faults .

Now when considering $f_k \neq 0$, and using the same constraints in (6a),(6c), while substituting (6b) by:

$$J_i - T_iB_i = 0 \tag{32}$$

The state estimation becomes:

$$\hat{x}_k = M_i\epsilon_k + x_k + N_iF_{f,i}f_k \tag{33}$$

The residuals signal that is defined by $r = y - \hat{y}$ becomes:

$$\begin{aligned} r_k &= C_ix_k + F_{f,i}f_k - (C_i\hat{x}_k) \\ &= C_i(x_k - \hat{x}_k) + F_{f,i}f_k \\ &= C_iM_i\epsilon_k + (C_iN_i + I)F_{f,i}f_k \end{aligned} \tag{34}$$

And the dynamic of the error variable ϵ becomes:

$$\epsilon_{k+1} = F_i\epsilon_k + (G_iF_{f,i} - T_iE_{f,i})f_k \tag{35}$$

Then the residuals generator has the following state space representation:

$$\begin{cases} \epsilon_{k+1} = F_i\epsilon_k + (G_iF_{f,i} - T_iE_{f,i})f_k \\ r_k = C_iM_i\epsilon_k + (C_iN_i + I)F_{f,i}f_k \end{cases} \tag{36}$$

It can be put in the generic form:

$$\begin{cases} \epsilon_{k+1} = \mathbb{A}_i\epsilon_k + \mathbb{B}_{f,i}f_k \\ r_k = \mathbb{C}_i\epsilon_k + \mathbb{D}_{f,i}f_k \end{cases} \tag{37}$$

where:

$$\begin{cases} \mathbb{A}_i = \mathbb{A}_{1,i} - \mathbb{L}_i\mathbb{A}_{2,i}, \quad \mathbb{B}_{f,i} = \mathbb{B}_{1f,i} - \mathbb{L}_i\mathbb{B}_{2f,i} \\ \mathbb{B}_{1f,i} = \mathbb{G}_{1,i}F_{f,i} - T_iE_{f,i}, \quad \mathbb{B}_{2f,i} = \mathbb{G}_{2,i}F_{f,i} \\ \mathbb{C}_i = C_iM_i, \quad \mathbb{D}_{f,i} = (C_iN_i + I)F_{f,i}, \quad \mathbb{L}_i = L_{2,i} \end{cases} \tag{38}$$

A. $\mathcal{H}_-$ fault sensitivity observer design

The $\mathcal{H}_-$ observer design problem in the presence of faults is expressed as an optimization problem by finding the gain matrix $\mathbb{L}_i$ such that the $\mathcal{H}_-$ norm $\|T_{rf_i}\|_-$ is maximized subject to β_i

Theorem 2: For a given switched LTI system with faults as defined in (37), there exist gain vectors $\mathbb{L}_i$ that satisfy the $\mathcal{H}_-$ observer design problem if there exits for $i, j \in \mathbb{S}$, symmetric positive matrices $\mathbb{P}_i$ and $\mathbb{P}_j$, matrices $\mathbb{U}_i$ and positive scalars β_i, such that the following LMIs are satisfied:

$$\begin{bmatrix} -\mathbb{P}_j - \mathbb{C}_i^T\mathbb{C}_i & -\mathbb{C}_i^T\mathbb{D}_{d,i} & \mathbb{A}_{1,i}^T\mathbb{P}_i + \mathbb{A}_{2,i}^T\mathbb{U}_i \\ \star & -\mathbb{D}_{f,i}^T\mathbb{D}_{f,i} + \beta_i^2 I & \mathbb{B}_{1,i}^T\mathbb{P}_i + \mathbb{B}_{2,i}^T\mathbb{U}_i \\ \star & \star & -\mathbb{P}_i \end{bmatrix} < 0 \tag{39}$$

Proof 4: Consider the switched Lyapunov function $V_k = \epsilon_k^T P_{\alpha(k)}\epsilon_k$ where $P_{\alpha(k)} > 0$. If such a function exist, and the following equality hold, then the The $\mathcal{H}_-$ fault sensitivity is guaranteed.

$$\epsilon_{k+1}^T P_{\alpha(k+1)}\epsilon_{k+1} - \epsilon_k^T P_{\alpha(k)}\epsilon_k - r_k^T r_k + \beta_i^2 f_k^T f_k < 0 \tag{40}$$

The rest of the proof is omitted here for lack of space, where steps of calculations are very similar to the bounded real lemma.

