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# Robustness of adaptive narrowband beamforming with respect to bandwidth

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## Abstract

This paper addresses the robustness of adaptive narrowband beamforming with respect to bandwidth based on the loss of performance in terms of Signal to Interference plus Noise Ratio (SINR). The criterion used by Zatman to define a narrowband environment, i.e. the ratio between the jammer plus noise covariance matrix and the noise eigenvalue, is studied from the point of view of a loss of SINR after narrowband beamforming under non narrowband conditions. Using theoretical results about the eigenvalues and eigenvectors of covariance matrices for signals closely spaced in frequency by Lee [5], it is shown that Zatman's criterion can be interpreted as an upper bound on the SINR loss which is nearly reached under certain conditions that are specified.

## Index Terms

Narrowband, array signal processing, adaptive beamforming, bandwidth, robustness, SINR, Direction Of Arrival.

**EDICS Category: SAM-BEAM, SAM-PERF**

## I. INTRODUCTION

Evolution of radar technology and full digitized arrays allow the short term use of wideband waveforms for different advantages, such as high range resolution, stealthness, etc. (see, e.g. [1]). This breakthrough is a new challenge in radar for which broadband time domain or frequency domain array processing algorithms allow one to improve performance over standard spatial beamforming based on narrowband

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assumptions [2]. However, this improvement is done at the price of an increased complexity. Therefore, to optimize the choice between narrowband or broadband beamforming algorithms we have to precisely evaluate the performance of standard narrowband algorithms when the waveform has a certain given bandwidth.

In [3], Zatman proposes a general definition of a narrowband signal environment which is often used as a reference for adaptive beamforming as well as for Direction Of Arrival (DOA) estimation. It is based on the second eigenvalue of the jammer plus noise covariance matrix. Thus, an environment is qualified as “narrowband” if this eigenvalue is smaller than 3dB above the noise level in the jammer plus noise covariance matrix. The author has shown by simulations that an increase of bandwidth leads to the second eigenvalue rising above the noise floor and a growth of the angular region in which the jammer denies coverage. Therefore, the decrease of performance of adaptive beamforming algorithms is related to the second eigenvalue in the jammer plus noise covariance matrix. However, he has not given an explicit relation between the second eigenvalue level and the beamformer’s performance losses. Moreover, he has considered a zero-bandwidth target whereas in practical applications, its bandwidth will often be non-zero like that of the jammer, which may also induce losses on the SINR. Those issues have been partially considered in [4] where the authors have proposed to define the ratio between the Signal to Interference + Noise Ratio (SINR) resulting from narrowband beamforming with non-zero bandwidth conditions to that resulting from the same processing with zero bandwidth conditions, as a criterion for narrowband beamforming. However, they have considered a jammer-free environment, which is not realistic for most radar applications.

In this paper, we propose to use the same criterion as in [4] to study the robustness of adaptive narrowband beamforming, in the presence of a target and a non-zero bandwidth jammer whose Directions of Arrival (DOAs) are assumed to be arbitrary. First, we derive the expression of the SINR for zero-bandwidth and non-zero bandwidth target models. Using theoretical results about the eigenvalues and eigenvectors of covariance matrices for signals closely spaced in frequency by Lee [5], we show that under the assumption of a small fractional bandwidth, both models lead to the same expression. Then, we relate the considered SINR ratio to the criterion proposed by Zatman to define a narrowband environment, i.e., the ratio between the second eigenvalue of the jammer plus noise covariance matrix and the noise eigenvalue. Thus, we show that the latter criterion can be interpreted as an upper bound on the SINR loss due to bandwidth, w.r.t. the target DOA, and derive sufficient conditions for which the upper bound is nearly reached.

This paper is organized as follows. The data model is given in Section II. Then, the considered

robustness criterion is given in Section III and detailed for zero-bandwidth and non-zero bandwidth target models. Finally, this criterion is related to Zatman's definition of narrowband in Section IV.

## II. DATA MODEL

We consider a radar system where the receive antenna is a Uniform Linear Array<sup>1</sup> (ULA) composed of  $N$  sensors. The transmitted waveform has carrier frequency  $f_0$  and the array is assumed to have half-wavelength spacing w.r.t. the carrier frequency. Then, consider an environment composed of one jammer, thermal noise and a target. The jammer is modelled by a non-zero bandwidth white stationary process with power  $\sigma_J^2$  and bandwidth  $B$  and the thermal noise by a spatially white complex process, with power  $\sigma_n^2$ . The jamming plus noise covariance matrix<sup>2</sup> is

$$\bar{\mathbf{R}} = \int_{-\frac{B}{2}}^{\frac{B}{2}} \frac{\sigma_J^2}{B} \phi_J(f_0 + f) \phi_J(f_0 + f)^H df + \sigma_n^2 \mathbf{I} \quad (1)$$

with

$$\phi_J(f) = \left[ 1 \quad e^{j\pi \frac{f}{f_0} u_J} \quad \dots \quad e^{j(N-1)\pi \frac{f}{f_0} u_J} \right]^T$$

where  $u_J = \sin(\theta_J)$  and  $\theta_J$  is the DOA of the jammer. Finally the target signal is modelled by a stationary process, with power  $\sigma_S^2$ , known DOA  $\theta_S$  and covariance matrix  $\bar{\mathbf{R}}_S$ .

## III. ROBUSTNESS CRITERION W.R.T. BANDWIDTH

The common performance measure of SINR is chosen to study the robustness of adaptive narrowband beamforming w.r.t. bandwidth. In practical cases, knowing whether a signal is narrowband or not is important to select the proper spatial processing. If the signal is narrowband, spatial processing alone is sufficient [6]. On the contrary, under non narrowband conditions, space-time or subband processing allows one to compensate for performance losses due to bandwidth, see e.g., [2, chap.6], [7]–[10].

