Mean Geometry for 2D random fields: level perimeter and level total curvature integrals

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MEAN GEOMETRY FOR 2D RANDOM FIELDS: LEVEL PERIMETER AND LEVEL TOTAL CURVATURE INTEGRALS

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Abstract. We introduce the level perimeter integral and the total curvature integral associated with a real valued function f defined on the plane $\mathbb{R}^2$ as integrals allowing to compute the perimeter of the excursion set of f above level t and the total (signed) curvature of its boundary for almost every level t. Thanks to the Gauss-Bonnet theorem, the total curvature is directly related to the Euler Characteristic of the excursion set. We show that the level perimeter and the total curvature integrals can be explicitly computed in two different frameworks: piecewise constant functions (also called here elementary functions) and smooth (at least C^2) functions. Considering 2D random fields (in particular considering shot noise random fields), we compute their mean perimeter and total curvature integrals, and this provides new explicit computations of the mean perimeter and Euler Characteristic densities of excursion sets, beyond the Gaussian framework.

Contents

1. Introduction

Considering a stationary 2-dimensional random field $X = (X(x))_{x\in\mathbb{R}^2}$ we are interested in statistically describing the geometry of its excursions sets

$$E_X(t) := \{X \geq t\} \subset \mathbb{R}^2,$$

according to a level $t \in \mathbb{R}$, in a given bounded open subset of $\mathbb{R}^2$. In the following U will denote an open bounded set or simply $\mathbb{R}^2$ when not bounded. We will focus on $E_X(t) \cap U = \{x \in U; X(x) \geq t\}$ as well as $\partial E_X(t) \cap U$ its boundary trace on U, where as usual $\partial E_X(t) = \overline{E_X(t)} \setminus \overset{o}{E_X}(t)$.

In view of the measurability of X, its excursion sets $E_X(t)$ are random Borel sets for all level $t \in \mathbb{R}$. When moreover X is a.s. upper semi-continuous, these random sets are a.s. closed (see [28]) and therefore $E_X(t) \cap \overline{U}$ is a compact random set. In dimension 2, the geometry of a compact "nice" set $K \subset \mathbb{R}^2$ with piecewise C^2 boundary ∂K may be described by three functionals: its area $\mathcal{L}(K)$, perimeter $\mathrm{Per}(K) = \mathcal{H}^1(\partial K)$, where $\mathcal{H}^1$ is the one-dimensional Hausdorff measure of the length of plane curves, and Euler Characteristic $\chi(K)$ that counts the number of connected components minus the number of holes. According to the Gauss-Bonnet Theorem, when ∂K is a disjoint finite union of closed curves, $\chi(K)$ is also equal to $\frac{1}{2\pi}\mathrm{TC}(\partial K)$, with $\mathrm{TC}(\partial K)$ the total curvature of the positively oriented curve ∂K (see precise definitions in Definition 1 and Theorem 1). Let us quote that these geometrical features are also used with different conventions according to the

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setting. For instance, in convex geometry, for K a convex body, intrinsic volumes, respectively Minkowski's functionals, are defined by $V_0(K) = \chi(K) = 1$, $V_1(K) = \frac{1}{2}\mathcal{H}^1(\partial K)$ and $V_2(K) = \mathcal{L}(K)$, respectively $W_2(K) = \frac{1}{2}\mathrm{TC}(\partial K) = \pi$, $W_1(K) = \frac{1}{2}\mathcal{H}^1(\partial K)$ and $W_0(K) = \mathcal{L}(K)$ (see [32]), while in differential geometry when K is a compact 2-dimensional submanifold with C^2 smooth boundary, Lipschitz Killing curvatures of K are defined by $C_0(K) = \frac{1}{2\pi}\mathrm{TC}(\partial K)$, $C_1(K) = \frac{1}{2}\mathcal{H}^1(\partial K)$ and $C_2(K) = \mathcal{L}(K)$ and may be extended to positive reach sets (see [33]).

When considering stationary random sets, it is natural to define corresponding mean density functionals. They are usually defined considering the limit behavior of a rescaled observation through a large window, let say rU, for r large. Actually, this procedure allows to remove boundary effects. We adopt a similar point of view in this paper, by removing boundary effect, using an open window U. Hence, for a bounded open set U we will focus on the mean area $\mathbb{E}\left(\mathcal{L}(E_X(t) \cap U)\right)$, the mean perimeter $\mathbb{E}\left(\mathrm{Per}(E_X(t), U)\right)$, and on the mean total curvature $\mathbb{E}\left(\mathrm{TC}(\partial E_X(t), U)\right)$ of excursion sets. Densities will then clearly appear as

$$\mathbb{E}\left(\mathcal{L}(E_X(t) \cap U)\right) = \overline{\mathcal{L}(E_X(t))}\mathcal{L}(U), \quad \mathbb{E}\left(\mathrm{Per}(E_X(t), U)\right) = \overline{\mathrm{Per}(E_X(t))}\mathcal{L}(U)$$

$$\text{and} \quad \mathbb{E}\left(\mathrm{TC}(\partial E_X(t), U)\right) = \overline{\chi(E_X(t))}\mathcal{L}(U).$$

As far as stationarity is involved, the mean area is not hard to find since

$$\begin{aligned} \mathbb{E}\left(\mathcal{L}(E_X(t) \cap U)\right) &= \int_U \mathbb{E}\left(\mathbf{1}_{X(x)\geq t}\right) dx \\ &= \mathcal{L}(U)\mathbb{P}(X(0) \geq t). \end{aligned}$$

It follows that an exact formula can be set up as soon as the distribution of $X(0)$ (that is the same as any $X(x)$ by stationarity) is known and $\overline{\mathcal{L}(E_X(t))} = \mathbb{P}(X(0) \geq t)$. Now establishing formulas for the mean perimeter and the mean total curvature or Euler Characteristic is more difficult and requires additional assumptions on the field. Computing the Euler Characteristic of excursion sets of random fields is a problem that has received much attention. Indeed, in many applications, the Euler Characteristic is a very useful index of the geometry of the field, as explained for instance in the review paper of R. Adler [2], or in the papers of K. Worsley [35] or [36] where applications in astrophysics or in brain imaging are mentioned.

Despite its "global" definition (the number of connected components minus the number of holes), the Euler Characteristic of an excursion set is in fact a purely local quantity related, by Morse theory, to the number of critical points of X in U, or, by the Gauss-Bonnet theorem, to the total curvature of the boundary of the excursion set. Here, we will extensively use this second equivalence to obtain explicit computations of the mean Euler Characteristic density of the excursion sets of some random fields.

In the framework of Gaussian random field, the first equivalence is usually used. For stationary isotropic Gaussian random fields, an explicit formula for any level t may be set for the expectation of the Euler Characteristic density, only depending on the variance and the second spectral moment of the field. This is an important result with many statistical applications. In particular, for large levels t, the Euler Characteristic gives a good approximation of the probability that the suprema of the field is greater than t and can therefore be used as a p-value: this is the Euler Characteristic heuristic (see [5] for instance). In a "tour de force", a Central Limit Theorem has recently been established in [17] that proves the accuracy of the estimation over only one sample path as the size of the observation is growing. There are also some interesting results apart from the Gaussian framework for χ^2, F and t-fields [35] as well as stable [3] or infinitely divisible random fields [4] for instance. A test of Gaussianity can therefore be set up using Euler Characteristic of level sets as proposed in [15]. However, most of general results rely on strong smoothness regularity assumptions and on conditional distribution densities that are often difficult to evaluate for non-Gaussian fields.

Now, in this paper, we will be particularly interested in another family of infinitely divisible random fields, that are not Gaussian, namely the *shot noise random fields*. A shot noise random field is defined on $\mathbb{R}^2$ by

$$\forall x \in \mathbb{R}^2, \quad X(x) = \sum_i g_{m_i}(x - x_i),$$

where the x_i are the points of an homogeneous Poisson point process of intensity λ in $\mathbb{R}^2$, and the m_i are "marks", independent of the Poisson point process. Such fields allow explicit computations and may appear, in view of asymptotic normality in high intensity [19], as a bridge between the Gaussian setting and the discrete models of stochastic geometry such as the Boolean ones [32]. Several results for the computation of the perimeter were obtained in our previous paper [11]. Since the study for all level t is often difficult we extend here our point of view of working in a weak framework by considering these geometric quantities as functions of the level t. Hence quantities of interest will be given, when it makes sense, by the *mean level perimeter integral* $\mathbb{E}\left(\mathrm{LP}_X(h, U)\right)$ and the *mean level total curvature integral* $\mathbb{E}\left(\mathrm{LTC}_X(h, U)\right)$ of X, where the

level perimeter integral and the *level total curvature integral* are defined, for h a continuous bounded function on $\mathbb{R}$, by

$$\mathrm{LP}_X(h,U) := \int_{\mathbb{R}} h(t)\mathrm{Per}(E_X(t),U)dt \tag{1}$$

and

$$\mathrm{LTC}_X(h,U) := \int_{\mathbb{R}} h(t)\mathrm{TC}(\partial E_X(t),U)dt. \tag{2}$$

Let us quote that this allows to get informations on the mean geometry of excursion sets for almost all levels t and hence to give insights on their evolution according to the level. This setting will allow us to recover and generalize some important results established in the two different frameworks of Boolean models in stochastic geometry and of smooth Gaussian random fields.

In Section 2 we propose a general definition of the level perimeter integral and of the level total curvature integral of a function, that allows to compute the perimeter and total curvature (and therefore the Euler Characteristic) of its excursion sets for almost every level. We introduce elementary functions as a particular case of piecewise constant functions (with piecewise smooth discontinuity set) in Section 3 and compute their level perimeter and total curvature integral. We give explicit expressions for elementary shot noise random fields, where the functions g_{m_i} are elementary functions. This allows us to generalize results of the literature (about the Boolean model [26], or about "random configurations" [14]). The last section is devoted to results on smooth (random) functions. In particular our weak framework allows us to get explicit formulas for some isotropic symmetric fields, recovering known Gaussian results.

Let us finally emphasize that we have made here the deliberate choice of not working in the weakest possible functional framework. Our goal is to work with piecewise smooth functions (like the indicator function of a set having a piecewise C^2 boundary for instance). But we believe some of our results can be extended to functions with a weakest regularity. Let us also mention the recent work of R. Lachièze-Rey in [21] and [22] that relates the Euler Characteristic to the three-point joint distribution of the random field. And also the even more recent paper [23] where R. Lachièze-Rey gives formulas for the Euler Characteristic of isotropic shot noise random field that are a.s. Morse functions.

