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# Mean Geometry for 2D random fields: Level perimeter and level total curvature integrals

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**Abstract:** We introduce the level perimeter integral and the total curvature integral associated with a real valued function  $f$  defined on the plane  $\mathbb{R}^2$  as integrals allowing to compute the perimeter of the excursion set of  $f$  above level  $t$  and the total (signed) curvature of its boundary for almost every level  $t$ . Thanks to the Gauss-Bonnet theorem, the total curvature is directly related to the Euler Characteristic of the excursion set. We show that the level perimeter and the total curvature integrals can be explicitly computed in two different frameworks: piecewise constant functions (also called here elementary functions) and smooth (at least  $C^2$ ) functions. Considering 2D random fields (in particular considering shot noise random fields), we compute their mean perimeter and total curvature integrals, and this provides new explicit computations of the mean perimeter and Euler Characteristic densities of excursion sets, beyond the Gaussian framework.

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**1. Introduction**

Considering a real-valued stationary 2-dimensional random field  $X = (X(x))_{x \in \mathbb{R}^2}$  we are interested in statistically describing the geometry of its excursions sets, defined for  $t \in \mathbb{R}$  by

$$E_X(t) := \{X \geq t\} \subset \mathbb{R}^2,$$

in a given bounded open subset of  $\mathbb{R}^2$ . In the following  $U$  will denote an open bounded set or simply  $\mathbb{R}^2$  when not bounded. We will focus on  $E_X(t) \cap U = \{x \in U; X(x) \geq t\}$  as well as  $\partial E_X(t) \cap U$  its boundary trace on  $U$ , where as usual  $\partial E_X(t) = \overline{E_X(t)} \setminus E_X(t)$ .

In view of the measurability of  $X$ , its excursion sets  $E_X(t)$  are random Borel sets for all level  $t \in \mathbb{R}$ . When moreover  $X$  is a.s. upper semi-continuous, these random sets are a.s. closed (see [28]) and therefore  $E_X(t) \cap \bar{U}$  is a compact random set. In dimension 2, the geometry of a compact “nice” set  $K \subset \mathbb{R}^2$  with piecewise  $C^2$  boundary  $\partial K$  may be described by three functionals: its area  $\mathcal{L}(K)$ , perimeter  $\text{Per}(K) = \mathcal{H}^1(\partial K)$ , where  $\mathcal{H}^1$  is the one-dimensional Hausdorff measure of the length of plane curves, and Euler Characteristic  $\chi(K)$  that counts the number of connected components minus the number of holes. According to the Gauss-Bonnet Theorem, when  $\partial K$  is a disjoint finite union of closed curves,  $\chi(K)$  is also equal to  $\frac{1}{2\pi} \text{TC}(\partial K)$ , with  $\text{TC}(\partial K)$  the total curvature of the positively oriented curve  $\partial K$  (see precise definitions in Definition 1 and Theorem 1). Let us quote that these geometrical features are also used with different conventions according to the setting. For instance, in convex geometry, for  $K$  a convex body, intrinsic volumes, respectively Minkowski’s functionals, are defined by  $V_0(K) = \chi(K) = 1$ ,  $V_1(K) = \frac{1}{2} \mathcal{H}^1(\partial K)$  and  $V_2(K) = \mathcal{L}(K)$ , respectively  $W_2(K) = \frac{1}{2} \text{TC}(\partial K) = \pi$ ,  $W_1(K) = \frac{1}{2} \mathcal{H}^1(\partial K)$  and  $W_0(K) = \mathcal{L}(K)$  (see [32]), while in differential geometry when  $K$  is a compact 2-dimensional submanifold with  $C^2$  smooth boundary, Lipschitz Killing curvatures of  $K$  are defined by  $C_0(K) = \frac{1}{2\pi} \text{TC}(\partial K)$ ,  $C_1(K) = \frac{1}{2} \mathcal{H}^1(\partial K)$  and  $C_2(K) = \mathcal{L}(K)$  and may be extended to positive reach sets (see [33]).

When considering stationary random sets, it is natural to define corresponding mean density functionals. They are usually defined considering the limit behavior of a rescaled observation through a large window, let say  $rU$ , for  $r$  large. Actually, this procedure allows to remove boundary effects. We adopt a similar point of view in this paper, by removing boundary effect, using an open window  $U$ . Hence, for a bounded open set  $U$  we will focus on the mean area  $\mathbb{E}(\mathcal{L}(E_X(t) \cap U))$ , the mean perimeter  $\mathbb{E}(\text{Per}(E_X(t), U))$ , and on the mean total curvature  $\mathbb{E}(\text{TC}(\partial E_X(t), U))$  of excursion sets. Densities will then clearly appear as

$$\mathbb{E}(\mathcal{L}(E_X(t) \cap U)) = \overline{\mathcal{L}(E_X(t))} \mathcal{L}(U), \quad \mathbb{E}(\text{Per}(E_X(t), U)) = \overline{\text{Per}(E_X(t))} \mathcal{L}(U)$$

$$\text{and} \quad \mathbb{E}(\text{TC}(\partial E_X(t), U)) = \overline{\chi(E_X(t))} \mathcal{L}(U).$$

As far as stationarity is involved, the mean area is not hard to find since

$$\begin{aligned} \mathbb{E}(\mathcal{L}(E_X(t) \cap U)) &= \int_U \mathbb{E}(\mathbf{1}_{X(x) \geq t}) dx \\ &= \mathcal{L}(U) \mathbb{P}(X(0) \geq t). \end{aligned}$$

It follows that an exact formula can be set up as soon as the distribution of  $X(0)$  (that is the same as any  $X(x)$  by stationarity) is known and  $\overline{\mathcal{L}(E_X(t))} = \mathbb{P}(X(0) \geq t)$ . Now establishing

formulas for the mean perimeter and the mean total curvature or Euler Characteristic is more difficult and requires additional assumptions on the field. Computing the Euler Characteristic of excursion sets of random fields is a problem that has received much attention. Indeed, in many applications, the Euler Characteristic is a very useful index of the geometry of the field, as explained for instance in the review paper of R. Adler [2], or in the papers of K. Worsley [35] or [36] where applications in astrophysics or in brain imaging are mentioned.

Despite its “global” definition (the number of connected components minus the number of holes), the Euler Characteristic of an excursion set is in fact a purely local quantity related, by Morse theory, to the number of critical points of  $X$  in  $U$ , or, by the Gauss-Bonnet theorem, to the total curvature of the boundary of the excursion set. Here, we will extensively use this second equivalence to obtain explicit computations of the mean Euler Characteristic density of the excursion sets of some random fields.

In the framework of Gaussian random field, the first equivalence is usually used. For stationary isotropic Gaussian random fields, an explicit formula for any level  $t$  may be set for the expectation of the Euler Characteristic density, only depending on the variance and on the second spectral moment of the field. This is an important result with many statistical applications. In particular, for large levels  $t$ , the Euler Characteristic gives a good approximation of the probability that the supremum of the field is greater than  $t$  and can therefore be used as a p-value: this is the Euler Characteristic heuristic (see [5] for instance). In a “tour de force”, a Central Limit Theorem has recently been established in [17] that proves the accuracy of the estimation over only one sample path as the size of the observation is growing. There are also some interesting results apart from the Gaussian framework for  $\chi^2$ ,  $F$  and  $t$ -fields [35] as well as stable [3] or infinitely divisible random fields [4] for instance. A test of Gaussianity can therefore be set up using Euler Characteristic of level sets as proposed in [15]. However, most of general results rely on strong smoothness regularity assumptions and on conditional distribution densities that are often difficult to evaluate for non-Gaussian fields.

Now, in this paper, we will be particularly interested in another family of infinitely divisible random fields, that are not Gaussian, namely the *shot noise random fields*. A shot noise random field is defined on  $\mathbb{R}^2$  by

$$\forall x \in \mathbb{R}^2, \quad X(x) = \sum_i g_{m_i}(x - x_i),$$

where the  $x_i$  are the points of an homogeneous Poisson point process of intensity  $\lambda$  in  $\mathbb{R}^2$ , and the  $m_i$  are “marks”, independent of the Poisson point process. Such fields allow explicit computations and may appear, in view of asymptotic normality in high intensity [19], as a bridge between the Gaussian setting and the discrete models of stochastic geometry such as the Boolean ones [32]. Several results for the computation of the perimeter were obtained in our previous paper [11]. Since the study for all level  $t$  is often difficult we extend here our point of view of working in a weak framework by considering these geometric quantities as functions of the level  $t$ . Hence quantities of interest will be given, when it makes sense, by the *mean level perimeter integral*  $\mathbb{E}(\text{LP}_X(h, U))$  and the *mean level total curvature integral*  $\mathbb{E}(\text{LTC}_X(h, U))$  of  $X$ , where the *level perimeter integral* and the *level total curvature integral* are defined, for  $h$  a continuous bounded function on  $\mathbb{R}$ , by

$$\text{LP}_X(h, U) := \int_{\mathbb{R}} h(t) \text{Per}(E_X(t), U) dt \quad (1)$$

and

$$\text{LTC}_X(h, U) := \int_{\mathbb{R}} h(t) \text{TC}(\partial E_X(t), U) dt. \quad (2)$$

Let us quote that this allows to get informations on the mean geometry of excursion sets for almost every level  $t$ , and hence to give insights on their evolution according to the level. This

setting will allow us to recover and generalize some important results established in the two different frameworks of Boolean models in stochastic geometry and of smooth Gaussian random fields.

In Section 2 we propose a general definition of the level perimeter integral and of the level total curvature integral of a function, that allows to compute the perimeter and the total curvature (and therefore the Euler Characteristic) of its excursion sets for almost every level. We introduce elementary functions as a particular case of piecewise constant functions (with piecewise smooth discontinuity set) in Section 3 and compute their level perimeter and total curvature integral. We give explicit expressions for elementary shot noise random fields, where the functions  $g_{m_i}$  are elementary functions. This allows us to generalize results of the literature (about the Boolean model [26], or about “random configurations” [14]). The last section is devoted to results on smooth (random) functions. In particular our weak framework allows us to get explicit formulas for some isotropic fields, recovering known Gaussian results.

Let us finally emphasize that we have made here the deliberate choice of not working in the weakest possible functional framework. Our goal is to work with piecewise smooth functions (like the indicator function of a set having a piecewise  $C^2$  boundary for instance). But we believe some of our results can be extended to functions with a weakest regularity. Let us also mention the recent work of R. Lachièze-Rey in [21] and [22] that relates the Euler Characteristic to the three-point joint distribution of the random field. And also the even more recent paper [23] where R. Lachièze-Rey gives formulas for the Euler Characteristic of isotropic shot noise random field that are a.s. Morse functions.

## 2. General framework

### 2.1. Sets of finite perimeter and finite total curvature

We consider a Borel set  $E$  of  $\mathbb{R}^2$  and the open set  $U \subset \mathbb{R}^2$ . We assume that the indicator function  $\mathbf{1}_E$  is a function in  $SBV(U)$ . Let us recall (see [7]) that a function  $f$  belongs to  $SBV(U)$ , the space of *special functions of bounded variation in  $U$*  if  $f \in L^1(U)$  and has its distributional derivative representable by a finite Radon measure in  $U$  that is

$$\int_U f(x) \frac{\partial \phi}{\partial x_l}(x) dx = - \int_U \phi(x) D_l f(dx) \quad \forall \phi \in C_c^1(U, \mathbb{R}), \quad \forall l = 1, 2$$

for some  $\mathbb{R}^2$ -valued measure  $Df = (D_1 f, D_2 f)$ , such that

$$Df = \nabla f \mathcal{L} + (f^+ - f^-) \nu_f \mathcal{H}^1 \llcorner J_f,$$

where

- $D^a f := \nabla f \mathcal{L}$  is the absolutely continuous part of the Radon measure  $Df$  with respect to the Lebesgue measure  $\mathcal{L}$  and  $\nabla f$  is the *approximate differential* of  $f$  (see [7] p.165 and Theorem 3.83 p.176).
- $D^j f := (f^+ - f^-) \nu_f \mathcal{H}^1 \llcorner J_f$  is the *jump part* of  $Df$ , with  $J_f$  the set of *approximate jump points* of  $f$  for which there exist two reals  $f^-(x) < f^+(x)$  and a direction  $\nu_f(x) \in S^1$  with

$$\lim_{\rho \rightarrow 0} \rho^{-2} \int_{B_\rho^+(x, \nu_f(x))} |f(y) - f^+(x)| dx = 0 \quad \text{and} \quad \lim_{\rho \rightarrow 0} \rho^{-2} \int_{B_\rho^-(x, \nu_f(x))} |f(y) - f^-(x)| dx = 0,$$

where  $B_\rho^+(x, \nu)$ , resp.  $B_\rho^-(x, \nu)$ , denotes the half-ball determined by  $\nu \in S^1$  i.e.  $\{y \in B_\rho(x); \langle y - x, \nu \rangle > 0\}$ , resp.  $\{y \in B_\rho(x); \langle y - x, \nu \rangle < 0\}$ . We denote here  $B_\rho(x)$  the ball of radius  $\rho$  and center  $x$ .

The *approximate discontinuity set* (see [7] Proposition 3.64 p.160)  $S_f$  is the set of points where  $f$  is not *approximately continuous*. It is a  $\mathcal{L}$ -negligible Borel set, countably  $\mathcal{H}^1$ -rectifiable with  $\mathcal{H}^1(S_f \setminus J_f) = 0$ , by Federer-Vol’pert Theorem ([7] Theorem 3.78 p.173).

This framework, used in our previous paper [11], is convenient to define the perimeter of a set  $E$  in  $U$  such that  $\mathbf{1}_E \in SBV(U)$  as

$$\text{Per}(E, U) := \|D\mathbf{1}_E\|(U) = \sup\left\{ \int_U \mathbf{1}_E \text{div} \varphi \, dx \mid \varphi \in C_c^1(U, \mathbb{R}^2), \|\varphi\|_\infty \leq 1 \right\} < +\infty.$$

It follows that denoting  $S_{\mathbf{1}_E}$  the approximate discontinuity set of  $\mathbf{1}_E$ , the set  $S_{\mathbf{1}_E} \cap U$  (its trace in  $U$ ) is included in  $\partial E \cap U$  the boundary of  $E$  in  $U$ . In order to get information linked with length and Euler Characteristic we make the stronger assumption that the boundary  $\partial E \cap U$  in  $U$  coincides with  $S_{\mathbf{1}_E} \cap U$  (which is equivalent to say that the discontinuity points are exactly the approximate discontinuity points in  $U$ ), and is a piecewise  $C^2$  plane curve.

Let us recall here some basic facts and definitions about plane curves, following [16]. We say that an oriented curve  $\Gamma$  is *piecewise regular* if it is simple (with possibly several closed simple curves) and piecewise  $C^2$  such that any point  $x \in \Gamma$  of the curve is else

- **regular:** one can find an arc-length  $C^2$  parametrization  $\gamma : (0, \varepsilon) \rightarrow \Gamma$  with  $x = \gamma(s)$  for some  $s \in (0, \varepsilon)$ , with  $\varepsilon > 0$ , and a normal vector  $\nu_\Gamma(x) = \gamma'(s)^\perp \in S^1$  with  $\gamma'(s)^\perp$  the  $+\frac{\pi}{2}$  rotation of the tangent vector  $\gamma'(s)$ . The normal cone (defined for positive reach set [33]) of  $\Gamma$  at  $x$  is given by  $\text{Nor}(\Gamma, x) = \{-\nu_\Gamma(x)\}$ . The signed curvature  $\kappa_\Gamma(x)$  of  $\Gamma$  at  $x = \gamma(s)$  is then defined as

$$\kappa_\Gamma(x) = \langle \gamma''(s), \nu_\Gamma(x) \rangle,$$

where  $\langle \cdot, \cdot \rangle$  is the usual Euclidean scalar product on  $\mathbb{R}^2$ . Note that since  $\gamma$  is an arc-length parametrization we have  $\mathcal{H}^1(\gamma(0, \varepsilon)) = \varepsilon$ .

- **corner:** one can find a simple continuous arc-length parametrization  $\gamma : (-\varepsilon, \varepsilon) \rightarrow \Gamma$  such that  $x = \gamma(0)$  with  $\gamma$  being  $C^2$  on  $(-\varepsilon, \varepsilon) \setminus \{0\}$  and  $\gamma'$  admits limits  $\gamma'(0^-) \in S^1$  and  $\gamma'(0^+) \in S^1$  at 0, with  $\nu_\Gamma^-(x) := \gamma'(0^-)^\perp$  and  $\nu_\Gamma^+(x) := \gamma'(0^+)^\perp$  linearly independent (no “cusp”) in  $S^1$  such that the normal cone of  $\Gamma$  at  $x$  is given by  $\text{Nor}(\Gamma, x) = \{-p\nu_\Gamma^-(x) - q\nu_\Gamma^+(x); p, q \geq 0\} \cap S^1$ . We then define  $\beta_\Gamma(x) \in (0, \pi)$  the angle of the cone  $\text{Nor}(\Gamma, x)$ , corresponding to the size of the jump of  $\nu_\Gamma$  at point  $x$  and  $\alpha_\Gamma(x) = \pm\beta_\Gamma(x) \in (-\pi, \pi)$  the *turning angle* at  $x$ , where the sign is given according to the orientation of the curve. Note that we also have  $\mathcal{H}^1(\gamma((-\varepsilon, \varepsilon))) = 2\varepsilon$ .

We denote  $\mathcal{R}_\Gamma$  the set of regular points and  $\mathcal{C}_\Gamma$  the set of corner points of the curve  $\Gamma$ .

**Definition 1** (Elementary set, Perimeter and Total curvature). *We say that a Borel set  $E$  is an elementary set of  $U$  if  $\mathbf{1}_E$  is in  $SBV(U)$  and its boundary  $\Gamma := \partial E$  coincides with  $S_{\mathbf{1}_E}$  in  $U$  with  $\Gamma \cap U$  a piecewise regular curve, positively oriented in such a way that the normals are oriented towards  $E$ , given by a finite union of simple connected curves (possibly closed), with curvature  $\kappa_\Gamma$  integrable on  $\mathcal{R}_\Gamma \cap U$ , that is  $\kappa_\Gamma \in L^1(\mathcal{R}_\Gamma \cap U, \mathcal{H}^1)$ , and with a finite number of corner points in  $U$ , that is  $\mathcal{H}^0(\mathcal{C}_\Gamma \cap U) < +\infty$ . It follows that the length of the curve in  $U$  is given by*

$$\mathcal{H}^1(\mathcal{R}_\Gamma \cap U) = \text{Per}(E, U),$$

and its absolute curvature in  $U$  is

$$\text{TaC}(\Gamma, U) := \int_{\mathcal{R}_\Gamma \cap U} |\kappa_\Gamma(x)| \mathcal{H}^1(dx) + \sum_{x \in \mathcal{C}_\Gamma \cap U} |\alpha_\Gamma(x)| < +\infty.$$

We then define the total curvature of  $\Gamma$  in  $U$  as

$$\text{TC}(\Gamma, U) := \int_{\mathcal{R}_\Gamma \cap U} \kappa_\Gamma(x) \mathcal{H}^1(dx) + \sum_{x \in \mathcal{C}_\Gamma \cap U} \alpha_\Gamma(x).$$

On Figure 1 we give examples of sets that are or are not elementary according to our definition.

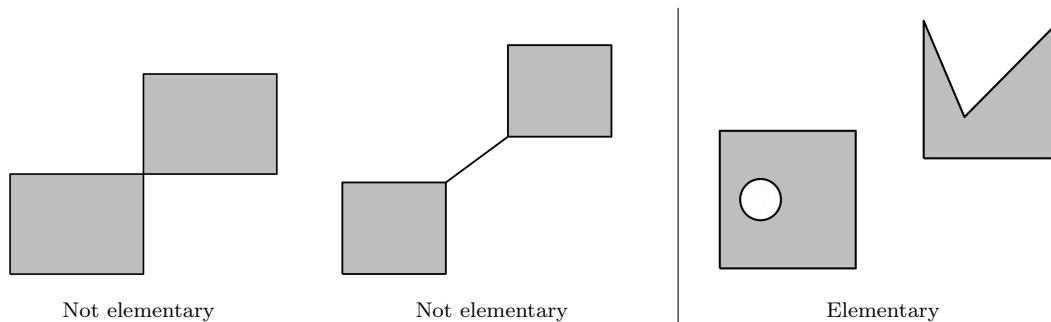


FIG 1. Left: two sets that are not elementary sets according to Definition 1: the first set has one multiple point; for the second set, its indicator function has a discontinuity set not equal to its approximate discontinuity set. Right: an elementary set.

Let us quote that for  $E$  an elementary set of  $U$ , setting  $f = \mathbf{1}_E \in \text{SBV}(U)$ , the jump part  $J_f$  of  $Df$  is equal to  $\mathcal{R}_{\partial E}$ , with  $(f^+(x), f^-(x), \nu_f(x)) = (1, 0, \nu_{\partial E}(x))$  for  $x \in J_f$  and  $\mathcal{S}_f$  is the disjoint union of  $\mathcal{R}_{\partial E} \cup \mathcal{C}_{\partial E}$ .

Notice that the definition of TaC is the same as the one introduced by Milnor in [27]. But here, in this work, we will pay a particular attention to the signed total curvature, and not to its absolute value. Actually, our definition of total curvature is the same as the one of Santaló in [29], Chapter 7. The total curvature is intrinsic, it doesn't depend on the parametrization of the curve. But it depends on its orientation: if we reverse the orientation of the curve then its total curvature is changed into its opposite.

In particular, when  $E$  is an elementary set of a bounded open set  $U$ , as soon as  $\partial E = \partial E^c$ , which is the case when  $E$  is closed or open for instance, its complementary  $E^c = \mathbb{R}^2 \setminus E$  is also an elementary set of  $U$ , with

$$\text{Per}(E^c, U) = \text{Per}(E, U), \quad \text{TaC}(\partial E^c, U) = \text{TaC}(\partial E, U) \quad \text{and} \quad \text{TC}(\partial E^c, U) = -\text{TC}(\partial E, U).$$

Note also that an elementary set in  $U$  is also an elementary set in any open subset  $U' \subset U$  and an elementary set of  $\mathbb{R}^2$  is of course elementary for any bounded open set  $U$ . Finally, let us quote that  $E$  is an elementary set in  $U$  if and only if  $E \cap U$  is an elementary set in  $U$ .

The link between the level total curvature and Euler Characteristic is given by the Gauss-Bonnet theorem.

**Theorem 1** (Gauss-Bonnet Theorem). *Let  $U$  be an open bounded set. Let  $E \subset U$  be an elementary set and a regular region (meaning that  $E = \overline{E}$ ). Then,  $S_{\mathbf{1}_E} = \partial E$  and*

- The perimeter of  $E$ , defined by

$$\text{Per}(E, U) := \|D\mathbf{1}_E\|(U) = \sup \left\{ \int_U \mathbf{1}_E \text{div} \varphi \, dx \mid \varphi \in C_c^1(U, \mathbb{R}^2), \|\varphi\|_\infty \leq 1 \right\} < +\infty,$$

is equal to

$$\mathcal{H}^1(\mathcal{R}_{\partial E}) = \mathcal{H}^1(\partial E).$$

- By Gauss-Bonnet Theorem, the Euler Characteristic of  $E$  is given by

$$\chi(E) = \frac{1}{2\pi} \text{TC}(\partial E, U),$$

where the total curvature of  $\partial E$  is equal to

$$\text{TC}(\partial E, U) = \int_{\mathcal{R}_{\partial E}} \kappa_{\partial E}(x) \mathcal{H}^1(dx) + \sum_{x \in \mathcal{C}_{\partial E}} \alpha_{\partial E}(x).$$

*Proof.* The first point follows from Gauss-Green Theorem (see Section 3.3 of [7]). For the second point we use that  $E$  being closed and elementary, its boundary must be a finite disjoint union of closed piecewise  $C^2$  curves so that we can apply Gauss-Bonnet Theorem stated in [16] p.274 for a regular region.  $\square$

**Remark 1:** It is a well-known result of differential geometry of plane curves that the total curvature of any regular simple closed curve is  $2\pi$  or  $-2\pi$  (depending on the orientation of the curve). This result is sometimes called Hopf's Umlaufsatz, or also the theorem of turning tangents ([16] p.396).

**Remark 2:** Note that when  $E \subset U$  is a regular region one has  $\partial E = \partial E^c$  and  $\text{TC}(\partial E^c, U) = -\text{TC}(\partial E, U)$ , that allows to consider both  $E$  or its complementary. In contrast, for Euler Characteristic, we have to consider the compact set  $\bar{U} \setminus \overset{\circ}{E}$ , and that yields  $\chi(\bar{U} \setminus \overset{\circ}{E}) = 1 - \chi(E)$ .

## 2.2. Level integrals for excursion sets

**Definition 2** (Level perimeter and total curvature integrals). *Let  $f$  be a real-valued function defined on  $U$  such that  $f \in \text{SBV}(U)$ . For  $t \in \mathbb{R}$ , we define the excursion set of  $f$  for the level  $t$  as*

$$E_f(t) := \{f \geq t\} \subset \mathbb{R}^2.$$

We assume that for almost every  $t \in \mathbb{R}$ , the set  $E_f(t)$  is an elementary set of  $U$  in the sense of Definition 1 and that  $t \mapsto \text{TaC}(\partial E_f(t), U)$  is an integrable function on  $\mathbb{R}$ . We then say that the function  $f$  is of special bounded variation and of finite level total curvature integral (on  $U$ ). Then, the level perimeter integral and the level total curvature integral of  $f$  are defined for any bounded continuous function  $h$  on  $\mathbb{R}$  by

$$\text{LP}_f(h, U) = \int_{\mathbb{R}} h(t) \text{Per}(E_f(t), U) dt \quad \text{and} \quad \text{LTC}_f(h, U) = \int_{\mathbb{R}} h(t) \text{TC}(\partial E_f(t), U) dt. \quad (3)$$

We simply denote  $V_f(U)$  for  $\text{LP}_f(1, U)$  ( $= \|Df\|(U)$  by the co-area formula [7]) and  $\text{LTC}_f(U)$  for  $\text{LTC}_f(1, U)$ .