Under non-zero bandwidth conditions, the SINR expression is given by

$$\text{SINR} = \frac{\mathbf{w}^H \bar{\mathbf{R}}_S \mathbf{w}}{\mathbf{w}^H \bar{\mathbf{R}} \mathbf{w}} \quad (2)$$

where  $\bar{\mathbf{R}}$  and  $\bar{\mathbf{R}}_S$  are defined in the previous Section and  $\mathbf{w}$  is a spatial filter. When a zero-bandwidth designed adaptive beamformer is computed under non-zero bandwidth conditions, its expression is given

<sup>1</sup>The results of this paper are easily extended to an arbitrary array geometry.

<sup>2</sup>As in [3], we consider a radar scenario, assuming that this covariance matrix does not contain a target signal's component and study steady-state performance.

by<sup>3</sup>

$$\mathbf{w}_{NZB} \propto \bar{\mathbf{R}}^{-1} \phi_S \quad (3)$$

and the SINR expression becomes, using (3) in (2)

$$\text{SINR} = \frac{\phi_S^H \bar{\mathbf{R}}^{-1} \bar{\mathbf{R}}_S \bar{\mathbf{R}}^{-1} \phi_S}{\phi_S^H \bar{\mathbf{R}}^{-1} \phi_S}. \quad (4)$$

In practical applications, the target and jammer bandwidth can often be assumed to be identical. Indeed, when receiving a non-zero bandwidth target signal, a bandpass filter is often applied to the data, thus reducing the jammer bandwidth to that of the signal. Therefore, we consider in the following a non-zero bandwidth target with the same bandwidth  $B$  as that of the jammer. However, to make the analysis of the full non-zero bandwidth case (i.e. non-zero bandwidth jammer and target) easier, we first consider the simpler case of a zero-bandwidth target model with a non-zero bandwidth jammer model. In the following, we detail SINR expression (4) depending on the target model.

#### A. Zero-bandwidth target case

Here, we assume that the target signal is modelled by a zero-bandwidth stationary process. Its covariance matrix is given by

$$\bar{\mathbf{R}}_S \stackrel{\text{def}}{=} \mathbf{R}_S = \sigma_S^2 \phi_S \phi_S^H \quad (5)$$

with

$$\phi_S = \left[ 1 \quad e^{j\pi u_S} \quad \dots \quad e^{j(N-1)\pi u_S} \right]^T \quad \text{where } u_S = \sin(\theta_S). \quad (6)$$

Injecting (5) in (4), we obtain

$$\text{SINR}_{NZB} = \sigma_S^2 \phi_S^H \bar{\mathbf{R}}^{-1} \phi_S. \quad (7)$$

#### B. Non-zero bandwidth target case

Here, we assume that the target is modelled by a non-zero bandwidth white stationary process with bandwidth  $B$ . Its covariance matrix may be written as<sup>4</sup>

$$\bar{\mathbf{R}}_S = \int_{-\frac{B}{2}}^{\frac{B}{2}} \frac{\sigma_S^2}{B} \phi_S(f_0 + f) \phi_S(f_0 + f)^H df \quad (8)$$

<sup>3</sup>We use the subscript  $NZB$  when the environment is non-zero bandwidth and  $ZB$  when it is zero-bandwidth.

<sup>4</sup>Note that in [4], a simpler rank-one target covariance matrix of the form  $\bar{\mathbf{R}}_S = \sigma_S^2 \bar{\phi}_S \bar{\phi}_S^H$  with  $\bar{\phi}_S \propto \int_{-\frac{B}{2}}^{\frac{B}{2}} \phi_S(f_0 + f) df$  was used.

with

$$\phi_S(f) = \left[ 1 \quad e^{j\pi \frac{f}{f_0} u_S} \quad \dots \quad e^{j(N-1)\pi \frac{f}{f_0} u_S} \right]^T.$$

Now we show that (4) can be approximated by the SINR expression obtained with a zero-bandwidth target (7) under the assumption of a small fractional bandwidth. More precisely, we prove in Appendix I the following result:

*Result 1:* When the second eigenvalue of the target covariance matrix  $\lambda_2$  satisfies  $\lambda_2 \ll \left(\frac{\sigma_J^2}{\sigma_n^2}\right)\sigma_S^2$ , the SINR appearing in criterion (4) is approximated by:

$$\text{SINR}_{NZB} = \sigma_S^2 \phi_S^H \bar{\mathbf{R}}^{-1} \phi_S. \quad (9)$$

This approximation is validated by extensive numerical comparisons, for arbitrary jammer DOAs and number of sensors. For instance, Fig. 1 compares actual SINR (4) to approximate one (9) as a function of the target DOA. The parameters are  $N = 10$ ,  $\sigma_J^2 = 30$  dB,  $\sigma_S^2 = 0$  dB,  $\sigma_n^2 = 0$  dB,  $u_J = 0.1$  (those parameters will be used in the following simulations) and the fractional bandwidth  $\frac{B}{f_0} = 0.15$ . We observe that the approximation is accurate except when the target DOA is close to array endfire. Then, to illustrate the influence of the fractional bandwidth, we plot in Fig. 2 the same expressions, as well as the term  $\rho = \frac{\lambda_2 \sigma_J^2}{\sigma_n^2 \sigma_S^2}$ , as a function of the fractional bandwidth, with  $u_S = 0.15$ . First, we note that the approximation remains accurate for fractional bandwidths up to about 0.15. Then, we observe that  $\lambda_2$  rapidly increases with the SINR and that the condition in Result 1 is sufficient but not necessary. Indeed, the approximation may be very accurate whereas the condition  $\frac{\lambda_2 \sigma_J^2}{\sigma_n^2 \sigma_S^2} \ll 1$  is not satisfied, due to the coarse upper-bounds used for the derivation of Result 1.