2. General framework

2.1. **Sets of finite perimeter and finite total curvature.** We consider a Borel set E of $\mathbb{R}^2$ and the open set $U \subset \mathbb{R}^2$. We assume that the indicator function $\mathbf{1}_E$ is a function in $SBV(U)$. Let us recall (see [7]) that a function f belongs to $SBV(U)$, the space of *special functions of bounded variation in* U if $f \in L^1(U)$ and has its distributional derivative representable by a finite Radon measure in U that is

$$\int_U f(x)\frac{\partial \phi}{\partial x_l}(x)\,dx = -\int_U \phi(x)\,D_l f(dx) \quad \forall \phi \in C_c^1(U,\mathbb{R}), \ \ \forall l = 1,2$$

for some $\mathbb{R}^2$-valued measure $Df = (D_1 f, D_2 f)$, is such that

$$Df = \nabla f \mathcal{L} + (f^+ - f^-)\nu_f \mathcal{H}^1 \llcorner J_f,$$

where

- $D^a f := \nabla f \mathcal{L}$ is the absolutely continuous part of the Radon measure Df with respect to the Lebesgue measure $\mathcal{L}$ and ∇f is the *approximate differential* of f (see [7] p.165 and Theorem 3.83 p.176).
- $D^j f := (f^+ - f^-)\nu_f \mathcal{H}^{n-1} \llcorner J_f$ is the *jump part* of Df, with J_f the set of *approximate jump points* of f for which there exist two reals $f^-(x) < f^+(x)$ and a direction $\nu_f(x) \in S^{n-1}$ with

  $$\lim_{\rho\to 0} \rho^{-2} \int_{B_\rho^+(x,\nu_f(x))} |f(y) - f^+(x)|\,dx = 0 \quad \text{and} \quad \lim_{\rho\to 0} \rho^{-2} \int_{B^-(x,\nu_f(x))} |f(y) - f^-(x)|\,dx = 0,$$

  where $B_\rho^+(x,\nu)$, resp. $B_\rho^-(x,\nu)$, denotes the half-ball determined by $\nu \in S^{n-1}$ *i.e.* $\{y \in B_\rho(x); \langle y - x, \nu\rangle > 0\}$, resp. $\{y \in B_\rho(x); \langle y - x, \nu\rangle < 0\}$. We denote here $B_\rho(x)$ the ball of radius ρ and center x.

The *approximate discontinuity set* (see [7] Proposition 3.64 p.160) S_f is the set of points where f is not *approximately continuous*. It is a $\mathcal{L}$-negligible Borel set, countably $\mathcal{H}^1$-rectifiable with $\mathcal{H}^1(S_f \setminus J_f) = 0$, by Federer-Vol'pert Theorem ([7] Theorem 3.78 p.173).

This framework, used in our previous paper [11], is convenient to define the perimeter of a set E in U such that $\mathbf{1}_E \in SBV(U)$ as

$$\mathrm{Per}(E,U) := \|D\mathbf{1}_E\|(U) = \sup\{\int_U \mathbf{1}_E \mathrm{div}\varphi dx | \varphi \in C^1_c(U,\mathbb{R}^2), \|\varphi\|_\infty\} < +\infty.$$

It follows that denoting $S_{\mathbf{1}_E}$ the approximate discontinuity set of $\mathbf{1}_E$, the set $S_{\mathbf{1}_E} \cap U$ (its trace in U) is included in $\partial E \cap U$ the boundary of E in U. In order to get information linked with length and Euler Characteristic we make the stronger assumption that the boundary $\partial E \cap U$ in U coincides with $S_{\mathbf{1}_E} \cap U$ (which is equivalent to say that the discontinuity points are exactly the approximate discontinuity points in U), and is a piecewise C^2 plane curve.

Let us recall here some basic facts and definitions about plane curves, following [16]. We say that an oriented curve Γ is *piecewise regular* if it is simple (with possibly several closed simple curves) and piecewise C^2 such that any point $x \in \Gamma$ of the curve can be else

- **regular:** one can find an arc-length C^2 parametrization $\gamma : (0,\varepsilon) \to \Gamma$ with $x = \gamma(s)$ for some $s \in (0,\varepsilon)$, with $\varepsilon > 0$, and a normal vector $\nu_\Gamma(x) = \gamma'(s)^\perp \in S^1$ with $\gamma'(s)^\perp$ the $+\frac{\pi}{2}$ rotation of the tangent vector $\gamma'(s)$. The normal cone (defined for positive reach set [33]) of Γ at x is given by $\mathrm{Nor}(\Gamma,x) = \{-\nu_\Gamma(x)\}$. The signed curvature $\kappa_\Gamma(x)$ of Γ at $x = \gamma(s)$ is then defined as
$$\kappa_\Gamma(x) = \langle \gamma''(s), \nu_\Gamma(x)\rangle,$$
where $\langle\cdot,\cdot\rangle$ is the usual Euclidean scalar product on $\mathbb{R}^2$. Note that since γ is an an arc-length parametrization we have $\mathcal{H}^1(\gamma(0,\varepsilon)) = \varepsilon$.
- **corner:** one can find a simple continuous arc-length parametrization $\gamma : (-\varepsilon,\varepsilon) \to \Gamma$ such that $x = \gamma(0)$ with γ being C^2 on $(-\varepsilon,\varepsilon) \setminus \{0\}$ and γ' admits limits $\gamma'(0^-) \in S^1$ and $\gamma'(0^+) \in S^1$ at 0, with $\nu^-_\Gamma(x) := \gamma'(0^-)^\perp$ and $\nu^+_\Gamma(x) := \gamma'(0^+)^\perp$ linearly independent (no "cusp") in S^1 such that the normal cone of Γ at x is given by $\mathrm{Nor}(\Gamma,x) = \{-p\nu^-_\Gamma(x) - q\nu^+_\Gamma(x); p,q \geq 0\} \cap S^1$. We then define $\beta_\Gamma(x) \in (0,\pi)$ the angle of the cone $\mathrm{Nor}(\Gamma,x)$, corresponding to the size of the jump of ν_Γ at point x and $\alpha_\Gamma(x) = \pm\beta_\Gamma(x) \in (-\pi,\pi)$ the *turning angle* at x, where the sign is given according to the orientation of the curve. Note that we also have $\mathcal{H}^1(\gamma((-\varepsilon,\varepsilon))) = 2\varepsilon$.

We note $\mathcal{R}_\Gamma$ the set of regular points and $\mathcal{C}_\Gamma$ the set of corner points of the curve Γ.

Definition 1 (Elementary set, Perimeter and Total curvature)**.** *We say that a Borel set E is an elementary set of U if $\mathbf{1}_E$ is in $SBV(U)$ and its boundary $\Gamma := \partial E$ coincides with $S_{\mathbf{1}_E}$ in U with $\Gamma \cap U$ a piecewise regular curve, positively oriented in such a way that the normals are oriented towards E, given by a finite union of simple connected curves (possibly closed), with curvature κ_Γ integrable on $\mathcal{R}_\Gamma \cap U$, that is $\kappa_\Gamma \in L^1(\mathcal{R}_\Gamma \cap U, \mathcal{H}^1)$, and with a finite number of corner points in U, that is $\mathcal{H}^0(\mathcal{C}_\Gamma \cap U) < +\infty$. It follows that the length of the curve in U is given by*

$$\mathcal{H}^1(\mathcal{R}_\Gamma \cap U) = \mathrm{Per}(E,U),$$

and its absolute curvature in U is

$$\mathrm{TaC}(\Gamma,U) := \int_{\mathcal{R}_\Gamma \cap U} |\kappa_\Gamma(x)| \mathcal{H}^1(dx) + \sum_{x \in \mathcal{C}_\Gamma \cap U} |\alpha_\Gamma(x)| < +\infty.$$

We then define the total curvature of Γ in U as

$$\mathrm{TC}(\Gamma,U) := \int_{\mathcal{R}_\Gamma \cap U} \kappa_\Gamma(x) \mathcal{H}^1(dx) + \sum_{x \in \mathcal{C}_\Gamma \cap U} \alpha_\Gamma(x).$$

Let us quote that for E an elementary set of U, setting $f = \mathbf{1}_E \in SBV(U)$, the jump part J_f of Df is equal to $\mathcal{R}_{\partial E}$, with $(f^+(x), f^-(x), \nu_f(x)) = (1,0,\nu_{\partial E}(x))$ for $x \in J_f$ and $\mathcal{S}_f$ is the disjoint union of $\mathcal{R}_{\partial E} \cup \mathcal{C}_{\partial E}$.

Notice that the definition of TaC is the same as the one introduced by Milnor in [27]. But here, in this work, we will pay a particular attention to the signed total curvature, and not to its absolute value. Actually, our definition of total curvature is the same as to the one of Santaló in [29], Chapter 7. The total curvature is intrinsic, it doesn't depend on the parametrization of the curve. But it depends on its orientation: if we reverse the orientation of the curve then its total curvature is changed into its opposite.

In particular, when E is an elementary set of a bounded open set U, as soon as $\partial E = \partial E^c$, which is the case when E is closed or open for instance, its complementary $E^c = \mathbb{R}^2 \setminus E$ is also an elementary set of U, with

$$\mathrm{Per}(E^c,U) = \mathrm{Per}(E,U),\ \mathrm{TaC}(\partial E^c,U) = \mathrm{TaC}(\partial E \cap U, U), \text{ and } \mathrm{TC}(\partial E^c,U) = -\mathrm{TC}(\partial E \cap U, U).$$

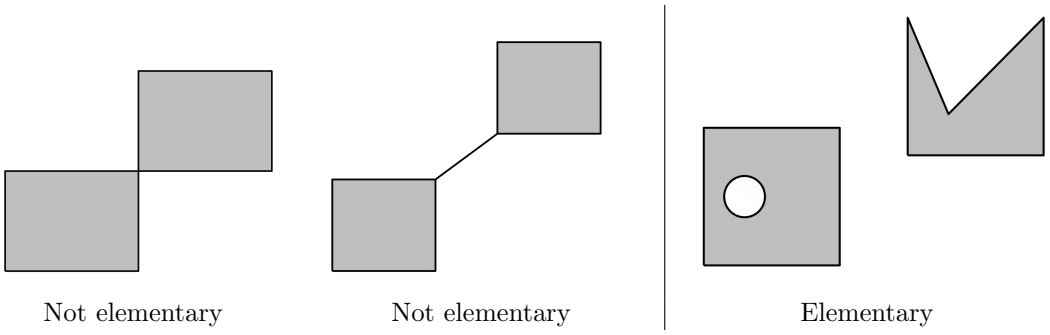

FIGURE 1. Left: two sets that are not elementary sets according to Definition 1: the first set has one multiple point; for the second set, its indicator function has a discontinuity set not equal to its approximate discontinuity set. Right: an elementary set.

Note also that an elementary set in U is also an elementary set in any open subset $U' \subset U$ and an elementary set of $\mathbb{R}^2$ is of course elementary for any bounded open set U. Finally, let us quote that E is an elementary set in U if and only if $E \cap U$ is an elementary set in U.

The link between the level total curvature and Euler Characteristic is given by the Gauss-Bonnet theorem.

Theorem 1 (Gauss-Bonnet Theorem)**.** *Let $E \subset U$ be an elementary set and a regular region (meaning that $E = \overline{\overset{o}{E}}$). Then, $S_{\mathbf{1}_E} = \partial E$ and*

- *the perimeter of E defined by*

$$\mathrm{Per}(E,U) := \|D\mathbf{1}_E\|(U) = \sup\{\int_U \mathbf{1}_E div\varphi dx \,|\, \varphi \in C^1_c(U,\mathbb{R}^2), \|\varphi\|_\infty\} < +\infty,$$

corresponds to

$$\mathcal{H}^1(\mathcal{R}_{\partial E}) = \mathcal{H}^1(\partial E).$$

- *By Gauss-Bonnet Theorem, the Euler Characteristic of E is given by*

$$\chi(E) = \frac{1}{2\pi}\mathrm{TC}(\partial E, U),$$

where the total curvature of ∂E is equal to

$$\mathrm{TC}(\partial E, U) = \int_{\mathcal{R}_{\partial E}} \kappa_{\partial E}(x)\mathcal{H}^1(dx) + \sum_{x \in \mathcal{C}_{\partial E}} \alpha_{\partial E}(x).$$

Proof. The first point follows from Gauss-Green Theorem (see Section 3.3 of [7]). For the second point we use that E being closed and elementary, its boundary must be composed by a finite disjoint union of closed piecewise C^2 curves so that we can apply Gauss-Bonnet Theorem stated in [16] p.274 for regular region. □

Remark 1: It is a well-known result of differential geometry of plane curves that the total curvature of any regular simple closed curve is 2π or -2π (depending on the orientation of the curve). This result is sometimes called Hopf's Umlaufsatz, or also the theorem of turning tangents ([16] p.396).