Let us remark that when  $t > \max_U f$ , then  $E_f(t) \cap U = \emptyset$  and therefore  $\text{Per}(E_f(t), U) = \text{TC}(\partial E_f(t), U) = 0$ . On the other hand, when  $t \leq \min_U f$ , then  $E_f(t) \cap U = U$ , and thus  $\partial E_f(t) \cap U = \emptyset$ . Therefore we also have  $\text{Per}(E_f(t), U) = \text{TC}(\partial E_f(t), U) = 0$ . This shows that the perimeter and total curvature are 0 for levels  $t$  outside the range of  $f$ .

Let us also notice that when  $H$  is a  $C^1$  diffeomorphism on  $\mathbb{R}$  with bounded derivative  $h = H'$ , by a simple change of variable, the function  $H \circ f$  is also of special variation and of finite level total curvature integral on  $U$  with  $E_{H \circ f}(t) = E_f(H^{-1}(t))$  so that

$$V_{H \circ f}(U) = \text{LP}_f(h, U) \quad \text{and} \quad \text{LTC}_{H \circ f}(U) = \text{LTC}_f(h, U).$$



**Link with Euler Integral.** Due to the additivity property of the Euler Characteristic

$$\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B),$$

it is natural to set up an integration theory with respect to Euler Characteristic [24, 34]. However, since  $\chi$  is only finitely additive, a careful choice of integrands must be done. This problem was tackled by defining the class of constructible functions [30], then extended by the class of “tame” real-valued functions in [9]. Following this framework, Bobrowski and Borman obtained in [13] the first probabilistic statement about the persistent homology generated by sublevel sets. We briefly recall the definitions used in [13] for comparison with our setting. When  $f$  is a real continuous function defined on a compact topological set  $S$ , it is said to be a tame function if the homotopy types of  $E_f(t) = \{f \geq t\}$  and  $\{f \leq t\}$  change only finitely many times as  $t$  varies over  $\mathbb{R}$ , and the Euler Characteristic of each set is always finite. For such a function, a lower and upper Euler integrals are defined by

$$\begin{aligned} \int_S f [d\chi] &= \int_0^{+\infty} (\chi(f \geq t) - \chi(f < -t)) dt \\ \int_S f [d\chi] &= \int_0^{+\infty} (\chi(f > t) - \chi(f \leq t)) dt, \end{aligned}$$

where  $\chi(f \geq t) = \chi(E_f(t))$ ,  $\chi(f < t) = \chi(S) - \chi(f \geq t)$ , etc. Note that when  $S = \bar{U}$  we always have  $\chi(f \geq t) = \chi(S) = 1$  for any  $t \leq \min_S f$  and thus  $t \mapsto \chi(f \geq t)$  is not integrable on  $\mathbb{R}$ , explaining the above definition of Euler integrals.

In contrast, we can simply define  $\text{LTC}_f(U)$  for  $f$  a function of special bounded variation and finite level total curvature on  $U$ . Of course,  $\frac{1}{2\pi} \text{TC}(\partial E_f(t), U)$  will not coincide with  $\chi(E_f(t) \cap U)$  when the excursion set is not included in the observation window in view of its boundary. See also an illustration of this fact on Figure 2. But it can be seen as a “modified” Euler Characteristic, in a sense very similar to the one used in the book of Adler and Taylor [6] or in the paper of Estrade and León [17], where critical points in  $U$  are only taken into account, and not the ones on the boundary of  $U$ . Moreover, when considering large domains (that is  $rU$  for  $r$  going to infinity) the total curvature (in expectation) will grow like  $r^2 \mathcal{L}(U)$  whereas the sum of the turning angles on  $\partial rU$  will (in expectation also) grow like  $r \mathcal{H}^1(\partial U)$ , being negligible for large  $r$ .

We will show how the perimeter and the level total curvature integrals can be explicitly computed in different situations and we will apply it for computing Perimeter and Euler Characteristic densities of stationary fields. The first situation is the one of sums of piecewise constant functions (also called elementary functions), and the second situation will be the case of smooth (at least  $C^2$ ) functions.

### 3. Elementary functions and shot noise random fields

#### 3.1. Elementary functions

We introduce the class of elementary functions that are piecewise constant functions of special bounded variation. In order to compute the total curvature of excursion sets we need to be more precise on the discontinuity set, which is  $\mathcal{H}^1$ -rectifiable under the only assumption of special bounded variation.

We first introduce some notations. For a point  $x \in U$  and a real positive number  $\rho$ , we recall that  $B_\rho(x)$  denotes the open ball of radius  $\rho$  and center  $x$ . For an oriented simple piecewise

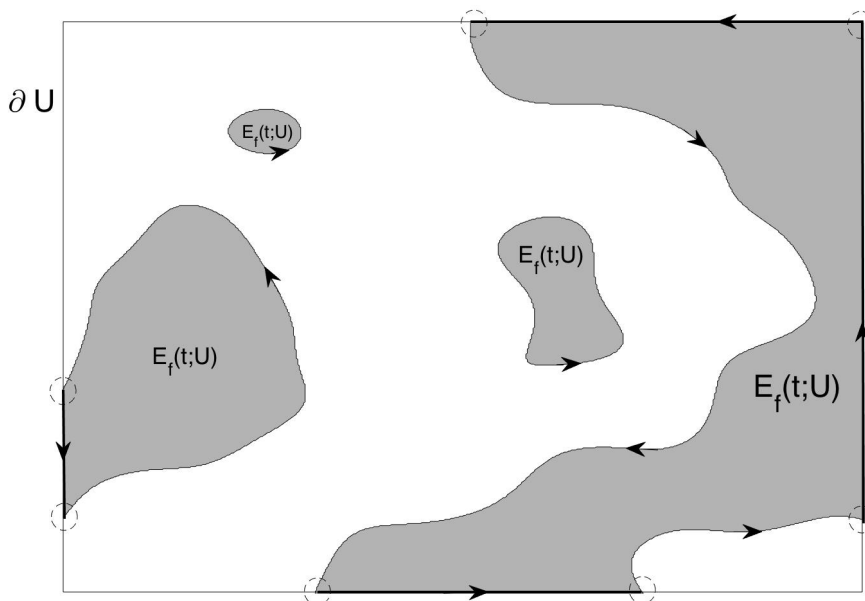


FIG 2. By the Gauss-Bonnet theorem, the Euler Characteristic of the excursion set  $E_f(t) \cap \bar{U}$  (in gray) is equal to the total curvature of its boundary in the open rectangular domain  $U$  plus the turning angles at the points where  $E_f(t) \cap \bar{U}$  meets  $\partial U$  and also plus  $\pi/2$  for each of the corners of the rectangle  $U$  that are in  $E_f(t)$ . All these special points are the ones marked by the small dashed circles on the figure.

$C^2$  curve  $\gamma$  with finite length, and a point  $x$  on  $\gamma$ , then for  $\rho$  small enough,  $B_\rho(x) \setminus \gamma$  is made of two connected components. These two “half-balls” are respectively denoted by  $B_\rho^+(x, \gamma)$  and  $B_\rho^-(x, \gamma)$ . The half-ball  $B_\rho^+(x, \gamma)$  is the component that is on the side of the normal  $\nu_\gamma$  to  $\gamma$ .

**Definition 3** (Elementary function). We say that a function  $f$  defined on  $U$  is an elementary function on  $U$  if  $f$  is a piecewise constant function in  $SBV(U)$  taking a finite number of values (meaning that  $f(U)$  is a finite subset of  $\mathbb{R}$ ), and if  $\mathcal{S}_f$  the discontinuity set of  $f$  can be decomposed in  $U$  into

$$\mathcal{S}_f \cap U = (\mathcal{R}_f \cup \mathcal{C}_f \cup \mathcal{I}_f) \cap U,$$

where  $(\mathcal{S}_f \setminus \mathcal{I}_f) \cap U$  is a piecewise regular curve and (see also Figure 3):

- $\mathcal{R}_f \cap U$  is the regular part of the discontinuity set of  $f$  in  $U$ : it is the finite and disjoint union of  $C^2$  simple connected curves having finite length and finite total curvature. More precisely, if  $x \in \mathcal{R}_f \cap U$ , then there exists  $\rho > 0$  such that  $\mathcal{S}_f \cap B_\rho(x)$  is a simple  $C^2$  oriented curve  $\gamma$  separating the ball  $B_\rho(x)$  in two half-balls  $B_\rho^+(x, \gamma)$  and  $B_\rho^-(x, \gamma)$ . Moreover there exist two real numbers  $f^+(x) > f^-(x)$  such that  $f(y) = f^+(x)$  for all  $y \in B_\rho^+(x, \gamma)$  and  $f(y) = f^-(x)$  for all  $y \in B_\rho^-(x, \gamma)$ . We also denote

$$\nu_f(x) = \nu_\gamma(x) \quad \text{and} \quad \kappa_f(x) = \kappa_\gamma(x) \quad \text{with} \quad \kappa_f \in L^1(\mathcal{R}_f, \mathcal{H}^1).$$

- $\mathcal{C}_f \cap U$  is the set of corner points of  $f$  in  $U$ : it is a finite set of points (meaning  $\mathcal{H}^0(\mathcal{C}_f) < +\infty$ , with  $\mathcal{H}^0$  the counting measure) such that if  $x \in \mathcal{C}_f \cap U$ , then there exists  $\rho > 0$  such that  $\mathcal{S}_f \cap B_\rho(x)$  is a simple piecewise  $C^2$  oriented curve  $\gamma$  having only one corner at  $x$ . We write  $\alpha_f(x) \in (-\pi, \pi)$  the turning angle of  $\gamma$  at  $x$ . As for regular points,  $\gamma$  separates

the ball  $B_\rho(x)$  in two half-balls  $B_\rho^+(x, \gamma)$  and  $B_\rho^-(x, \gamma)$ , and moreover there exist two real numbers  $f^+(x) > f^-(x)$  such that  $f(y) = f^+(x)$  for all  $y \in B_\rho^+(x, \gamma)$  and  $f(y) = f^-(x)$  for all  $y \in B_\rho^-(x, \gamma)$ . The turning angle at such a corner point is denoted

$$\alpha_f(x) = \alpha_\gamma(x).$$

- $\mathcal{I}_f \cap U$  is the set of intersection points of  $f$  in  $U$ : it is a finite set of points (meaning  $\mathcal{H}^0(\mathcal{I}_f) < +\infty$ ) such that for  $x \in \mathcal{I}_f \cap U$ , then there exists  $\rho > 0$  such that  $\mathcal{S}_f \cap B_\rho(x)$  is the union of two different simple and oriented  $C^2$  curves  $\gamma_1$  and  $\gamma_2$  such that  $\{x\} = \gamma_1 \cap \gamma_2$  and such that the intersection is non-degenerate (meaning that  $\nu_{\gamma_1}(x)$  and  $\nu_{\gamma_2}(x)$  are not colinear). Each curve separates the ball in two half-balls, and there exist 4 real numbers  $f^-(x) \leq f_+^-(x), f_+^-(x) \leq f^+(x)$  with at least 3 different values, such that  $f = f^-(x)$  on  $B_\rho^-(x, \gamma_1) \cap B_\rho^-(x, \gamma_2)$ ;  $f = f_+^-(x)$  on  $B_\rho^-(x, \gamma_1) \cap B_\rho^+(x, \gamma_2)$ ;  $f = f_+^-(x)$  on  $B_\rho^+(x, \gamma_1) \cap B_\rho^-(x, \gamma_2)$  and  $f = f^+(x)$  on  $B_\rho^+(x, \gamma_1) \cap B_\rho^+(x, \gamma_2)$ . And we define

$$\begin{aligned} \beta_f(x) &= d_{S^1}(\nu_{\gamma_1}(x), \nu_{\gamma_2}(x)) \\ &= \min(|\text{Arg } \nu_{\gamma_1}(x) - \text{Arg } \nu_{\gamma_2}(x)|, 2\pi - |\text{Arg } \nu_{\gamma_1}(x) - \text{Arg } \nu_{\gamma_2}(x)|) \in (0, \pi), \end{aligned}$$

the geodesic distance between  $\nu_{\gamma_1}(x)$  and  $\nu_{\gamma_2}(x)$  on  $S^1$ .

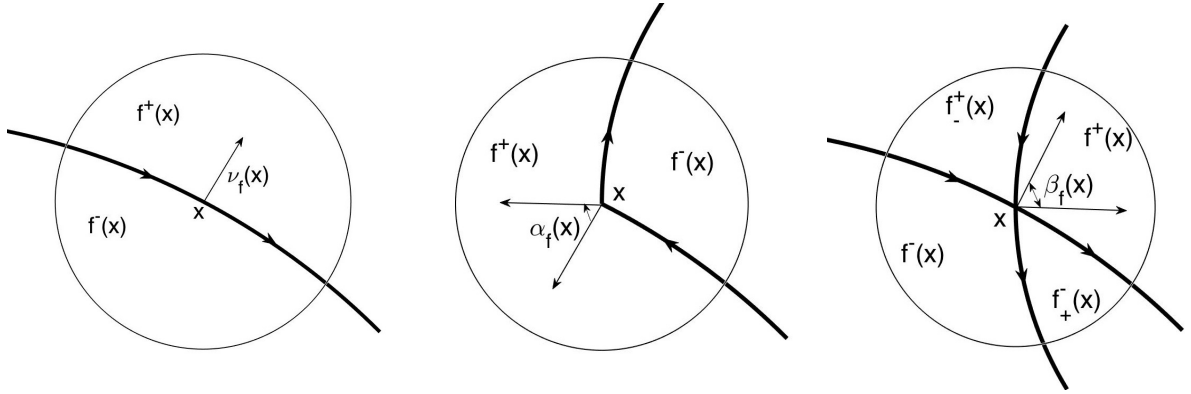


FIG 3. The three types of points of the discontinuity set of an elementary function. From left to right: a regular point, a corner point and an intersection point.

**Proposition 1.** *If  $f$  is an elementary function on  $U$ , then for all  $t \in \mathbb{R}$ ,  $E_f(t)$  is an elementary set of  $U$  in the sense of Definition 1. Moreover,  $f$  is of finite total variation and finite level total curvature integral on  $U$  with*

$$V_f(U) = \|Df(U)\| = \int_{\mathcal{R}_f \cap U} [f^+(x) - f^-(x)] \mathcal{H}^1(dx) \text{ and} \quad (4)$$

$$\begin{aligned} \text{LTaC}_f(U) &= \int_{\mathcal{R}_f \cap U} [f^+(x) - f^-(x)] |\kappa_f(x)| \mathcal{H}^1(dx) + \sum_{x \in \mathcal{C}_f \cap U} [f^+(x) - f^-(x)] |\alpha_f(x)| \\ &+ \sum_{x \in \mathcal{I}_f \cap U} [f^+(x) - \max(f_+^-(x), f_-^-(x)) + \min(f_+^-(x), f_-^-(x)) - f^-(x)] \beta_f(x) < +\infty. \end{aligned} \quad (5)$$

If  $h$  is a bounded continuous function on  $\mathbb{R}$ , and  $H$  is a primitive of  $h$  (for instance  $H(t) = \int_0^t h(u) du$ ), then the level perimeter integral and the total curvature integral of  $f$  are given by

$$\text{LP}_f(h, U) = \int_{\mathcal{R}_f \cap U} [H(f^+(x)) - H(f^-(x))] \mathcal{H}^1(dx) \quad (6)$$

$$\begin{aligned} \text{LTC}_f(h, U) &= \int_{\mathcal{R}_f \cap U} [H(f^+(x)) - H(f^-(x))] \kappa_f(x) \mathcal{H}^1(dx) + \sum_{x \in \mathcal{C}_f \cap U} [H(f^+(x)) - H(f^-(x))] \alpha_f(x) \\ &\quad + \sum_{x \in \mathcal{I}_f \cap U} [H(f^+(x)) + H(f^-(x)) - H(f_+^+(x)) - H(f_-^-(x))] \beta_f(x). \end{aligned} \quad (7)$$

In particular, when  $h = 1$ , we get  $\text{LP}_f(1, U) = V_f(U)$  and

$$\begin{aligned} \text{LTC}_f(U) &= \int_{\mathcal{R}_f \cap U} [f^+(x) - f^-(x)] \kappa_f(x) \mathcal{H}^1(dx) + \sum_{x \in \mathcal{C}_f \cap U} [f^+(x) - f^-(x)] \alpha_f(x) \\ &\quad + \sum_{x \in \mathcal{I}_f \cap U} [f^+(x) + f^-(x) - f_+^+(x) - f_-^-(x)] \beta_f(x). \end{aligned} \quad (8)$$

*Proof.* We assume that  $m = \text{Card}(f(U)) \geq 2$ . Otherwise, if  $m = 1$ , then for all  $t \in \mathbb{R}$ ,  $E_f(t) \cap U = \emptyset$  or  $U$  and therefore  $\text{Per}(E_f(t), U) = \text{TaC}(\partial E_f(t), U) = \text{TC}(\partial E_f(t), U) = 0$ . In the following we denote the values of  $f$  in  $U$  by  $v_1 < \dots < v_m$  and set  $v_0 = -\infty$ . We first remark that  $E_f(t) \cap U = \emptyset$  for  $t > v_m$ ,  $E_f(t) \cap U = U$  for  $t \leq v_1$  and  $E_f(t) \cap U = E_f(v_i) \cap U$  for  $v_{i-1} < t \leq v_i$  and  $2 \leq i \leq m$ . The set of discontinuity points is given by  $\mathcal{S}_f \cap U = \cup_{i=2}^m \partial E_f(v_i) \cap U$ .

Let us prove that  $E_f(v_i)$  is an elementary set in  $U$  for  $2 \leq i \leq m$ . Since  $f \in \text{SBV}(U)$  is piecewise constant, one can find  $\varphi_i$  a  $C^1$  real function such that  $\mathbf{1}_{E_f(v_i) \cap U} = \varphi_i \circ f$ . By chain rule (see Theorem 3.96 of [7]), it follows that  $\mathbf{1}_{E_f(v_i) \cap U} \in \text{SBV}(U)$  and therefore  $\mathbf{1}_{E_f(v_i)} \in \text{SBV}(U)$ . Now let us prove that the boundary  $\Gamma_i := \partial E_f(v_i) \cap U$  coincides with the approximate discontinuity points of  $\mathbf{1}_{E_f(v_i) \cap U}$  and is a piecewise regular curve. Since  $\Gamma_i \subset \mathcal{S}_f$ , we can write

$$\Gamma_i = (\Gamma_i \cap \mathcal{R}_f) \cup (\Gamma_i \cap \mathcal{C}_f) \cup (\Gamma_i \cap \mathcal{I}_f).$$

Note that for  $x \in \Gamma_i$ , one has  $f^+(x) \geq v_i$  and  $f^-(x) < v_i$  and  $x$  is also an approximate discontinuity point of  $\mathbf{1}_{E_f(v_i) \cap U}$ . Moreover  $\Gamma_i \cap (\mathcal{R}_f \cup \mathcal{C}_f)$  is a piecewise regular curve with length  $\mathcal{H}^1(\Gamma_i \cap (\mathcal{R}_f \cup \mathcal{C}_f)) = \mathcal{H}^1(\Gamma_i \cap \mathcal{R}_f) = \text{Per}(E_f(v_i), U) < +\infty$  and for  $x \in \Gamma_i \cap \mathcal{R}_f$  we get  $\kappa_{\Gamma_i}(x) = \kappa_f(x)$ , while for  $x \in \Gamma_i \cap \mathcal{C}_f \subset \mathcal{C}_{\Gamma_i}$  we get  $\alpha_{\Gamma_i}(x) = \alpha_f(x)$ .

When  $x \in \Gamma_i \cap \mathcal{I}_f$ , we have an intersection point of  $f$ , and it may become a corner point or a regular point for  $\Gamma_i$ . Indeed, using the same notations for intersection points as in Definition 3, we may write  $x \in \gamma_1 \cap \gamma_2$  with the two simple  $C^2$  curves  $\gamma_1$  and  $\gamma_2$  being such that  $(\gamma_1 \cup \gamma_2) \cap B_\rho(x) = \mathcal{S}_f \cap B_\rho(x)$ . If  $v_i \leq \min(f_+^+(x), f_-^-(x))$ , then  $\Gamma_i \cap B_\rho(x) = (\gamma_1 \cap B_\rho^-(x, \gamma_2)) \cup (\gamma_2 \cap B_\rho^-(x, \gamma_1))$  and  $x \in \mathcal{C}_{\Gamma_i}$  with  $\alpha_{\Gamma_i}(x) = -\beta_f(x)$ . Without loss of generality we may assume that  $f_+^+(x) \leq f_-^-(x)$ . If  $f_+^+(x) < v_i \leq f_-^-(x)$  then  $\Gamma_i \cap B_\rho(x) = \gamma_2$  and therefore  $x$  is not a corner point of  $\Gamma_i$  but a regular point  $\mathcal{H}^1$ -negligible with  $\kappa_{\Gamma_i}(x) = \kappa_{\gamma_1}(x)$ . If  $v_i > \max(f_+^+(x), f_-^-(x))$ , then  $\Gamma_i \cap B_\rho(x) = (\gamma_1 \cap B_\rho^+(x, \gamma_2)) \cup (\gamma_2 \cap B_\rho^+(x, \gamma_1))$  and  $x$  is again a corner point of  $\Gamma_i$  with  $\alpha_{\Gamma_i}(x) = \beta_f(x)$ . Finally,  $\Gamma_i$  is a piecewise regular curve and  $E_f(v_i)$  is an elementary set in  $U$  with  $\text{Per}(E_f(v_i)) = \mathcal{H}^1(\mathcal{R}_f \cap \Gamma_i)$  and

$$\begin{aligned} \text{TaC}(\Gamma_i, U) &= \int_{\mathcal{R}_f \cap \Gamma_i} |\kappa_f(x)| \mathcal{H}^1(dx) + \sum_{x \in \Gamma_i \cap \mathcal{C}_f} |\alpha_f(x)| \\ &\quad + \sum_{x \in \Gamma_i \cap \mathcal{I}_f} \beta_f(x) \left( \mathbf{1}_{v_i > \max(f_+^+(x), f_-^-(x))} + \mathbf{1}_{v_i \leq \min(f_+^+(x), f_-^-(x))} \right) < +\infty, \end{aligned}$$

while

$$\begin{aligned} \text{TC}(\Gamma_i, U) &= \int_{\mathcal{R}_f \cap \Gamma_i} \kappa_f(x) \mathcal{H}^1(dx) + \sum_{x \in \Gamma_i \cap \mathcal{C}_f} \alpha_f(x) \\ &\quad + \sum_{x \in \Gamma_i \cap \mathcal{I}_f} \beta_f(x) \left( \mathbf{1}_{v_i > \max(f_+^-(x), f_-^-(x))} - \mathbf{1}_{v_i \leq \min(f_+^-(x), f_-^-(x))} \right). \end{aligned}$$

It follows that for all  $t \in \mathbb{R}$ , the set  $E_f(t)$  is an elementary set in  $U$  and that  $t \mapsto \text{TaC}(\partial E_f(t), U)$  is an integrable function on  $\mathbb{R}$ , as a step function with compact support. Hence  $f$  is of special bounded variation and finite level curvature integral on  $U$ .