B. $\mathcal{H}_-$ synthesis problem

As it has been proposed in [14], if $\mathbb{D}_{f,i} = 0$ then $\mathcal{H}_-$ index is always zero, and the LMI (39) is infeasible (diagonal element is positive), and for $\mathbb{D}_{f,i} \neq 0$, the optimization problem would give $\beta_i = \|\mathbb{D}_{f,i}\|$

To avoid this restriction, solutions have been presented in [14] and [10], it consists on adding fictive weighting filters to the design: by adding an auxiliary direct channel to the system, and then multiply by a high pass filter $\mathcal{F}_H$ as it is shown in figure (1)

For $\mathbb{D}_{f,i} = \begin{bmatrix} \mathbb{D}_{1f,i} \\ 0 \end{bmatrix}$, a suitable matrix is $\mathbb{D}_{add,i} = \begin{bmatrix} 0 \\ \epsilon_i I \end{bmatrix}$.

The high pass filter $\mathcal{F}_H$ is a weighting filter that is used to raise up the high-frequency response, so that minimum

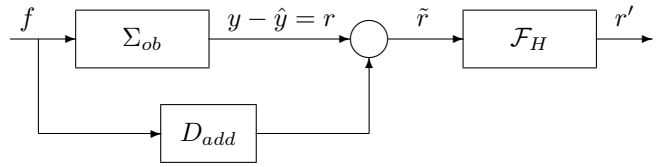

Fig. 1. $\mathcal{H}_-$ Loop Shaping with Additive filter weighting

singular value of the whole system occurs near the low-frequency region. $\mathcal{F}_H$ has the following transfer function:

$$F_{H,i}(s) = \left(\frac{s/\omega_{1,i} + 1}{s/\omega_{2,i} + 1}\right)^{m_i} \tag{41}$$

where $\omega_{1,i} < \omega_{2,i}$ and m_i the order of the filter.

The parameters $\omega_{1,i}$, $\omega_{2,i}$ and m_i are chosen such that the transfer function $T_{rf_i}(s)$ has the desired shape. This procedure is analog to the loop shaping method in the standard $\mathcal{H}_\infty$ problem.

The observer with the additive filter becomes:

$$\begin{cases} \epsilon_{k+1} &= \mathbb{A}_i\epsilon_k + \mathbb{B}_{d,i}w_k + \mathbb{B}_{f,i}f_k \\ r_k &= y_k - \hat{y}_k = \mathbb{C}_i\epsilon_k + \mathbb{D}_{d,i}w_k + \mathbb{D}_{f,i}f_k \\ \tilde{r}_k &= (y - \hat{y}) + \mathbb{D}_{add,i}f \\ x_{k+1}^h &= A_{h,i}x_k^h + B_{h,i}\tilde{r}_k \\ r'_k &= C_{h,i}x_k^h + D_{h,i}\tilde{r}_k \end{cases} \tag{42}$$

where $F_{H,i}(q^{-1}) := \left[\begin{array}{c|c} A_{h,i} & B_{h,i} \\ \hline C_{h,i} & D_{h,i} \end{array}\right]$ is a discrete time realization of the high pass filter $\mathcal{F}_{H_i}$.

From equations (42), an augmented residual is deduced:

$$\begin{cases} \begin{bmatrix} \epsilon_{k+1} \\ x_{k+1}^h \end{bmatrix} &= \begin{bmatrix} \mathbb{A}_{1,i} - L_i\mathbb{A}_{2,i} & 0 \\ B_{h,i}\mathbb{C}_i & A_{h,i} \end{bmatrix} \begin{bmatrix} \epsilon_k \\ x_k^h \end{bmatrix} \\ & \quad + \begin{bmatrix} \mathbb{B}_{1f,i} - L_i\mathbb{B}_{2f,i} \\ B_{h,i}(\mathbb{D}_{f,i} + \mathbb{D}_{add,i}) \end{bmatrix} f_k \\ r'_k &= \begin{bmatrix} D_{h_i}\mathbb{A}_{2,i} & C_{h_i} \end{bmatrix} \begin{bmatrix} \epsilon_k \\ x_k^h \end{bmatrix} \\ & \quad + \begin{bmatrix} D_{h,i}(\mathbb{D}_{f,i} + \mathbb{D}_{add,i}) \end{bmatrix} f_k \end{cases} \tag{43}$$