Remark 2: Note that when $E \subset U$ is a regular region one has $\partial E = \partial E^c$ and $\mathrm{TC}(\partial E^c, U) = -\mathrm{TC}(\partial E, U)$, that allows to consider both E or its complementary. In contrast, for Euler Characteristic, we have to consider the compact set $\overline{U} \setminus \overset{o}{E}$, and that yields $\chi(\overline{U} \setminus \overset{o}{E}) = 1 - \chi(E)$.

2.2. **Level integrals for excursion sets.**

Definition 2 (Level perimeter and total curvature integrals)**.** *Let f be a real-valued function defined on U such that $f \in SBV(U)$. For $t \in \mathbb{R}$, we define the excursion set of f for the level t as*

$$E_f(t) := \{f \geq t\} \subset \mathbb{R}^2.$$

We assume that for almost every $t \in \mathbb{R}$, the set $E_f(t)$ is an elementary set of U in the sense of Definition 1 and that $t \mapsto \mathrm{TaC}(\partial E_f(t), U)$ is an integrable function on $\mathbb{R}$. We then say that the function f is of special bounded variation and of finite level total curvature integral (on U). Then, the level perimeter integral and the level total curvature integral of f are defined for any bounded continuous function h on $\mathbb{R}$ by

$$\mathrm{LP}_f(h,U) = \int_{\mathbb{R}} h(t)\mathrm{Per}(E_f(t),U)\,dt \text{ and } \mathrm{LTC}_f(h,U) = \int_{\mathbb{R}} h(t)\mathrm{TC}(\partial E_f(t),U)\,dt. \tag{3}$$

We simply denote $V_f(U)$ *for* $\mathrm{LP}_f(1,U)$ *(*$= \|Df\|(U)$ *by the co-area formula* [7]*) and* $\mathrm{LTC}_f(U)$ *for* $\mathrm{LTC}_f(1,U)$.

Let us remark that when $t > \max_U f$, then $E_f(t) \cap U = \emptyset$ and therefore $\mathrm{Per}(E_f(t), U) = \mathrm{TC}(\partial E_f(t), U) = 0$. On the other hand, when $t \leq \min_U f$, then $E_f(t) \cap U = U$, and thus $\partial E_f(t) \cap U = \emptyset$. Therefore we also have $\mathrm{Per}(E_f(t), U) = \mathrm{TC}(\partial E_f(t), U) = 0$. This shows that the perimeter and total curvature are 0 for levels t outside the range of f.

Let us also notice that when H is a C^1 diffeomorphism on $\mathbb{R}$ with bounded derivative $h = H'$, by a simple change of variable, the function $H \circ f$ is also of special variation and of finite level total curvature integral on U with $E_{H\circ f}(t) = E_f(H^{-1}(t))$ so that

$$V_{H\circ f}(U) = \mathrm{LP}_f(h, U) \text{ and } \mathrm{LTC}_{H\circ f}(U) = \mathrm{LTC}_f(h, U).$$

Link with Euler Integral. Due to the additivity property of the Euler Characteristic

$$\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B),$$

it is natural to set up an integration theory with respect to Euler Characteristic [24, 34]. However, since χ is only finitely additive, a careful choice of integrands must be done. This problem was tackled by defining the class of constructible functions [30], then extended by the class of "tame" real-valued functions in [9]. Following this framework, Bobrowski and Borman obtained in [13] the first probabilistic statement about the persistent homology generated by sublevel sets. We briefly recall the definitions used in [13] for comparison with our setting. When f is a real continuous function defined on a compact topological set S, it is said to be a tame function if the homotopy types of $E_f(t) = \{f \geq t\}$ and $\{f \leq t\}$ change only finitely many times as t varies over $\mathbb{R}$, and the Euler Characteristic of each set is always finite. For such a function, a lower and upper Euler integrals are defined by

$$\begin{aligned} \int_S f \lfloor d\chi \rfloor &= \int_0^{+\infty} (\chi(f \geq t) - \chi(f < -t))\, dt \\ \int_S f \lceil d\chi \rceil &= \int_0^{+\infty} (\chi(f > t) - \chi(f \leq t))\, dt, \end{aligned}$$

where $\chi(f \geq t) = \chi(E_f(t))$, $\chi(f < t) = \chi(S) - \chi(f \geq t)$, etc. Note that when $S = \overline{U}$ we always have $\chi(f \geq t) = \chi(S) = 1$ for any $t \leq \min_S f$ and thus $t \mapsto \chi(f \geq t)$ is not integrable on $\mathbb{R}$, explaining the above definition of Euler integrals.

In contrast, we can simply define $\mathrm{LTC}_f(U)$ for f a function of special bounded variation and finite level total curvature on U. Of course, $\frac{1}{2\pi}\mathrm{TC}(\partial E_f(t), U)$ will not coincide with $\chi(E_f(t) \cap U)$ when the excursion set is not included in the observation window in view of its boundary. But it can be seen as a "modified" Euler Characteristic, in a sense very similar to the one used in the book of Adler and Taylor [6] or in the paper of Estrade and León [17], where critical points in U are only taken into account, and not the ones on the boundary of U. Moreover, when considering large domains (that is rU for r going to infinity) the total curvature (in expectation) will grow like $r^2\mathcal{L}(U)$ whereas the sum of the turning angles on ∂rU will (in expectation also) grow like $r\mathcal{H}^1(\partial U)$, being negligible for large r.

We will show how the perimeter and the level total curvature integrals can be explicitly computed in different situations and we will apply it for computing Perimeter and Euler Characteristic densities of stationary fields. The first situation is the one of sums of piecewise constant functions (also called elementary functions), and the second situation will be the case of smooth (at least C^2) functions.

3. Elementary functions and shot noise random fields

3.1. **Elementary functions.** We introduce the class of elementary functions that are piecewise constant functions of special bounded variation. In order to compute the total curvature of excursion sets we need to be more precise on the discontinuity set, which is $\mathcal{H}^1$-rectifiable under the only assumption of special bounded variation.

We first introduce some notations. For a point $x \in U$ and a real positive number ρ, we recall that $B_\rho(x)$ denotes the open ball of radius ρ and center x. For an oriented simple piecewise C^2 curve γ with finite length, and a point x on γ, then for ρ small enough, $B_\rho(x) \setminus \gamma$ is made of two connected components. These two "half-balls" are respectively denoted by $B^+_\rho(x, \gamma)$ and $B^-_\rho(x, \gamma)$. The half-ball $B^+_\rho(x, \gamma)$ is the component that is on the side of the normal ν_γ to γ.

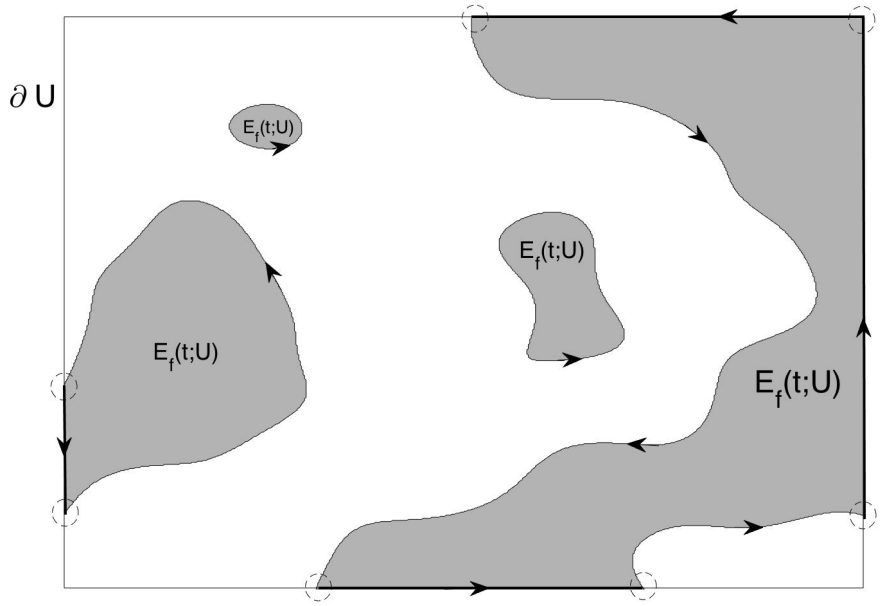

FIGURE 2. By the Gauss-Bonnet theorem, the Euler Characteristic of the excursion set $E_f(t) \cap \overline{U}$ (in gray) is equal to the total curvature of its boundary in the open rectangular domain U plus the turning angles at the points where $E_f(t) \cap \overline{U}$ meets ∂U and also plus $\pi/2$ for each of the corners of the rectangle U that are in $E_f(t)$. All these special points are the ones marked by the small dashed circles on the figure.

Definition 3 (Elementary function)**.** *We say that a function f defined on U is an elementary function on U if f is a piecewise constant function in $SBV(U)$ taking a finite number of values (meaning that $f(U)$ is a finite subset of $\mathbb{R}$), and if $\mathcal{S}_f$, the discontinuity set of f can be decomposed in U as*

$$\mathcal{S}_f \cap U = (\mathcal{R}_f \cup \mathcal{C}_f \cup \mathcal{I}_f) \cap U,$$

where $(\mathcal{S}_f \setminus \mathcal{I}_f) \cap U$ is a piecewise regular curve and (see also Figure 3):

- *$\mathcal{R}_f \cap U$ is the regular part of the discontinuity set of f in U: it is the finite and disjoint union of C^2 simple connected curves having finite length and finite total curvature. More precisely, if $x \in \mathcal{R}_f \cap U$, then there exists $\rho > 0$ such that $\mathcal{S}_f \cap B_\rho(x)$ is a simple C^2 oriented curve γ separating the ball $B_\rho(x)$ in two half-balls $B^+_\rho(x,\gamma)$ and $B^-_\rho(x,\gamma)$. Moreover there exist two real numbers $f^+(x) > f^-(x)$ such that $f(y) = f^+(x)$ for all $y \in B^+_\rho(x,\gamma)$ and $f(y) = f^-(x)$ for all $y \in B^-_\rho(x,\gamma)$. We also denote*

$$\nu_f(x) = \nu_\gamma(x) \quad \text{and} \quad \kappa_f(x) = \kappa_\gamma(x) \text{ with } \kappa_f \in L^1(\mathcal{R}_f, \mathcal{H}^1).$$