Now, let  $h$  be a continuous bounded function on  $\mathbb{R}$  and  $H$  a primitive of  $h$ . Since  $f \in SBV(U)$  we already know by Theorem 1 of [11] that (6) holds. Moreover,

$$\int_{\mathbb{R}} h(t) \text{TaC}(\partial E_f(t), U) dt = \sum_{i=2}^m \int_{v_{i-1}}^{v_i} h(t) \text{TaC}(\Gamma_i, U) dt = \sum_{i=2}^m [H(v_i) - H(v_{i-1})] \text{TaC}(\Gamma_i, U).$$

Then, using the above formula for  $\text{TaC}(\Gamma_i, U)$ , we get the sum of three terms. The first one is given by

$$\sum_{i=2}^m [H(v_i) - H(v_{i-1})] \int_{\mathcal{R}_f \cap \Gamma_i} |\kappa_f(x)| \mathcal{H}^1(dx) = \int_{\mathcal{R}_f} |\kappa_f(x)| \sum_{i=2}^m [H(v_i) - H(v_{i-1})] \mathbf{1}_{\Gamma_i}(x) \mathcal{H}^1(dx).$$

When  $x \in \mathcal{S}_f \cap U = \cup_{i=2}^m \Gamma_i$ , we denote  $i(x)$ , (resp.  $j(x) \geq i(x)$ ), the minimal (resp. maximal) index  $i = 2, \dots, m$  such that  $x \in \Gamma_i$  and  $f^-(x) := v_{i(x)-1}$ , (resp.  $f^+(x) := v_{j(x)}$ ). It follows that

$$\begin{aligned} \sum_{i=2}^m [H(v_i) - H(v_{i-1})] \mathbf{1}_{\Gamma_i}(x) &= \sum_{i=i(x)}^{j(x)} [H(v_i) - H(v_{i-1})] \\ &= H(v_{j(x)}) - H(v_{i(x)-1}) \\ &= H(f^+(x)) - H(f^-(x)). \end{aligned}$$

Therefore the first term is

$$\int_{\mathcal{R}_f \cap U} [H(f^+(x)) - H(f^-(x))] |\kappa_f(x)| \mathcal{H}^1(dx).$$

Similarly, the second term is equal to

$$\sum_{i=2}^m [H(v_i) - H(v_{i-1})] \sum_{x \in \Gamma_i \cap \mathcal{C}_f} |\alpha_f(x)| = \sum_{x \in \mathcal{C}_f \cap U} [H(f^+(x)) - H(f^-(x))] |\alpha_f(x)|.$$

Finally, the third one is equal to

$$\begin{aligned} &\sum_{i=2}^m [H(v_i) - H(v_{i-1})] \sum_{x \in \Gamma_i \cap \mathcal{I}_f} |\beta_f(x)| \left( \mathbf{1}_{v_i > \max(f_+^-(x), f_-^-(x))} + \mathbf{1}_{v_i \leq \min(f_+^-(x), f_-^-(x))} \right) \\ &= \sum_{x \in \mathcal{I}_f \cap U} \beta_f(x) \left( \sum_{i=l(x)+1}^{j(x)} [H(v_i) - H(v_{i-1})] + \sum_{i=i(x)}^{k(x)} [H(v_i) - H(v_{i-1})] \right), \end{aligned}$$

where we have introduced  $k(x)$  and  $l(x)$  with  $i(x) - 1 \leq k(x) \leq l(x) \leq j(x)$  such that  $\min(f^+(x), f^-(x)) = v_{k(x)}$  and  $\max(f^+(x), f^-(x)) = v_{l(x)}$ , with the convention that  $\sum_{i=l(x)+1}^{j(x)} = 0$  if  $l(x) = j(x)$  and  $\sum_{i=i(x)}^{k(x)} = 0$  if  $k(x) = i(x) - 1$ . It follows that this third term is equal to

$$\begin{aligned} & \sum_{x \in \mathcal{I}_f \cap U} \beta_f(x) (H(v_{j(x)}) - H(v_{l(x)}) + (H(v_{k(x)}) - H(v_{i(x)-1}))) \\ &= \sum_{x \in \mathcal{I}_f} \beta_f(x) (H(f^+(x)) - H(\max(f^+(x), f^-(x))) + H(\min(f^+(x), f^-(x))) - H(f^-(x))). \end{aligned}$$

In particular, for  $h = 1$ , we obtain Formula (5). The same computations as above give the result (7) for  $\text{LTC}_f(h, U)$ . Indeed now, the third term is equal to

$$\begin{aligned} & \sum_{i=2}^m [H(v_i) - H(v_{i-1})] \sum_{x \in \Gamma_i \cap \mathcal{I}_f} \beta_f(x) \left( \mathbf{1}_{v_i > \max(f^+(x), f^-(x))} - \mathbf{1}_{v_i \leq \min(f^+(x), f^-(x))} \right) \\ &= \sum_{x \in \mathcal{I}_f} \beta_f(x) \left( \sum_{i=l(x)+1}^{j(x)} [H(v_i) - H(v_{i-1})] - \sum_{i=i(x)}^{k(x)} [H(v_i) - H(v_{i-1})] \right) \\ &= \sum_{x \in \mathcal{I}_f} \beta_f(x) (H(f^+(x)) + H(f^-(x)) - H(f^+(x)) - H(f^-(x))). \end{aligned}$$

Taking again  $h = 1$  we obtain (8).  $\square$

The above proposition gives the formula for the level total curvature of an elementary function on  $U$ . Now, since we will be interested in shot-noise random fields that are obtained by summing elementary functions, we need to have also a formula for the level total curvature of a sum of elementary functions on  $U$ . This is the aim of the following proposition.

**Proposition 2.** *Let  $f, g$  be elementary functions on  $U$  such that their respective discontinuity sets  $\mathcal{S}_f$  and  $\mathcal{S}_g$  intersect only at a finite number of regular points in  $U$ , that is  $\mathcal{S}_f \cap \mathcal{S}_g \cap U = \mathcal{R}_f \cap \mathcal{R}_g \cap U$  with  $\mathcal{H}^0(\mathcal{S}_f \cap \mathcal{S}_g \cap U) < +\infty$ . We moreover assume that the intersections are non-degenerate in  $U$ , meaning that if  $x \in \mathcal{R}_f \cap \mathcal{R}_g \cap U$ , then  $\nu_f(x)$  and  $\nu_g(x)$  are not colinear. Then  $f + g$  is also an elementary function on  $U$  with*

- $\mathcal{R}_{f+g} \cap U = (\mathcal{R}_f \Delta \mathcal{R}_g) \cap U = (\mathcal{R}_f \cup \mathcal{R}_g \setminus (\mathcal{R}_f \cap \mathcal{R}_g)) \cap U$  and
  - if  $x \in \mathcal{R}_{f+g} \cap \mathcal{R}_f \cap U$ , then  $\kappa_{f+g}(x) = \kappa_f(x)$  with  $(f+g)^+(x) = f^+(x) + g(x)$  and  $(f+g)^-(x) = f^-(x) + g(x)$ ;
  - if  $x \in \mathcal{R}_{f+g} \cap \mathcal{R}_g \cap U$ , then  $\kappa_{f+g}(x) = \kappa_g(x)$  with  $(f+g)^+(x) = f(x) + g^+(x)$  and  $(f+g)^-(x) = f(x) + g^-(x)$ ;
- $\mathcal{C}_{f+g} \cap U = (\mathcal{C}_f \cup \mathcal{C}_g) \cap U$  is a disjoint union and
  - if  $x \in \mathcal{C}_f \cap U$ , then  $\alpha_{f+g}(x) = \alpha_f(x)$  with  $(f+g)^+(x) = f^+(x) + g(x)$  and  $(f+g)^-(x) = f^-(x) + g(x)$ ,
  - if  $x \in \mathcal{C}_g \cap U$ , then  $\alpha_{f+g}(x) = \alpha_g(x)$  with  $(f+g)^+(x) = f(x) + g^+(x)$  and  $(f+g)^-(x) = f(x) + g^-(x)$ ;
- $\mathcal{I}_{f+g} \cap U = (\mathcal{I}_f \cup \mathcal{I}_g \cup (\mathcal{R}_f \cap \mathcal{R}_g)) \cap U$  is a disjoint union and

- if  $x \in \mathcal{I}_f \cap U$ , then  $\beta_{f+g}(x) = \beta_f(x)$  with  $(f+g)^+(x) = f^+(x) + g(x)$ ,  $(f+g)^-(x) = f^-(x) + g(x)$ ,  $(f+g)_+^-(x) = f_+^-(x) + g(x)$  and  $(f+g)_-^+(x) = f_-^+(x) + g(x)$
- if  $x \in \mathcal{I}_g \cap U$ , then  $\beta_{f+g}(x) = \beta_g(x)$  with  $(f+g)^+(x) = f(x) + g^+(x)$ ,  $(f+g)^-(x) = f(x) + g^-(x)$ ,  $(f+g)_+^-(x) = f(x) + g_+^-(x)$  and  $(f+g)_-^+(x) = f(x) + g_-^+(x)$
- if  $x \in \mathcal{R}_f \cap \mathcal{R}_g \cap U$  then  $\beta_{f+g}(x) = d_{S^1}(\nu_f(x), \nu_g(x)) \in (0, \pi)$  with  $(f+g)^+(x) = f^+(x) + g^+(x)$ ,  $(f+g)^-(x) = f^-(x) + g^-(x)$  and  $\{(f+g)_-^+(x), (f+g)_+^-(x)\} = \{f^+(x) + g^-(x), f^-(x) + g^+(x)\}$ .

Moreover, we have that

$$V_{f+g}(U) = V_f(U) + V_g(U) \quad \text{and} \quad \text{LTC}_{f+g}(U) = \text{LTC}_f(U) + \text{LTC}_g(U), \quad (9)$$

while

$$\begin{aligned} \text{LTaC}_{f+g}(U) &= \text{LTaC}_f(U) + \text{LTaC}_g(U) \\ &+ \sum_{x \in \mathcal{R}_f \cap \mathcal{R}_g \cap U} \left( (f+g)^+(x) - \max((f+g)_-^+(x), (f+g)_+^-(x)) \right. \\ &\quad \left. + \min((f+g)_-^+(x), (f+g)_+^-(x) - (f+g)^-(x)) \right) \beta_{f+g}(x) \\ &\leq \text{LTaC}_f(U) + \text{LTaC}_g(U) + 2\pi \sum_{x \in \mathcal{R}_f \cap \mathcal{R}_g \cap U} [(f^+(x) - f^-(x)) + (g^+(x) - g^-(x))]. \end{aligned} \quad (10)$$

*Proof.* Since  $f$  and  $g$  are both piecewise constant functions in  $SBV(U)$ , then  $f+g$  is also a piecewise constant function in  $SBV(U)$ , and its discontinuity set  $\mathcal{S}_{f+g}$  satisfies  $\mathcal{S}_{f+g} \cap U \subset (\mathcal{S}_f \cup \mathcal{S}_g) \cap U$ . We now need to show that a point in  $\mathcal{S}_{f+g} \cap U$  is else a regular point, a corner or an intersection point in the sense of Definition 3.

If  $x \in (\mathcal{S}_f \setminus \mathcal{S}_g) \cap U$ , then we have three cases:

- if  $x \in (\mathcal{R}_f \setminus \mathcal{S}_g) \cap U$ , then  $x \in \mathcal{R}_{f+g} \cap U$  with  $\kappa_{f+g}(x) = \kappa_f(x)$  and  $(f+g)^+(x) = f^+(x) + g(x)$  and  $(f+g)^-(x) = f^-(x) + g(x)$ .
- if  $x \in (\mathcal{C}_f \setminus \mathcal{S}_g) \cap U$ , then  $x \in \mathcal{C}_{f+g} \cap U$  with  $\alpha_{f+g}(x) = \alpha_f(x)$  and  $(f+g)^+(x) = f^+(x) + g(x)$  and  $(f+g)^-(x) = f^-(x) + g(x)$ .
- if  $x \in (\mathcal{I}_f \setminus \mathcal{S}_g) \cap U$ , then  $x \in \mathcal{I}_{f+g} \cap U$  with  $\beta_{f+g}(x) = \beta_f(x)$  and  $(f+g)^+(x) = f^+(x) + g(x)$ ,  $(f+g)^-(x) = f^-(x) + g(x)$ ,  $(f+g)_+^-(x) = f_+^-(x) + g(x)$  and  $(f+g)_-^+(x) = f_-^+(x) + g(x)$ .

The same symmetric formulas hold when  $x \in (\mathcal{S}_g \setminus \mathcal{S}_f) \cap U$ . Now, when  $x \in \mathcal{S}_f \cap \mathcal{S}_g \cap U = \mathcal{R}_f \cap \mathcal{R}_g \cap U$ , since we made the hypothesis that the intersection points are non-degenerate, then  $x$  becomes an intersection point of  $f+g$ , that is  $x \in \mathcal{I}_{f+g}$  and moreover  $\beta_{f+g}(x) = d_{S^1}(\nu_f(x), \nu_g(x))$ ,  $(f+g)^+(x) = f^+(x) + g^+(x)$ ,  $(f+g)^-(x) = f^-(x) + g^-(x)$ , and  $\{(f+g)_-^+(x), (f+g)_+^-(x)\} = \{f^+(x) + g^-(x), f^-(x) + g^+(x)\}$ .

Finally, having identified the discontinuity set of  $f+g$ , since we assume that  $\mathcal{H}^0(\mathcal{R}_f \cap \mathcal{R}_g \cap U) < +\infty$  we deduce that  $(\mathcal{R}_{f+g} \cup \mathcal{C}_{f+g}) \cap U$  is a piecewise regular curve and

$$\begin{aligned} \int_{\mathcal{R}_{f+g} \cap U} |\kappa_{f+g}(x)| \mathcal{H}^1(dx) &= \int_{\mathcal{R}_f \cap U} |\kappa_f(x)| \mathcal{H}^1(dx) + \int_{\mathcal{R}_g \cap U} |\kappa_g(x)| \mathcal{H}^1(dx) < +\infty, \\ \mathcal{H}^0(\mathcal{C}_{f+g} \cap U) &= \mathcal{H}^0(\mathcal{C}_f \cap U) + \mathcal{H}^0(\mathcal{C}_g \cap U) < +\infty \\ \text{and } \mathcal{H}^0(\mathcal{I}_{f+g} \cap U) &= \mathcal{H}^0(\mathcal{I}_f \cap U) + \mathcal{H}^0(\mathcal{I}_g \cap U) + \mathcal{H}^0(\mathcal{R}_f \cap \mathcal{R}_g \cap U) < +\infty. \end{aligned}$$

This finishes to prove that  $f+g$  is an elementary function. We also notice in particular that when  $x \in \mathcal{R}_f \cap \mathcal{R}_g \cap U$ , one has  $(f+g)^+(x) + (f+g)^-(x) = (f+g)_-^+(x) + (f+g)_+^-(x) = f^+(x) + g^+(x) + f^-(x) + g^-(x)$ , such that applying the result of the previous proposition (Equation (8)) we can obtain the formula (9) for the level total curvature of  $f+g$ , while the result for the variation comes from the fact that  $\mathcal{H}^1(\mathcal{R}_f \cap \mathcal{R}_g \cap U) = 0$ .  $\square$

**Remark:** Formula (9) says that the total variation and the total curvature of a sum of two elementary functions with prescribed intersection of discontinuity sets are the sum of their total variation and their total curvature. This result is quite striking, but we have to underline that it does not hold in general. This will appear clearly in the next section where we will consider smooth functions  $f$  and where the formula for the level perimeter and the total curvature integral is obviously non linear in  $f$ .

### 3.2. Level perimeter and level total curvature of an elementary shot noise random field

We consider here a shot-noise random field defined on  $\mathbb{R}^2$  by

$$\forall x \in \mathbb{R}^2, \quad X_\Phi(x) = \sum_{i \in I} g_{m_i}(x - x_i),$$

where  $\Phi = \{(x_i, m_i)\}_{i \in I}$  is a Poisson point process on  $\mathbb{R}^2 \times \mathbb{R}^d$ , defined on a probability space  $(\Omega, \mathcal{A}, \mathbb{P})$ , of intensity  $\lambda \mathcal{L} \times F$ , with  $\lambda > 0$  real,  $\mathcal{L}$  the Lebesgue measure on  $\mathbb{R}^2$  and  $F$  a probability measure on  $\mathbb{R}^d$ . Note that equivalently, we may define  $\Phi$  as an independently *marked Poisson point process* where  $\{x_i\}_i$  is an homogeneous Poisson point process of intensity  $\lambda$  and the  $m_i$  are “marks”, following a law  $F(dm)$  on  $\mathbb{R}^d$  (with  $d \geq 1$ ) and independent of the Poisson point process  $\{x_i\}_i$ . Let  $g : \mathbb{R}^2 \times \mathbb{R}^d \rightarrow \mathbb{R}$  be a measurable function such that the functions  $g_m := g(\cdot, m)$  satisfy

$$\int_{\mathbb{R}^2 \times \mathbb{R}^d} |g_m(x)| dx F(dm) < +\infty. \quad (11)$$

Then, the random field  $X_\Phi$  is well defined as an almost surely locally integrable function on  $\mathbb{R}^2$  (see [11]). Note that moreover, the random field  $X_\Phi$  is stationary. We will first give formulas for the level perimeter and total curvature integrals of  $X_\Phi$  on an open bounded set  $U$  in the case where the  $g_m$  are elementary functions on  $\mathbb{R}^2$ , then we will compute their expectation. Finally we will give explicit results in the case of weighted indicator functions of random sets, obtained from a deterministic compact elementary set and regular region  $D$ , by random rotation and dilation. Specific computations for disks with  $D = D(0, 1)$  and squares with  $D = [0, 1]^2$  are linked with some recent results on Boolean models.

Throughout the rest of this section we also assume that for  $F$ -almost every  $m \in \mathbb{R}^d$ , the  $g_m$  are elementary functions on  $\mathbb{R}^2$ , with compact support and such that

$$\int_{\mathbb{R}^d} V_{g_m}(\mathbb{R}^2) F(dm) < +\infty \quad \text{and} \quad \int_{\mathbb{R}^d} \text{LTaC}_{g_m}(\mathbb{R}^2) F(dm) < +\infty, \quad (12)$$

where  $V_{g_m}(\mathbb{R}^2)$  and  $\text{LTaC}_{g_m}(\mathbb{R}^2)$  are defined by (4) and (5) choosing  $U = \mathbb{R}^2$ . For  $F$ -almost every  $m$ ,  $g_m$  is assumed to have a compact support, that can be included in a square  $[-T_m, T_m]^2$  with  $T_m \in \mathbb{R}_+$ , and its maximal value  $\|g_m\|_\infty = \max_{[-T_m, T_m]^2} |g_m|$  is finite. We will assume moreover that

$$\int_{\mathbb{R}^d} T_m^2 F(dm) < +\infty \quad \text{and} \quad \int_{\mathbb{R}^d} \|g_m\|_\infty F(dm) < +\infty. \quad (13)$$

Note that the first assumption of (13) implies that there is a.s. only a finite random number of  $g_m$ , denoted by  $N(U)$ , contributing to the values of  $X_\Phi$  on the bounded open set  $U \subset (-T, T)^2$ , for some  $T > 0$ . Actually, it is clear that

$$N(U) \leq \#\{i; \bar{U} \cap (x_i + [-T_{m_i}, T_{m_i}]^2) \neq \emptyset\} \leq \#\{i; \|x_i\|_\infty \leq (T_{m_i} + T)\},$$



with  $\|x\|_\infty = \max(|x_1|, |x_2|)$ , for  $x \in \mathbb{R}^2$ . It follows that  $\mathbb{E}(N(U)) \leq \lambda \int_{\mathbb{R}^d} 4(T_m + T)^2 F(dm)$ . Since  $F$  is a finite measure, under (13), we get

$$\mathbb{E}(N(U)) < +\infty. \quad (14)$$

In the following we will use the notation  $\tau_x$  to denote the translation of  $x$  in  $\mathbb{R}^2$  (i.e.  $\tau_x y = y + x$  for all  $y \in \mathbb{R}^2$ ). We will also denote  $\Phi_i = \Phi \setminus \{(x_i, m_i)\}$  for  $i \in I$ ,  $\Phi_{ij} = \Phi \setminus \{(x_i, m_i), (x_j, m_j)\}$  for  $i \neq j$  in  $I$ , and their associated shot noise random fields

$$\forall i, \quad X_{\Phi_i}(x) = \sum_{k: k \neq i} g_{m_k}(x - x_k) \quad \text{and} \quad \forall j \neq i, \quad X_{\Phi_{ij}}(x) = \sum_{k: k \neq i, k \neq j} g_{m_k}(x - x_k).$$

**Theorem 2.** *Assume that for  $F$ -almost every  $m \in \mathbb{R}^d$ , the function  $g_m$  is an elementary function on  $\mathbb{R}^2$  (in the sense of Definition 3) satisfying (11), (12) and (13), and such that*

$$\int_{\mathbb{R}^d} \mathcal{H}^0(\mathcal{S}_{g_m} \setminus \mathcal{R}_{g_m}) F(dm) < +\infty \quad (15)$$

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} \int_{\mathbb{R}^2} \mathcal{H}^0(\mathcal{R}_{g_{m'}} \cap \tau_x \mathcal{R}_{g_m}) dx F(dm) F(dm') < +\infty \quad (16)$$

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} \int_{\mathbb{R}^2} \mathcal{H}^0(\{y \in \mathcal{R}_{g_{m'}} \cap \tau_x \mathcal{R}_{g_m}; \nu_{g_{m'}}(y) = \pm \nu_{g_m}(y - x)\}) dx F(dm) F(dm') = 0. \quad (17)$$

Then, almost surely, for all bounded open set  $U \subset \mathbb{R}^2$ ,  $X_\Phi$  is an elementary function on  $U$  and its discontinuity set on  $U$  is given by  $\mathcal{S}_{X_\Phi} \cap U$  where  $\mathcal{S}_{X_\Phi} = \mathcal{R}_{X_\Phi} \cup \mathcal{C}_{X_\Phi} \cup \mathcal{I}_{X_\Phi}$ , with

- $\mathcal{R}_{X_\Phi} = \left( \bigcup_i \tau_{x_i} \mathcal{R}_{g_{m_i}} \right) \setminus \left( \bigcup_{i, j \neq i} \tau_{x_i} \mathcal{R}_{g_{m_i}} \cap \tau_{x_j} \mathcal{R}_{g_{m_j}} \right)$ , and if  $x \in \mathcal{R}_{X_\Phi} \cap U$ , there exists a unique  $i$  such that  $x \in \tau_{x_i} \mathcal{R}_{g_{m_i}}$  with  $\kappa_{X_\Phi}(x) = \kappa_{g_{m_i}}(x - x_i)$  and

$$X_\Phi^+(x) = g_{m_i}^+(x - x_i) + X_{\Phi_i}(x) \quad \text{and} \quad X_\Phi^-(x) = g_{m_i}^-(x - x_i) + X_{\Phi_i}(x);$$

- $\mathcal{C}_{X_\Phi} = \bigcup_i \tau_{x_i} \mathcal{C}_{g_{m_i}}$ , and if  $x \in \mathcal{C}_{X_\Phi} \cap U$ , there exists a unique  $i$  such that  $x \in \tau_{x_i} \mathcal{C}_{g_{m_i}}$  with  $\alpha_{X_\Phi}(x) = \alpha_{g_{m_i}}(x - x_i)$  and

$$X_\Phi^+(x) = g_{m_i}^+(x - x_i) + X_{\Phi_i}(x) \quad \text{and} \quad X_\Phi^-(x) = g_{m_i}^-(x - x_i) + X_{\Phi_i}(x);$$

- $\mathcal{I}_{X_\Phi} = \left( \bigcup_i \tau_{x_i} \mathcal{I}_{g_{m_i}} \right) \cup \left( \bigcup_{i, j \neq i} \tau_{x_i} \mathcal{R}_{g_{m_i}} \cap \tau_{x_j} \mathcal{R}_{g_{m_j}} \right)$  and if  $x \in \mathcal{I}_{X_\Phi} \cap U$ , only two situations occur

– there exists a unique  $i$  such that  $x \in \tau_{x_i} \mathcal{I}_{g_{m_i}}$ , with  $\beta_{X_\Phi}(x) = \beta_{g_{m_i}}(x - x_i)$  and

$$\begin{aligned} X_\Phi^+(x) &= g_{m_i}^+(x - x_i) + X_{\Phi_i}(x), & X_\Phi^-(x) &= g_{m_i}^-(x - x_i) + X_{\Phi_i}(x), \\ X_{\Phi_+}(x) &= g_{m_i}^+(x - x_i) + X_{\Phi_i}(x), & X_{\Phi_-}(x) &= g_{m_i}^-(x - x_i) + X_{\Phi_i}(x); \end{aligned}$$

– or there exists a unique pair  $\{i, j\}$  with  $i \neq j$  such that  $x \in \tau_{x_i} \mathcal{R}_{g_{m_i}} \cap \tau_{x_j} \mathcal{R}_{g_{m_j}}$  with

$$\beta_{X_\Phi}(x) = d_{S^1}(\nu_{g_{m_i}}(x - x_i), \nu_{g_{m_j}}(x - x_j)) \in (0, \pi)$$

and

$$\begin{aligned} X_{\Phi}^+(x) &= g_{m_i}^+(x - x_i) + g_{m_j}^+(x - x_j) + X_{\Phi_{ij}}(x), \\ X_{\Phi}^-(x) &= g_{m_i}^-(x - x_i) + g_{m_j}^-(x - x_j) + X_{\Phi_{ij}}(x), \\ &\{X_{\Phi}^+(x), X_{\Phi}^-(x)\} \\ &= \{g_{m_i}^+(x - x_i) + g_{m_j}^-(x - x_j) + X_{\Phi_{ij}}(x), g_{m_i}^-(x - x_i) + g_{m_j}^+(x - x_j) + X_{\Phi_{ij}}(x)\}. \end{aligned}$$

In particular, a.s.

$$V_{X_{\Phi}}(U) = \sum_i V_{g_{m_i}}(\tau_{-x_i}U) \quad \text{and} \quad \text{LTC}_{X_{\Phi}}(U) = \sum_i \text{LTC}_{g_{m_i}}(\tau_{-x_i}U),$$

with

$$V_{g_{m_i}}(\tau_{-x_i}U) = \int_{U \cap \tau_{x_i} \mathcal{R}_{g_{m_i}}} [g_{m_i}^+(x - x_i) - g_{m_i}^-(x - x_i)] \mathcal{H}^1(dx),$$

and

$$\begin{aligned} \text{LTC}_{g_{m_i}}(\tau_{-x_i}U) &= \int_{U \cap \tau_{x_i} \mathcal{R}_{g_{m_i}}} [g_{m_i}^+(x - x_i) - g_{m_i}^-(x - x_i)] \kappa_{g_{m_i}}(x - x_i) \mathcal{H}^1(dx) \\ + \sum_{x \in U \cap \tau_{x_i} \mathcal{C}_{g_{m_i}}} &[g_{m_i}^+(x - x_i) - g_{m_i}^-(x - x_i)] \alpha_{g_{m_i}}(x - x_i) \\ + \sum_{x \in U \cap \tau_{x_i} \mathcal{I}_{g_{m_i}}} &[g_{m_i}^+(x - x_i) + g_{m_i}^-(x - x_i) - g_{m_i}^+(x - x_i) - g_{m_i}^-(x - x_i)] \beta_{g_{m_i}}(x - x_i). \end{aligned}$$

*Proof.* Since it is sufficient to prove the result for all rectangles  $U = (a_1, b_1) \times (a_2, b_2)$  with  $a_1, a_2, b_1, b_2 \in \mathbb{Q}$ , we only have to prove it holds almost surely on some fixed rectangle  $U = (a_1, b_1) \times (a_2, b_2)$ , with  $a_1 < b_1$  and  $a_2 < b_2$ . Let us quote that we already have proven that  $X_{\Phi} \in SBV(U)$  in Theorem 2 of [11] in a more general framework. However we need here to be more precise on its discontinuity set  $\mathcal{S}_{X_{\Phi}} \subset \cup_i \tau_{x_i} \mathcal{S}_{g_{m_i}}$ .