### C. Expression of the robustness criterion

Here, we introduce a robustness criterion defined by the ratio between the non-zero bandwidth SINR (for both zero-bandwidth and non-zero bandwidth targets with  $\frac{B}{f_0} \ll 1$ ) and the zero-bandwidth SINR [4]. This criterion allows one to quantify the loss in SINR due to the increase of bandwidth of the environment (see, e.g. [3] for illustrations of this loss in SINR). Its expression is

$$r = \frac{\text{SINR}_{NZB}}{\text{SINR}_{ZB}} \quad (10)$$

where  $\text{SINR}_{NZB}$  is given by (7) and (9) and  $\text{SINR}_{ZB}$  is the optimal SINR under zero-bandwidth conditions equal to

$$\text{SINR}_{ZB} = \sigma_S^2 \phi_S^H \mathbf{R}^{-1} \phi_S$$

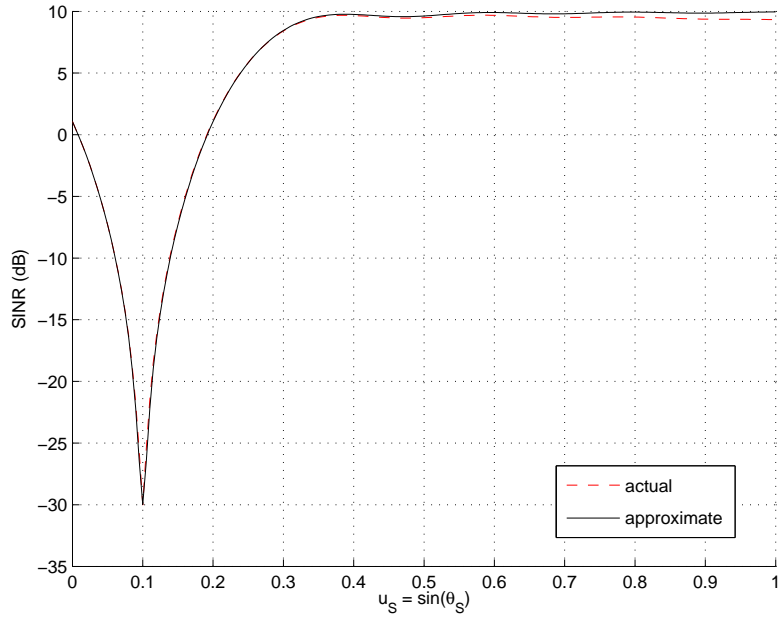


Fig. 1. Actual SINR (4) and approximate SINR (9), as a function of the target DOA.

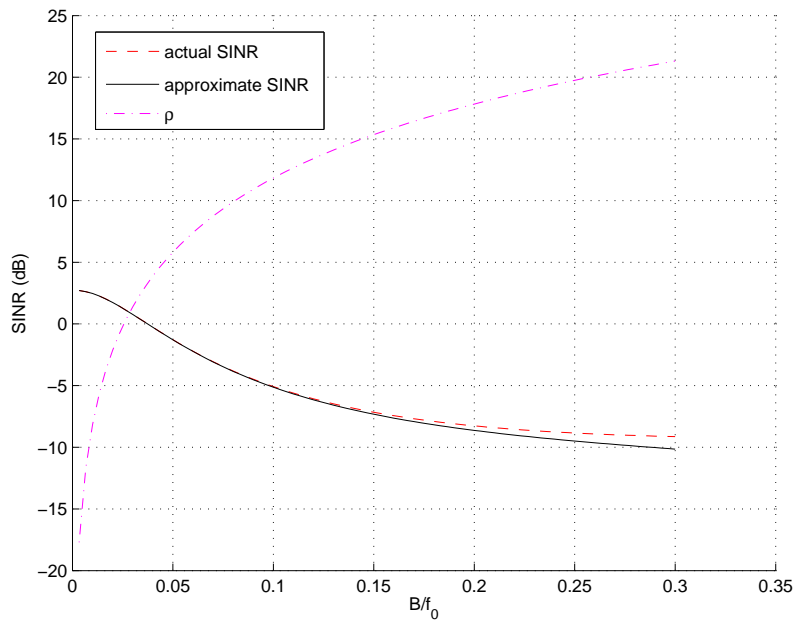


Fig. 2. Actual SINR (4), approximate SINR (9) and  $\rho = \frac{\lambda_2 \sigma_I^2}{\sigma_n^2 \sigma_S^2}$  as a function of the fractional bandwidth.

where

$$\mathbf{R} = \sigma_J^2 \phi_J \phi_J^H + \sigma_n^2 \mathbf{I} \quad (11)$$

is the zero-bandwidth jamming plus noise covariance matrix, and  $\phi_J = \phi_J(f_0)$  is the zero-bandwidth jammer steering vector. Because  $r < 1$ ,  $r^{-1}$  will denote the SINR loss throughout the paper. Using the expressions of  $\text{SINR}_{NZB}$  and  $\text{SINR}_{ZB}$  in (10), we obtain the detailed form of the chosen robustness criterion

$$r = \frac{\phi_S^H \bar{\mathbf{R}}^{-1} \phi_S}{\phi_S^H \mathbf{R}^{-1} \phi_S}. \quad (12)$$

#### IV. RELATION BETWEEN THE SINR LOSS AND ZATMAN'S DEFINITION OF NARROWBAND

Now, we want to relate SINR ratio (12) to the ratio between the second eigenvalue of the jammer plus noise covariance matrix and the noise eigenvalue, proposed by Zatman for defining a narrowband environment.

##### A. Upper bound on the SINR loss

Assuming that the fractional bandwidth is small, we prove in the conditions of Zatman [3] the following result:

*Result 2:* In the presence of a zero-bandwidth target and a non-zero bandwidth jammer, the ratio between the second eigenvalue of the jammer plus noise covariance matrix and the noise eigenvalue is an upper bound on the SINR loss  $r^{-1}$  of the optimal adaptive beamformer due to bandwidth w.r.t. the target DOA.