- *$\mathcal{C}_f \cap U$ is the set of corner points of f in U: it is a finite set of points (meaning $\mathcal{H}^0(\mathcal{C}_f) < +\infty$, with $\mathcal{H}^0$ the counting measure) such that if $x \in \mathcal{C}_f \cap U$, then there exists $\rho > 0$ such that $\mathcal{S}_f \cap B_\rho(x)$ is a simple piecewise C^2 oriented curve γ having only one corner at x. We write $\alpha_f(x) \in (-\pi, \pi)$ the turning angle of γ at x. As for regular points, γ separates the ball $B_\rho(x)$ in two half-balls $B^+_\rho(x,\gamma)$ and $B^-_\rho(x,\gamma)$, and moreover there exist two real numbers $f^+(x) > f^-(x)$ such that $f(y) = f^+(x)$ for all $y \in B^+_\rho(x,\gamma)$ and $f(y) = f^-(x)$ for all $y \in B^-_\rho(x,\gamma)$. The turning angle at such a corner point is denoted*

$$\alpha_f(x) = \alpha_\gamma(x).$$

- *$\mathcal{I}_f \cap U$ is the set of intersection points of f in U: it is a finite set of points (meaning $\mathcal{H}^0(\mathcal{I}_f) < +\infty$) such that for $x \in \mathcal{I}_f \cap U$, then there exists $\rho > 0$ such that $\mathcal{S}_f \cap B_\rho(x)$ is the union of two different simple and oriented C^2 curves γ_1 and γ_2 such that $\{x\} = \gamma_1 \cap \gamma_2$ and such that the intersection is non-degenerate (meaning that $\nu_{\gamma_1}(x)$ and $\nu_{\gamma_2}(x)$ are not colinear). Each curve separates the ball in two half-balls, and there exist 4 real numbers $f^-(x) \le f^+_-(x), f^-_+(x) \le f^+(x)$ with at least 3 different values, such that $f = f^-(x)$ on $B^-_\rho(x,\gamma_1) \cap B^-_\rho(x,\gamma_2)$; $f = f^-_+(x)$ on $B^-_\rho(x,\gamma_1) \cap B^+_\rho(x,\gamma_2)$; $f = f^+_-(x)$*

on $B_\rho^+(x,\gamma_1)\cap B_\rho^-(x,\gamma_2)$ *and* $f=f^+(x)$ *on* $B_\rho^+(x,\gamma_1)\cap B_\rho^+(x,\gamma_2)$. *And we define*

$$\begin{aligned}\beta_f(x) &= d_{S^1}(\nu_{\gamma_1}(x),\nu_{\gamma_2}(x))\\ &= \min(|\mathrm{Arg}\,\nu_{\gamma_1}(x)-\mathrm{Arg}\,\nu_{\gamma_2}(x)|, 2\pi-|\mathrm{Arg}\,\nu_{\gamma_1}(x)-\mathrm{Arg}\,\nu_{\gamma_2}(x)|)\in(0,\pi),\end{aligned}$$

the geodesic distance between $\nu_{\gamma_1}(x)$ *and* $\nu_{\gamma_2}(x)$ *on* S^1.

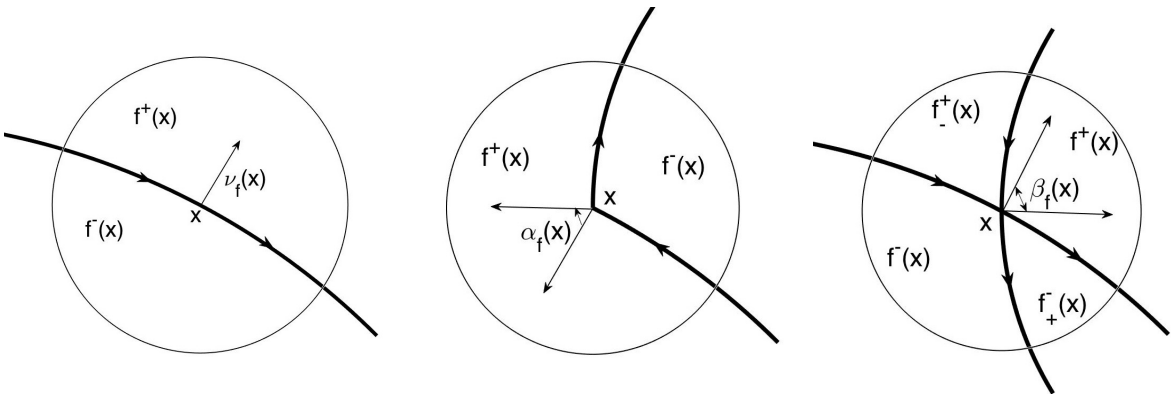

FIGURE 3. The three types of points of the discontinuity set of an elementary function. From left to right: a regular point, a corner point and an intersection point.

Proposition 1. *If* f *is an elementary function on* U, *then for all* $t\in\mathbb{R}$, $E_f(t)$ *is an elementary set of* U *in the sense of Definition 1. Moreover,* f *is of special bounded variation and finite level total curvature integral on* U *with*

$$V_f(U) = \|Df(U)\| = \int_{\mathcal{R}_f\cap U}[f^+(x)-f^-(x)]\mathcal{H}^1(dx) \text{ and} \tag{4}$$

$$\begin{aligned}\mathrm{LTaC}_f(U) &= \int_{\mathcal{R}_f\cap U}[f^+(x)-f^-(x)]|\kappa_f(x)|\mathcal{H}^1(dx)+\sum_{x\in\mathcal{C}_f\cap U}[f^+(x)-f^-(x)]|\alpha_f(x)|\\ &+\sum_{x\in\mathcal{I}_f\cap U}[f^+(x)-\max(f_-^+(x),f_-^-(x))+\min(f_-^+(x),f_-^-(x))-f^-(x)]\beta_f(x)<+\infty.\end{aligned} \tag{5}$$

If h *is a bounded continuous function on* $\mathbb{R}$, *and* H *is a primitive of* h *(for instance* $H(t)=\int_0^t h(u)\,du$*), then the level perimeter integral and the total curvature integral of* f *are given by*

$$\mathrm{LP}_f(h,U) = \int_{\mathcal{R}_f\cap U}[H(f^+(x))-H(f^-(x))]\mathcal{H}^1(dx) \tag{6}$$

$$\begin{aligned}\mathrm{LTC}_f(h,U) &= \int_{\mathcal{R}_f\cap U}[H(f^+(x))-H(f^-(x))]\kappa_f(x)\mathcal{H}^1(dx)+\sum_{x\in\mathcal{C}_f\cap U}[H(f^+(x))-H(f^-(x))]\alpha_f(x)\\ &+\sum_{x\in\mathcal{I}_f\cap U}[H(f^+(x))+H(f^-(x))-H(f_-^+(x))-H(f_+^-(x))]\beta_f(x).\end{aligned} \tag{7}$$

In particular, when $h=1$, *we get* $\mathrm{LP}_f(1,U)=V_f(U)$ *and*

$$\begin{aligned}\mathrm{LTC}_f(U) &= \int_{\mathcal{R}_f\cap U}[f^+(x)-f^-(x)]\kappa_f(x)\mathcal{H}^1(dx)+\sum_{x\in\mathcal{C}_f\cap U}[f^+(x)-f^-(x)]\alpha_f(x)\\ &+\sum_{x\in\mathcal{I}_f\cap U}[f^+(x)+f^-(x)-f_-^+(x)-f_+^-(x)]\beta_f(x).\end{aligned} \tag{8}$$

Proof. We assume that $m=\mathrm{Card}(f(U))\geq 2$. Otherwise, if $m=1$, then for all $t\in\mathbb{R}$, $E_f(t)\cap U=\emptyset$ or U and therefore $\mathrm{Per}(E_f(t),U)=\mathrm{TaC}(\partial E_f(t),U)=\mathrm{TC}(\partial E_f(t),U)=0$. In the following we denote the values of f in U by $v_1<\ldots<v_m$ and set $v_0=-\infty$. We first remark that $E_f(t)\cap U=\emptyset$ for $t>v_m$, $E_f(t)\cap U=U$ for $t\leq v_1$ and $E_f(t)\cap U=E_f(v_i)\cap U$ for $v_{i-1}<t\leq v_i$ and $2\leq i\leq m$. The set of discontinuity points is given by $\mathcal{S}_f\cap U=\cup_{i=2}^m\partial E_f(v_i)\cap U$.

Let us prove that $E_f(v_i)$ is an elementary set in U for $2 \leq i \leq m$. Since $f \in SBV(U)$ is piecewise constant, one can find φ_i a C^1 real function such that $\mathbf{1}_{E_f(v_i)\cap U} = \varphi_i \circ f$. By chain rule (see Theorem 3.96 of [7]), it follows that $\mathbf{1}_{E_f(v_i)\cap U} \in SBV(U)$ and therefore $\mathbf{1}_{E_f(v_i)} \in SBV(U)$. Now let us prove that the boundary $\Gamma_i := \partial E_f(v_i) \cap U$ coincides with the approximate discontinuity points of $\mathbf{1}_{E_f(v_i)\cap U}$ and is a piecewise regular curve. Since $\Gamma_i \subset \mathcal{S}_f$, we can write

$$\Gamma_i = (\Gamma_i \cap \mathcal{R}_f) \cup (\Gamma_i \cap \mathcal{C}_f) \cup (\Gamma_i \cap \mathcal{I}_f)\,.$$

Note that for $x \in \Gamma_i$, one has $f^+(x) \geq v_i$ and $f^-(x) < v_i$ and x is also an approximate discontinuity point of $\mathbf{1}_{E_f(v_i)\cap U}$. Moreover $\Gamma_i \cap (\mathcal{R}_f \cup \mathcal{C}_f)$ is a piecewise regular curve with length $\mathcal{H}^1(\Gamma_i \cap (\mathcal{R}_f \cup \mathcal{C}_f)) = \mathcal{H}^1(\Gamma_i \cap \mathcal{R}_f) = \mathrm{Per}(E_f(v_i), U) < +\infty$ and for $x \in \Gamma_i \cap \mathcal{R}_f$ we get $\kappa_{\Gamma_i}(x) = \kappa_f(x)$, while for $x \in \Gamma_i \cap \mathcal{C}_f \subset \mathcal{C}_{\Gamma_i}$ we get $\alpha_{\Gamma_i}(x) = \alpha_f(x)$.
When $x \in \Gamma_i \cap \mathcal{I}_f$, we have an intersection point of f, and it may become a corner point or a regular point for Γ_i. Indeed, using the same notations for intersection points as in Definition 3, we may write $x \in \gamma_1 \cap \gamma_2$ with the two simple C^2 curves γ_1 and γ_2 being such that $(\gamma_1 \cup \gamma_2) \cap B_\rho(x) = \mathcal{S}_f \cap B_\rho(x)$. If $v_i \leq \min(f_-^+(x), f_+^-(x))$, then $\Gamma_i \cap B_\rho(x) = (\gamma_1 \cap B_\rho^-(x,\gamma_2)) \cup (\gamma_2 \cap B_\rho^-(x,\gamma_1))$ and $x \in \mathcal{C}_{\Gamma_i}$ with $\alpha_{\Gamma_i}(x) = -\beta_f(x)$. Without loss of generality we may assume that $f_-^+(x) \leq f_+^-(x)$. If $f_-^+(x) < v_i \leq f_+^-(x)$ then $\Gamma_i \cap B_\rho(x) = \gamma_2$ and therefore x is not a corner point of Γ_i but a regular point $\mathcal{H}^1$-negligible with $\kappa_{\Gamma_i}(x) = \kappa_{\gamma_1}(x)$. If $v_i > \max(f_-^+(x), f_-^-(x))$, then $\Gamma_i \cap B_\rho(x) = (\gamma_1 \cap B_\rho^+(x,\gamma_2)) \cup (\gamma_2 \cap B_\rho^+(x,\gamma_1))$ and x is again a corner point of Γ_i with $\alpha_{\Gamma_i}(x) = \beta_f(x)$. Finally, Γ_i is a piecewise regular curve and $E_f(v_i)$ is an elementary set in U with $\mathrm{Per}(E_f(v_i)) = \mathcal{H}^1(\mathcal{R}_f \cap \Gamma_i)$ and

$$\begin{aligned}\mathrm{TaC}(\Gamma_i, U) &= \int_{\mathcal{R}_f\cap\Gamma_i} |\kappa_f(x)|\mathcal{H}^1(dx) + \sum_{x\in\Gamma_i\cap\mathcal{C}_f} |\alpha_f(x)| \\ &\quad + \sum_{x\in\Gamma_i\cap\mathcal{I}_f} \beta_f(x)\left(\mathbb{1}_{v_i>\max(f_-^+(x),f_-^-(x))} + \mathbb{1}_{v_i\leq\min(f_-^+(x),f_-^-(x))}\right) < +\infty,\end{aligned}$$

while

$$\begin{aligned}\mathrm{TC}(\Gamma_i, U) &= \int_{\mathcal{R}_f\cap\Gamma_i} \kappa_f(x)\mathcal{H}^1(dx) + \sum_{x\in\Gamma_i\cap\mathcal{C}_f} \alpha_f(x) \\ &\quad + \sum_{x\in\Gamma_i\cap\mathcal{I}_f} \beta_f(x)\left(\mathbb{1}_{v_i>\max(f_-^+(x),f_-^-(x))} - \mathbb{1}_{v_i\leq\min(f_-^+(x),f_-^-(x))}\right).\end{aligned}$$

It follows that for all $t \in \mathbb{R}$, the set $E_f(t)$ is an elementary set in U and that $t \mapsto \mathrm{TaC}(\partial E_f(t), U)$ is an integrable function on $\mathbb{R}$, as a step function with compact support. Hence f is of special bounded variation and finite level curvature integral on U.