Theorem (2) is then applied to (43), which yields to the following theorem:

Theorem 3: The $\mathcal{H}_-$/stable FDO for the augmented system in (43) is guaranteed, if there exist for $i, j \in \mathbb{S}$ and given positive reel scalars β_i, matrices U_{a_i} and symmetric positive matrices P_i^a and P_j^a s.t. the following LMIs hold:

$$\left[\begin{array}{c|c|c} -P_j^a - C_{a_i}^T C_{a_i} & -C_{a_i}^T D_{af_i} & A_{0_i}^T P_i^a + C_{0_i}^T U_{a_i}^T \\ \hline \star & \begin{array}{c} -D_{af_i}^T D_{af_i} \\ +\beta_i^2 I \end{array} & \begin{array}{c} B_{f0_i}^T P_i^a \\ +F_{f0_i}^T U_{a_i}^T \end{array} \\ \hline \star & \star & -P_i^a \end{array}\right] < 0 \tag{44a}$$

$$\begin{bmatrix} -P_j^a & 0 & A_{0_i}^T P_i^a + C_{0_i}^T U_{a_i}^T \\ \star & 0 & B_{f0_i}^T P_i^a + F_{f0_i}^T U_{a_i}^T \\ \star & \star & -P_i^a \end{bmatrix} \leq 0 \tag{44b}$$

where the filters gain are $L_i = I_0^T P_{a_i}^{-1} U_{a_i}$, and the matrices A_{0_i}, B_{f0_i}, C_{0_i}, F_{f0_i} and I_0 are:

$$A_{0_i} = \begin{bmatrix} \mathbb{A}_{1,i} & 0 \\ B_{h,i}\mathbb{C}_i & A_{h,i} \end{bmatrix}, \; I_0 = \begin{bmatrix} -I \\ 0 \end{bmatrix}, \quad C_{0_i} = \begin{bmatrix} \mathbb{A}_{2,i} & 0 \end{bmatrix},$$

$$B_{f0_i} = \begin{bmatrix} \mathbb{B}_{1f,i} \\ B_{h,i}(\mathbb{D}_{f,i} + \mathbb{D}_{add,i}) \end{bmatrix}, \qquad F_{f0_i} = \begin{bmatrix} \mathbb{B}_{2f,i} & 0 \end{bmatrix} \tag{44c}$$

Proof 5: Let x_k^a be the augmented state of the augmented system, $x_k^a = \begin{bmatrix} \epsilon_k^T & x_k^{hT} \end{bmatrix}^T$. The corresponding augmented switched Lyapunov function $V_k^a = \epsilon_k^T P_{\alpha(k)}^a \epsilon_k$ is considered. The $\mathcal{H}_-$ objective for the FDO design are guaranteed by the inequalities:

$$x_{k+1}^{aT} P_{\alpha(k+1)}^a x_{k+1}^a - x_k^{aT} P_{\alpha(k)}^a x_k^a - r_k^T r_k + \beta_i^2 f_k^T f_k < 0 \tag{45}$$

and with similar steps of calculations as in the proof of *Theorem (1)*, the LMIs in (44a) are the deduced.

On the other hand, in order to guaranty the stability of the observer, the SLF must be positive and decreasing, which is expressed by the following inequality:

$$x_{k+1}^{aT} P_{\alpha(k+1)}^a x_{k+1}^a - x_k^{aT} P_{\alpha(k)}^a x_k^a < 0 \tag{46}$$

which yields to the LMI (44b). □

Following the steps 1), 2) and 3) of the algorithm given in section III, the observer gain in step 4) is then computed using numerical optimization tool by maximizing the LMIS in (44) subject to β_i. And finally compute the unknown matrices of the observer as in step 5).

Remark 4: The $P_{\alpha(k)}^a$ matrices in the inequality (45) are not required to be sign definite (positive definite), in the sens of the H_- index for the fault detection.

However, these matrices should be positive definite and this is a necessary condition in order to get positive Switched Lyapunov Functions, and hence proofing the stability of the observer (in fault free case) and the FDO.