##### Proof

Using the derivation given in Appendix I and based on the results of [5], the non-zero bandwidth jammer covariance matrix  $\bar{\mathbf{R}} - \sigma_n^2 \mathbf{I}$  can be approximated by a rank-two matrix<sup>5</sup> where its largest two eigenvalues and the associated eigenvector are respectively  $\mu_1 - \sigma_n^2 \approx N\sigma_J^2$ ,  $\mu_2 - \sigma_n^2$  and  $\mathbf{u}_1 \approx \frac{\phi_J}{\sqrt{N}}$  and  $\mathbf{u}_2$ . Then, using (11),  $\mathbf{R} - \sigma_n^2 \mathbf{I} = N\sigma_J^2 \frac{\phi_J}{\sqrt{N}} \frac{\phi_J^H}{\sqrt{N}}$ . Consequently, we have:

$$\bar{\mathbf{R}} \approx \mathbf{R} + (\mu_2 - \sigma_n^2) \mathbf{u}_2 \mathbf{u}_2^H.$$

Using the matrix inversion lemma, we obtain:

$$\bar{\mathbf{R}}^{-1} \approx \mathbf{R}^{-1} - \mathbf{R}^{-1} \mathbf{u}_2 \left( \frac{1}{\mu_2 - \sigma_n^2} + \mathbf{u}_2^H \mathbf{R}^{-1} \mathbf{u}_2 \right)^{-1} \mathbf{u}_2^H \mathbf{R}^{-1}. \quad (13)$$

<sup>5</sup>Note that this assumption has been justified in [3] by the empirical observation that the eigenvalues of  $\bar{\mathbf{R}}$  overtake the noise floor one at a time when the bandwidth is increased.



Incorporating (13) in (12), we obtain:

$$r \approx 1 - \frac{|\phi_S^H \mathbf{R}^{-1} \mathbf{u}_2|^2}{\phi_S^H \mathbf{R}^{-1} \phi_S} \times \frac{1}{\left(\frac{1}{\mu_2 - \sigma_n^2} + \mathbf{u}_2^H \mathbf{R}^{-1} \mathbf{u}_2\right)}. \quad (14)$$

A lower bound of this SINR ratio (14) w.r.t. the target DOA is obtained by considering unconstrained steering vector  $\phi_S$ . In this case, (14) is minimized when the term  $\frac{|\phi_S^H \mathbf{R}^{-1} \mathbf{u}_2|^2}{\phi_S^H \mathbf{R}^{-1} \phi_S}$  is maximized, i.e., with  $\phi_S \propto \mathbf{u}_2$ . Using,  $\mathbf{u}_2^H \mathbf{R}^{-1} \mathbf{u}_2 \approx \frac{1}{\sigma_n^2}$  derived from  $\mathbf{u}_2^H \frac{\phi_J}{\sqrt{N}} \approx \mathbf{u}_2^H \mathbf{u}_1 = 0$ , this associated lower bound on the SINR ratio  $r$  is equal to<sup>6</sup>

$$r_{lb} = \frac{\sigma_n^2}{\mu_2}. \quad (15)$$

■

Consequently, the ratio between the noise eigenvalue and the second eigenvalue of the jammer plus noise covariance matrix can be interpreted as a lower bound on the SINR ratio  $r$  or conversely, the ratio between the second eigenvalue of the jammer plus noise covariance matrix and the noise eigenvalue as an upper bound on the SINR loss. From this result, we deduce an upper bound of the SINR loss  $r^{-1}$  in the presence of a narrowband jammer, in the sense of Zatman's definition. Thus, when the second eigenvalue of the jammer plus noise covariance matrix is smaller than 3dB above the noise eigenvalue, Result 2 proves that the SINR loss will be smaller than 3dB for arbitrary target and jammer DOAs. Indeed, if  $\mu_2 \leq 2\sigma_n^2$ , we have:

$$1 \leq \text{SINR loss} = \frac{1}{r} \leq \frac{1}{r_{lb}} = \frac{\mu_2}{\sigma_n^2} \leq 2.$$

For a non-zero bandwidth target, we have shown in Section III that the SINR expression could be precisely approximated by the SINR expression in the presence of a zero-bandwidth target, under the assumption that  $\lambda_2 \ll \left(\frac{\sigma_n^2}{\sigma_j^2}\right)\sigma_S^2$  where  $\lambda_2$  is the second eigenvalue of the target covariance matrix (see Result 1). Therefore, Result 2 is also valid for a non-zero bandwidth target, under the latter assumption.

After having given a general relation between Zatman's criterion and the SINR loss  $r^{-1}$ , we now want to give sufficient conditions for which the upper bound  $r_{lb}^{-1}$  is nearly reached for a certain DOA of the target.

<sup>6</sup>Note that an approximation of  $\mathbf{u}_2$  is given by the derivative of  $\phi_J(f)$  w.r.t.  $f$ , orthogonalized by  $\phi_J$ ,  $\mathbf{u}_2 \approx \left( \mathbf{I} - \frac{\phi_J \phi_J^H}{N} \right) \frac{d\phi_J(f)}{df} \Big|_{f=f_0} + o\left(\frac{B}{f_0}\right)$  [5]. Since there is no target DOA for which  $\phi_S$  is proportional to this vector, the lower bound on the SINR ratio can not be reached.

### B. Derivation of sufficient conditions for which the upper bound is nearly reached

For ease of notations, we only consider the zero-bandwidth target model. However, as we have already noted, the analysis remains valid in the presence of a non-zero bandwidth target under the conditions given by Result 1.

Our aim is to analyze the SINR ratio given by (12). To proceed, we first use the approximation introduced in [3] and justified by the analysis of Lee [5] applied to the spectral representation of stationary bandlimited signals (see Appendix I), to replace the jammer covariance by a rank-two matrix, under the assumption of a small fractional bandwidth. Then, we make the second assumption that the array is composed of many sensors<sup>7</sup> which allows one to derive limit expressions of the considered SINR ratio.