Now, let h be a continuous bounded function on $\mathbb{R}$ and H a primitive of h. Since $f \in SBV(U)$ we already know by Theorem 1 of [11] that (6) holds. Moreover,

$$\int_{\mathbb{R}} h(t)\mathrm{TaC}(\partial E_f(t), U)\,dt = \sum_{i=2}^m \int_{v_{i-1}}^{v_i} h(t)\mathrm{TaC}(\Gamma_i, U)\,dt = \sum_{i=2}^m [H(v_i) - H(v_{i-1})]\mathrm{TaC}(\Gamma_i, U).$$

Then, using the above formula for $\mathrm{TaC}(\Gamma_i, U)$, we get the sum of three terms. The first one is given by

$$\sum_{i=2}^m [H(v_i) - H(v_{i-1})] \int_{\mathcal{R}_f\cap\Gamma_i} |\kappa_f(x)|\mathcal{H}^1(dx) = \int_{\mathcal{R}_f} |\kappa_f(x)| \sum_{i=2}^m [H(v_i) - H(v_{i-1})]\mathbb{1}_{\Gamma_i}(x)\mathcal{H}^1(dx).$$

When $x \in \mathcal{S}_f \cap U = \cup_{i=2}^m \Gamma_i$, we denote $i(x)$, (resp. $j(x) \geq i(x)$), the minimal (resp. maximal) index $i = 2, \ldots, m$ such that $x \in \Gamma_i$ and $f^-(x) := v_{i(x)-1}$, (resp. $f^+(x) := v_{j(x)}$). It follows that

$$\begin{aligned}\sum_{i=2}^m [H(v_i) - H(v_{i-1})]\mathbb{1}_{\Gamma_i}(x) &= \sum_{i=i(x)}^{j(x)} [H(v_i) - H(v_{i-1})] \\ &= H(v_{j(x)}) - H(v_{i(x)-1}) \\ &= H(f^+(x)) - H(f^-(x)).\end{aligned}$$

Therefore the first term is

$$\int_{\mathcal{R}_f\cap U} [H(f^+(x)) - H(f^-(x))]\,|\kappa_f(x)|\,\mathcal{H}^1(dx).$$

Similarly, the second term is equal to

$$\sum_{i=2}^{m}[H(v_i)-H(v_{i-1})]\sum_{x\in\Gamma_i\cap\mathcal{C}_f}|\alpha_f(x)| = \sum_{x\in\mathcal{C}_f\cap U}[H(f^+(x))-H(f^-(x))]\,|\alpha_f(x)|.$$

Finally, the third one is equal to

$$\sum_{i=2}^{m}[H(v_i)-H(v_{i-1})]\sum_{x\in\Gamma_i\cap\mathcal{I}_f}|\beta_f(x)|\left(\mathbb{1}_{v_i>\max(f_-^+(x),f_-^-(x))}+\mathbb{1}_{v_i\leq\min(f_-^+(x),f_-^-(x))}\right)$$
$$=\sum_{x\in\mathcal{I}_f\cap U}\beta_f(x)\left(\sum_{i=l(x)+1}^{j(x)}[H(v_i)-H(v_{i-1})]+\sum_{i=i(x)}^{k(x)}[H(v_i)-H(v_{i-1})]\right),$$

where we have introduced $k(x)$ and $l(x)$ with $i(x)-1\leq k(x)\leq l(x)\leq j(x)$ such that $\min(f_-^+(x),f_-^-(x))=v_{k(x)}$ and $\max(f_-^+(x),f_-^-(x))=v_{l(x)}$, with the convention that $\sum_{i=l(x)+1}^{j(x)}=0$ if $l(x)=j(x)$ and $\sum_{i=i(x)}^{k(x)}=0$ if $k(x)=i(x)-1$. It follows that this third term is equal to

$$\sum_{x\in\mathcal{I}_f\cap U}\beta_f(x)\left(H(v_{j(x)})-H(v_{l(x)})+(H(v_{k(x)})-H(v_{i(x)-1}))\right)$$
$$=\sum_{x\in\mathcal{I}_f}\beta_f(x)\left(H(f^+(x))-H(\max(f_-^+(x),f_-^-(x)))+H(\min(f_-^+(x),f_-^-(x)))-H(f^-(x))\right).$$

In particular, for $h=1$, we obtain Formula (5). The same computations as above give the result (7) for $\mathrm{LTC}_f(h,U)$.

Indeed now, the third term is equal to

$$\sum_{i=2}^{m}[H(v_i)-H(v_{i-1})]\sum_{x\in\Gamma_i\cap\mathcal{I}_f}\beta_f(x)\left(\mathbb{1}_{v_i>\max(f_-^+(x),f_-^-(x))}-\mathbb{1}_{v_i\leq\min(f_-^+(x),f_-^-(x))}\right)$$
$$=\sum_{x\in\mathcal{I}_f}\beta_f(x)\left(\sum_{i=l(x)+1}^{j(x)}[H(v_i)-H(v_{i-1})]-\sum_{i=i(x)}^{k(x)}[H(v_i)-H(v_{i-1})]\right)$$
$$=\sum_{x\in\mathcal{I}_f}\beta_f(x)\left(H(f^+(x))+H(f^-(x))-H(f_-^+(x))-H(f_+^-(x))\right)$$

Taking again $h=1$ we obtain (8). □

The above proposition gives the formula for the level total curvature of an elementary function on U. Now, since we will be interested in shot-noise random fields that are obtained by summing elementary functions, we need to have also a formula for the level total curvature of a sum of elementary functions on U. This is the aim of the following proposition.

Proposition 2. *Let f,g be elementary functions on U such that their respective discontinuity sets $\mathcal{S}_f$ and $\mathcal{S}_g$ intersect only at a finite number of regular points in U, that is $\mathcal{S}_f\cap\mathcal{S}_g\cap U=\mathcal{R}_f\cap\mathcal{R}_g\cap U$ with $\mathcal{H}^0(\mathcal{S}_f\cap\mathcal{S}_g\cap U)<+\infty$. We moreover assume that the intersections are non-degenerate in U, meaning that if $x\in\mathcal{R}_f\cap\mathcal{R}_g\cap U$, then $\nu_f(x)$ and $\nu_g(x)$ are not colinear. Then $f+g$ is also an elementary function on U with*

- $\mathcal{R}_{f+g}\cap U=(\mathcal{R}_f\Delta\mathcal{R}_g)\cap U=(\mathcal{R}_f\cup\mathcal{R}_g\setminus(\mathcal{R}_f\cap\mathcal{R}_g))\cap U$ *and*
  - *if $x\in\mathcal{R}_{f+g}\cap\mathcal{R}_f\cap U$, then $\kappa_{f+g}(x)=\kappa_f(x)$ with $(f+g)^+(x)=f^+(x)+g(x)$ and $(f+g)^-(x)=f^-(x)+g(x)$;*
  - *if $x\in\mathcal{R}_{f+g}\cap\mathcal{R}_g\cap U$, then $\kappa_{f+g}(x)=\kappa_g(x)$ with $(f+g)^+(x)=f(x)+g^+(x)$ and $(f+g)^-(x)=f(x)+g^-(x)$;*
- $\mathcal{C}_{f+g}\cap U=(\mathcal{C}_f\cup\mathcal{C}_g)\cap U$ *is a disjoint union and*
  - *if $x\in\mathcal{C}_f\cap U$, then $\alpha_{f+g}(x)=\alpha_f(x)$ with $(f+g)^+(x)=f^+(x)+g(x)$ and $(f+g)^-(x)=f^-(x)+g(x)$,*
  - *if $x\in\mathcal{C}_g\cap U$, then $\alpha_{f+g}(x)=\alpha_g(x)$ with $(f+g)^+(x)=f(x)+g^+(x)$ and $(f+g)^-(x)=f(x)+g^-(x)$;*
- $\mathcal{I}_{f+g}\cap U=(\mathcal{I}_f\cup\mathcal{I}_g\cup(\mathcal{R}_f\cap\mathcal{R}_g))\cap U$ *is a disjoint union and*
  - *if $x\in\mathcal{I}_f\cap U$, then $\beta_{f+g}(x)=\beta_f(x)$ with $(f+g)^+(x)=f^+(x)+g(x)$, $(f+g)^-(x)=f^-(x)+g(x)$, $(f+g)_+^-(x)=f_+^-(x)+g(x)$ and $(f+g)_-^+(x)=f_-^+(x)+g(x)$*

- *if* $x \in \mathcal{I}_g \cap U$, *then* $\beta_{f+g}(x) = \beta_g(x)$ *with* $(f+g)^+(x) = f(x)+g^+(x)$, $(f+g)^-(x) = f(x)+g^-(x)$, $(f+g)^-_+(x) = f(x)+g^-_+(x)$ *and* $(f+g)^+_-(x) = f(x)+g^+_-(x)$
- *if* $x \in \mathcal{R}_f \cap \mathcal{R}_g \cap U$ *then* $\beta_{f+g}(x) = d_{S^1}(\nu_f(x), \nu_g(x)) \in (0,\pi)$ *with* $(f+g)^+(x) = f^+(x)+g^+(x)$, $(f+g)^-(x) = f^-(x)+g^-(x)$ *and* $\{(f+g)^+_-(x), (f+g)^-_+(x)\} = \{f^+(x)+g^-(x), f^-(x)+g^+(x)\}$.