Let us first remark that when  $A_{\Phi}$  is a finite set of points of  $\mathbb{R}^2$  depending on the marked Poisson point process  $\Phi = \{(x_i, m_i)\}$ , as soon as  $\mathbb{E}(\mathcal{H}^0(A_{\Phi})) < +\infty$ , one has,

$$\bigcup_j A_{\Phi_j} \cap \tau_{x_j} \mathcal{S}_{g_{m_j}} = \emptyset \quad \text{a.s.}$$

This follows from the fact that, by Slivnyak-Mecke formula (see [8] Theorem 1.4.5),

$$\begin{aligned} \mathbb{E} \left( \mathcal{H}^0 \left( \bigcup_j A_{\Phi_j} \cap \tau_{x_j} \mathcal{S}_{g_{m_j}} \right) \right) &\leq \lambda \int_{\mathbb{R}^2 \times \mathbb{R}^d} \mathbb{E}(\mathcal{H}^0(A_{\Phi} \cap \tau_x \mathcal{S}_{g_m})) dx F(dm) \\ &\leq \lambda \mathbb{E}(\mathcal{H}^0(A_{\Phi})) \int_{\mathbb{R}^d} \mathcal{L}(\mathcal{S}_{g_m}) F(dm) = 0, \end{aligned}$$

since  $\mathcal{L}(\mathcal{S}_{g_m}) = 0$ , using Fubini Theorem and the translation invariance of both  $\mathcal{H}^0$  and  $\mathcal{L}$ .

Our first assumption (15) implies that

$$\bigcup_{i,j \neq} \tau_{x_i} (\mathcal{S}_{g_{m_i}} \setminus \mathcal{R}_{g_{m_i}}) \cap \tau_{x_j} \mathcal{S}_{g_{m_j}} \cap U = \emptyset \quad \text{a.s.} \quad (18)$$

Indeed, taking  $A_\Phi = \bigcup_i \tau_{x_i} (\mathcal{S}_{g_{m_i}} \setminus \mathcal{R}_{g_{m_i}}) \cap U$ , Campbell formula (see [8] Theorem 1.4.3) ensures that

$$\begin{aligned} \mathbb{E}(\mathcal{H}^0(A_\Phi)) &\leq \lambda \int_{\mathbb{R}^2 \times \mathbb{R}^d} \mathcal{H}^0(\tau_x(\mathcal{S}_{g_m} \setminus \mathcal{R}_{g_m}) \cap U) dx F(dm) \\ &\leq \lambda \mathcal{L}(U) \int_{\mathbb{R}^d} \mathcal{H}^0(\mathcal{S}_{g_m} \setminus \mathcal{R}_{g_m}) F(dm) < +\infty. \end{aligned}$$

Then, (18) follows from the preceding remark since

$$\bigcup_{i,j \neq} \tau_{x_i} (\mathcal{S}_{g_{m_i}} \setminus \mathcal{R}_{g_{m_i}}) \cap \tau_{x_j} \mathcal{S}_{g_{m_j}} \cap U = \bigcup_j A_{\Phi_j} \cap \tau_{x_j} \mathcal{S}_{g_{m_j}}.$$

The second assumption (16) will ensure both that

$$\mathcal{H}^0 \left( \bigcup_{i,j \neq} \tau_{x_i} \mathcal{R}_{g_{m_i}} \cap \tau_{x_j} \mathcal{R}_{g_{m_j}} \cap U \right) < \infty \quad \text{a.s.}, \quad (19)$$

and

$$\bigcup_{i,j,k \neq} \tau_{x_i} \mathcal{R}_{g_{m_i}} \cap \tau_{x_j} \mathcal{R}_{g_{m_j}} \cap \tau_{x_k} \mathcal{S}_{g_{m_k}} \cap U = \emptyset \quad \text{a.s.} \quad (20)$$

Here we set  $A_\Phi = \bigcup_{i,j \neq} \tau_{x_i} \mathcal{R}_{g_{m_i}} \cap \tau_{x_j} \mathcal{R}_{g_{m_j}} \cap U$ . Using again Slivnyak-Mecke formula and Campbell formula we obtain that

$$\begin{aligned} \mathbb{E}(\mathcal{H}^0(A_\Phi)) &\leq \lambda^2 \int_{\mathbb{R}^2 \times \mathbb{R}^d} \int_{\mathbb{R}^2 \times \mathbb{R}^d} \mathcal{H}^0(\tau_x \mathcal{R}_{g_m} \cap \tau_{x'} \mathcal{R}_{g_{m'}} \cap U) dx F(dm) dx' F(dm') \\ &\leq \lambda^2 \mathcal{L}(U) \int_{\mathbb{R}^d \times \mathbb{R}^d} \int_{\mathbb{R}^2} \mathcal{H}^0(\mathcal{R}_{g_{m'}} \cap \tau_x \mathcal{R}_{g_m}) dx F(dm) F(dm') < +\infty. \end{aligned}$$

It follows that  $\mathcal{H}^0(A_\Phi) < +\infty$  a.s. and  $\bigcup_k A_{\Phi_k} \cap \tau_{x_k} \mathcal{S}_{g_{m_k}} = \emptyset$  a.s. .

Finally and similarly, the last assumption ensures that

$$\{y \in \bigcup_{i,j \neq} \tau_{x_i} \mathcal{R}_{g_{m_i}} \cap \tau_{x_j} \mathcal{R}_{g_{m_j}} \cap U; \nu_{g_{m_i}}(y - x_i) = \pm \nu_{g_{m_j}}(y - x_j)\} = \emptyset \quad \text{a.s.} \quad (21)$$

This follows from the fact that the expected  $\mathcal{H}^0$  measure of this set is zero, according to Slivnyak-Mecke formula and Campbell formula, Fubini Theorem and translation invariance.

Now let us consider the random variable  $N(U)$  counting the number of functions  $g_{m_i}$  contributing to the values of  $X_\Phi$  on  $U$  and recall that, under the assumption that  $\int_{\mathbb{R}^d} T_m^2 F(dm) < +\infty$ ,  $N(U)$  is a.s. finite, as a consequence of (14).

We now will prove the result of Theorem 2 by induction on the value of  $N(U)$  once we have fixed an almost sure realization ensuring the previous conditions.

For  $N(U) = 0$  there is nothing to prove since  $X_\Phi = 0$  on  $U$  in this case. Let us assume the result holds when  $N(U) = n \geq 0$  and let us prove it for  $N(U) = n + 1$ . We can assume that there exists  $(x_i, m_i)$  such that  $\tau_{-x_i} g_{m_i}$  contributes to the values of  $X$  on  $U$  and write  $X_\Phi = X_{\Phi_i} + \tau_{-x_i} g_{m_i}$ . The number of functions contributing to  $X_{\Phi_i}$  is given by  $N(U) - 1$  so we can use our induction to state that  $X_{\Phi_i}$  is an elementary function on  $U$  with discontinuity set  $\mathcal{S}_{X_{\Phi_i}} \cap U$  where

$$\mathcal{S}_{X_{\Phi_i}} = \bigcup_{j;j \neq i} \tau_{x_j} \mathcal{S}_{g_{m_j}}, \quad \text{with} \quad \mathcal{R}_{X_{\Phi_i}} = \left( \bigcup_{j;j \neq i} \tau_{x_j} \mathcal{R}_{g_{m_j}} \right) \setminus \left( \bigcup_{j,k \neq i; j \neq i, k \neq i} \tau_{x_j} \mathcal{R}_{g_{m_j}} \cap \tau_{x_k} \mathcal{R}_{g_{m_k}} \right),$$

$$\mathcal{C}_{X_{\Phi_i}} = \bigcup_{j:j \neq i} \tau_{x_j} \mathcal{C}_{g_{m_j}} \quad \text{and} \quad \mathcal{I}_{X_{\Phi_i}} = \left( \bigcup_{j:j \neq i} \tau_{x_j} \mathcal{I}_{g_{m_j}} \right) \cup \left( \bigcup_{j,k \neq i, k \neq i} \tau_{x_j} \mathcal{R}_{g_{m_j}} \cap \tau_{x_k} \mathcal{R}_{g_{m_k}} \right).$$

Then the discontinuity points are given by  $\mathcal{S}_{X_{\Phi}} \cap U$  with  $\mathcal{S}_{X_{\Phi}} \subset \mathcal{S}_{X_{\Phi_i}} \cup \tau_{x_i} \mathcal{S}_{g_{m_i}} = \bigcup_j \tau_{x_j} \mathcal{S}_{g_{m_j}}$  with

$$\mathcal{S}_{X_{\Phi_i}} \cap \tau_{x_i} (\mathcal{S}_{g_{m_i}} \setminus \mathcal{R}_{g_{m_i}}) \cap U \subset \bigcup_{j:j \neq i} \tau_{x_j} \mathcal{S}_{g_{m_j}} \cap \tau_{x_i} (\mathcal{S}_{g_{m_i}} \setminus \mathcal{R}_{g_{m_i}}) \cap U = \emptyset$$

by (18). Moreover,

$$\mathcal{S}_{X_{\Phi_i}} \setminus \mathcal{R}_{X_{\Phi_i}} \subset \bigcup_{j:j \neq i} \tau_{x_j} (\mathcal{S}_{g_{m_j}} \setminus \mathcal{R}_{g_{m_j}}) \cup \bigcup_{j,k \neq i, k \neq i} \tau_{x_j} \mathcal{R}_{g_{m_j}} \cap \tau_{x_k} \mathcal{R}_{g_{m_k}},$$

with again  $\bigcup_{j:j \neq i} \tau_{x_j} (\mathcal{S}_{g_{m_j}} \setminus \mathcal{R}_{g_{m_j}}) \cap \tau_{x_i} \mathcal{S}_{g_{m_i}} \cap U = \emptyset$  by (18) and  $\bigcup_{j,k \neq i, k \neq i} \tau_{x_j} \mathcal{R}_{g_{m_j}} \cap \tau_{x_k} \mathcal{R}_{g_{m_k}} \cap \tau_{x_i} \mathcal{S}_{g_{m_i}} \cap U = \emptyset$  by (20). Therefore, we may conclude that

$$\mathcal{S}_{X_{\Phi_i}} \cap \tau_{x_i} \mathcal{S}_{g_{m_i}} \cap U \subset \mathcal{R}_{X_{\Phi_i}} \cap \tau_{x_i} \mathcal{R}_{g_{m_i}} \cap U,$$

with  $\mathcal{H}^0(\mathcal{R}_{X_{\Phi_i}} \cap \tau_{x_i} \mathcal{R}_{g_{m_i}} \cap U) < +\infty$  by (19). Finally let us quote that the intersections are non-degenerate in view of (21). Hence, according to Proposition 2,  $X_{\Phi}$  is an elementary function on  $U$ . Moreover, let fix a point  $x \in U$ . If  $x \in \mathcal{R}_{X_{\Phi_i}} \setminus \tau_{x_i} \mathcal{S}_{g_{m_i}}$  we get  $X_{\Phi}^{\pm}(x) = X_{\Phi_i}^{\pm}(x) + g_{m_i}(x - x_i)$  and by induction there exists a unique  $j \neq i$  such that  $x \in \tau_{x_j} \mathcal{R}_{g_{m_j}}$  so that  $X_{\Phi}^{\pm}(x) = X_{\Phi_{ij}}(x) + g_{m_j}^{\pm}(x - x_j) + g_{m_i}(x - x_i) = X_{\Phi_j}(x) + g_{m_j}^{\pm}(x - x_j)$  and  $\kappa_{X_{\Phi}}(x) = \kappa_{g_{m_j}}(x - x_j)$ . Similarly, if  $x \in \tau_{x_i} \mathcal{R}_{g_{m_i}} \setminus \mathcal{S}_{X_{\Phi_i}}$  we get  $X_{\Phi}^{\pm}(x) = X_{\Phi_i}(x) + g_{m_i}^{\pm}(x - x_i)$  and  $\kappa_{X_{\Phi}}(x) = \kappa_{g_{m_i}}(x - x_i)$ . In the same way, by induction and using the fact that  $\mathcal{C}_{X_{\Phi}}$  is the disjoint union of  $\tau_{x_j} \mathcal{C}_{g_{m_j}}$  we obtain that if  $x \in \mathcal{C}_{X_{\Phi}}$ , there exists a unique  $i$  such that  $x \in \tau_{x_i} \mathcal{C}_{g_{m_i}}$  and  $\alpha_{X_{\Phi}}(x) = \alpha_{g_{m_i}}(x - x_i)$  with  $X_{\Phi}^{\pm}(x) = g_{m_i}^{\pm}(x - x_i) + X_{\Phi_i}(x)$ . Finally  $\mathcal{I}_{X_{\Phi}}$  is the disjoint union of  $\mathcal{I}_{X_{\Phi_i}}$ ,  $\tau_{x_i} \mathcal{I}_{g_{m_i}}$  and  $\mathcal{R}_{X_{\Phi_i}} \cap \tau_{x_i} \mathcal{R}_{g_{m_i}}$ . By induction,  $\mathcal{I}_{X_{\Phi_i}} = \bigcup_{j:j \neq i} \tau_{x_j} \mathcal{I}_{g_{m_j}} \cup \bigcup_{j,k \neq i, k \neq i} \tau_{x_j} \mathcal{R}_{g_{m_j}} \cap \tau_{x_k} \mathcal{R}_{g_{m_k}}$  and  $\mathcal{R}_{X_{\Phi_i}} \cap \tau_{x_i} \mathcal{R}_{g_{m_i}} = \bigcup_{j:j \neq i} \tau_{x_j} \mathcal{R}_{g_{m_j}} \cap \tau_{x_i} \mathcal{R}_{g_{m_i}}$  where unions are all disjoint. Hence, grouping the terms we get the result.  $\square$

**Theorem 3.** *Under the hypothesis of Theorem 2, assuming moreover that*

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} \int_{\mathbb{R}^2} (\|g_m\|_{\infty} + \|g_{m'}\|_{\infty}) \mathcal{H}^0(\mathcal{R}_{g_{m'}} \cap \tau_x \mathcal{R}_{g_m}) dx F(dm) F(dm') < +\infty, \quad (22)$$

then the random variables  $V_{X_{\Phi}}(U)$ ,  $\text{LTaC}_{X_{\Phi}}(U)$  and  $\text{LTC}_{X_{\Phi}}(U)$  have finite expectation for any bounded open set  $U$ . Moreover one has

$$\mathbb{E}(V_{X_{\Phi}}(U)) = \lambda \mathcal{L}(U) \int_{\mathbb{R}^d} V_{g_m}(\mathbb{R}^2) F(dm) \quad \text{and} \quad \mathbb{E}(\text{LTC}_{X_{\Phi}}(U)) = \lambda \mathcal{L}(U) \int_{\mathbb{R}^d} \text{LTC}_{g_m}(\mathbb{R}^2) F(dm),$$

with

$$\begin{aligned} V_{g_m}(\mathbb{R}^2) &= \int_{\mathcal{R}_{g_m}} [g_m^+(z) - g_m^-(z)] \mathcal{H}^1(dz) \\ \text{LTC}_{g_m}(\mathbb{R}^2) &= \int_{\mathcal{R}_{g_m}} [g_m^+(z) - g_m^-(z)] \kappa_{g_m}(z) \mathcal{H}^1(dz) + \sum_{z \in \mathcal{C}_{g_m}} [g_m^+(z) - g_m^-(z)] \alpha_{g_m}(z) \\ &\quad + \sum_{z \in \mathcal{I}_{g_m}} [g_m^+(z) + g_m^-(z) - g_m^{\pm}(z) - g_m^{\mp}(z)] \beta_{g_m}(z). \end{aligned}$$

It follows that for a.e.  $t \in \mathbb{R}$ , the random variables  $\text{Per}(E_{X_\Phi}(t), U)$  and  $\text{TC}(\partial E_{X_\Phi}(t), U)$  have finite expectation such that for any  $h$  continuous bounded function

$$\mathbb{E}(\text{LP}_{X_\Phi}(h, U)) = \int_{\mathbb{R}} h(t) \mathbb{E}(\text{Per}(E_{X_\Phi}(t), U)) dt$$

$$\text{and } \mathbb{E}(\text{LTC}_{X_\Phi}(h, U)) = \int_{\mathbb{R}} h(t) \mathbb{E}(\text{TC}(\partial E_{X_\Phi}(t), U)) dt,$$

with

$$\begin{aligned} \mathbb{E}(\text{LP}_{X_\Phi}(h, U)) &= \lambda \mathcal{L}(U) \int_{\mathbb{R}^d} \int_{\mathcal{R}_{g_m}} \int_{g_m^-(x)}^{g_m^+(x)} \mathbb{E}(h(X_\Phi(0) + s)) ds \mathcal{H}^1(dx) F(dm) \\ \mathbb{E}(\text{LTC}_{X_\Phi}(h, U)) &= \lambda \mathcal{L}(U) \int_{\mathbb{R}^d} (R(h, m) + C(h, m) + I(h, m)) F(dm), \end{aligned}$$

where

$$\begin{aligned} R(h, m) &= \int_{\mathcal{R}_{g_m}} \int_{g_m^-(x)}^{g_m^+(x)} \mathbb{E}(h(X_\Phi(0) + s)) ds \kappa_{g_m}(x) \mathcal{H}^1(dx), \\ C(h, m) &= \sum_{x \in \mathcal{C}_{g_m}} \int_{g_m^-(x)}^{g_m^+(x)} \mathbb{E}(h(X_\Phi(0) + s)) ds \alpha_{g_m}(x), \end{aligned}$$

and  $I(h, m) = I^{(1)}(h, m) + I^{(2)}(h, m)$ , with

$$I^{(1)}(h, m) = \sum_{x \in \mathcal{I}_{g_m}} \left( \int_{g_m^+(x)}^{g_m^-(x)} \mathbb{E}(h(X_\Phi(0) + s)) ds - \int_{g_m^-(x)}^{g_m^+(x)} \mathbb{E}(h(X_\Phi(0) + s)) ds \right) \beta_{g_m}(x)$$

$$\begin{aligned} \text{and } I^{(2)}(h, m) &= \frac{\lambda}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^2} \sum_{z \in \tau_x \mathcal{R}_{g_m} \cap \mathcal{R}_{g_{m'}}} d_{S^1}(\nu_{g_m}(z-x), \nu_{g_{m'}}(z)) \\ &\quad \times \int_{g_m^-(z-x)}^{g_m^+(z-x)} \mathbb{E}(h(X_\Phi(0) + s + g_{m'}^+(z)) - h(X_\Phi(0) + s + g_m^-(z))) ds dx F(dm'). \end{aligned}$$

*Proof.* First note that the results on the perimeter follow from Theorem 2 and Proposition 1 of [11] since in view of (11) and (12) we have  $\int_{\mathbb{R}^d} \|g_m\|_{BV(\mathbb{R}^d)} F(dm) < +\infty$ . Moreover, according to (10), one has a.s.

$$\text{LTaC}_{X_\Phi}(U) \leq \sum_i \text{LTaC}_{g_{m_i}}(\tau_{-x_i} U) + 2\pi \sum_{i,j \neq} \mathcal{H}^0(\tau_{x_i} \mathcal{R}_{g_{m_i}} \cap \tau_{x_j} \mathcal{R}_{g_{m_j}} \cap U) (\|g_{m_i}\|_\infty + \|g_{m_j}\|_\infty).$$

By Campbell formula,

$$\mathbb{E} \left( \sum_i \text{LTaC}_{g_{m_i}}(\tau_{-x_i} U) \right) = \int_{\mathbb{R}^2 \times \mathbb{R}^d} \text{LTaC}_{g_m}(\tau_{-x} U) \lambda dx F(dm).$$

Hence, by Fubini Theorem,

$$\mathbb{E} \left( \sum_i \text{LTaC}_{g_{m_i}}(\tau_{-x_i} U) \right) = \lambda \mathcal{L}(U) \int_{\mathbb{R}^d} \text{LTaC}_{g_m}(\mathbb{R}^2) F(dm) < +\infty.$$

Moreover, by Slivnyak-Mecke formula,

$$\begin{aligned}
 & \mathbb{E} \left( \sum_{i,j \neq} \mathcal{H}^0 \left( \tau_{x_i} \mathcal{R}_{g_{m_i}} \cap \tau_{x_j} \mathcal{R}_{g_{m_j}} \cap U \right) (\|g_{m_i}\|_\infty + \|g_{m_j}\|_\infty) \right) \\
 &= \int_{\mathbb{R}^d \times \mathbb{R}^d} \int_{\mathbb{R}^2 \times \mathbb{R}^2} \mathcal{H}^0 \left( \tau_x \mathcal{R}_{g_m} \cap \tau_{x'} \mathcal{R}_{g_{m'}} \cap U \right) (\|g_m\|_\infty + \|g_{m'}\|_\infty) \lambda^2 dx dx' F(dm) F(dm') \\
 &= \lambda^2 \mathcal{L}(U) \int_{\mathbb{R}^d \times \mathbb{R}^d} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \mathbf{1}_{z \in \mathcal{R}_{g_{m'}} \cap \tau_x \mathcal{R}_{g_m}} (\|g_m\|_\infty + \|g_{m'}\|_\infty) \mathcal{H}^0(dz) dx F(dm) F(dm') \\
 &= \lambda^2 \mathcal{L}(U) \int_{\mathbb{R}^d \times \mathbb{R}^d} \int_{\mathbb{R}^2} (\|g_m\|_\infty + \|g_{m'}\|_\infty) \mathcal{H}^0(\mathcal{R}_{g_{m'}} \cap \tau_x \mathcal{R}_{g_m}) dx F(dm) F(dm') < +\infty,
 \end{aligned}$$

by assumption (22). It follows that  $\mathbb{E}(\text{LTaC}_{X_\Phi}(U)) < +\infty$  and therefore

$$\mathbb{E} \left( \int_{\mathbb{R}} |\text{TC}(\partial E_{X_\Phi}(t), U)| dt \right) < +\infty.$$

So for a.e.  $t \in \mathbb{R}$ , the random variable  $\text{TC}(\partial E_{X_\Phi}(t), U)$  has a finite expectation and  $t \mapsto \mathbb{E}(\text{TC}(\partial E_{X_\Phi}(t), U))$ , is a function in  $L^1(\mathbb{R})$ . Note that since  $\text{LTC}_{X_\Phi}(U) = \sum_i \text{LTC}_{g_{m_i}}(\tau_{-x_i} U)$  we simply get by Campbell formula and Fubini Theorem that

$$\mathbb{E}(\text{LTC}_{X_\Phi}(U)) = \lambda \mathcal{L}(U) \int_{\mathbb{R}^d} \text{LTC}_{g_m}(\mathbb{R}^2) F(dm).$$

Now, let  $h$  be a continuous bounded function with primitive denoted by  $H$ . We already know that a.s.  $\text{LTC}_{X_\Phi}(h, U) = \int_{\mathbb{R}} h(t) \text{TC}(\partial E_{X_\Phi}(t), U) dt$  may be written as the sum of three terms  $R_h + C_h + I_h$  with finite expectation under our assumptions. By Fubini Theorem, it follows that

$$\mathbb{E}(\text{LTC}_{X_\Phi}(h, U)) = \int_{\mathbb{R}} h(t) \mathbb{E}(\text{TC}(\partial E_{X_\Phi}(t), U)) dt = \mathbb{E}(R_h) + \mathbb{E}(C_h) + \mathbb{E}(I_h).$$

For the first term we get

$$\begin{aligned}
 \mathbb{E}(R_h) &= \mathbb{E} \left( \int_{\mathcal{R}_{X_\Phi} \cap U} [H(X_\Phi^+(z)) - H(X_\Phi^-(x))] \kappa_{X_\Phi}(z) \mathcal{H}^1(dz) \right) \\
 &= \mathbb{E} \left( \sum_i \int_{\tau_{x_i} \mathcal{R}_{g_{m_i}} \cap U} [H(X_{\Phi_i}(z) + g_{m_i}^+(z - x_i)) - H(X_{\Phi_i}(z) + g_{m_i}^-(z - x_i))] \kappa_{g_{m_i}}(z - x_i) \mathcal{H}^1(dz) \right) \\
 &= \int_{\mathbb{R}^2 \times \mathbb{R}^d} \int_{\tau_x \mathcal{R}_{g_m} \cap U} \mathbb{E} \left( [H(X_\Phi(z) + g_m^+(z - x)) - H(X_\Phi(z) + g_m^-(z - x))] \kappa_{g_m}(z - x) \mathcal{H}^1(dz) \right) \lambda dx F(dm),
 \end{aligned}$$

by Slivnyak-Mecke formula. Using the translation invariance of both  $\mathcal{H}^1$  and  $\mathcal{L}$  and the stationarity of  $X_\Phi$  we get that

$$\begin{aligned}
 \mathbb{E}(R_h) &= \lambda \mathcal{L}(U) \int_{\mathbb{R}^d} \int_{\mathcal{R}_{g_m}} \int_{g_m^-(z)}^{g_m^+(z)} \mathbb{E}(h(X_\Phi(0) + s)) \kappa_{g_m}(z) ds \mathcal{H}^1(dz) F(dm) \quad (23) \\
 &= \lambda \mathcal{L}(U) \int_{\mathbb{R}^d} R(h, m) F(dm).
 \end{aligned}$$