1) *Approximation of covariance matrix:* We use the following approximation:

$$\tilde{\mathbf{R}} = \frac{\sigma_J^2 \phi_{J,1} \phi_{J,1}^H + \sigma_J^2 \phi_{J,2} \phi_{J,2}^H}{2} + \sigma_n^2 \mathbf{I} \quad (16)$$

with

$$\begin{aligned} \phi_{J,1} &= \phi_J(f_0 - \Delta f) \\ \phi_{J,2} &= \phi_J(f_0 + \Delta f) \end{aligned}$$

where  $\Delta f = \frac{B}{2\sqrt{3}}$ . We have validated this second approximation by extensive numerical comparisons. However, we note that the rank-two approximation leads to significant errors in the SINR expression, when the fractional bandwidth is too large. Indeed, in that case, the effective rank of the covariance matrix is larger than 2 and the approximation is not justified anymore. However, simulations show that for fractional bandwidths up to  $\frac{B}{f_0} = 0.3$ , with the chosen parameters, this approximation remains acceptable. It is illustrated in Fig. 3 where we plot expression (7) with or without approximation of the covariance matrix according to (16) for  $\frac{B}{f_0} = 0.3$  as a function of the target DOA and in Fig. 4 for  $u_S = 0.15$  as a function of the fractional bandwidth.

We observe that errors due to this second approximation are very small. Therefore, (16) may be used for analysis of the non-zero bandwidth case at small fractional bandwidths.

2) *Derivation of limit expressions of the SINR ratio:* The approximation of  $\bar{\mathbf{R}}$  in (12) by  $\tilde{\mathbf{R}}$  allows one to derive a closed-form expression of  $\text{SINR}_{NZB}$  (7) and then of  $r$ . Indeed, after a double application

<sup>7</sup>This assumption is justified in most radar applications, for which high spatial resolution is required.

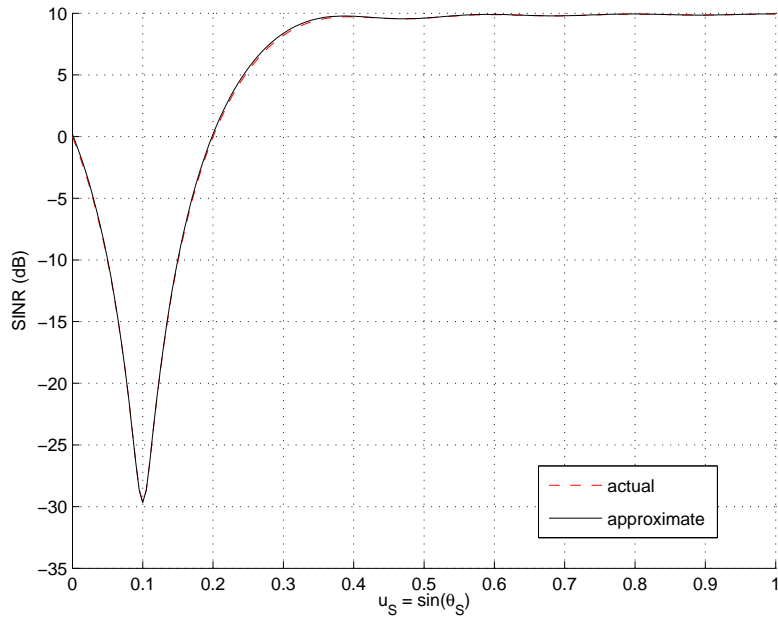


Fig. 3. Actual SINR (7) with or without approximation of the jammer plus noise covariance matrix by (16) as a function of the target DOA

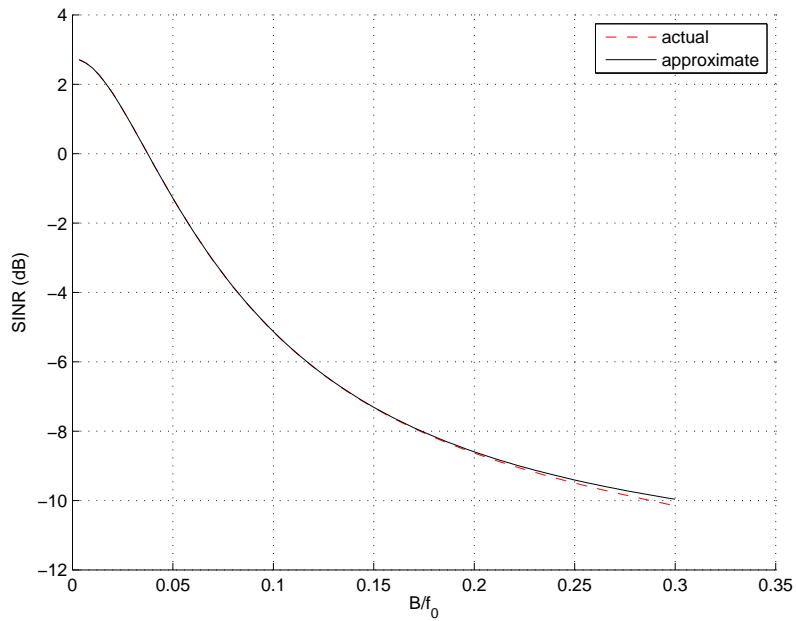


Fig. 4. Actual SINR (7) with or without approximation of the jammer plus noise covariance matrix by (16) as a function of the fractional bandwidth

of the matrix inversion lemma, we can write

$$\begin{aligned}\phi_S^H \tilde{\mathbf{R}}^{-1} \phi_S &= \frac{N}{\sigma_n^2} - \frac{|\phi_S^H \phi_{J,2}|^2}{\sigma_n^4 \beta} - \frac{|\phi_S^H \phi_{J,1}|^2}{\sigma_n^4 \alpha} \\ &\quad - \frac{|\phi_S^H \phi_{J,2}|^2 |\phi_{J,2}^H \phi_{J,1}|^2}{\sigma_n^8 \beta^2 \alpha} \\ &\quad + \frac{2}{\sigma_n^6 \beta \alpha} \text{Re}\{(\phi_S^H \phi_{J,2})(\phi_{J,1}^H \phi_S)(\phi_{J,2}^H \phi_{J,1})\}\end{aligned}$$

with  $\beta = \frac{2}{\sigma_J^2} + \frac{N}{\sigma_n^2}$ ,  $\alpha = \beta - \frac{|\phi_{J,1}^H \phi_{J,2}|^2}{\sigma_n^4 \beta}$  and