V. Example

Consider the LTI switched MIMO system of the form (1) with the following matrices:

$$A_1 = \begin{bmatrix} 0.96 & 0.04 & 0.02 \\ 0.012 & 0.60 & 0.64 \\ -0.06 & 0.08 & 0.94 \end{bmatrix}, \; E_{f_1} = \begin{bmatrix} 0.06 \\ 1.2 \\ 0.18 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0.96 & 0.04 & 0.02 \\ 0.012 & 0.60 & 0.14 \\ -0.02 & 0.08 & 0.94 \end{bmatrix}, E_{f_2} = \begin{bmatrix} 0.07 \\ 1.4 \\ 0.49 \end{bmatrix},$$

$$C_{1,2} = \begin{bmatrix} 0.4 & 0 & 0 \\ 0 & .12 & 0 \end{bmatrix}; \quad F_{f_{1,2}} = \begin{bmatrix} 0.01 \end{bmatrix}.$$

The initial conditions are $x(0) = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}^T$.

The system is observable. A stable $\mathcal{H}_-$ FD minimal order observer can be designed.

First, define the matrices E_i, and compute T_i and K_i:

$$E_1 = E_2 = \begin{bmatrix} 10 & 10 & 4 \\ -200 & 1 & -10 \end{bmatrix}; \text{rank} \begin{pmatrix} E_{1\&2} \\ C_{1\&2} \end{pmatrix} = 3$$

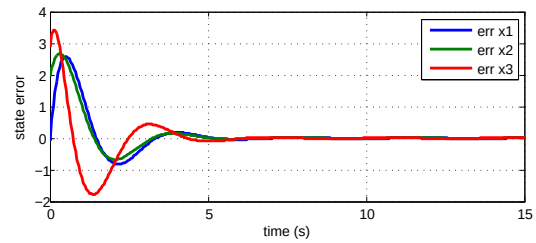

Fig. 2. State estimation errors in fault-free case

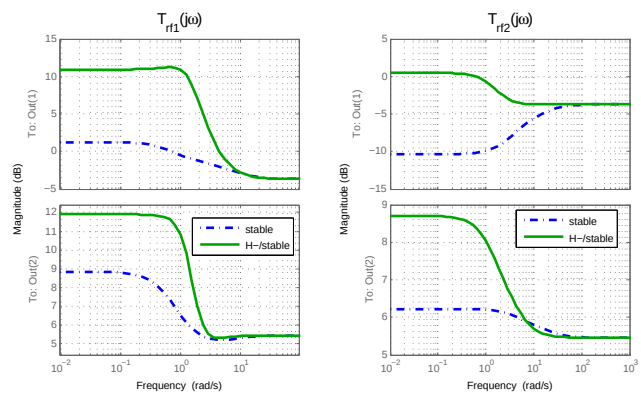

Fig. 3. Sensitivity plots $T_{rf_1}(j\omega)$(right) and $T_{rf_2}(j\omega)$ (left)

Then define the loop shaping matrices and weighting filter: suitable $D_{add,i}$ matrices are chosen for the additive filter. Let

$$D_{add,1} = D_{add,2} = \begin{bmatrix} 0.046 & 0.04 \end{bmatrix}^T,$$

$$\text{and} \quad F_{H,1}(s) = F_{H,2}(s) = \left(\frac{s/.095 + 1}{s/8 + 1} \right)^2$$

Using Matlab optimization tools such YALMIP or SeDuMi, the set of LMIs can be numerically solved.

The plots in figure (2) show that the state estimation error converge to zero in the fault free case.

Finally, the faults or residuals sensitivity are plot in figure (3). It could be noticed that the fault sensitivity with the FDO in low frequencies has been enhanced, compared to the simple stability observer design SO. In addition, this was done locally on low frequencies, while noticing that the global minima of the sensitivity functions are located in high frequencies. These results prove the utility of the loop shaping method applied in the proposed method.

VI. Conclusion and Further Work

In this paper, LMI conditions have been proposed to design a reduced order state estimator and a fault detection observer for LTI switched systems.

First the condition of existence of the generalized observer are stated and proved. Then two design objectives are formulated using LMIs: 1) stabilize the observer and 2) maximize the H_--index from the faults to the residuals. The results are obtained using switched Lyapunov functions in order to avoid conservatism introduced by single quadratic Lyapunov functions.