Similarly, for the second term we get

$$\begin{aligned}
 \mathbb{E}(C_h) &= \mathbb{E} \left( \sum_{z \in \mathcal{C}_{X_\Phi} \cap U} [H(X_\Phi^+(z)) - H(X_\Phi^-(z))] \alpha_{X_\Phi}(z) \right) \\
 &= \mathbb{E} \left( \sum_i \mathbf{1}_{z \in \tau_{x_i} \mathcal{C}_{g_{m_i}} \cap U} [H(X_{\Phi_i}(z) + g_{m_i}^+(z - x_i)) - H(X_{\Phi_i}(z) + g_{m_i}^-(z - x_i))] \alpha_{g_{m_i}}(z - x_i) \mathcal{H}^1(dz) \right) \\
 &= \lambda \mathcal{L}(U) \int_{\mathbb{R}^d} \sum_{z \in \mathcal{C}_{g_m}} \mathbb{E}([H(X_\Phi(0) + g_m^+(z)) - H(X_\Phi(0) + g_m^-(z))]) \alpha_{g_m}(z) F(dm) \\
 &= \lambda \mathcal{L}(U) \int_{\mathbb{R}^d} C(h, m) F(dm), \quad (24)
 \end{aligned}$$

with  $C(h, m) = \sum_{z \in \mathcal{C}_{g_m}} \int_{g_m^+(z)}^{g_m^-(z)} \mathbb{E}(h(X_\Phi(0) + s)) ds \alpha_{g_m}(z)$ . Finally, the last term may be itself decomposed in two terms, say  $I_h = I_h^{(1)} + I_h^{(2)}$ . With similar computations we get that  $\mathbb{E}(I_h^{(1)})$  is equal to

$$\begin{aligned}
 \lambda \mathcal{L}(U) \int_{\mathbb{R}^d} \sum_{z \in \mathcal{I}_{g_m}} \left( \int_{g_m^+(z)}^{g_m^-(z)} \mathbb{E}(h(X_\Phi(0) + s)) ds - \int_{g_m^-(z)}^{g_m^+(z)} \mathbb{E}(h(X_\Phi(0) + s)) ds \right) \beta_{g_m}(z) F(dm) \\
 = \lambda \mathcal{L}(U) \int_{\mathbb{R}^d} I^{(1)}(h, m) F(dm). \quad (25)
 \end{aligned}$$

Since  $\tau_{x_i} \mathcal{R}_{g_{m_i}} \cap \tau_{x_j} \mathcal{R}_{g_{m_j}} = \tau_{x_j} \mathcal{R}_{g_{m_j}} \cap \tau_{x_i} \mathcal{R}_{g_{m_i}}$ , writing for  $z \in \tau_{x_i} \mathcal{R}_{g_{m_i}} \cap \tau_{x_j} \mathcal{R}_{g_{m_j}}$

$$\Delta_{m_i, m_j} h(t, z - x_i, z - x_j) = \int_{g_{m_j}^-(z - x_j)}^{g_{m_j}^+(z - x_j)} [h(t + g_{m_i}^+(z - x_i) + s) - h(t + g_{m_i}^-(z - x_i) + s)] ds,$$

the term  $\mathbb{E}(I_h^{(2)})$  is equal to

$$\begin{aligned}
 \mathbb{E} \left( \frac{1}{2} \sum_{i \neq j} \sum_{z \in \tau_{x_i} \mathcal{R}_{g_{m_i}} \cap \tau_{x_j} \mathcal{R}_{g_{m_j}} \cap U} \Delta_{m_i, m_j} h(X_{\Phi_{ij}}(z), z - x_i, z - x_j) d_{S^1}(\nu_{g_{m_i}}(z - x_i), \nu_{g_{m_j}}(z - x_j)) \right) \\
 = \frac{1}{2} \iint \sum_{z \in \tau_x \mathcal{R}_{g_m} \cap \tau_{x'} \mathcal{R}_{g_{m'}} \cap U} \mathbb{E}(\Delta_{m, m'} h(X_\Phi(z), z - x, z - x')) \\
 \times d_{S^1}(\nu_{g_m}(z - x), \nu_{g_{m'}}(z - x')) \lambda^2 dx F(dm) dx' F(dm'),
 \end{aligned}$$

by Slivnyak-Mecke formula. By change of variables, translation invariance of  $\mathcal{H}^0$  and  $\mathcal{L}$ , and stationarity of  $X_\Phi$ , we get that  $\mathbb{E}(I_h^{(2)})$  is equal to

$$\frac{\lambda^2}{2} \mathcal{L}(U) \int_{\mathbb{R}^d \times \mathbb{R}^d} \int_{\mathbb{R}^2} \sum_{z \in \tau_x \mathcal{R}_{g_m} \cap \tau_{x'} \mathcal{R}_{g_{m'}}} \mathbb{E}(\Delta_{m, m'} h(X_\Phi(0), z - x, z)) d_{S^1}(\nu_{g_m}(z - x), \nu_{g_{m'}}(z)) dx F(dm) F(dm').$$

Now, let us quote that choosing  $h_u(s) = e^{ius}$  for  $u, s \in \mathbb{R}$  we have that

$$\mathbb{E}(\text{LP}_{X_\Phi}(h_u, \mathbb{R}^2)) = \lambda \mathcal{L}(U) \mathbb{E} \left( e^{iu X_\Phi(0)} \right) \int_{\mathbb{R}^d} \text{LP}_{g_m}(h_u, \mathbb{R}^2) F(dm), \quad (26)$$

while (23)+(24)+(25) is equal to

$$\lambda \mathcal{L}(U) \mathbb{E} \left( e^{iuX_\Phi(0)} \right) \int_{\mathbb{R}^d} \text{LTC}_{g_m}(h_u, \mathbb{R}^2) F(dm). \quad (27)$$

Therefore

$$\begin{aligned} \mathbb{E}(\text{LTC}_{X_\Phi}(h_u, U)) &= \int_{\mathbb{R}} e^{iut} \mathbb{E}(\text{TC}(\partial E_{X_\Phi}(t), U)) dt = \lambda \mathcal{L}(U) \mathbb{E} \left( e^{iuX_\Phi(0)} \right) \int_{\mathbb{R}^d} [\text{LTC}_{g_m}(h_u, \mathbb{R}^2) + \\ &\frac{\lambda}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^2} \sum_{z \in \tau_x \mathcal{R}_{g_m} \cap \mathcal{R}_{g_{m'}}} (e^{iug_{m'}^+(z)} - e^{iug_{m'}^-(z)}) \int_{g_m^-(z-x)}^{g_m^+(z-x)} e^{ius} d_{S^1}(\nu_{g_m}(z-x), \nu_{g_{m'}}(z)) ds dx F(dm')] F(dm) \end{aligned} \quad (28)$$

□

### 3.3. Explicit computations

In this section, we will give some explicit computations of the mean level total curvature function of elementary shot noise random fields. These results generalize the results of Decreusefond et al. [14] obtained for indicator functions of a square, and also the known results on the Boolean model (that correspond to the excursion set of level  $t = 1$ ). We will also show some numerical simulations.

Let us first recall that for shot noise random fields, the characteristic function of  $X_\Phi(0)$  is explicit and given by (see for instance [10])

$$\forall u \in \mathbb{R}, \quad \mathbb{E}(e^{iuX_\Phi(0)}) = \exp \left( \lambda \int_{\mathbb{R}^2 \times \mathbb{R}^d} (e^{iug_m(x)} - 1) dx F(dm) \right).$$

In this section we consider  $D$  an elementary compact subset of  $\mathbb{R}^2$  and regular region so that it has a piecewise smooth boundary given by  $\Gamma = \partial D$  a finite union of positively oriented closed simple curves, piecewise  $C^2$  and of finite total curvature, i.e.  $\text{TaC}(\Gamma, \mathbb{R}^2) < +\infty$ . Note that by Gauss-Bonnet Theorem we have

$$\text{TC}(\Gamma, \mathbb{R}^2) = 2\pi\chi(D).$$

We will focus on the case where the marks are of the form  $m = (b, r, \theta) \in \mathbb{R} \times [0, +\infty) \times [0, 2\pi] \subset \mathbb{R}^d$  with  $d = 3$ , with distribution  $F(dm) = F_B(db)F_R(dr)F_\Theta(d\theta)$  and functions  $g_m$  given by

$$\forall x \in \mathbb{R}^2, \quad g_m(x) = b \mathbf{1}_{R_\theta r D}(x),$$

where  $rD$  is the dilation of  $D$  by the factor  $r$ , and  $R_\theta$  denotes the rotation of angle  $\theta$ . We will denote  $b^+ = \max(b, 0)$ ,  $b^- = \min(0, b)$  such that for  $x \in \partial R_\theta r D$ , we have  $g_m^+(x) = b^+$  and  $g_m^-(x) = b^-$  with  $g_m^+(x) - g_m^-(x) = b^+ - b^- = |b|$  and  $b = (b^+ - b^-) \text{sgn}(b)$ .

We denote by  $B$  (resp.  $B^+ = \max(B, 0)$  and  $B^- = \min(B, 0)$ ),  $R$  and  $\Theta$  independent random variables with distributions  $F_B$ ,  $F_R$  and  $F_\Theta$ . We will mainly focus on the case where  $\Theta$  is uniform on  $[0, 2\pi]$ , that is  $F_\Theta(d\theta) = \frac{1}{2\pi} \mathbf{1}_{[0, 2\pi]} d\theta$  for random shapes with uniform rotation; or on the case where  $\Theta = 0$  a.s., that is  $F_\Theta = \delta_0$  corresponding to simpler marks  $m = (b, r)$ .

**Theorem 4.** *We assume that*

$$\mathbb{E}(|B|) < +\infty \quad \text{and} \quad \mathbb{E}(R^2) < +\infty.$$



We denote the mean perimeter and the mean area of  $R_\theta rD$  by

$$\bar{p} := \int_{\mathbb{R}^2} \text{Per}(R_\theta rD) F_R(dr) F_\Theta(d\theta) = \text{Per}(D) \mathbb{E}(R)$$

$$\text{and } \bar{a} := \int_{\mathbb{R}^2} \mathcal{L}(R_\theta rD) F_R(dr) F_\Theta(d\theta) = \mathcal{L}(D) \mathbb{E}(R^2).$$

Then  $X_\Phi$  satisfies the assumptions of Theorem 2 and 3 and

$$\mathbb{E}(\text{LP}_{X_\Phi}(U)) = \lambda \mathcal{L}(U) \mathbb{E}(|B|) \bar{p} \quad \text{and} \quad \mathbb{E}(\text{LTC}_{X_\Phi}(U)) = 2\pi \lambda \mathcal{L}(U) \mathbb{E}(B) \chi(D).$$

Moreover, when  $F_\Theta$  is the uniform law on  $[0, 2\pi]$ , then, for all  $u \neq 0$ ,

$$\begin{aligned} \int_{\mathbb{R}} e^{iut} \mathbb{E}(\text{Per}(E_{X_\Phi}(t), U)) dt &= \lambda \mathcal{L}(U) \mathbb{E}(e^{iuX_\Phi(0)}) \frac{\widehat{F}_{B^+}(u) - \widehat{F}_{B^-}(u)}{iu} \bar{p} \\ \int_{\mathbb{R}} e^{iut} \mathbb{E}(\text{TC}(\partial E_{X_\Phi}(t), U)) dt &= \lambda \mathcal{L}(U) \mathbb{E}(e^{iuX_\Phi(0)}) \\ &\quad \times \frac{1}{iu} \left( 2\pi (\widehat{F}_B(u) - 1) \chi(D) + \frac{\lambda}{2} (\widehat{F}_{B^+}(u) - \widehat{F}_{B^-}(u))^2 \bar{p}^2 \right), \end{aligned}$$

where  $\widehat{F}_B$ , respectively  $\widehat{F}_{B^+}$  or  $\widehat{F}_{B^-}$ , denotes the characteristic function of  $B$ , respectively  $B^+$  or  $B^-$ . In the case where  $B = 1 (= B^+)$  a.s. we deduce that  $\forall k \in \mathbb{N}, \forall t \in (k, k+1]$ ,

$$\begin{aligned} \mathbb{E}(\text{Per}(E_{X_\Phi}(t), U)) &= \lambda \mathcal{L}(U) e^{-\lambda \bar{a}} \frac{(\lambda \bar{a})^k}{k!} \quad \text{and} \\ \mathbb{E}(\text{TC}(\partial E_{X_\Phi}(t), U)) &= 2\pi \lambda \mathcal{L}(U) e^{-\lambda \bar{a}} \frac{(\lambda \bar{a})^k}{k!} \left( \chi(D) - \frac{\lambda}{4\pi} \bar{p}^2 + \frac{\bar{p}^2}{4\pi \bar{a}} k \right). \end{aligned} \quad (29)$$

*Proof.* Since  $\mathcal{H}^1(R_\theta r\Gamma) = r\mathcal{H}^1(\Gamma)$  and  $\text{TaC}(R_\theta r\Gamma, \mathbb{R}^2) = \text{TaC}(\Gamma, \mathbb{R}^2)$ , the  $g_m$ 's are elementary functions with, for  $F(dm)$  almost every  $m = (b, r, \theta)$ , writing  $L = \mathcal{H}^1(\Gamma) = \text{Per}(D)$ , we have

$$V_{g_m}(\mathbb{R}^2) = |b| r L \quad \text{and} \quad \text{LTaC}(g_m, \mathbb{R}^2) = |b| \text{TaC}(\Gamma, \mathbb{R}^2).$$

Let us remark that Equation (11) becomes

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^2} |g_m(x)| dx F(dm) = \mathbb{E}(|B|) \bar{a} < +\infty.$$

Equation (12) is easily checked since

$$\begin{aligned} \int_{\mathbb{R}^d} V_{g_m}(\mathbb{R}^2) F(dm) &= \mathbb{E}(|B|) \mathbb{E}(R) L < +\infty \quad \text{and} \\ \int_{\mathbb{R}^d} \text{LTaC}_{g_m}(\mathbb{R}^2) F(dm) &= \mathbb{E}(|B|) \text{TaC}(\Gamma, \mathbb{R}^2) < +\infty. \end{aligned}$$

Assumption (13) also follows from the fact that  $\bar{a} < +\infty$  and  $\mathbb{E}(|B|) < +\infty$ . For (15), let us remark that  $\mathcal{I}_{g_m} = \emptyset$  and  $\mathcal{C}_{g_m} = R_\theta r\mathcal{C}_\Gamma$  such that  $\mathcal{H}^0(\mathcal{S}_{g_m} \setminus \mathcal{R}_{g_m}) = \mathcal{H}^0(\mathcal{C}_\Gamma)$ , ensuring (15), by assumption on  $\Gamma$ .

Now, in order to check (16) and (17) we need a kind of *kinematic formula*.

**Lemma 1** (Kinematic formula). *Let  $f$  be a measurable non-negative function on  $[-\pi, \pi]^2$ , periodically extended on  $\mathbb{R}^2$ , and let  $\Gamma_1, \Gamma_2$  be two piecewise regular simple closed curves. Let  $I_{\Gamma_1, \Gamma_2}(f)$  be defined by*

$$I_{\Gamma_1, \Gamma_2}(f) := \int_{\mathbb{R}^2} \sum_{z \in \Gamma_1 \cap \tau_x \Gamma_2} f(\text{Arg } \nu_{\Gamma_1}(z), \text{Arg } \nu_{\Gamma_2}(z - x)) dx.$$

Then

$$I_{\Gamma_1, \Gamma_2}(f) = \int_0^{L_1} \int_0^{L_2} f(\theta_1(s_1) + \frac{\pi}{2}, \theta_2(s_2) + \frac{\pi}{2}) |\sin(\theta_1(s_1) - \theta_2(s_2))| ds_1 ds_2, \quad (30)$$

where the curve  $\Gamma_1$  (resp.  $\Gamma_2$ ) of length  $L_1 = \mathcal{H}^1(\Gamma_1)$  (resp.  $L_2 = \mathcal{H}^1(\Gamma_2)$ ), is parametrized by  $s_1 \mapsto \gamma_1(s_1)$  (resp. by  $s_2 \mapsto \gamma_2(s_2)$ ) where  $s_1$  (resp.  $s_2$ ) is arc length, and  $\theta_1(s_1) = \text{Arg } \gamma_1'(s_1)$  (resp.  $\theta_2(s_2) = \text{Arg } \gamma_2'(s_2)$ ).

*Proof.* A heuristic proof of this result is given in the book of Santaló [29]. But it can be made fully rigorous using the co-area formula for Lipschitz mappings. Indeed, if  $F : \mathbb{R}^2 \mapsto \mathbb{R}^2$  is Lipschitz, then for any non-negative measurable function  $v$  on  $\mathbb{R}^2$ , the co-area formula (see [18] or [7]) states that

$$\int_{\mathbb{R}^2} v(y) |\det(DF(y))| dy = \int_{\mathbb{R}^2} \left( \int_{F^{-1}(x)} v(t) \mathcal{H}^0(dt) \right) dx = \int_{\mathbb{R}^2} \sum_{t \in F^{-1}(x)} v(t) dx.$$

Now, we use this formula with  $F$  given by  $F(s_1, s_2) = \gamma_1(s_1) - \gamma_2(s_2)$  for  $(s_1, s_2) \in [0, L_1] \times [0, L_2]$  (and extended to  $\mathbb{R}^2$  by periodicity for instance), and  $v$  given by  $v(s_1, s_2) = f(\theta_1(s_1) + \frac{\pi}{2}, \theta_2(s_2) + \frac{\pi}{2}) \mathbf{1}_{(s_1, s_2) \in [0, L_1] \times [0, L_2]}$ . Since  $\Gamma_1$  and  $\Gamma_2$  are piecewise regular closed curves, they have only a finite number of corner points, and therefore  $\gamma_1$  and  $\gamma_2$  are differentiable except at a finite number of points, and we can compute at a regular point  $y = (s_1, s_2)$

$$|\det(DF(y))| = |\det(\gamma_1'(s_1), \gamma_2'(s_2))| = |\sin(\theta_1(s_1) - \theta_2(s_2))|.$$

And we also notice that if  $x \in \mathbb{R}^2$  and  $t = (s_1, s_2) \in F^{-1}(x)$ , then  $x = \gamma_1(s_1) - \gamma_2(s_2)$  and therefore  $z = \gamma_1(s_1)$  belongs to both  $\Gamma_1$  and  $\tau_x \Gamma_2$ , and  $\gamma_2(s_2) = z - x$ . This concludes the proof of the lemma.  $\square$

In particular, taking  $f = 1$ , it follows that  $I_{\Gamma_1, \Gamma_2}(1) \leq L_1 L_2$ . Note also that we moreover have the exact formula

$$\int_0^{2\pi} \int_0^{2\pi} I_{R_\theta \Gamma_1, R_{\theta'} \Gamma_2}(1) d\theta d\theta' = 2\pi \times \int_0^{2\pi} |\sin(\theta)| d\theta L_1 L_2 = 2\pi \times 4L_1 L_2,$$

according to a generalization of Poincaré's Formula (see [29] for instance). Using the fact that  $\Gamma$  is a finite disjoint union of closed curves we obtain that  $I_{R_\theta r \Gamma, R_{\theta'} r' \Gamma}(1) \leq L^2 r r'$ , since the length of  $R_\theta r \Gamma$ , resp.  $R_{\theta'} r' \Gamma$ , is  $rL$ , resp.  $r'L$ , with  $L = \mathcal{H}^1(\Gamma)$  the length of  $\Gamma$  and

$$\int_0^{2\pi} \int_0^{2\pi} I_{R_\theta r \Gamma, R_{\theta'} r' \Gamma}(1) d\theta d\theta' = 2\pi \times 4L^2 r r'.$$

It follows that

$$\begin{aligned} & \int_{\mathbb{R}^d \times \mathbb{R}^d} \int_{\mathbb{R}^2} \mathcal{H}^0(\mathcal{R}_{g_{m'}} \cap \tau_x \mathcal{R}_{g_m}) dx F(dm) F(dm') \\ &= \int_{\mathbb{R}_+^2} \int_{[0, 2\pi]^2} I_{R_\theta r \Gamma, R_{\theta'} r' \Gamma}(1) F_\Theta(d\theta) F_\Theta(d\theta') F_R(dr) F_R(dr') \\ &\leq 4L^2 \left( \int_{\mathbb{R}_+} r F_r(dr) \right)^2 = 4\bar{p}^2, \end{aligned}$$

with  $\bar{p} = L \mathbb{E}(R)$  the mean perimeter, proving (16).

Moreover, for  $f(\theta_1, \theta_2) = \mathbf{1}_{\theta_1 \equiv \theta_2} + \mathbf{1}_{\theta_1 \equiv \theta_2 + \pi}$ , where  $\equiv$  stands for equality modulo  $2\pi$ , we clearly have

$$I_{\Gamma_1, \Gamma_2}(f) = 0,$$

in view of (30). Since  $\Gamma$  is a finite disjoint union of closed curves, it follows that

$$\begin{aligned} & \int_{\mathbb{R}^d \times \mathbb{R}^d} \int_{\mathbb{R}^2} \mathcal{H}^0(\{y \in \mathcal{R}_{g_{m'}} \cap \tau_x \mathcal{R}_{g_m}; \nu_{g_{m'}}(y) = \pm \nu_{g_m}(y-x)\}) dx F(dm) F(dm') \\ &= \int_{\mathbb{R}_+^2} \int_{[0, 2\pi]^2} I_{R_\theta r \Gamma, R_{\theta'}, r' \Gamma}(f) F_\Theta(d\theta) F_\Theta(d\theta') F_R(dr) F_R(dr') = 0, \end{aligned}$$

so that (17) holds. Therefore we get the statement of Theorem 2. Moreover, we also get

$$\begin{aligned} & \int_{\mathbb{R}^d \times \mathbb{R}^d} \int_{\mathbb{R}^2} (\|g_m\|_\infty + \|g_{m'}\|_\infty) \mathcal{H}^0(\mathcal{R}_{g_{m'}} \cap \tau_x \mathcal{R}_{g_m}) dx F(dm) F(dm') \\ & \leq 4\bar{p}^2 \int_{\mathbb{R} \times \mathbb{R}} (|b| + |b'|) F_B(db) F_B(db') = 8\bar{p}^2 \mathbb{E}(|B|) < +\infty, \end{aligned}$$

and (22) is also satisfied so that Theorem 3 holds.