$$\begin{aligned}\phi_S^H \phi_{J,1} &= e^{-j(N-1)x_1} \frac{\sin(Nx_1)}{\sin(x_1)} \\ \phi_S^H \phi_{J,2} &= e^{-j(N-1)x_2} \frac{\sin(Nx_2)}{\sin(x_2)} \\ \phi_{J,2}^H \phi_{J,1} &= e^{-j2(N-1)\Delta x} \frac{\sin(2N\Delta x)}{\sin(2\Delta x)}\end{aligned}$$

where

$$\begin{aligned}x_1 &= x_0 + \Delta x \\ x_2 &= x_0 - \Delta x \\ x_0 &= \frac{\pi}{2}(u_S - u_J) \\ \Delta x &= \frac{\pi}{2} \frac{\Delta f}{f_0} u_J.\end{aligned}\tag{17}$$

Then after a Taylor series expansion, using  $N \gg 1$  and  $\frac{\Delta f}{f_0} \ll 1$  under constraint  $N \frac{\Delta f}{f_0} \ll 1$ , and  $\frac{\sigma_J^2}{\sigma_n^2} \gg 1$ , we can write for  $x_0 \neq 0$

$$\phi_S^H \tilde{\mathbf{R}}^{-1} \phi_S \approx \phi_S^H \mathbf{R}^{-1} \phi_S - \frac{b^2}{4a} \left( \frac{3\sigma_J^2 \Delta x^2}{N^3 \sigma_n^2 \sigma_J^2 \Delta x^2 + 3\sigma_n^4} \right)$$

with

$$\begin{cases} a = \frac{\sin^2(Nx_0)}{\sin^2(x_0)} \\ b = \frac{N \sin(2Nx_0)}{\sin^2(x_0)} - \frac{\sin^2(Nx_0) \sin(2x_0)}{\sin^4(x_0)}. \end{cases}\tag{18}$$

We deduce an approximate expression of the proposed criterion:

$$r \approx 1 - \frac{\frac{b^2}{4a} \left( \frac{3\sigma_J^2 \Delta x^2}{N^3 \sigma_n^2 \sigma_J^2 \Delta x^2 + 3\sigma_n^4} \right)}{\phi_S^H \mathbf{R}^{-1} \phi_S}.\tag{19}$$

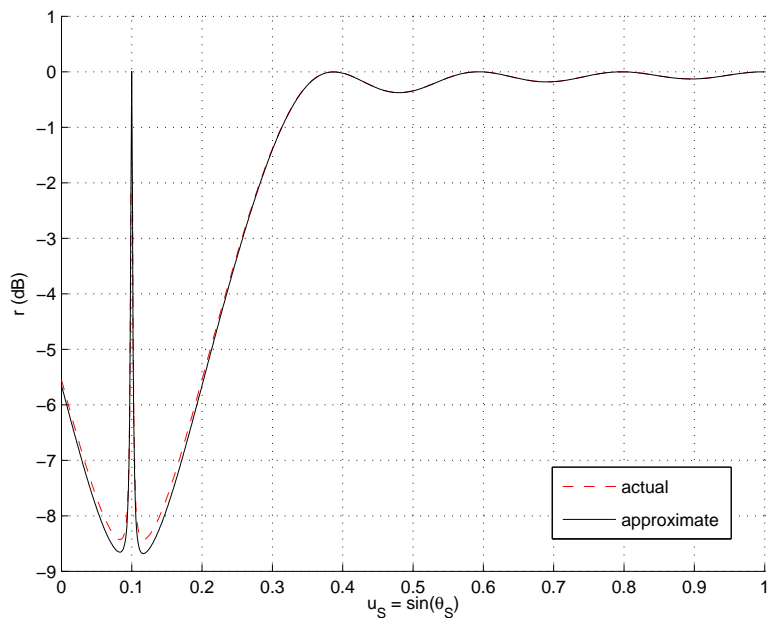


Fig. 5. Actual SINR ratio (12) and approximate SINR ratio (19), as a function of the target DOA.

In order to validate the latter equation, we now compare in Fig. 5 approximate relation (19) with actual one (12) w.r.t. the target DOA. The fractional bandwidth is  $\frac{B}{f_0} = 0.1$ .

First, we observe that the approximate plot is an accurate estimate of  $r$ . We notice that when the target is in the vicinity of the jammer, the SINR losses  $r^{-1}$  increase until a target position close to that of the jammer. Then, from that position to the jammer one, the SINR losses quickly decrease. When the target and jammer DOAs are equal, the losses have nearly vanished (actual ratio (12) is equal to  $\frac{\sigma_n^2 + N\sigma_j^2}{\mu_2} \approx 1$  [5] where  $\mu_1$  is the largest eigenvalue of  $\bar{\mathbf{R}}$ ).

Next, we want to estimate the 'worst-case' SINR ratio  $r_{min}$  w.r.t. the target DOA and relate it to the lower bound  $r_{lb}$  (15). We prove the following result:

*Result 3:* Under the assumptions that  $\frac{B}{f_0} \ll 1$  and  $N \gg 1$  under constraint  $N\frac{B}{f_0} \ll 1$ , the maximal SINR loss  $r_{min}^{-1}$  w.r.t. the target DOA nearly reaches the upper bound  $r_{lb}^{-1}$  equal to the ratio between the second eigenvalue of the jammer plus noise covariance matrix and the noise eigenvalue.