Possible extensions should concern robust observation, fault tolerant control and robust control.

APPENDIX

Denote:

$$\begin{cases} V_1 = \begin{bmatrix} I_q & -T_iA_i\Gamma_i^+ \\ 0_q & \Gamma_i^\perp \\ 0 & \Gamma_i\Gamma_i^+ \end{bmatrix}, \qquad V_2 = \begin{bmatrix} I_q & 0 \\ -\Gamma_i^+\Pi_{1,i} & I_n \end{bmatrix} \\ V_3 = \begin{bmatrix} I_q & -sI_q & 0 \\ 0 & I_q & 0 \\ 0 & 0 & I_p \end{bmatrix}, V_4 = \begin{bmatrix} -T_i & -sK_i \\ -C_i & sI_p - X_i \\ 0 & I_p \end{bmatrix} \\ X_i = \begin{matrix} I_p + C_iA_iC_i^+ \\ (\text{or } C_iA_iC_i^+) \end{matrix}, \qquad \breve{E}_i = T_j - K_iC_i \end{cases} \tag{47}$$

V_1 is of rank $n+q$ and V_4 is of rank $n+p$ from (**A.2**), V_2 and V_3 are regular matrices of rank $n+q$ and $p+2q$ respectively.

Consider two matrices $A : n \times m$ and $B : m \times p$, the Sylvester's inequality states:

$$\begin{aligned} rank(A) + rank(B) - m &\leq rank(AB) \\ &\leq min\left(rank(A), rank(B)\right) \end{aligned} \tag{48}$$

Then under assumptions (**A.1**) and (**A.2**), the following equalities hold:

$$\begin{aligned} n = rank\begin{pmatrix} zI_n - A_i \\ C_i \end{pmatrix} &= rank\left(V_4\begin{pmatrix} zI_n - A_i \\ C_i \end{pmatrix}\right) \\ &= rank\begin{pmatrix} T_iA_i - zT_i - zK_iC_i \\ C_iA_i - X_iC_i \\ C_i \end{pmatrix} \\ = rank\begin{pmatrix} T_iA_i - zE_i \\ C_i \text{ (or } 0) \\ C_i \end{pmatrix} &= rank\begin{pmatrix} T_iA_i - zE_i \\ C_i \end{pmatrix} \end{aligned} \tag{49a}$$

$$\begin{aligned} rank\begin{pmatrix} zI_q & T_iA_i \\ I_q & E_i \\ 0 & C_i \end{pmatrix} &= rank\left(V_3\begin{bmatrix} zI_q & T_iA_i \\ I_q & E_i \\ 0 & C_i \end{bmatrix}\right) \\ = rank\begin{pmatrix} 0 & T_iA_i - zE_i \\ I_q & E \\ 0 & C_i \end{pmatrix} &= rank\begin{pmatrix} T_iA_i - zE_i \\ C_i \end{pmatrix} + q \end{aligned} \tag{49b}$$

$$\begin{aligned} rank\begin{pmatrix} zI_q & T_iA_i \\ I_q & E_i \\ 0 & C_i \end{pmatrix} &= rank\left(V_1\begin{bmatrix} zI_q & T_iA_i \\ I_q & E_i \\ 0 & C_i \end{bmatrix}V_2\right) \\ &= rank\begin{pmatrix} zI_q - T_iA_i\Gamma_i^+\Pi_{1,i} & 0 \\ \Gamma_i^\perp\Pi_{1,i} & 0 \\ 0 & \Gamma_i \end{pmatrix} \\ &= rank\begin{pmatrix} zI_q - T_iA_i\Gamma_i^+\Pi_{1,i} \\ \Gamma_i^\perp\Pi_{1,i} \end{pmatrix} + n \\ &= rank\begin{pmatrix} zI_q - \mathbb{A}_{1,i} \\ \mathbb{A}_{2,i} \end{pmatrix} + n \end{aligned} \tag{49c}$$

Finally, using (49a) in (49b), then (49c), the following equivalence holds:

$$rank\begin{pmatrix} zI_n - A_i \\ C_i \end{pmatrix} = n \Leftrightarrow rank\begin{pmatrix} zI_q - \mathbb{A}_{1,i} \\ \mathbb{A}_{2,i} \end{pmatrix} = q \tag{50}$$

□

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