Moreover, we have that

$$V_{f+g}(U) = V_f(U) + V_g(U) \text{ and } \mathrm{LTC}_{f+g}(U) = \mathrm{LTC}_f(U) + \mathrm{LTC}_g(U), \tag{9}$$

while

$$\begin{aligned} \mathrm{LTaC}_{f+g}(U) &= \mathrm{LTaC}_f(U) + \mathrm{LTaC}_g(U) \\ &\quad + \sum_{x \in \mathcal{R}_f \cap \mathcal{R}_g \cap U} \Big((f+g)^+(x) - \max((f+g)^+_-(x), (f+g)^-_+(x)) \\ &\qquad + \min((f+g)^+_-(x), (f+g)^-_+(x) - (f+g)^-(x)) \Big) \beta_{f+g}(x) \\ &\leq \mathrm{LTaC}_f(U) + \mathrm{LTaC}_g(U) + 2\pi \sum_{x \in \mathcal{R}_f \cap \mathcal{R}_g \cap U} [(f^+(x) - f^-(x)) + (g^+(x) - g^-(x))]. \end{aligned} \tag{10}$$

Proof. Since f and g are both piecewise constant functions in $SBV(U)$, then $f+g$ is also a piecewise constant function in $SBV(U)$, and its discontinuity set $\mathcal{S}_{f+g}$ satisfies $\mathcal{S}_{f+g} \cap U \subset (\mathcal{S}_f \cup \mathcal{S}_g) \cap U$. We now need to show that a point in $\mathcal{S}_{f+g} \cap U$ is else a regular point, a corner or an intersection point in the sense of Definition 3. If $x \in (\mathcal{S}_f \setminus \mathcal{S}_g) \cap U$, then we have three cases:
- if $x \in (\mathcal{R}_f \setminus \mathcal{S}_g) \cap U$, then $x \in \mathcal{R}_{f+g} \cap U$ with $\kappa_{f+g}(x) = \kappa_f(x)$ and $(f+g)^+(x) = f^+(x) + g(x)$ and $(f+g)^-(x) = f^-(x) + g(x)$.
- if $x \in (\mathcal{C}_f \setminus \mathcal{S}_g) \cap U$, then $x \in \mathcal{C}_{f+g} \cap U$ with $\alpha_{f+g}(x) = \alpha_f(x)$ and $(f+g)^+(x) = f^+(x) + g(x)$ and $(f+g)^-(x) = f^-(x) + g(x)$.
- if $x \in (\mathcal{I}_f \setminus \mathcal{S}_g) \cap U$, then $x \in \mathcal{I}_{f+g} \cap U$ with $\beta_{f+g}(x) = \beta_f(x)$ and $(f+g)^+(x) = f^+(x)+g(x)$, $(f+g)^-(x) = f^-(x) + g(x)$, $(f+g)^-_+(x) = f^-_+(x) + g(x)$ and $(f+g)^+_-(x) = f^+_-(x) + g(x)$.

The same symmetric formulas hold when $x \in (\mathcal{S}_g \setminus \mathcal{S}_f) \cap U$. Now, when $x \in \mathcal{S}_f \cap \mathcal{S}_g \cap U = \mathcal{R}_f \cap \mathcal{R}_g \cap U$, since we made the hypothesis that the intersection points are non-degenerate, then x becomes an intersection point of $f+g$, that is $x \in \mathcal{I}_{f+g}$ and moreover $\beta_{f+g}(x) = d_{S^1}(\nu_f(x), \nu_g(x))$, $(f+g)^+(x) = f^+(x) + g^+(x)$, $(f+g)^-(x) = f^-(x) + g^-(x)$, and $\{(f+g)^+_-(x), (f+g)^-_+(x)\} = \{f^+(x)+g^-(x), f^-(x)+g^+(x)\}$.

Finally, having identified the discontinuity set of $f+g$, since we assume that $\mathcal{H}^0(\mathcal{R}_f \cap \mathcal{R}_g \cap U) < +\infty$ we deduce that $(\mathcal{R}_{f+g} \cup \mathcal{C}_{f+g}) \cap U$ is a piecewise regular curve and

$$\begin{aligned} \int_{\mathcal{R}_{f+g} \cap U} |\kappa_{f+g}(x)| \mathcal{H}^1(dx) &= \int_{\mathcal{R}_f \cap U} |\kappa_f(x)| \mathcal{H}^1(dx) + \int_{\mathcal{R}_g \cap U} |\kappa_g(x)| \mathcal{H}^1(dx) < +\infty, \\ \mathcal{H}^0(\mathcal{C}_{f+g} \cap U) &= \mathcal{H}^0(\mathcal{C}_f \cap U) + \mathcal{H}^0(\mathcal{C}_g \cap U) < +\infty \\ \text{and} \quad \mathcal{H}^0(\mathcal{I}_{f+g} \cap U) &= \mathcal{H}^0(\mathcal{I}_f \cap U) + \mathcal{H}^0(\mathcal{I}_g \cap U) + \mathcal{H}^0(\mathcal{R}_f \cap \mathcal{R}_g \cap U) < +\infty. \end{aligned}$$

This finishes to prove that $f+g$ is an elementary function. We also notice in particular that when $x \in \mathcal{R}_f \cap \mathcal{R}_g \cap U$, one has $(f+g)^+(x) + (f+g)^-(x) = (f+g)^+_-(x) + (f+g)^-_+(x) = f^+(x) + g^+(x) + f^-(x) + g^-(x)$, such that applying the result of the previous proposition (Equation (8)) we can obtain the formula (9) for the level total curvature of $f+g$, while the result for the variation comes from the fact that $\mathcal{H}^1(\mathcal{R}_f \cap \mathcal{R}_g \cap U) = 0$. □

Remark: Formula (9) says that the total variation and the total curvature of a sum of two elementary functions with prescribed intersection of discontinuity sets are the sum of their total variation and their total curvature. This result is quite striking, but we have to underline that it does not hold in general. This will appear clearly in the next section where we will consider smooth functions f and where the formula for the level perimeter and the total curvature integral is obviously non linear in f.

3.2. **Level perimeter and total curvature of an elementary shot noise random field.** We consider here a shot-noise random field defined on $\mathbb{R}^2$ by

$$\forall x \in \mathbb{R}^2, \quad X_\Phi(x) = \sum_{i \in I} g_{m_i}(x - x_i),$$

where $\Phi = \{(x_i, m_i)\}_{i \in I}$ is a Poisson point process on $\mathbb{R}^2 \times \mathbb{R}^d$, defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, of intensity $\lambda \mathcal{L} \times F$, with $\lambda > 0$ real, $\mathcal{L}$ the Lebesgue measure on $\mathbb{R}^2$ and F a probability measure on $\mathbb{R}^d$. Note that equivalently, we may define Φ as an independently *marked Poisson point process* where $\{x_i\}_i$ is an homogeneous Poisson point process of intensity λ and the m_i are "marks", following a law $F(dm)$ on $\mathbb{R}^d$ (with

$d \geq 1$) and independent of the Poisson point process $\{x_i\}_i$. Let $g : \mathbb{R}^2 \times \mathbb{R}^d \to \mathbb{R}$ be a measurable function such that the functions $g_m := g(\cdot, m)$ satisfy

$$\int_{\mathbb{R}^2 \times \mathbb{R}^d} |g_m(x)|\, dx\, F(dm) < +\infty. \tag{11}$$

Then, the random field X_Φ is well defined as an almost surely locally integrable function on $\mathbb{R}^2$ (see [11]). Note that moreover, the random field X_Φ is stationary. We will first give formulas for the level perimeter and total curvature integrals of X_Φ on an open bounded set U in the case where the g_m are elementary functions on $\mathbb{R}^2$, then we will compute their expectation. Finally we will give explicit results in the case of weighted indicator functions of random sets, obtained from a deterministic compact elementary set and regular region D, by random rotation and dilation. Specific computations for disks with $D = D(0,1)$ and squares with $D = [0,1]^2$ are linked with some recent results on Boolean models.

Throughout the rest of this section we also assume that for F-almost every $m \in \mathbb{R}^d$, the g_m are elementary functions on $\mathbb{R}^2$, with compact support and such that

$$\int_{\mathbb{R}^d} V_{g_m}(\mathbb{R}^2) F(dm) < +\infty \quad \text{and} \quad \int_{\mathbb{R}^d} \mathrm{LTaC}_{g_m}(\mathbb{R}^2) F(dm) < +\infty, \tag{12}$$

where $V_{g_m}(\mathbb{R}^2)$ and $\mathrm{LTaC}_{g_m}(\mathbb{R}^2)$ are defined by (4) and (5) choosing $U = \mathbb{R}^2$ and write $\mathrm{LP}_{g_m}(\cdot, \mathbb{R}^2)$ and $\mathrm{LTC}_{g_m}(\cdot, \mathbb{R}^2)$, the level total curvature integral of g_m in the whole space $\mathbb{R}^2$. For F-almost every m, g_m is assumed to have a compact support, that can be included in a square $[-T_m, T_m]^2$ with $T_m \in \mathbb{R}_+$, and its maximal value $\|g_m\|_\infty = \max_{[-T_m,T_m]^2} |g_m|$ is finite. We will assume moreover that

$$\int_{\mathbb{R}^d} T_m^2\, F(dm) < +\infty \quad \text{and} \quad \int_{\mathbb{R}^d} \|g_m\|_\infty\, F(dm) < +\infty. \tag{13}$$

Note that the first assumption of (13) implies that there is only a finite random number of g_m, denoted by $N(U)$, contributing to the values of X_Φ on the bounded open set $U \subset (-T, T)^2$, for some $T > 0$. Actually, it is clear that

$$N(U) \leq \#\{i; \overline{U} \cap \left(x_i + [-T_{m_i}, T_{m_i}]^2\right) \neq \emptyset\} \leq \#\{i; \|x_i\|_\infty \leq (T_{m_i} + T)\},$$

with $\|x\|_\infty = \max(|x_1|, |x_2|)$, for $x \in \mathbb{R}^2$. It follows that $\mathbb{E}(N(U)) \leq \lambda \int_{\mathbb{R}^d} 4(T_m + T)^2 F(dm)$. Since F is a finite measure, under (13), we get

$$\mathbb{E}(N(U)) < +\infty. \tag{14}$$

In the following we will use the notation τ_x to denote the translation of x in $\mathbb{R}^2$ (i.e. $\tau_x y = y + x$ for all $y \in \mathbb{R}^2$). We will also denote $\Phi_i = \Phi \setminus \{(x_i, m_i)\}$ for $i \in I$, $\Phi_{ij} = \Phi \setminus \{(x_i, m_i), (x_j, m_j)\}$ for $i \neq j$ in I, and their associated shot noise random fields

$$\forall i, \quad X_{\Phi_i}(x) = \sum_{k; k \neq i} g_{m_k}(x - x_k) \quad \text{and} \quad \forall j \neq i, \quad X_{\Phi_{ij}}(x) = \sum_{k; k \neq i, k \neq j} g_{m_k}(x - x_k).$$

Theorem 2. *Assume that for F-almost every $m \in \mathbb{R}^d$, the function g_m is an elementary function on $\mathbb{R}^2$ (in the sense of Definition 3) satisfying* (11), (12) *and* (13), *and such that*

$$\int_{\mathbb{R}^d} \mathcal{H}^0\left(\mathcal{S}_{g_m} \setminus \mathcal{R}_{g_m}\right) F(dm) < +\infty \tag{15}$$

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} \int_{\mathbb{R}^2} \mathcal{H}^0\left(\mathcal{R}_{g_{m'}} \cap \tau_x \mathcal{R}_{g_m}\right) dx F(dm) F(dm') < +\infty \tag{16}$$

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} \int_{\mathbb{R}^2} \mathcal{H}^0\left(\left\{y \in \mathcal{R}_{g_{m'}} \cap \tau_x \mathcal{R}_{g_m}; \nu_{g_{m'}}(y) = \pm \nu_{g_m}(y - x)\right\}\right) dx F(dm) F(dm') = 0. \tag{17}$$