### Proof

Noticing that the minimum SINR ratio  $r_{min}$  is reached when  $0 < |x_0| \ll 1$ , we can use a Taylor series

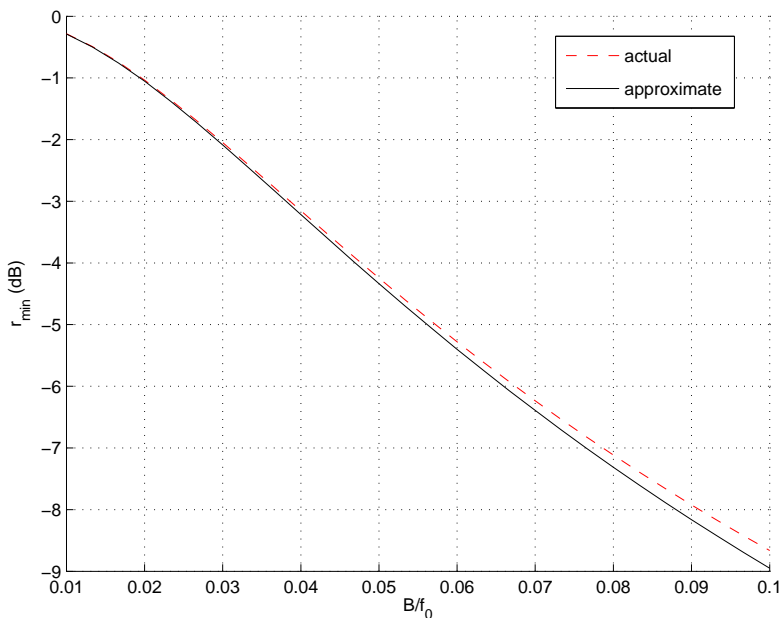


Fig. 6. Actual and approximate (21) values of  $r_{min}$ , as a function of the fractional bandwidth.

expansion of term  $\frac{b^2}{a}$  in (19), and obtain after a few algebraic manipulations summarised in Appendix II

$$\frac{b^2}{a} \approx \frac{4}{9} N^6 x_0^2. \quad (20)$$

Then, by noticing that  $\phi_S^H \mathbf{R}^{-1} \phi_S \approx \frac{N^3 x_0^2}{3\sigma_n^2}$  for  $\frac{\sigma_J^2}{\sigma_n^2} \gg 1$ ,  $0 < |x_0| \ll 1$  and using (17), we obtain the following approximation of the minimum value of  $r$ :

$$r_{min} \approx \frac{\sigma_n^2}{\frac{N^3}{3} \sigma_J^2 \Delta x^2 + \sigma_n^2} = \frac{\sigma_n^2}{\frac{N^3}{3} \sigma_J^2 \frac{\pi^2}{48} \left(\frac{B}{f_0}\right)^2 u_J^2 + \sigma_n^2}. \quad (21)$$

We remark that  $\frac{N^3}{3} \sigma_J^2 \frac{\pi^2}{48} \left(\frac{B}{f_0}\right)^2 u_J^2$  represents the first order Taylor series expansion of the second eigenvalue of the jammer's noise free covariance matrix (deduced from [3, rels 27 and 28] for  $\frac{NB}{f_0} \ll 1$ ), so that the denominator of (21) approximates  $\mu_2$ . Finally, taking the inverse of (21) completes the proof. ■

In order to observe the influence of the fractional bandwidth w.r.t. minimal SINR ratio  $r_{min}$ , we now plot in Fig. 6 this one as a function of  $\frac{B}{f_0}$ .

First, we note that approximate expression (21) is a very accurate estimate of the actual value  $r_{min}$

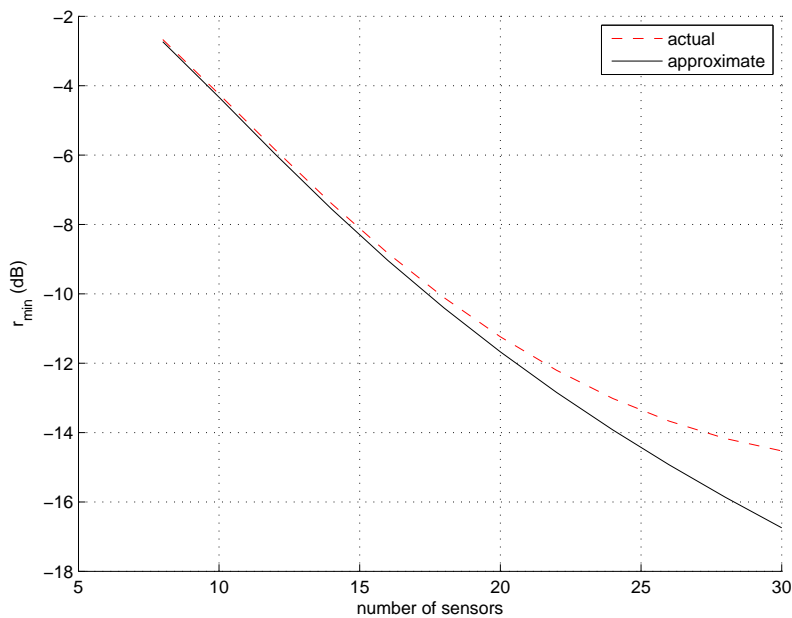


Fig. 7. Actual and approximate (21) values of  $r_{min}$ , as a function of the number of sensors.

obtained from (12), except for important fractional bandwidths. Second, we observe that  $r_{min}$  rapidly decreases when the fractional bandwidth increases. Then, we analyze the influence of the number of sensors on  $r_{min}$ . Fig. 7 shows the value of this criterion for the actual and approximate expressions, for different values of  $N$  and  $\frac{B}{f_0} = 0.05$ .

We observe that the approximation given by (21) is very accurate, except for high values of  $N$ . This can be explained by the fact that the series expansion done previously is valid under the hypothesis that  $N \frac{\Delta f}{f_0} \ll 1$ .

## V. CONCLUSION

In this paper, the robustness of adaptive narrowband beamforming with respect to bandwidth has been studied where the criterion of the loss of performance of the standard narrowband processing in terms of SINR, under the assumption of a non-zero bandwidth environment, w.r.t. the narrowband case has been proposed. Using results about the eigenvalues and eigenvectors of the covariance matrix for signals closely spaced in frequency, this SINR loss has been related to the ratio between the second eigenvalue of the jammer plus noise covariance matrix and the noise eigenvalue. Thus, it has been shown that under the assumption of a small fractional bandwidth, the SINR loss is upper bounded by the ratio between

the second eigenvalue of the jammer plus noise covariance matrix and the noise eigenvalue, for both zero-bandwidth and non-zero bandwidth target models. Then, sufficient conditions for which the upper bound of the SINR loss is nearly reached have been given.