Note that, we simply have

$$\mathbb{E}(V_{X_\Phi}(U)) = \lambda \mathcal{L}(U) \int_{\mathbb{R}^d} V_{g_m}(\mathbb{R}^2) F(dm) = \lambda \mathcal{L}(U) \mathbb{E}(|B|) \mathbb{E}(R) \mathcal{H}^1(\Gamma),$$

and

$$\mathbb{E}(\text{LTC}_{X_\Phi}(U)) = \lambda \mathcal{L}(U) \int_{\mathbb{R}^d} \text{LTC}_{g_m}(\mathbb{R}^2) F(dm) = \lambda \mathcal{L}(U) \mathbb{E}(B) \text{TC}(\Gamma, \mathbb{R}^2),$$

using the fact that  $\text{LTC}_{g_m}(\mathbb{R}^2) = (b^+ - b^-) \times \text{sgn}(b) \times \text{TC}(\Gamma, \mathbb{R}^2) = b \text{TC}(\Gamma, \mathbb{R}^2)$ . Since  $\text{TC}(\Gamma, \mathbb{R}^2) = 2\pi \chi(D)$ , we obtain the first general statement. For  $u \neq 0$ , we can explicitly compute the characteristic function of  $X_\Phi(0)$ , given by

$$\mathbb{E}(e^{iu X_\Phi(0)}) = \exp\left(\lambda \int_{\mathbb{R}^d \times \mathbb{R}^2} (e^{iug_m(x)} - 1) F(dm) dx\right) = e^{\lambda \bar{a} (\widehat{F}_B(u) - 1)},$$

where  $\widehat{F}_B(u) = \mathbb{E}(e^{iuB}) = \int_{\mathbb{R}} e^{iub} F_B(db)$  is the characteristic function of  $B$ . In particular for  $B = 1$  a.s.  $X_\Phi(0)$  is a Poisson random variable of parameter  $\lambda \bar{a}$ . We also have, writing  $h_u$  as previously,

$$\begin{aligned} \text{LP}_{g_m}(h_u, \mathbb{R}^2) &= \int_{b^-}^{b^+} e^{iut} \mathcal{H}^1(R_\theta r \Gamma) dt = r \mathcal{H}^1(\Gamma) \frac{e^{iub^+} - e^{iub^-}}{iu} \\ \text{LTC}_{g_m}(h_u, \mathbb{R}^2) &= \int_{b^-}^{b^+} e^{iut} \text{sgn}(b) \text{TC}(R_\theta r \Gamma, \mathbb{R}^2) dt = \text{TC}(\Gamma, \mathbb{R}^2) \frac{e^{iub^+} - e^{iub^-}}{iu} \text{sgn}(b), \end{aligned}$$

so that

$$\int_{\mathbb{R}^d} \text{LP}_{g_m}(h_u, \mathbb{R}^2) F(dm) = \mathbb{E}(R) \mathcal{H}^1(\Gamma) \frac{\widehat{F}_{B^+}(u) - \widehat{F}_{B^-}(u)}{iu}$$

and

$$\int_{\mathbb{R}^d} \text{LTC}_{g_m}(h_u, \mathbb{R}^2) F(dm) = \text{TC}(\Gamma, \mathbb{R}^2) \frac{\widehat{F}_B(u) - 1}{iu}.$$

Therefore

$$\mathbb{E}(\text{LP}_{X_\Phi}(h_u, U)) = \int_{\mathbb{R}} e^{iut} \mathbb{E}(\mathcal{H}^1(\partial E_{X_\Phi}(t, U))) dt = \lambda \mathcal{L}(U) \mathbb{E}(e^{iuX_\Phi(0)}) \mathbb{E}(R) \mathcal{H}^1(\Gamma) \frac{\widehat{F}_{B^+}(u) - \widehat{F}_{B^-}(u)}{iu}.$$

Finally, let us remark that for  $f(\theta_1, \theta_2) = \min(|\theta_1 - \theta_2|, 2\pi - |\theta_1 - \theta_2|)$  (distance between two angles) and  $f_\pi(\theta_1, \theta_2) = f(\pi + \theta_1, \theta_2)$ , we get

$$\begin{aligned} & \int_{\mathbb{R}^d} \int_{\mathbb{R}^2 \times \mathbb{R}^d} (e^{iug_{m'}^+(z)} - e^{iug_{m'}^-(z)}) \left( \int_{g_m^-(z-x)}^{g_m^+(z-x)} e^{ius} ds \right) \\ & \times \sum_{z \in \tau_x \mathcal{R}_{g_m} \cap \mathcal{R}_{g_{m'}}} d_{S^1}(\nu_{g_m}(z-x), \nu_{g_{m'}}(z)) dx F(dm) F(dm') \\ = & \frac{(\int_0^{+\infty} (e^{iub} - 1) F_B(db))^2 + (\int_{-\infty}^0 (1 - e^{iub}) F_B(db))^2}{iu} \\ & \times \int_{\mathbb{R}_+^2} \int_{[0, 2\pi]^2} I_{R_\theta r \Gamma, R_{\theta'} r' \Gamma}(f) F_\Theta(d\theta) F_\Theta(d\theta') F_R(dr) F_R(dr') \\ + & \frac{(\int_0^{+\infty} (e^{iub} - 1) F_B(db)) (\int_{-\infty}^0 (1 - e^{iub'}) F_B(db'))}{iu} \\ & \times \int_{\mathbb{R}_+^2} \int_{[0, 2\pi]^2} I_{R_\theta r \Gamma, R_{\theta'} r' \Gamma}(f_\pi) F_\Theta(d\theta) F_\Theta(d\theta') F_R(dr) F_R(dr') \end{aligned}$$

This last expression may not be simple to compute. However, assuming from now on that  $F_\Theta(d\theta) = \frac{1}{2\pi} \mathbf{1}_{[0, 2\pi]} d\theta$ , we obtain by (30), for two simple closed curves  $\Gamma_1, \Gamma_2$ ,

$$\begin{aligned} & \int_{[0, 2\pi]^2} I_{R_\theta \Gamma_1, R_{\theta'} \Gamma_2}(f) F_\Theta(d\theta) F_\Theta(d\theta') = \frac{1}{2\pi} \int_0^{2\pi} \int_0^{L_1} \int_0^{L_2} |\sin(\theta_1(s_1) - \theta_2(s_2) - \theta)| \\ & \times \min(|\theta_1(s_1) - \theta_2(s_2) - \theta|, 2\pi - |\theta_1(s_1) - \theta_2(s_2) - \theta|) ds_1 ds_2 d\theta \\ = & L_1 L_2. \end{aligned}$$

Similarly we have

$$\int_{[0, 2\pi]^2} I_{R_\theta \Gamma_1, R_{\theta'} \Gamma_2}(f_\pi) F_\Theta(d\theta) F_\Theta(d\theta') = L_1 L_2.$$

Since  $\Gamma$  is a finite disjoint union of such closed curves, we get

$$\int_{[0, 2\pi]^2} I_{R_\theta r \Gamma, R_{\theta'} r' \Gamma}(f) F_\Theta(d\theta) F_\Theta(d\theta') = \int_{[0, 2\pi]^2} I_{R_\theta r \Gamma, R_{\theta'} r' \Gamma}(f_\pi) F_\Theta(d\theta) F_\Theta(d\theta') = L^2 r r'.$$

It follows that for uniform rotations, we have

$$\mathbb{E}(\text{LTC}_{X_\Phi}(h_u, U)) = \lambda \mathcal{L}(U) \mathbb{E}(e^{iuX_\Phi(0)}) \frac{1}{iu} \left( (\widehat{F}_B(u) - 1) \text{TC}(\Gamma, \mathbb{R}^2) + \frac{\lambda}{2} (\widehat{F}_{B^+}(u) - \widehat{F}_{B^-}(u))^2 \bar{p}^2 \right).$$

When moreover  $B = 1$  a.s.,  $B^+ = 1$  and  $B^- = 0$  a.s., we can deduce an exact formula for the mean level total curvature function of  $X_\Phi$ . Actually, in this case  $X_\Phi(0)$  follows a Poisson law of parameter  $\lambda \bar{u}$  and  $\frac{\widehat{F}_{B^+}(u) - \widehat{F}_{B^-}(u)}{iu} = \frac{\widehat{F}_B(u) - 1}{iu}$  is the characteristic function of a uniform random variable  $Z$  on  $[0, 1]$ , while  $\widehat{F}_B(u) \frac{\widehat{F}_B(u) - 1}{iu}$  is the characteristic function of  $Z + 1$ . Then, considering  $Z$  independent from  $X_\Phi(0)$  we recognize

$$\mathbb{E}(\text{LP}_{X_\Phi}(h_u, U)) = \lambda \mathcal{L}(U) \mathbb{E} \left( e^{iu[X_\Phi(0) + Z]} \right) \mathbb{E}(R) \mathcal{H}^1(\Gamma), \text{ and}$$

$$\mathbb{E}(\text{LTC}_{X_\Phi}(h_u, U)) = \lambda \mathcal{L}(U) \left[ \mathbb{E} \left( e^{iu[X_\Phi(0)+Z]} \right) \left( \text{TC}(\Gamma, \mathbb{R}^2) - \frac{\lambda}{2} \bar{p}^2 \right) + \mathbb{E} \left( e^{iu[X_\Phi(0)+Z+1]} \right) \frac{\lambda}{2} \bar{p}^2 \right].$$

Note also that since here the random field  $X_\Phi$  has integer values, then  $\forall k \in \mathbb{N}$ ,

$$\forall t \in (k, k+1], \quad \{x \in U; X_\Phi(x) \geq t\} = \{x \in U; X_\Phi(x) \geq k+1\},$$

and therefore

$$\mathbb{E}(\text{Per}(E_{X_\Phi}(t), U)) = \mathbb{E}(\text{Per}(E_{X_\Phi}(k+1), U)) \quad \text{and} \quad \mathbb{E}(\text{TC}(\partial E_{X_\Phi}(t), U)) = \mathbb{E}(\text{TC}(\partial E_{X_\Phi}(k+1), U)).$$

Hence we may conclude that  $\forall k \in \mathbb{N}, \forall t \in (k, k+1]$

$$\begin{aligned} \mathbb{E}(\text{Per}(E_{X_\Phi}(t), U)) &= \lambda \mathcal{L}(U) e^{-\lambda \bar{a}} \frac{(\lambda \bar{a})^k}{k!} \bar{p} \quad \text{and} \\ \mathbb{E}(\text{TC}(\partial E_{X_\Phi}(t), U)) &= \lambda \mathcal{L}(U) e^{-\lambda \bar{a}} \frac{(\lambda \bar{a})^k}{k!} \left( \text{TC}(\Gamma, \mathbb{R}^2) - \frac{\lambda}{2} \bar{p}^2 + \frac{\bar{p}^2}{2\bar{a}} k \right). \end{aligned} \quad (31)$$

Note that, thanks to Gauss-Bonnet Theorem, we have  $\text{TC}(\Gamma, \mathbb{R}^2) = 2\pi\chi(D)$ , so that we may rewrite this as in (29). □

Let us remark that Formula (29) only involves the Euler Characteristic, the mean perimeter and the mean area of the shapes. When  $k = 0$ , we find the formula of the mean Euler Characteristic density of a rotation invariant Boolean model as obtained by Mecke and Wagner in [26] and by Mecke in [25] stating that

$$\lim_{r \rightarrow +\infty} \frac{\mathbb{E}(\chi(\{x \in rU; X(x) \geq 1\}))}{\pi \mathcal{L}(rU)} = e^{-\lambda m_0(D)} (\lambda m_2(D) - \lambda^2 m_1(D)^2),$$

with  $m_0(D) = \bar{a}$ ,  $m_1(D) = \bar{p}/2\pi$  and  $m_2(D) = \chi(D)/\pi$ . Actually, following the framework of [31] for Boolean models, we can define some volume densities for excursion sets as, for all  $k \in \mathbb{N}$  and  $t \in (k, k+1]$ ,

$$\begin{aligned} \overline{\mathcal{L}(E_X(t))} &:= \mathbb{P}(X(0) \geq k+1) = 1 - e^{-(\lambda \bar{a})} \sum_{l=0}^k \frac{(\lambda \bar{a})^l}{l!} \\ \overline{\text{Per}(E_X(t))} &= e^{-(\lambda \bar{a})} \frac{(\lambda \bar{a})^k}{k!} (\lambda \bar{p}) \\ \overline{\chi(E_X(t))} &= e^{-(\lambda \bar{a})} \frac{(\lambda \bar{a})^k}{k!} \left( \lambda \chi(D) - \frac{1}{4\pi} (\lambda \bar{p})^2 + \frac{\bar{p}^2}{4\pi \bar{a}} k \right), \end{aligned}$$

recovering the results of Boolean model for  $k = 0$  in dimension 2 (see p.389 of [31]). The typical behavior of  $\overline{\chi(E_X(k))}$ , as a function of  $k \in \mathbb{N} \setminus \{0\}$ , is the following:

- It starts, when  $k$  is small, by being negative. This is explained by the fact that  $\{X_\Phi \geq k\}$  is essentially made of one big connected component with many small holes in it. In particular the minimum value is achieved for an integer denoted  $k_-$ . The explicit value of  $k_-$  can be computed from Equation (31). The formula is not very nice, but it has a simple asymptotic behavior when  $\lambda$  is large, since then we have

$$k_- = \lambda \bar{a} - \sqrt{\lambda \bar{a}} + O(1).$$

- Then, after  $k_-$ , the mean level total curvature  $\mathbb{E}(\text{TC}(\partial E_{X_\Phi}(k, U)))$  increases and it crosses 0 in the interval that contains  $k_0$  where

$$k_0 = \lambda \bar{a} - \frac{2\pi \bar{a}}{\bar{p}^2} \chi(D) = \lambda \bar{a} + O(1).$$

For this level, there are as many connected components as holes.

- After  $k_0$ , the mean level total curvature is positive and it increases till a value  $k_+$  and afterwards it decreases and goes to 0 as  $k$  goes to infinity. As for  $k_-$  the value of  $k_+$  is explicit, and its asymptotic behavior when  $\lambda$  is large is

$$k_+ = \lambda \bar{a} + \sqrt{\lambda \bar{a}} + O(1).$$

**Example 1: Random disks**

We assume here that  $D = D(0, 1)$  is a disk of radius 1, and that  $B = 1$  a.s. In this case we have  $\text{TC}(\partial D) = 2\pi\chi(D) = 2\pi$ ,  $\bar{p} = 2\pi\mathbb{E}(R)$  and  $\bar{a} = \pi\mathbb{E}(R^2)$ . Note also that since  $R_\theta rD = rD$  for all  $\theta$ , whatever  $F_\Theta$  is, the shot noise random field has the same law as the one with marks given by  $\tilde{m} = (b, r) \in [0, +\infty)^2 \subset \mathbb{R}^2$ , with distribution  $G(d\tilde{m}) = F_B(db)F_R(dr)$ . An example of such a random field with comparisons between the theoretical value of  $\mathbb{E}(\text{TC}(\partial E_{X_\Phi}(t), U))/2\pi$  of Euler Characteristic and an empirical estimate on a large domain are shown on Figure 4. The caption of the figure gives the practical and technical details of the simulation.

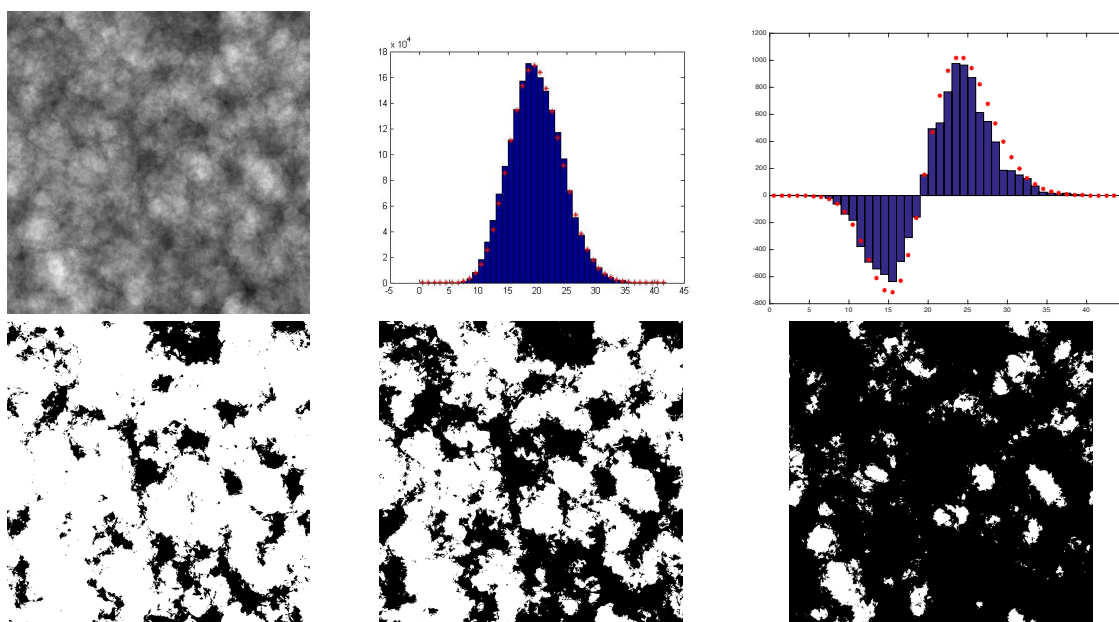


FIG 4. Shot noise random field with indicator functions of random disks. This sample has been obtained using Matlab, with a domain of size  $2000 \times 2000$  pixels, a Poisson point process of intensity  $\lambda = 0.001$ , and random disks of radius  $R = 50$  or  $R = 100$  (each with probability 0.5). Top middle and right figure: empirical Perimeter and Euler Characteristic as a function of the level  $t$  (computed thanks to the Matlab functions `bwperim` and `bweuler`), compared with the theoretical values (red stars) of Equation (29). Bottom line: Three excursion sets corresponding respectively from left to right to the level  $t = 15$ ,  $t = 19$  (that is the “critical level” where the Euler Characteristic turns from negative to positive) and  $t = 25$ .

Let us quote that we can also compute the mean level total curvature for a non isotropic shape. This is the case of squares for instance, as developed in the following example.

**Example 2: Random squares**

We assume here that  $D$  is a square of side length 1 and  $\Theta = 0$  a.s. with  $F(dm) = F_B(db)F_R(dr)\delta_0(d\theta)$  or equivalently that marks are given by  $\tilde{m} = (b, r) \in \mathbb{R} \times [0, +\infty) \subset \mathbb{R}^2$ , with distribution  $G(d\tilde{m}) = F_B(db)F_R(dr)$ . In this case,  $\Gamma = \partial D$  is made of four line segments, with  $\text{TC}(\Gamma, \mathbb{R}^2) = 2\pi\chi(D) = 2\pi$ ,  $\bar{p} = 4\mathbb{E}(R)$  and  $\bar{a} = \mathbb{E}(R^2)$ . On the boundary of a square, the curvature is 0, and it has four corner points with a turning angle equal to  $\pi/2$ . Now, considering (28), we get for  $u \neq 0$  and  $h_u(t) = e^{iut}$ ,

$$\int_{\mathbb{R}^d} \text{LTC}_{g_m}(h_u)F(dm) = \text{TC}(\Gamma, \mathbb{R}^2) \frac{\widehat{F}_B(u) - 1}{iu}.$$

For the second term, according to the kinematic formula (30), the only remaining terms are for  $\theta_1(s_1) = \theta_2(s_2) \pm \frac{\pi}{2}$  for which

$$f(\theta_1(s_1), \theta_2(s_2)) = \min(|\theta_1(s_1) - \theta_2(s_2)|, 2\pi - |\theta_1(s_1) - \theta_2(s_2)|) = \frac{\pi}{2} = f_\pi(\theta_1(s_1), \theta_2(s_2)).$$

It follows that

$$I_{\Gamma_1, \Gamma_2}(f) = \frac{\pi}{2} \times 8r_1 \times r_2 = I_{\Gamma_1, \Gamma_2}(f_\pi),$$

and therefore

$$\int_{\mathbb{R}^2} \sum_{z \in \tau_x \mathcal{R}_{g_m} \cap \mathcal{R}_{g_{m'}}} d_{S^1}(\nu_{g_m}(z-x), \nu_{g_{m'}}(z)) dx = \frac{\pi}{2} \times 8r \times r'.$$

Hence,

$$\begin{aligned} & \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^2 \times \mathbb{R}^d} \sum_{z \in \tau_x \mathcal{R}_{g_m} \cap \mathcal{R}_{g_{m'}}} (e^{iug_{m'}^+(z)} - e^{iug_{m'}^-(z)}) \\ & \times \left( \int_{g_m^+(z-x)}^{g_m^-(z-x)} e^{ius} ds \right) d_{S^1}(\nu_{g_m}(z-x), \nu_{g_{m'}}(z)) dx F(dm)F(dm') \\ & = \frac{(\widehat{F}_{B^+}(u) - \widehat{F}_{B^-}(u))^2}{iu} 2\pi \int_{\mathbb{R}_+^2} rr' F_R(dr)F_R(dr') = 2\pi \frac{(\widehat{F}_{B^+}(u) - \widehat{F}_{B^-}(u))^2 \bar{p}^2}{iu 16}. \end{aligned}$$

Finally, in this case

$$\begin{aligned} & \int_{\mathbb{R}} e^{iut} \mathbb{E}(\text{TC}(\partial E_{X_\Phi}(t), U)) dt \\ & = \lambda \mathcal{L}(U) \mathbb{E}(e^{iuX_\Phi(0)}) \frac{1}{iu} \left( (\widehat{F}_B(u) - 1) \text{TC}(\Gamma, \mathbb{R}^2) + 2\pi \lambda (\widehat{F}_{B^+}(u) - \widehat{F}_{B^-}(u))^2 \frac{\bar{p}^2}{16} \right). \end{aligned}$$

For  $B = 1$  a.s., inverting as previously, we obtain

$$\forall k \in \mathbb{N}, \forall t \in (k, k+1], \quad \frac{1}{2\pi} \mathbb{E}(\text{TC}(\partial E_{X_\Phi}(t), U)) = \lambda \mathcal{L}(U) e^{-\lambda \bar{a}} \frac{(\lambda \bar{a})^k}{k!} \left( 1 - \frac{\lambda}{16} \bar{p}^2 + \frac{\bar{p}^2}{16 \bar{a}} k \right). \quad (32)$$

It is illustrated on Figure 5. This formula generalizes one of the results of Decreusefond et al. [14]. Actually, considering the Boolean model made of squares of constant size  $R = 2\varepsilon$  a.s. for some  $\varepsilon > 0$ , we get for  $k = 0$ , and  $a > 0$ ,

$$\forall t \in (0, 1], \quad \frac{1}{2\pi} \mathbb{E}(\text{TC}(\partial E_{X_\Phi}(t), (0, a)^2)) = \lambda a^2 e^{-\lambda(2\varepsilon)^2} (1 - \lambda(2\varepsilon)^2),$$

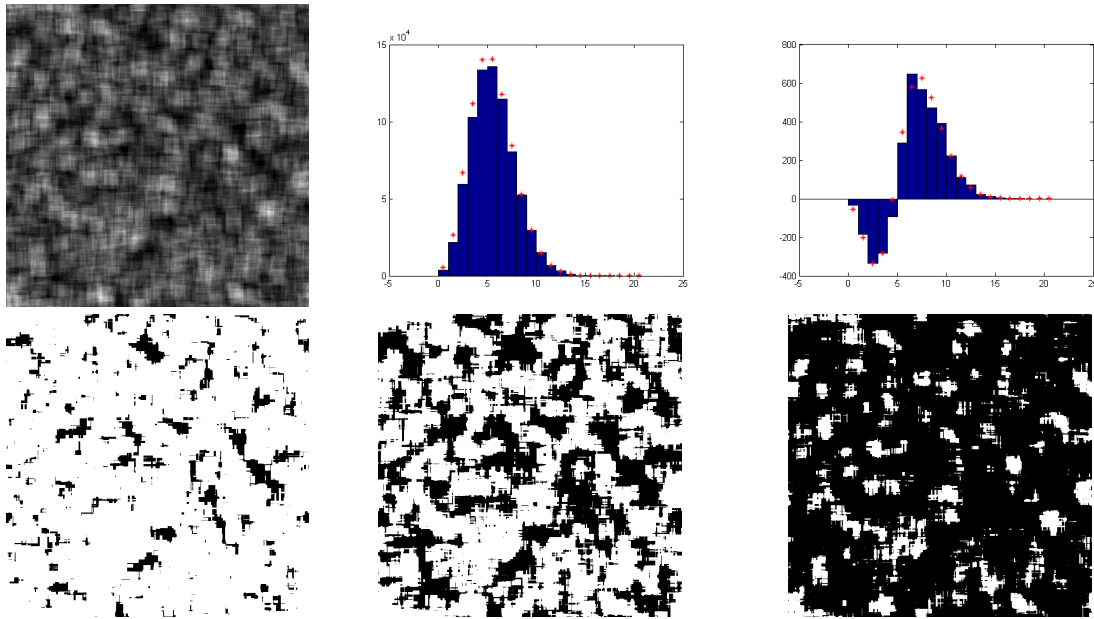


FIG 5. Shot noise random field with indicator functions of random squares. This sample has been obtained on a domain of size  $2000 \times 2000$  pixels, a Poisson point process of intensity  $\lambda = 0.005$ , and random squares of fixed side length  $R = 100$ . Top middle and right figures: empirical Perimeter and Euler Characteristic as functions of the level  $t$ , compared with the theoretical value (red stars) of Equation (32). Bottom line: Three excursion sets corresponding respectively from left to right to the level  $t = 3$ ,  $t = 5$  (that is the “critical level” where the Euler Characteristic turns from negative to positive) and  $t = 8$ .

that corresponds to the mean Euler Characteristic of the Boolean model in dimension 2, considered in the torus of size  $a > 0$  in Theorem 11 of [14].

#### 4. Level integrals of smooth functions and random fields

##### 4.1. The case of smooth functions

In the previous section, we have computed the level total curvature of elementary functions, that are piecewise constant functions (that are in particular not continuous). Here, we now consider the case of smooth (at least  $C^2$ ) functions. The question of the link between these two cases will be discussed in a future paper.

In the sequel, for  $f$  a  $C^2$  function we denote by  $\nabla f$  its gradient vector and by  $D^2 f$  its Hessian matrix.