Then, almost surely, for all bounded open set $U \subset \mathbb{R}^2$, X_Φ is an elementary function on U and its discontinuity set on U is given by $\mathcal{S}_{X_\Phi} \cap U$ where $\mathcal{S}_{X_\Phi} = \mathcal{R}_{X_\Phi} \cup \mathcal{C}_{X_\Phi} \cup \mathcal{I}_{X_\Phi}$, with

- $\mathcal{R}_{X_\Phi} = \left(\bigcup_i \tau_{x_i} \mathcal{R}_{g_{m_i}}\right) \setminus \left(\bigcup_{i,j \neq} \tau_{x_i} \mathcal{R}_{g_{m_i}} \cap \tau_{x_j} \mathcal{R}_{g_{m_j}}\right)$, *and if $x \in \mathcal{R}_{X_\Phi} \cap U$, there exists a unique i such that $x \in \tau_{x_i} \mathcal{R}_{g_{m_i}}$ with $\kappa_{X_\Phi}(x) = \kappa_{g_{m_i}}(x - x_i)$ and*

$$X_\Phi{}^+(x) = g_{m_i}^+(x - x_i) + X_{\Phi_i}(x) \text{ and } X_\Phi{}^-(x) = g_{m_i}^-(x - x_i) + X_{\Phi_i}(x);$$

- $\mathcal{C}_{X_\Phi} = \bigcup_i \tau_{x_i}\mathcal{C}_{g_{m_i}}$, *and if* $x \in \mathcal{C}_{X_\Phi} \cap U$, *there exists a unique* i *such that* $x \in \tau_{x_i}\mathcal{C}_{g_{m_i}}$ *with* $\alpha_{X_\Phi}(x) = \alpha_{g_{m_i}}(x - x_i)$ *and*
$$X_\Phi{}^+(x) = g^+_{m_i}(x - x_i) + X_{\Phi_i}(x) \text{ and } X_\Phi{}^-(x) = g^-_{m_i}(x - x_i) + X_{\Phi_i}(x);$$
- $\mathcal{I}_{X_\Phi} = \left(\bigcup_i \tau_{x_i}\mathcal{I}_{g_{m_i}}\right) \cup \left(\bigcup_{i,j\neq} \tau_{x_i}\mathcal{R}_{g_{m_i}} \cap \tau_{x_j}\mathcal{R}_{g_{m_j}}\right)$ *and if* $x \in \mathcal{I}_{X_\Phi} \cap U$, *only two situations occur*
  - *there exists a unique* i *such that* $x \in \tau_{x_i}\mathcal{I}_{g_{m_i}}$, *with* $\beta_{X_\Phi}(x) = \beta_{g_{m_i}}(x - x_i)$ *and*
$$X_\Phi{}^+(x) = g^+_{m_i}(x - x_i) + X_{\Phi_i}(x), \quad X_\Phi{}^-(x) = g^-_{m_i}(x - x_i) + X_{\Phi_i}(x),$$
$$X_{\Phi^-_+}(x) = g_{m_i}{}^-_+(x - x_i) + X_{\Phi_i}(x), \quad X_{\Phi^+_-}(x) = g_{m_i}{}^+_-(x - x_i) + X_{\Phi_i}(x);$$
  - *or there exists a unique pair* $\{i, j\}$ *with* $i \neq j$ *such that* $x \in \tau_{x_i}\mathcal{R}_{g_{m_i}} \cap \tau_{x_j}\mathcal{R}_{g_{m_j}}$ *with*
$$\beta_{X_\Phi}(x) = d_{S^1}(\nu_{g_{m_i}}(x - x_i), \nu_{g_{m_j}}(x - x_j)) \in (0, \pi)$$

$$X_\Phi{}^+(x) = g^+_{m_i}(x - x_i) + g^+_{m_j}(x - x_j) + X_{\Phi_{ij}}(x), \quad X_\Phi{}^-(x) = g^-_{m_i}(x - x_i) + g^-_{m_j}(x - x_j) + X_{\Phi_{ij}}(x),$$
$$\{X_{\Phi^+_-}(x), X_{\Phi^-_+}(x)\} = \{g^+_{m_i}(x - x_i) + g^-_{m_j}(x - x_j) + X_{\Phi_{ij}}(x), g^-_{m_i}(x - x_i) + g^+_{m_j}(x - x_j) + X_{\Phi_{ij}}(x)\}.$$

In particular, a.s.
$$V_{X_\Phi}(U) = \sum_i V_{g_{m_i}}(\tau_{-x_i}U) \text{ and } \mathrm{LTC}_{X_\Phi}(U) = \sum_i \mathrm{LTC}_{g_{m_i}}(\tau_{-x_i}U),$$
with
$$V_{g_{m_i}}(\tau_{-x_i}U) = \int_{U\cap\tau_{x_i}\mathcal{R}_{g_{m_i}}} [g^+_{m_i}(x - x_i) - g^-_{m_i}(x - x_i)]\,\mathcal{H}^1(dx),$$
and
$$\begin{aligned}\mathrm{LTC}_{g_{m_i}}(\tau_{-x_i}U) &= \sum_i \int_{U\cap\tau_{x_i}\mathcal{R}_{g_{m_i}}} [g^+_{m_i}(x - x_i) - g^-_{m_i}(x - x_i)]\kappa_{g_{m_i}}(x - x_i)\,\mathcal{H}^1(dx)\\ &+ \sum_i \sum_{x\in U\cap\tau_{x_i}\mathcal{C}_{g_{m_i}}} [g^+_{m_i}(x - x_i) - g^-_{m_i}(x - x_i)]\alpha_{g_{m_i}}(x - x_i)\\ &+ \sum_i \sum_{x\in U\cap\tau_{x_i}\mathcal{I}_{g_{m_i}}} [g^+_{m_i}(x - x_i) + g^-_{m_i}(x - x_i) - g_{m_i}{}^+_-(x - x_i) - g_{m_i}{}^-_+(x - x_i)]\beta_{g_{m_i}}(x - x_i).\end{aligned}$$

Proof. Since it is sufficient to prove the result for all rectangles $U = (a_1, b_1) \times (a_2, b_2)$ with $a_1, a_2, b_1, b_2 \in \mathbb{Q}$, we only have to prove it holds almost surely on some fixed rectangle $U = (a_1, b_1) \times (a_2, b_2)$, with $a_1 < b_1$ and $a_2 < b_2$. Let us quote that we already have proven that $X_\Phi \in SBV(U)$ in Theorem 2 of [11] in a more general framework. However we need here to be more precise on its discontinuity set $\mathcal{S}_{X_\Phi} \subset \cup_i \tau_{x_i}\mathcal{S}_{g_{m_i}}$.

Let us first remark that when A_Φ is a finite set of points of $\mathbb{R}^2$ depending on the marked Poisson point process $\Phi = \{(x_i, m_i)\}$, as soon as $\mathbb{E}(\mathcal{H}^0(A_\Phi)) < +\infty$, one has,
$$\bigcup_j A_{\Phi_j} \cap \tau_{x_j}\mathcal{S}_{g_{m_j}} = \emptyset \quad \text{a.s..}$$
This follows from the fact that, by Slivnyak-Mecke formula (see [8] Theorem 1.4.5),
$$\begin{aligned}\mathbb{E}\left(\mathcal{H}^0\left(\bigcup_j A_{\Phi_j} \cap \tau_{x_j}\mathcal{S}_{g_{m_j}}\right)\right) &\leq \lambda \int_{\mathbb{R}^2\times\mathbb{R}^d} \mathbb{E}\left(\mathcal{H}^0(A_\Phi \cap \tau_x\mathcal{S}_{g_m})\right) dx F(dm)\\ &\leq \lambda\mathbb{E}\left(\mathcal{H}^0(A_\Phi)\right) \int_{\mathbb{R}^d} \mathcal{L}\left(\mathcal{S}_{g_m}\right) F(dm) = 0,\end{aligned}$$
since $\mathcal{L}\left(\mathcal{S}_{g_m}\right) = 0$, using Fubini Theorem and translation invariance of both $\mathcal{H}^0$ and $\mathcal{L}$.

Our first assumption (15) implies that
$$\bigcup_{i,j\neq} \tau_{x_i}\left(\mathcal{S}_{g_{m_i}} \setminus \mathcal{R}_{g_{m_i}}\right) \cap \tau_{x_j}\mathcal{S}_{g_{m_j}} \cap U = \emptyset \quad \text{a.s..} \tag{18}$$

Actually, taking $A_\Phi = \bigcup_i \tau_{x_i}\left(\mathcal{S}_{g_{m_i}} \setminus \mathcal{R}_{g_{m_i}}\right) \cap U$, Campbell formula (see [8] Theorem 1.4.3) ensures that

$$\begin{aligned}\mathbb{E}\left(\mathcal{H}^0\left(A_\Phi\right)\right) &\leq \lambda \int_{\mathbb{R}^2\times\mathbb{R}^d} \mathcal{H}^0\left(\tau_x\left(\mathcal{S}_{g_m} \setminus \mathcal{R}_{g_m}\right) \cap U\right) dx F(dm) \\ &\leq \lambda \mathcal{L}(U) \int_{\mathbb{R}^d} \mathcal{H}^0\left(\mathcal{S}_{g_m} \setminus \mathcal{R}_{g_m}\right) F(dm) < +\infty.\end{aligned}$$

Then, (18) follows from the preceding remark since $\bigcup_{i,j\neq} \tau_{x_i}\left(\mathcal{S}_{g_{m_i}} \setminus \mathcal{R}_{g_{m_i}}\right) \cap \tau_{x_j}\mathcal{S}_{g_{m_j}} \cap U = \bigcup_j A_{\Phi_j} \cap \tau_{x_j}\mathcal{S}_{g_{m_j}}$.

The second assumption (16) will ensure both that

$$\mathcal{H}^0\left(\bigcup_{i,j\neq} \tau_{x_i}\mathcal{R}_{g_{m_i}} \cap \tau_{x_j}\mathcal{R}_{g_{m_j}} \cap U\right) < \infty \quad \text{a.s.}, \tag{19}$$

and

$$\bigcup_{i,j,k\neq} \tau_{x_i}\mathcal{R}_{g_{m_i}} \cap \tau_{x_j}\mathcal{R}_{g_{m_j}} \cap \tau_{x_k}\mathcal{S}_{g_{m_k}} \cap U = \emptyset \quad \text{a.s..} \tag{20}$$

Here we set $A_\Phi = \bigcup_{i,j\neq} \tau_{x_i}\mathcal{R}_{g_{m_i}} \cap \tau_{x_j}\mathcal{R}_{g_{m_j}} \cap U$. Using again Slivnyak-Mecke formula and Campbell formula we obtain that

$$\begin{aligned}\mathbb{E}(\mathcal{H}^0(A_\Phi)) &\leq \lambda^2 \int_{\mathbb{R}^2\times\mathbb{R}^d}\int_{\mathbb{R}^2\times\mathbb{R}^d} \mathcal{H}^0\left(\tau_x\mathcal{R}_{g_m} \cap \tau_{x'}\mathcal{R}_{g_{m'}} \cap U\right) dx F(dm) dx' F(dm') \\ &\leq \lambda^2 \mathcal{L}(U) \int_{\mathbb{R}^d\times\mathbb{R}^d}\int_{\mathbb{R}^2} \mathcal{H}^0\left(\mathcal{R}_{g_{m'}} \cap \tau_x\mathcal{R}_{g_m}\right) dx F(dm) F(dm') < +\infty.\end{aligned}$$

It follows that $\mathcal{H}^0(A_\Phi) < +\infty$ a.s. and $\bigcup_k A_{\Phi_k} \cap \tau_{x_k}\mathcal{S}_{g_{m_k}} = \emptyset$ a.s. .