#### APPENDIX I: PROOF OF RESULT 1

First, consider the EVD of  $\bar{\mathbf{R}}_S$ :

$$\bar{\mathbf{R}}_S = \sum_{n=1}^N \lambda_n \mathbf{v}_n \mathbf{v}_n^H.$$

Using the spectral representation  $x_t = \int_{-\frac{B}{2}}^{\frac{B}{2}} e^{j2\pi ft} d\mu(f)$  of the complex envelope of the wide-sense stationary bandlimited Gaussian target signals, approximated by  $x_t \approx \sum_{l=0}^{L-1} a_l e^{j2\pi f_l t}$  with  $f_l = (-L + 2l + 1)\frac{B}{2L}$ ,  $L \gg 1$  and  $(a_l)_{l=0, \dots, L-1}$  uncorrelated Gaussian random variables with  $E\{|a_l|^2\} = \frac{\sigma_S^2}{L}$ ,  $\bar{\mathbf{R}}_S$  can be approximated as the spatial covariance associated with a discrete sum of zero-bandwidth signals closely spaced in frequency for low fractional bandwidths:

$$\bar{\mathbf{R}}_S \approx \sum_{l=0}^{L-1} \frac{\sigma_S^2}{L} \phi_S(f_0 + f_l) \phi_S^H(f_0 + f_l).$$

Consequently, the results of [5] apply. In particular

$$\mathbf{v}_1 = \frac{\phi_S}{\sqrt{N}} + O\left(\frac{B}{f_0}\right)$$

and

$$\begin{aligned} \lambda_1 &= \text{tr}(\bar{\mathbf{R}}_S) + O\left(\frac{B}{f_0}\right), \\ \lambda_n &= O\left[\left(\frac{B}{f_0}\right)^{2(n-1)}\right] \quad \text{for } n > 1 \end{aligned}$$

Therefore, for  $\frac{B}{f_0} \ll 1$ ,  $\mathbf{v}_1 \approx \frac{\phi_S}{\sqrt{N}}$  and  $\lambda_1 \approx N\sigma_S^2$  and the non-zero bandwidth target covariance matrix  $\bar{\mathbf{R}}_S$  can be approximated by the rank-two matrix

$$\bar{\mathbf{R}}_S \approx \sigma_S^2 \phi_S \phi_S^H + \lambda_2 \mathbf{v}_2 \mathbf{v}_2^H. \quad (22)$$

Injecting (22) in (4), we obtain

$$\text{SINR} \approx \sigma_S^2 \phi_S^H \bar{\mathbf{R}}^{-1} \phi_S \left( 1 + \frac{\lambda_2 |\phi_S^H \bar{\mathbf{R}}^{-1} \mathbf{v}_2|^2}{\sigma_S^2 (\phi_S^H \bar{\mathbf{R}}^{-1} \phi_S)^2} \right). \quad (23)$$



Now, we derive a sufficient condition to neglect the second term in (23). To proceed, let consider the following ratio

$$\frac{\lambda_2 |\phi_S^H \bar{\mathbf{R}}^{-1} \mathbf{v}_2|^2}{\sigma_S^2 (\phi_S^H \bar{\mathbf{R}}^{-1} \phi_S)^2}$$

and compare it to unity. Using the Cauchy-Schwartz inequality, we have

$$\frac{\lambda_2 |\phi_S^H \bar{\mathbf{R}}^{-1} \mathbf{v}_2|^2}{\sigma_S^2 (\phi_S^H \bar{\mathbf{R}}^{-1} \phi_S)^2} \leq \frac{\lambda_2 \mathbf{v}_2^H \bar{\mathbf{R}}^{-1} \mathbf{v}_2}{\sigma_S^2 \phi_S^H \bar{\mathbf{R}}^{-1} \phi_S}.$$

Then, since  $\mathbf{v}_2^H \bar{\mathbf{R}}^{-1} \mathbf{v}_2 \leq \frac{1}{\mu_N} \leq \frac{1}{\sigma_n^2}$  and  $\phi_S^H \bar{\mathbf{R}}^{-1} \phi_S \geq \frac{N}{\mu_1}$  where  $\mu_1$  and  $\mu_N$  are given by the EVD of  $\bar{\mathbf{R}} = \sum_{n=1}^N \mu_n \mathbf{u}_n \mathbf{u}_n^H$ , we obtain:

$$\frac{\lambda_2 |\phi_S^H \bar{\mathbf{R}}^{-1} \mathbf{v}_2|^2}{\sigma_S^2 (\phi_S^H \bar{\mathbf{R}}^{-1} \phi_S)^2} \leq \frac{\lambda_2 \mu_1}{N \sigma_S^2 \sigma_n^2}.$$

A sufficient condition to neglect the second term in (9) is therefore that

$$\frac{\lambda_2 \mu_1}{N \sigma_S^2 \sigma_n^2} \ll 1$$

and since  $\mu_1 \approx N \sigma_J^2$  for  $\frac{B}{f_0} \ll 1$ , this condition becomes

$$\lambda_2 \ll \left(\frac{\sigma_n^2}{\sigma_J^2}\right) \sigma_S^2$$

which proves Result 1.

## APPENDIX II: PROOF OF (20)

With

$$\frac{b^2}{a} = \frac{4 \sin^2(Nx_0)}{\sin^2(x_0)} \left( \frac{N}{\tan(Nx_0)} - \frac{1}{\tan(x_0)} \right)^2$$

deduced from (18), (20) is straightforwardly obtained from a third order expansion of  $\tan(Nx_0)$  and  $\tan(x_0)$  in  $Nx_0$  and  $x_0$  respectively, under the assumption that  $x_0 \ll 1$  and  $N \gg 1$  with constraint  $Nx_0 \ll 1$ .

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