**Proposition 3.** *Let  $f$  be a smooth (at least  $C^2$ ) function on an open set containing  $\bar{U}$ . Then  $f$  is of special bounded variation and of finite level total curvature integral on  $U$  with*

$$V_f(U) = \int_U \|\nabla f(x)\| dx \quad \text{and} \quad \text{LTaC}_f(U) \leq \int_U \|D^2 f(x)\|_2 dx,$$

where  $\|\cdot\|_2$  is the matrix norm subordinated to the Euclidean norm. Moreover, for  $h$  a bounded



continuous function on  $\mathbb{R}$ , the level perimeter and total curvature integrals of  $f$  are given by

$$\text{LP}_f(h, U) = \int_U h(f(x)) \|\nabla f(x)\| dx \quad \text{and}$$

$$\text{LTC}_f(h, U) = - \int_U h(f(x)) \frac{D^2 f(x) \cdot (\nabla f(x)^\perp, \nabla f(x)^\perp)}{\|\nabla f(x)\|^2} \mathbf{1}_{\|\nabla f(x)\| > 0} dx,$$

where if  $A = (a_{ij})_{1 \leq i, j \leq 2}$  is a  $2 \times 2$  symmetric matrix and  $y = (y_1, y_2) \in \mathbb{R}^2$ , we use the notation

$$A \cdot (y, y) = {}^t y A y = a_{11} y_1^2 + a_{22} y_2^2 + 2a_{12} y_1 y_2.$$

*Proof.* Note that since  $f$  is  $C^1$  on an open set containing  $\bar{U}$  we clearly have that  $f \in \text{SBV}(U)$  and thus  $E_f(t)$  is of finite perimeter in  $U$  for a.e.  $t \in \mathbb{R}$  by the co-area formula (see Theorem 3.40 [7]). Moreover, since  $f$  is actually  $C^2$  on an open set containing  $\bar{U}$ , by Morse-Sard theorem (see [20] p.69 for instance), the set of critical values of  $f$  has measure 0 in  $\mathbb{R}$ . Hence for a.e.  $t \in \mathbb{R}$ , for all points  $x \in U$  such that  $f(x) = t$  then  $\nabla f(x) \neq 0$ . Let  $t$  be such a non-critical value. It follows that  $\partial E_f(t) \cap U = \{x \in U; f(x) = t\}$ , and we clearly have  $S_{\mathbf{1}_{E_f(t)}} = \partial E_f(t) \cap U$  is a regular curve with finite length such that for any  $x \in \partial E_f(t) \cap U$  one can find a parametrization  $\gamma$  given by an implicit form  $f(\gamma(s)) = t$ , with  $x = \gamma(s)$ , normal vector  $\gamma'(s)^\perp = \nabla f(x) / \|\nabla f(x)\|$  and curvature given by

$$\kappa_f(x) = - \frac{D^2 f(x) \cdot (\nabla f^\perp(x), \nabla f^\perp(x))}{\|\nabla f(x)\|^3}. \quad (33)$$

Therefore

$$\begin{aligned} \text{TaC}(\partial E_f(t), U) &= \int_{\partial E_f(t) \cap U} |\kappa_f(x)| \mathcal{H}^1(dx) < +\infty \\ \text{and } \text{TC}(\partial E_f(t), U) &= \int_{\partial E_f(t) \cap U} \kappa_f(x) \mathcal{H}^1(dx). \end{aligned}$$

The fact that  $\partial E_f(t) \cap U$  is given by a finite union of simple connected curves comes from the compactness of  $\{x \in \bar{U}; f(x) = t\}$  in which it is included. Hence we have already proven that for a.e.  $t \in \mathbb{R}$ , the set  $E_f(t)$  is an elementary set of  $U$ . Now let us prove that  $t \mapsto \text{TaC}(\partial E_f(t), U)$  is integrable on  $\mathbb{R}$ . Let us define  $|\kappa_f|$  as a measurable non-negative function on  $U$  by setting  $|\kappa_f(x)| = +\infty$  if  $\nabla f(x) = 0$  and  $|\kappa_f(x)| = \left| \frac{D^2 f(x) \cdot (\nabla f^\perp(x), \nabla f^\perp(x))}{\|\nabla f(x)\|^3} \right|$ , otherwise. Now, let us recall the co-area formula for Lipschitz mappings (see [18] p.117 for instance): for any non-negative measurable or  $\mathcal{L}$ -integrable function  $g$ , the function  $t \mapsto \int_{\partial E_f(t) \cap U} g(x) \mathcal{H}^1(dx)$  is measurable and

$$\int_U g(x) \|\nabla f\|(x) dx = \int_{\mathbb{R}} \int_{\partial E_f(t) \cap U} g(x) \mathcal{H}^1(dx) dt.$$

Taking  $g = 1$ , we recover the co-area formula:

$$\int_U \|\nabla f\|(x) dx = \int_{\mathbb{R}} \mathcal{H}^1(\partial E_f(t) \cap U) dt = V_f(U),$$

while for  $g = |\kappa_f|$ ,

$$\begin{aligned} \text{LTaC}_f(U) &= \int_{\mathbb{R}} \int_{\partial E_f(t) \cap U} |\kappa_f(x)| \mathcal{H}^1(dx) = \int_U |\kappa_f(x)| \|\nabla f(x)\| dx \\ &\leq \int_U \|D^2 f(x)\|_2 dx, \end{aligned}$$

in view of (33). This concludes the proof that  $f$  is of special bounded variation and finite level total curvature integral on  $U$  and, for  $h$  a bounded continuous function on  $\mathbb{R}$ , using again twice the co-area formula but now with  $\max(g, 0)$  and  $-\min(g, 0)$  for  $g = h \circ f$  or  $g = (h \circ f) \kappa_f$ , and subtracting we get

$$\begin{aligned} \text{LP}_f(h) &= \int_{\mathbb{R}} h(f(x)) \|\nabla f(x)\| dx \\ \text{LTC}_f(h) &= \int_U h(f(x)) \kappa_f(x) \|\nabla f(x)\| dx = - \int_U h(f(x)) \frac{D^2 f(x) \cdot (\nabla f(x)^\perp, \nabla f(x)^\perp)}{\|\nabla f(x)\|^2} \mathbf{1}_{\|\nabla f(x)\| > 0} dx, \end{aligned}$$

in view of (33). □

#### 4.2. A general result for smooth stationary random fields

In this section we consider a smooth stationary random field  $X$  defined on  $\mathbb{R}^2$ . Let us introduce some notations for the derivatives of  $X$ . A point  $x \in \mathbb{R}^2$  is defined by its two coordinates  $x = (x_1, x_2)$  and we denote for  $i, j = 1, 2$

$$X_i := \frac{\partial X}{\partial x_i} \quad \text{and} \quad X_{ij} := \frac{\partial^2 X}{\partial x_i \partial x_j}.$$

With these notations it follows that  $\nabla X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$  and  $D^2 X = \begin{pmatrix} X_{11} & X_{12} \\ X_{12} & X_{22} \end{pmatrix}$ . Note that since  $X$  is stationary, for any  $x \in \mathbb{R}^2$ ,

$$(X(x), \nabla X(x), D^2 X(x)) \stackrel{d}{=} (X(0), \nabla X(0), D^2 X(0)).$$

When  $X$ ,  $\nabla X$  and  $D^2 X$  have also finite second order moment,  $X(x)$  and  $\nabla X(x)$  are not correlated, as well as  $\nabla X(x)$  and  $D^2 X(x)$  (see [1] p.31 for instance). This is very useful for Gaussian fields since it implies that  $\nabla X(x)$  is independent from  $(X(x), D^2 X(x))$ .

Using the result of Section 4.1 and the stationarity of  $X$ , we have the following formula.

**Theorem 5.** *Let  $X$  be a stationary  $C^2$  random field on  $\mathbb{R}^2$ , such that  $X(0), X_i(0), X_{ii}(0)$  and  $X_{12}(0)$  have finite expectations for  $i = 1, 2$ . Then, a.s.,  $X \in \text{SBV}(U)$  with  $DX = \nabla X \mathcal{L}$  and for a.e.  $t \in \mathbb{R}$ , the random variables  $\text{Per}(E_X(t), U)$  and  $\text{TC}(\partial E_X(t), U)$  have finite expectation such that for all  $h$  bounded continuous function on  $\mathbb{R}$ , one has*

$$\begin{aligned} \mathbb{E}(\text{LP}_X(h, U)) &= \int_{\mathbb{R}} h(t) \mathbb{E}(\text{Per}(E_X(t), U)) dt \\ \text{and} \quad \mathbb{E}(\text{LTC}_X(h, U)) &= \int_{\mathbb{R}} h(t) \mathbb{E}(\text{TC}(\partial E_X(t), U)) dt, \end{aligned}$$

with

$$\begin{aligned} \mathbb{E}(\text{LP}_X(h, U)) &= \mathcal{L}(U) \mathbb{E}(h(X(0)) \|\nabla X(0)\|) \\ \mathbb{E}(\text{LTC}_X(h, U)) &= -\mathcal{L}(U) \mathbb{E}\left(h(X(0)) \frac{D^2 X(0) \cdot (\nabla X(0)^\perp, \nabla X(0)^\perp)}{\|\nabla X(0)\|^2} \mathbf{1}_{\|\nabla X(0)\| > 0}\right) \end{aligned}$$

It follows that when the field  $X$  is isotropic, i.e.  $X \circ A \stackrel{fdd}{=} X$  for all orthogonal matrix  $A$ , then the above formulas reduce to,  $\forall i = 1, 2$ ,

$$\begin{aligned}\mathbb{E}(\text{LP}_X(h, U)) &= \frac{\pi}{2} \mathcal{L}(U) \mathbb{E}(h(X(0)) |X_i(0)|) \quad \text{and} \\ \mathbb{E}(\text{LTC}_X(h, U)) &= \mathcal{L}(U) \left( \overline{\text{LTC}_X^{(I)}}(h) + \overline{\text{LTC}_X^{(D)}}(h) \right)\end{aligned}$$

where

$$\begin{aligned}\overline{\text{LTC}_X^{(I)}}(h) &= -\mathbb{E}(h(X(0))X_{ii}(0)\mathbf{1}_{\|\nabla X(0)\|>0}) \\ \overline{\text{LTC}_X^{(D)}}(h) &= 4\mathbb{E}\left(h(X(0))\frac{X_1(0)X_2(0)}{\|\nabla X(0)\|^2}X_{12}(0)\mathbf{1}_{\|\nabla X(0)\|>0}\right).\end{aligned}$$

In particular, taking  $h = 1$  we have  $V_X(U) = \frac{\pi}{2} \mathcal{L}(U) \mathbb{E}(|X_i(0)|)$ . It follows that when  $X(0)$  admits a density  $p_{X(0)}$ , we get for almost every  $t \in \mathbb{R}$ ,  $\forall i = 1, 2$ ,

$$\begin{aligned}\mathbb{E}(\text{Per}(E_X(t), U)) &= \frac{\pi}{2} \mathcal{L}(U) \mathbb{E}(|X_i(0)| |X(0) = t) p_{X(0)}(t) \\ \mathbb{E}(\text{TC}(\partial E_X(t), U)) &= \mathcal{L}(U) [-\mathbb{E}(X_{ii}(0)\mathbf{1}_{\|\nabla X(0)\|>0} |X(0) = t) \\ &\quad + 4\mathbb{E}\left(\frac{X_1(0)X_2(0)}{\|\nabla X(0)\|^2}X_{12}(0)\mathbf{1}_{\|\nabla X(0)\|>0} |X(0) = t\right)] p_{X(0)}(t).\end{aligned}$$

**Remark:** Let us quote that assuming that the field  $X$  is isotropic and symmetric, i.e.  $X \stackrel{fdd}{=} -X$ , we get  $\mathbb{E}(\text{LTC}_X(h, U)) = 0$  for any even bounded function  $h$  and in particular for  $h = 1$ .

If moreover  $(X(0), X_1(0), X_2(0), X_{12}(0)) \stackrel{d}{=} (X(0), -X_1(0), X_2(0), X_{12}(0))$  we easily get  $\overline{\text{LTC}_X^{(D)}}(h) = 0$  any bounded function  $h$ . This is in particular the case for isotropic fields satisfying  $\nabla X(0)$  independent from  $(X(0), X_{12}(0))$  like centered stationary isotropic Gaussian random fields.

*Proof.* According to Proposition 3, since  $X$  is a.s.  $C^2$ , it is a.s. of special bounded variation and of finite level curvature integral on  $U$  with

$$V_X(U) = \int_U \|\nabla X(x)\| dx \quad \text{and} \quad \text{LTaC}_X(U) \leq \int_U \|D^2 X(x)\|_2 dx \quad \text{a.s.}$$

Since  $X$  is stationary, the finite expectation assumption implies that  $V_X(U)$  and  $\text{LTaC}_X(U)$  are non-negative random variables with finite expectation. By Fubini's Theorem, this implies that  $(t, \omega) \mapsto \text{Per}(E_X(t), U) \in L^1(\mathbb{R} \times \Omega)$  and  $(t, \omega) \mapsto \text{TaC}(\partial E_X(t), U) \in L^1(\mathbb{R} \times \Omega)$  so that we also have  $(t, \omega) \mapsto \text{TC}(\partial E_X(t), U) \in L^1(\mathbb{R} \times \Omega)$ . Moreover, a.s., for any  $h$  a bounded continuous function on  $\mathbb{R}$ ,

$$\text{LP}_X(h, U) = \int_{\mathbb{R}} h(t) \text{Per}(E_X(t), U) dt \quad \text{and} \quad \text{LTC}_X(h, U) = \int_{\mathbb{R}} h(t) \text{TC}(\partial E_X(t), U) dt,$$

with

$$\begin{aligned}\text{LP}_X(h, U) &= \int_U h(X(x)) \|\nabla X(x)\| dx \quad \text{and} \\ \text{LTC}_X(h, U) &= - \int_U h(X(x)) \frac{D^2 X(x) \cdot (\nabla X(x)^\perp, \nabla X(x)^\perp)}{\|\nabla X(x)\|^2} \mathbf{1}_{\|\nabla X(x)\|>0} dx.\end{aligned}$$

Hence, Fubini's Theorem yields the first result for expectation, while the second one is a straightforward consequence from the stationarity of  $X$ .

Under the assumption that the field is isotropic, we can exploit further on this formula. First let us recall that by Taylor formula, since  $X$  is a.s.  $C^2$  we have a.s. for all  $x, z \in \mathbb{R}^2$ ,

$$X(x+z) = X(x) + \langle \nabla X(x), z \rangle + \frac{1}{2} D^2 X(x).(z, z) + o_{\|z\| \rightarrow 0}(\|z\|^2).$$

In particular we obtain that, for any orthogonal matrix  $A$ ,

$$\nabla(X \circ A)(x) = {}^t A \nabla X(Ax) \text{ and } D^2(X \circ A)(x) = {}^t A (D^2 X)(Ax) A.$$

Since  $X \circ A \stackrel{fdd}{=} X$ , we deduce that

$$(X(x), \nabla X(x), D^2 X(x)) \stackrel{d}{=} (X(Ax), {}^t A \nabla X(Ax), {}^t A (D^2 X)(Ax) A),$$

and specifying to  $x = 0$ , it follows that

$$(X(0), \nabla X(0), D^2 X(0)) \stackrel{d}{=} (X(0), {}^t A \nabla X(0), {}^t A (D^2 X)(0) A). \quad (34)$$

Hence  $(X(0), \nabla X(0)) \stackrel{d}{=} (X(0), {}^t A \nabla X(0))$  and for any  $\theta \in [0, 2\pi)$ , denoting  $u(\theta) = (\cos \theta, \sin \theta) \in S^1$ , one has  $(X(0), X_i(0)) \stackrel{d}{=} (X(0), \langle u(\theta), \nabla X(0) \rangle)$ , according to orthogonal invariance (see Proposition 4.8 of [12]). Moreover  $\|\nabla X(0)\| = \frac{1}{4} \int_0^{2\pi} |\langle u(\theta), \nabla X(0) \rangle| d\theta$  so that we may deduce that

$$\mathbb{E}(h(X(0)) \|\nabla X(0)\|) = \frac{1}{4} \int_0^{2\pi} \mathbb{E}(h(X(0)) |\langle u(\theta), \nabla X(0) \rangle|) d\theta = \frac{\pi}{2} \mathbb{E}(h(X(0)) |X_i(0)|),$$

and the result for  $\mathbb{E}(\text{LP}_X(h, U))$  follows. Now, let us introduce the random variable  $\Theta$  with values in  $2\pi\mathbb{T}$  (identified with  $[0, 2\pi)$ ), such that

$$\nabla X(0) = \begin{pmatrix} X_1(0) \\ X_2(0) \end{pmatrix} = \|\nabla X(0)\| \begin{pmatrix} \cos \Theta \\ \sin \Theta \end{pmatrix}.$$

For any  $\theta \in [0, 2\pi)$  let us consider a rotation matrix  $R_\theta = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$  and a reflexion matrix  $S_\theta = \begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{pmatrix}$ . Let us quote that  $R_{-\theta} \nabla X(0) = {}^t R_\theta \nabla X(0) = \|\nabla X(0)\| \begin{pmatrix} \cos(\Theta - \theta) \\ \sin(\Theta - \theta) \end{pmatrix}$ ,  $S_\theta \nabla X(0) = {}^t S_\theta \nabla X(0) = \|\nabla X(0)\| \begin{pmatrix} \cos(\theta - \Theta) \\ \sin(\theta - \Theta) \end{pmatrix}$ . From (34) with  $A = S_\theta$  for  $\theta = \pi/2$  we have that

$$(X(0), \|\nabla X(0)\|, \Theta, X_{11}(0), X_{22}(0), X_{12}(0)) \stackrel{d}{=} (X(0), \|\nabla X(0)\|, \pi/2 - \Theta, X_{22}(0), X_{11}(0), X_{12}(0)). \quad (35)$$

But

$$\begin{aligned} & \mathbb{E} \left( h(X(0)) \frac{D^2 X(0).(\nabla X(0)^\perp, \nabla X(0)^\perp)}{\|\nabla X(0)\|^2} \mathbf{1}_{\|\nabla X(0)\| > 0} \right) \\ &= \mathbb{E} (h(X(0)) (X_{11}(0) \sin^2 \Theta + X_{22}(0) \cos^2 \Theta - 2X_{12}(0) \sin \Theta \cos \Theta) \mathbf{1}_{\|\nabla X(0)\| > 0}). \end{aligned}$$

Note that

$$\mathbb{E} (h(X(0)) X_{11}(0) \sin^2 \Theta \mathbf{1}_{\|\nabla X(0)\| > 0}) = \mathbb{E} (h(X(0)) X_{11}(0) \mathbf{1}_{\|\nabla X(0)\| > 0}) - \mathbb{E} (h(X(0)) X_{11}(0) \cos^2 \Theta \mathbf{1}_{\|\nabla X(0)\| > 0}),$$

so that

$$\begin{aligned} & \mathbb{E} \left( h(X(0)) [X_{11}(0) \sin^2 \Theta + X_{22}(0) \cos^2 \Theta] \mathbf{1}_{\|\nabla X(0)\|>0} \right) \\ &= \mathbb{E} \left( h(X(0)) X_{11}(0) \mathbf{1}_{\|\nabla X(0)\|>0} \right) + \mathbb{E} \left( h(X(0)) [X_{22}(0) - X_{11}(0)] \cos^2 \Theta \mathbf{1}_{\|\nabla X(0)\|>0} \right). \end{aligned}$$

From (35),

$$\mathbb{E} \left( h(X(0)) [X_{22}(0) - X_{11}(0)] \cos^2 \Theta \mathbf{1}_{\|\nabla X(0)\|>0} \right) = -\mathbb{E} \left( h(X(0)) [X_{22}(0) - X_{11}(0)] \sin^2 \Theta \mathbf{1}_{\|\nabla X(0)\|>0} \right),$$

and hence

$$\mathbb{E} \left( h(X(0)) [X_{22}(0) - X_{11}(0)] \cos^2 \Theta \mathbf{1}_{\|\nabla X(0)\|>0} \right) = \mathbb{E} \left( h(X(0)) \frac{X_{22}(0) - X_{11}(0)}{2} \cos(2\Theta) \mathbf{1}_{\|\nabla X(0)\|>0} \right).$$

Now, from (34) with  $A = R_\theta$  for  $\theta = \pi/4$ ,

$$(X(0), \|\nabla X(0)\|, \Theta, X_{12}(0)) \stackrel{d}{=} (X(0), \|\nabla X(0)\|, \Theta - \pi/4, \frac{X_{22}(0) - X_{11}(0)}{2}),$$

and it follows that

$$(X(0), \|\nabla X(0)\|, \Theta, \frac{X_{22}(0) - X_{11}(0)}{2}) \stackrel{d}{=} (X(0), \|\nabla X(0)\|, \Theta + \pi/4, X_{12}(0)). \quad (36)$$

Therefore,

$$\mathbb{E} \left( h(X(0)) [X_{22}(0) - X_{11}(0)] \cos^2 \Theta \mathbf{1}_{\|\nabla X(0)\|>0} \right) = -\mathbb{E} \left( h(X(0)) X_{12}(0) \sin(2\Theta) \mathbf{1}_{\|\nabla X(0)\|>0} \right).$$

The general result follows, by remarking that in view of (35)

$$\mathbb{E} \left( h(X(0)) X_{11}(0) \mathbf{1}_{\|\nabla X(0)\|>0} \right) = \mathbb{E} \left( h(X(0)) X_{22}(0) \mathbf{1}_{\|\nabla X(0)\|>0} \right).$$

Moreover, when  $X(0)$  admits a density  $p_{X(0)}$ , we can further write

$$\begin{aligned} \mathbb{E}(h(X(0)) | X_i(0) |) &= \int_{\mathbb{R}} h(t) \mathbb{E}(|X_i(0)| | X(0) = t) p_{X(0)}(t) dt, \\ \overline{\text{LTC}}_X^{(I)}(h) &= - \int_{\mathbb{R}} h(t) \mathbb{E}(X_{ii}(0) \mathbf{1}_{\|\nabla X(0)\|>0} | X(0) = t) p_{X(0)}(t) dt, \\ \overline{\text{LTC}}_X^{(D)}(h) &= 4 \int_{\mathbb{R}} h(t) \mathbb{E} \left( \frac{X_1(0) X_2(0)}{\|\nabla X(0)\|^2} X_{12}(0) \mathbf{1}_{\|\nabla X(0)\|>0} | X(0) = t \right) p_{X(0)}(t) dt \end{aligned}$$

such that for any continuous bounded function  $h$ ,

$$\begin{aligned} \int_{\mathbb{R}} h(t) \mathbb{E}(\text{TC}(\partial E_X(t, U))) dt &= \int_{\mathbb{R}} h(t) \mathbb{E}(|X_i(0)| | X(0) = t) p_{X(0)}(t) dt \quad \text{and} \\ \int_{\mathbb{R}} h(t) \mathbb{E}(\text{TC}(\partial E_X(t, U))) dt &= \int_{\mathbb{R}} h(t) \mathcal{L}(U) \left[ - \mathbb{E}(X_{ii}(0) \mathbf{1}_{\|\nabla X(0)\|>0} | X(0) = t) \right. \\ &\quad \left. + 4 \mathbb{E} \left( \frac{X_1(0) X_2(0)}{\|\nabla X(0)\|^2} X_{12}(0) \mathbf{1}_{\|\nabla X(0)\|>0} | X(0) = t \right) \right] p_{X(0)}(t) dt, \end{aligned}$$

implying the stated equalities for almost every  $t \in \mathbb{R}$ .  $\square$

**Remark:** A closely related result is given in Corollary 2.3 of [23] that should rewrite in our stationary setting, under additional assumption on  $X$ , as

$$\mathbb{E} \left( \int_{\mathbb{R}} h(t) \overline{\chi(E_X(t))} dt \right) = -\mathbb{E} \left( h(X(0)) \left[ \sum_{i=1}^2 \mathbf{1}_{\nabla X(0) \in Q_i} X_{ii}(0) \right] + h'(X(0)) \left[ \sum_{i=1}^2 \mathbf{1}_{\nabla X(0) \in Q_i} X_i(0)^2 \right] \right),$$

for  $h : \mathbb{R} \rightarrow \mathbb{R}$  a  $C^1$  function with compact support,  $Q_1 = \{x = (x_1, x_2); x_2 < x_1 < 0\}$  and  $Q_2 = \{x = (x_1, x_2); x_1 < x_2 < 0\}$ , and where  $\chi(E_X(t))$  stands for an Euler Characteristic density. Under the assumption that  $X$  is also isotropic, according to Remark 2.5 of [23],

$$\mathbb{E} (h'(X(0)) \mathbf{1}_{\nabla X(0) \in Q_i} X_i(0)^2) = \frac{\pi - 2}{16\pi} \mathbb{E} (h'(X(0)) \|\nabla X(0)\|^2).$$

Note that by stationarity

$$\begin{aligned} \mathbb{E} (h'(X(0)) X_i(0)^2) &= \int_0^1 \mathbb{E} (h'(X(te_i)) X_i(te_i)^2) dt = \mathbb{E} \left( \int_0^1 h'(X(te_i)) X_i(te_i)^2 dt \right) \\ &= -\mathbb{E} \left( \int_0^1 h(X(te_i)) X_{ii}(te_i) dt \right) = -\mathbb{E} (h(X(0)) X_{ii}(0)), \end{aligned}$$

integrating by parts and using the stationarity.

Now let us assume that  $\|\nabla X(0)\| > 0$  a.s. with  $\nabla X(0)$  independent from  $(X(0), X_{ii}(0))$ . Then

$$\mathbb{E} (h(X(0)) \mathbf{1}_{\nabla X(0) \in Q_i} X_{ii}(0)) = \mathbb{E} \left( h(X(0)) \mathbf{1}_{\Theta \in \pi + \frac{\pi}{4}(i-1, i)} X_{ii}(0) \right) = \frac{1}{8} \mathbb{E} (h(X(0)) X_{ii}(0)),$$

since  $\Theta$  has a uniform law on  $[0, 2\pi]$  by isotropy. It follows that

$$\mathbb{E} \left( \int_{\mathbb{R}} h(t) \overline{\chi(E_X(t))} dt \right) = -\sum_{i=1}^2 \left( \frac{1}{8} - \frac{\pi - 2}{8\pi} \right) \mathbb{E} (h(X(0)) X_{ii}(0)) = \frac{1}{2\pi \mathcal{L}(U)} \int_{\mathbb{R}} h(t) \mathbb{E}(\text{TC}(\partial E_X(t), U)) dt,$$

by Theorem 5, since it is also assumed that  $\|\nabla X(0)\| > 0$  a.s.

**Example** Let  $X$  be a stationary centered  $C^2$  isotropic Gaussian random field. Then,  $X_1(0)$  being independent from  $X(0)$ , we get  $\mathbb{E}(|X_1(0)| | X(0)) = \mathbb{E}(|X_1(0)|) = \sqrt{\frac{2\lambda_2}{\pi}}$ , where  $\lambda_2 = \text{Var}(X_1(0)) > 0$  denotes the second spectral moment. Moreover,

$$\mathbb{E}(X_{11}(0) | X(0)) = \frac{\text{Cov}(X(0), X_{11}(0))}{\text{Var}(X(0))} X(0) = \frac{\partial_1^2 \rho_X(0)}{\rho_X(0)} X(0) = \frac{-\lambda_2}{\sigma^2} X(0),$$

where  $\rho_X(x) = \text{Cov}(X(x), X(0))$  and  $\sigma^2 = \rho_X(0)$ , while by independence

$$\mathbb{E} \left( \frac{X_1(0) X_2(0)}{\|\nabla X(0)\|^2} X_{12}(0) | X(0) \right) = \mathbb{E} \left( \frac{X_1(0) X_2(0)}{\|\nabla X(0)\|^2} \right) \mathbb{E}(X_{12}(0) | X(0)) = 0.$$

Hence, since  $\mathbb{P}(\|\nabla X(0)\| = 0) = 0$ , we get in this case that, for almost every  $t \in \mathbb{R}$ ,

$$\mathbb{E}(\text{Per}(E_X(t), U)) = \mathcal{L}(U) \sqrt{\frac{\pi \lambda_2}{2}} \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{t^2}{2\sigma^2}} \quad (37)$$

$$\mathbb{E}(\text{TC}(\partial E_X(t), U)) = \mathcal{L}(U) \frac{\lambda_2}{\sigma^2} t \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{t^2}{2\sigma^2}}. \quad (38)$$

Let us emphasize that this last expression is exactly the formula obtained for  $2\pi\mathbb{E}(\chi(E_X(t) \cap \bar{U}))$ , stated for all  $t \in \mathbb{R}$ , under additional assumptions on  $X$  (see (3.2.8) of [2] for instance), where  $\chi$  denotes the DT (Differential Topology) Characteristic of the set.

Examples of such stationary isotropic random fields with comparisons between the theoretical values of  $\mathbb{E}(\text{TC}(\partial E_X(t), U))/(2\pi\mathcal{L}(U))$  (corresponding to the mean Euler Characteristic density of excursion sets) and an empirical estimate of Euler Characteristic on the square of fixed size  $[0, 1]$  are shown on Figures 6 and 7 (with  $\sigma^2 = 1$  and  $\lambda_2 = 2T^2$ ). The captions of the figures give the practical and technical details of the simulations. Note that in view of the covariance functions, a scaling relation may be set between  $T$  and the size of the square, explaining the convergence without boundary effects as  $T$  increases.

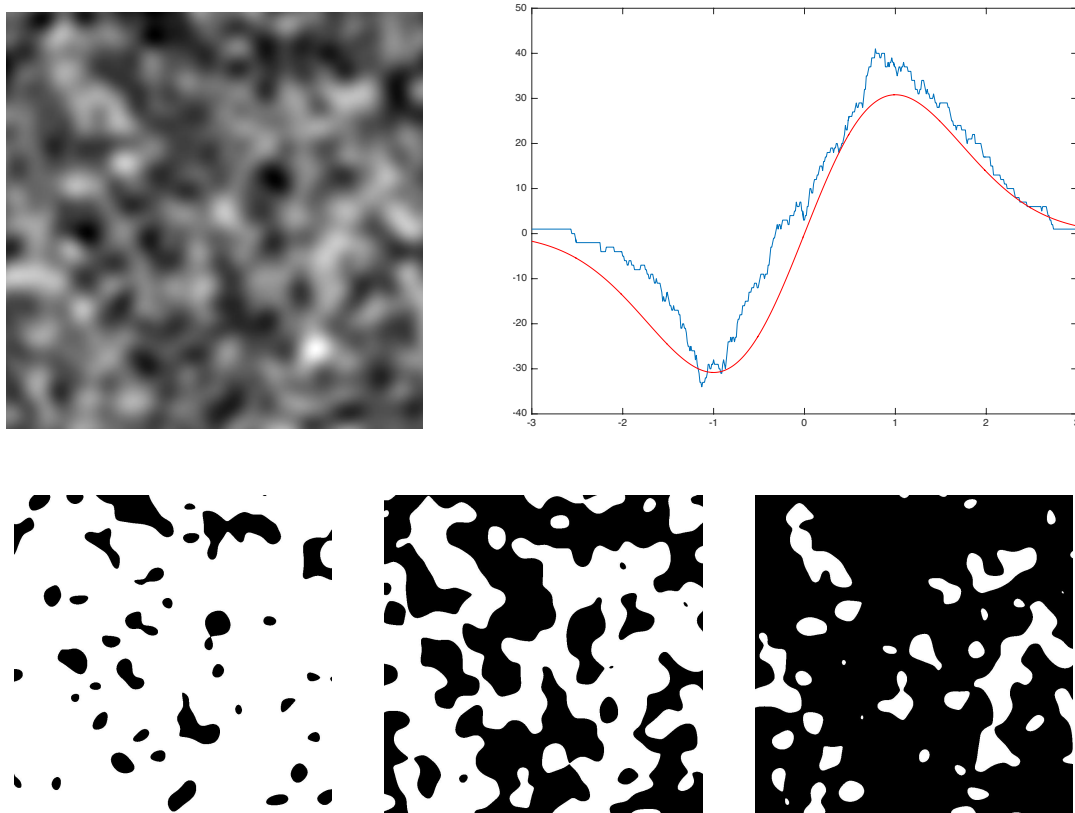


FIG 6. Gaussian random field with covariance  $\rho(x) = e^{-T^2\|x\|^2}$  for  $T = 10$ . This sample has been obtained using Matlab, with a discretized domain of size  $2^{10} \times 2^{10}$  pixels, using circulant embedding matrix. Top right figure: empirical Euler Characteristic as a function of the level  $t$  (computed thanks to the Matlab function `bweuler`), compared with the theoretical value (red one) of Equation (38). Bottom line: Three excursion sets corresponding respectively from left to right to the level  $t = -1$ ,  $t = 0$  (that is the “critical level” where the Euler Characteristic turns from negative to positive), and  $t = 1$ .

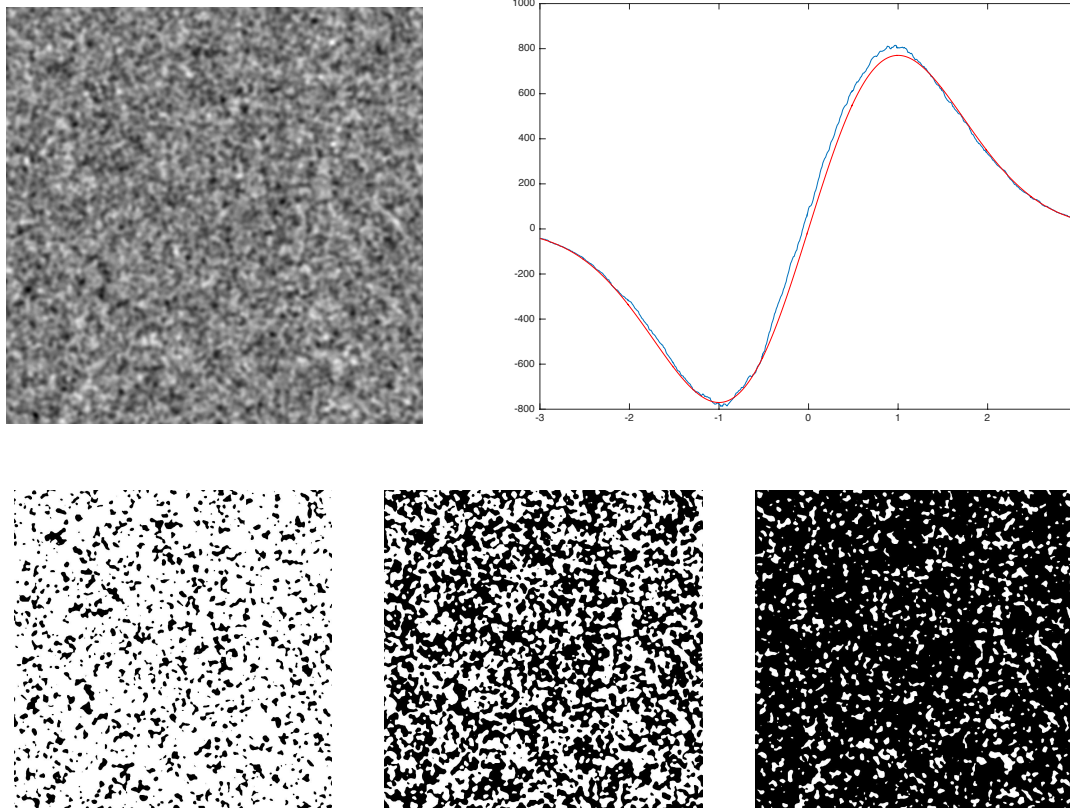


FIG 7. Gaussian random field with covariance  $\rho(x) = e^{-T^2\|x\|^2}$  for  $T = 100$ . This sample has been obtained using Matlab, with a discretized domain of size  $2^{10} \times 2^{10}$  pixels, using circulant embedding matrix. Top right figure: empirical Euler Characteristic as a function of the level  $t$  (computed thanks to the Matlab function `bweuler`), compared with the theoretical value (red one) of Equation (38). Bottom line: Three excursion sets corresponding respectively from left to right to the level  $t = -1$ ,  $t = 0$  (that is the “critical level” where the Euler Characteristic turns from negative to positive), and  $t = 1$ .

#### 4.3. Smooth shot noise random fields

As in Section 3.2, we consider here a shot noise random field defined on  $\mathbb{R}^2$  by

$$\forall x \in \mathbb{R}^2, \quad X_{\Phi}(x) = \sum_{i \in I} g_{m_i}(x - x_i),$$

where  $\Phi = \{(x_i, m_i)\}_{i \in I}$  is a Poisson point process on  $\mathbb{R}^2 \times \mathbb{R}^d$  of intensity  $\lambda \mathcal{L} \times F$ , with  $\lambda > 0$  real,  $\mathcal{L}$  the Lebesgue measure on  $\mathbb{R}^2$  and  $F$  a probability measure on  $\mathbb{R}^d$ . In order to get explicit formulas we have to make an assumption of isotropy. Moreover, since smooth shot noise random fields do not always admit a probability density (we have discussed this through several examples in our first paper [10]) we have to work with their characteristic functions.

**Theorem 6.** *Let assume that  $g : \mathbb{R}^2 \times \mathbb{R}^d \rightarrow \mathbb{R}$  is a measurable function such that for  $F$ -almost*



every  $m \in \mathbb{R}^d$  the functions  $g_m := g(\cdot, m)$  are  $C^3$  on  $\mathbb{R}^2$  satisfying

$$\int_{\mathbb{R}^2 \times \mathbb{R}^d} |D^{\mathbf{j}} g_m(x)| dx F(dm) < +\infty, \quad (39)$$

for all  $\mathbf{j} = (j_1, j_2) \in \mathbb{N}^2$  with  $|\mathbf{j}| = j_1 + j_2 \leq 3$  and where  $D^{\mathbf{j}} g_m = \frac{\partial^{|\mathbf{j}|} g_m}{\partial x_1^{j_1} \partial x_2^{j_2}}$ . Then  $X_\Phi$  is a.s. a stationary  $C^2$  field such that  $D^2 X$  has finite expectation, ensuring the assumptions of Theorem 5.

We assume moreover that  $m = (\tilde{m}, \theta) \in \mathbb{R}^{d-1} \times [0, 2\pi] \subset \mathbb{R}^d$  and  $F(dm) = G(d\tilde{m})F_\Theta(d\theta)$ , with  $F_\Theta$  the uniform law on  $[0, 2\pi]$ , and

$$g(\cdot, m) = g(\cdot, (\tilde{m}, \theta)) = \tilde{g}(R_\theta \cdot, \tilde{m}),$$

for  $F$ -a.e.  $m$ , with  $\tilde{g}(S_0 \cdot, \tilde{m}) = \tilde{g}(\cdot, \tilde{m})$ , recalling that  $S_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$  and  $R_\theta = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$ . Then  $X_\Phi$  is isotropic. It follows that, for all  $u \in \mathbb{R}$  by, for all  $j = 1, 2$ ,

$$\begin{aligned} \mathbb{E}(\text{LP}_{X_\Phi}(h_u, U)) &= -\frac{\mathcal{L}(U)}{2} \int_0^{+\infty} \frac{1}{v} \frac{\partial \varphi_{X_\Phi(0), \partial_j X_\Phi(0)}(u, v)}{\partial v} dv \\ \overline{\text{LTC}_{X_\Phi}^{(I)}}(h_u) &= -\lambda \mathbb{E} \left( e^{iu X_\Phi(0)} \right) \left( \int_{\mathbb{R}^d} \int_{\mathbb{R}^2} \partial_j^2 g_m(x) e^{iug_m(x)} dx F(dm) \right), \end{aligned}$$

where  $h_u(t) = e^{iut}$ ,

$$\varphi_{X_\Phi(0), \partial_j X_\Phi(0)}(u, v) = \mathbb{E} \left( e^{iu X_\Phi(0) + iv \partial_j X_\Phi(0)} \right) = \exp \left( \lambda \int_{\mathbb{R}^d \times \mathbb{R}^2} [e^{i[ug_m(x) + v \partial_j g_m(x)]} - 1] F(dm) dx \right),$$

and the notation  $\partial_j$ , respectively  $\partial_j^2$ , stands for  $\frac{\partial}{\partial x_j}$ , respectively  $\frac{\partial^2}{\partial x_j^2}$ .

**Remark:** Note that when  $g(A \cdot, m) = g(\cdot, m)$ , for all orthogonal matrix  $A$ ,  $X_\Phi$  has the same law as the shot noise random field given with marks  $\tilde{m} \in \mathbb{R}^{d-1}$  of law  $G(d\tilde{m})$ .

*Proof.* Following similar arguments as in Proposition 3 of [10], (39) will ensure that  $X_\Phi$  is a.s. a stationary  $C^2$  field such that  $X$ ,  $\nabla X$  and  $D^2 X$  have finite expectations, and we can differentiate under the sum. In particular, for all  $j = 1, 2$ ,

$$\partial_j X_\Phi(x) = \sum_{i \in I} \partial_j g_{m_i}(x - x_i) \quad \text{and} \quad \partial_j^2 X_\Phi(x) = \sum_{i \in I} \partial_j^2 g_{m_i}(x - x_i).$$

Hence the general formula of Theorem 5 is valid for  $X_\Phi$ .

Under the additional assumption on the kernel, we can prove isotropy. Actually, for any  $k \geq 1$ ,  $u_1, \dots, u_k \in \mathbb{R}$  and  $y_1, \dots, y_k \in \mathbb{R}^2$ , one has

$$\mathbb{E} \left( e^{i \sum_{j=1}^k u_j X_\Phi(y_j)} \right) = \exp \left( \lambda \int_{\mathbb{R}^d} \int_{\mathbb{R}^2} (e^{i \sum_{j=1}^k u_j g_m(y_j - x)} - 1) dx F(dm) \right).$$

Hence, for any orthogonal matrix  $A$ , by the change of variables  $x = Ay$ ,

$$\mathbb{E} \left( e^{i \sum_{j=1}^k u_j X_\Phi(Ay_j)} \right) = \exp \left( \lambda \int_{\mathbb{R}^d} \int_{\mathbb{R}^2} (e^{i \sum_{j=1}^k u_j g_m(A(y_j - y))} - 1) dy F(dm) \right)$$

Note that there exists  $\theta_0 \in [0, 2\pi)$  such that  $A = S_{\theta_0}$  or  $A = R_{\theta_0}$ . But for  $m = (\tilde{m}, \theta)$ ,

$$g_m \circ S_{\theta_0} = \tilde{g}_{\tilde{m}} \circ R_{\theta} S_{\theta_0} = \tilde{g}_{\tilde{m}} \circ S_0 R_{\theta} S_{\theta_0} = \tilde{g}_{\tilde{m}} \circ R_{-\theta-\theta_0}.$$

Since  $F_{\Theta}$  is the uniform law, we may assume without loss of generality that

$$\begin{aligned} \mathbb{E} \left( e^{i \sum_{j=1}^k u_j X_{\Phi}(A y_j)} \right) &= \exp \left( \lambda \int_{\mathbb{R}^d} \int_{\mathbb{R}^2} (e^{i u \sum_{j=1}^k u_j g_m(R_{\theta_0}(y_j - y))} - 1) dy F(dm) \right) \\ &= \exp \left( \lambda \int_{\mathbb{R}^d} \int_{\mathbb{R}^2} (e^{i u \sum_{j=1}^k u_j g_m(y_j - y)} - 1) dy F(dm) \right) = \mathbb{E} \left( e^{i \sum_{j=1}^k u_j X_{\Phi}(y_j)} \right) \end{aligned}$$

using the fact that  $R_{\theta} R_{\theta_0} = R_{\theta_0 + \theta}$  and  $F_{\Theta}$  uniform.

It follows that by Theorem 5, for any  $h$  bounded continuous function on  $\mathbb{R}$ , one has for all  $j = 1, 2$ ,

$$\mathbb{E}(\text{LP}_{X_{\Phi}}(h, U)) = \frac{\pi}{2} \mathcal{L}(U) \mathbb{E}(h(X_{\Phi}(0)) |\partial_j X_{\Phi}(0)|)$$

and

$$\overline{\text{LTC}_{X_{\Phi}}^{(I)}(h)} = -\mathbb{E}(h(X_{\Phi}(0)) \partial_j^2 X_{\Phi}(0)).$$

Taking  $h_u = e^{iu}$  for  $u \in \mathbb{R}$  we obtain

$$\begin{aligned} \mathbb{E}(\text{LP}_{X_{\Phi}}(h_u, U)) &= \frac{\pi}{2} \mathcal{L}(U) \mathbb{E}(e^{iu X_{\Phi}(0)} |\partial_j X_{\Phi}(0)|) \\ &= \frac{\pi}{2} \mathcal{L}(U) \times -\frac{1}{\pi} \int_0^{+\infty} \frac{1}{v} \frac{\partial \varphi_{X_{\Phi}(0), \partial_j X_{\Phi}(0)}(u, v)}{\partial v} dv, \end{aligned}$$

according to Proposition 2 of [11], using the fact that  $\mathbb{E}(|\partial_j X_{\Phi}(0)|) < +\infty$  and  $\varphi_{X_{\Phi}(0), \partial_j X_{\Phi}(0)}(u, v) = \varphi_{X_{\Phi}(0), \partial_j X_{\Phi}(0)}(u, -v)$  by isotropy. Moreover,

$$\overline{\text{LTC}_{X_{\Phi}}^{(I)}(h_u)} = -\mathbb{E} \left( e^{iu X_{\Phi}(0)} \partial_j^2 X_{\Phi}(0) \right) = i \frac{\partial \varphi_{X_{\Phi}(0), \partial_j^2 X_{\Phi}(0)}(u, 0)}{\partial v},$$

where  $\varphi_{X_{\Phi}(0), \partial_j^2 X_{\Phi}(0)}$  is the characteristic function of  $(X_{\Phi}(0), \partial_j^2 X_{\Phi}(0))$  given by, for  $(u, v) \in \mathbb{R}^2$

$$\begin{aligned} \varphi_{X_{\Phi}(0), \partial_j^2 X_{\Phi}(0)}(u, v) &= \mathbb{E} \left( e^{i[u X_{\Phi}(0) + v \partial_j^2 X_{\Phi}(0)]} \right) \\ &= \exp \left( \lambda \int_{\mathbb{R}^d} \int_{\mathbb{R}^2} (e^{i u g_m(x) + i v \partial_j^2 g_m(x)} - 1) dx F(dm) \right). \end{aligned}$$

Then,

$$\overline{\text{LTC}_{X_{\Phi}}^{(I)}(h_u)} = -\lambda \left( \int_{\mathbb{R}^d} \int_{\mathbb{R}^2} \partial_j^2 g_m(x) e^{i u g_m(x)} dx F(dm) \right) \varphi_{X_{\Phi}}(u),$$

with  $\varphi_{X_{\Phi}} = \varphi_{X_{\Phi}(0), \partial_j^2 X_{\Phi}(0)}(\cdot, 0)$  the characteristic function of  $X_{\Phi}(0)$ .  $\square$

## References

- [1] R. J. Adler. *The Geometry of Random Field*. John Wiley & Sons, 1981.
- [2] R. J. Adler. On excursion sets, tube formulas and maxima of random fields. *Annals of Applied Probability*, 10(1):1–74, 2000.

- [3] R. J. Adler, G. Samorodnitsky, and J. E. Taylor. Excursion sets of three classes of stable random fields. *Adv. in Appl. Probab.*, 42(2):293–318, 2010.
- [4] R. J. Adler, G. Samorodnitsky, and J. E. Taylor. High level excursion set geometry for non-Gaussian infinitely divisible random fields. *The Annals of Probability*, 41(1):134–169, 2013.
- [5] R. J. Adler, E. Subag, and J. E. Taylor. Rotation and scale space random fields and the Gaussian kinematic formula. *Ann. Statist.*, 40(6):2910–2942, 2012.
- [6] R. J. Adler and J. E. Taylor. *Random fields and geometry*. Springer Monographs in Mathematics. Springer, New York, 2007.
- [7] L. Ambrosio, N. Fusco, and D. Pallara. *Functions of bounded variation and free discontinuity problems*. Oxford university press, 2000.
- [8] F. Baccelli and B. Błaszczyszyn. *Stochastic Geometry and Wireless Networks, Volume I — Theory*, volume 3, No 3–4 of *Foundations and Trends in Networking*. NoW Publishers, 2009.
- [9] Y. Baryshnikov and R. Ghrist. Euler integration over definable functions. *Proc. Natl. Acad. Sci. USA*, 107(21):9525–9530, 2010.
- [10] H. Biermé and A. Desolneux. Crossings of smooth shot noise processes. *Annals of Applied Probability*, 22(6):2240–2281, 2012.
- [11] H. Biermé and A. Desolneux. On the perimeter of excursion sets of shot noise random fields. *The Annals of Probability*, 44(1):521–543, 2016.
- [12] M. Bilodeau and D. Brenner. *Theory of multivariate statistics*. Springer Texts in Statistics. Springer-Verlag, New York, 1999.
- [13] O. Bobrowski and M. S. Borman. Euler integration of Gaussian random fields and persistent homology. *Journal of Topology and Analysis*, 4(1):49–70, 2012.
- [14] L. Decreusefond, E. Ferraz, H. Randriambololona, and A. Vergne. Simplicial homology of random configurations. *Advances in Applied Probability*, 46(2):325–347, 2014.
- [15] E. Di Bernardino, A. Estrade, and J. R. León. A test of Gaussianity based on the Euler Characteristic of excursion sets. *Electron. J. Stat.*, 11(1):843–890, 2017.
- [16] M.P. Do Carmo. *Differential Geometry of Curves and Surfaces*. Prentice-Hall, 1976.
- [17] A. Estrade and J. R. León. A central limit theorem for the Euler characteristic of a Gaussian excursion set. *The Annals of Probability*, 44(6):3849–3878, 2016.
- [18] L.C. Evans and R.F. Gariepy. *Measure theory and fine properties of functions*. Studies in Advanced Mathematics. CRC Press, 1992.
- [19] L. Heinrich and V. Schmidt. Normal convergence of multidimensional shot noise and rates of this convergence. *Advances in Applied Probability*, 17(4):709–730, 1985.
- [20] M. Hirsch. *Differential Topology*. Springer-Verlag, 1976.
- [21] R. Lachièze-Rey. Bicovariograms and Euler characteristic I. Regular sets. Technical report, MAP5, Univ. Paris Descartes, 2015. <https://arxiv.org/abs/1510.00501>.
- [22] R. Lachièze-Rey. Bicovariograms and Euler characteristic II. Random fields excursions. Technical report, MAP5, Univ. Paris Descartes, 2015. <https://arxiv.org/abs/1510.00502>.
- [23] R. Lachièze-Rey. Two-dimensional Kac-Rice formula. Application to shot noise processes excursions. Technical report, MAP5, Univ. Paris Descartes, 2016. <https://arxiv.org/abs/1510.00502>.
- [24] R. D. MacPherson. Chern classes for singular algebraic varieties. *Ann. of Math. (2)*, 100:423–432, 1974.
- [25] K. R. Mecke. Exact moments of curvature measures in the Boolean model. *Journal of Statistical Physics*, 102(5-6):1343–1381, 2001.
- [26] K. R. Mecke and H. Wagner. Euler characteristic and related measures for random geometric sets. *Journal of Statistical Physics*, 64(3-4):843–850, 1991.
- [27] J.W. Milnor. On the total curvature of knots. *The Annals of Mathematics*, 52(2):248–257,

- 1950.
- [28] I. Molchanov. *Theory of Random Sets*. Probability and its applications. Springer, 1st edition. edition, 2005.
  - [29] L.A. Santaló. *Integral Geometry and Geometric Probability*. Cambridge University Press, second edition edition, 2004.
  - [30] P. Schapira. Operations on constructible functions. *J. Pure Appl. Algebra*, 72(1):83–93, 1991.
  - [31] R. Schneider and W. Weil. *Stochastic and integral geometry*. Probability and its Applications. Springer-Verlag, Berlin, 2008.
  - [32] D. Stoyan, W. S. Kendall, and J. Mecke. *Stochastic geometry and its applications*. Wiley Series in Probability and Mathematical Statistics: Applied Probability and Statistics. John Wiley & Sons, Ltd., Chichester, 1987. With a foreword by D. G. Kendall.
  - [33] C. Thäle. 50 years sets with positive reach—a survey. *Surv. Math. Appl.*, 3:123–165, 2008.
  - [34] O. Y. Viro. Some integral calculus based on Euler Characteristic. In *Topology and geometry—Rohlin Seminar*, volume 1346 of *Lecture Notes in Math.*, pages 127–138. Springer, Berlin, 1988.
  - [35] K. J. Worsley. Local Maxima and the Expected Euler Characteristic of Excursion Sets of Chi2, F and t Fields. *Advances in Applied Probability*, 26:13–42, 3 1994.
  - [36] K. J. Worsley. The geometry of random images. *Chance*, 9(1):27–40, 1996.