Finally and similarly, the last assumption ensures that

$$\{y \in \bigcup_{i,j\neq} \tau_{x_i}\mathcal{R}_{g_{m_i}} \cap \tau_{x_j}\mathcal{R}_{g_{m_j}} \cap U; \nu_{g_{m_i}}(y - x_i) = \pm\nu_{g_{m_j}}(y - x_j)\} = \emptyset \quad \text{a.s..} \tag{21}$$

This follows from the fact that the expected $\mathcal{H}^0$ measure of this set is zero, according to Slivnyak-Mecke formula and Campbell formula, Fubini Theorem and translation invariance.

Now let us consider the random variable $N(U)$ counting the number of functions g_{m_i} contributing to the values of X_Φ on U and recall that under the assumption that $\int_{\mathbb{R}^d} T_m^2 F(dm) < +\infty$, $N(U)$ is a.s. finite, as a consequence of (14).

We now will prove the result of Theorem 2 by induction on the value of $N(U)$ once we have fixed an almost sure realization ensuring the previous configuration.
For $N(U) = 0$ there is nothing to prove since $X_\Phi = 0$ on U in this case. Let us assume the result holds when $N(U) = n \geq 0$ and let us prove it for $N(U) = n + 1$. We can assume that there exists (x_i, m_i) such that $\tau_{-x_i} g_{m_i}$ contributes to the values of X on U and write $X_\Phi = X_{\Phi_i} + \tau_{-x_i} g_{m_i}$. The number of functions contributing to X_{Φ_i} is given by $N(U) - 1$ so we can use our induction to state that X_{Φ_i} is an elementary function on U with discontinuity set $\mathcal{S}_{X_{\Phi_i}} \cap U$ where

$$\mathcal{S}_{X_{\Phi_i}} = \bigcup_{j;j\neq i} \tau_{x_j}\mathcal{S}_{g_{m_j}}, \quad \text{with} \quad \mathcal{R}_{X_{\Phi_i}} = \left(\bigcup_{j;j\neq i} \tau_{x_j}\mathcal{R}_{g_{m_j}}\right) \setminus \left(\bigcup_{j,k\neq;j\neq i,k\neq i} \tau_{x_j}\mathcal{R}_{g_{m_j}} \cap \tau_{x_k}\mathcal{R}_{g_{m_k}}\right),$$

$$\mathcal{C}_{X_{\Phi_i}} = \bigcup_{j;j\neq i} \tau_{x_j}\mathcal{C}_{g_{m_j}} \quad \text{and} \quad \mathcal{I}_{X_{\Phi_i}} = \left(\bigcup_{j;j\neq i} \tau_{x_j}\mathcal{I}_{g_{m_j}}\right) \bigcup \left(\bigcup_{j,k\neq;j\neq i,k\neq i} \tau_{x_j}\mathcal{R}_{g_{m_j}} \cap \tau_{x_k}\mathcal{R}_{g_{m_k}}\right).$$

Then the discontinuity points are given by $\mathcal{S}_{X_\Phi} \cap U$ with $\mathcal{S}_{X_\Phi} \subset \mathcal{S}_{X_{\Phi_i}} \cup \tau_{x_i}\mathcal{S}_{g_{m_i}} = \bigcup_j \tau_{x_j}\mathcal{S}_{g_{m_j}}$ with

$$\mathcal{S}_{X_{\Phi_i}} \cap \tau_{x_i}\left(\mathcal{S}_{g_{m_i}} \setminus \mathcal{R}_{g_{m_i}}\right) \cap U \subset \bigcup_{j;j\neq i} \tau_{x_j}\mathcal{S}_{g_{m_j}} \cap \tau_{x_i}\left(\mathcal{S}_{g_{m_i}} \setminus \mathcal{R}_{g_{m_i}}\right) \cap U = \emptyset$$

by (18). Moreover,

$$\mathcal{S}_{X_{\Phi_i}} \setminus \mathcal{R}_{X_{\Phi_i}} \subset \bigcup_{j;j\neq i} \tau_{x_j}\left(\mathcal{S}_{g_{m_j}} \setminus \mathcal{R}_{g_{m_j}}\right) \cup \bigcup_{j,k\neq;j\neq i,k\neq i} \tau_{x_j}\mathcal{R}_{g_{m_j}} \cap \tau_{x_k}\mathcal{R}_{g_{m_k}},$$

with again $\bigcup_{j;j\neq i}\tau_{x_j}\left(\mathcal{S}_{g_{m_j}}\setminus\mathcal{R}_{g_{m_j}}\right)\cap\tau_{x_i}\mathcal{S}_{g_{m_i}}\cap U=\emptyset$ by (18) and $\bigcup_{j,k\neq;j\neq i,k\neq i}\tau_{x_j}\mathcal{R}_{g_{m_j}}\cap\tau_{x_k}\mathcal{R}_{g_{m_k}}\cap\tau_{x_i}\mathcal{S}_{g_{m_i}}\cap U=\emptyset$ by (20). Therefore, we may conclude that

$$\mathcal{S}_{X_{\Phi_i}}\cap\tau_{x_i}\mathcal{S}_{g_{m_i}}\cap U\subset\mathcal{R}_{X_{\Phi_i}}\cap\tau_{x_i}\mathcal{R}_{g_{m_i}}\cap U,$$

with $\mathcal{H}^0\left(\mathcal{R}_{X_{\Phi_i}}\cap\tau_{x_i}\mathcal{R}_{g_{m_i}}\cap U\right)<+\infty$ by (19). Finally let us quote that the intersections are non-degenerate in view of (21). Hence, according to Proposition 2, X_Φ is an elementary function on U. Moreover, let fix a point $x\in U$. If $x\in\mathcal{R}_{X_{\Phi_i}}\setminus\tau_{x_i}\mathcal{S}_{g_{m_i}}$ we get $X_\Phi^\pm(x)=X_{\Phi_i}^\pm(x)+g_{m_i}(x-x_i)$ and by induction there exists a unique $j\neq i$ such that $x\in\tau_{x_j}\mathcal{R}_{g_{m_j}}$ so that $X_\Phi^\pm(x)=X_{\Phi_{ij}}(x)+g_{m_j}^\pm(x-x_j)+g_{m_i}(x-x_i)=X_{\Phi_j}(x)+g_{m_j}^\pm(x-x_j)$ and $\kappa_{X_\Phi}(x)=\kappa_{g_{m_j}}(x-x_j)$. Similarly, if $x\in\tau_{x_i}\mathcal{R}_{g_{m_i}}\setminus\mathcal{S}_{X_{\Phi_i}}$ we get ${X_\Phi}^\pm(x)={X_{\Phi_i}}(x)+g_{m_i}^\pm(x-x_i)$ and $\kappa_{X_\Phi}(x)=\kappa_{g_{m_i}}(x-x_i)$. In the same way, by induction and using the fact that $\mathcal{C}_{X_\Phi}$ is the disjoint union of $\tau_{x_j}\mathcal{C}_{g_{m_j}}$ we obtain that if $x\in\mathcal{C}_{X_\Phi}$, there exists a unique i such that $x\in\tau_{x_i}\mathcal{C}_{g_{m_i}}$ and $\alpha_{X_\Phi}(x)=\alpha_{g_{m_i}}(x-x_i)$ with ${X_\Phi}^\pm(x)=g_{m_i}^\pm(x-x_i)+X_{\Phi_i}(x)$. Finally $\mathcal{I}_{X_\Phi}$ is the disjoint union of $\mathcal{I}_{X_{\Phi_i}}$, $\tau_{x_i}\mathcal{I}_{g_{m_i}}$ and $\mathcal{R}_{X_{\Phi_i}}\cap\tau_{x_i}\mathcal{R}_{g_{m_i}}$. By induction, $\mathcal{I}_{X_{\Phi_i}}=\bigcup_{j;j\neq i}\tau_{x_j}\mathcal{I}_{g_{m_j}}\cup\bigcup_{j,k\neq;j\neq i,k\neq i}\tau_{x_j}\mathcal{R}_{g_{m_j}}\cap\tau_{x_k}\mathcal{R}_{g_{m_k}}$ and $\mathcal{R}_{X_i}\cap\tau_{x_i}\mathcal{R}_{g_{m_i}}=\bigcup_{j;j\neq i}\tau_{x_j}\mathcal{R}_{g_{m_j}}\cap\tau_{x_i}\mathcal{R}_{g_{m_i}}$ where unions are all disjoint. Hence, grouping the terms we get the result. □

Theorem 3. *Under the hypothesis of Theorem 2, assuming moreover that*

$$\int_{\mathbb{R}^d\times\mathbb{R}^d}\int_{\mathbb{R}^2}(\|g_m\|_\infty+\|g_{m'}\|_\infty)\mathcal{H}^0\left(\mathcal{R}_{g_{m'}}\cap\tau_x\mathcal{R}_{g_m}\right)dxF(dm)F(dm')<+\infty, \tag{22}$$

then the random variables $V_{X_\Phi}(U)$, $\mathrm{LTaC}_{X_\Phi}(U)$ and $\mathrm{LTC}_{X_\Phi}(U)$ have finite expectation for any bounded open set U. Moreover one has

$$\mathbb{E}(V_{X_\Phi}(U))=\lambda\mathcal{L}(U)\int_{\mathbb{R}^d}V_{g_m}(\mathbb{R}^2)F(dm)\ \text{ and }\ \mathbb{E}(\mathrm{LTC}_{X_\Phi}(U))=\lambda\mathcal{L}(U)\int_{\mathbb{R}^d}\mathrm{LTC}_{g_m}(\mathbb{R}^2)F(dm),$$

with

$$\begin{aligned}
V_{g_m}(\mathbb{R}^2) &= \int_{\mathcal{R}_{g_m}}[{g_m}^+(z)-{g_m}^-(z)]\mathcal{H}^1(dz)\\
\mathrm{LTC}_{g_m}(\mathbb{R}^2) &= \int_{\mathcal{R}_{g_m}}[{g_m}^+(z)-{g_m}^-(z)]\kappa_{g_m}(z)\mathcal{H}^1(dz)+\sum_{z\in\mathcal{C}_{g_m}}[{g_m}^+(z)-{g_m}^-(z)]\alpha_{g_m}(z)\\
&\quad+\sum_{z\in\mathcal{I}_{g_m}}[{g_m}^+(z)+{g_m}^-(z)-{g_m}_-^+(z)-{g_m}_+^-(z)]\beta_{g_m}(z).
\end{aligned}$$

It follows that for a.e. $t\in\mathbb{R}$, the random variables $\mathrm{Per}(E_{X_\Phi}(t),U)$ and $\mathrm{TC}(\partial E_{X_\Phi}(t),U)$ have also finite expectation such that for any h continuous bounded function

$$\mathbb{E}(\mathrm{LP}_{X_\Phi}(h,U))=\int_{\mathbb{R}}h(t)\mathbb{E}\left(\mathrm{Per}(E_{X_\Phi}(t),U)\right)dt\ \text{ and }\ \mathbb{E}(\mathrm{LTC}_{X_\Phi}(h,U))=\int_{\mathbb{R}}h(t)\mathbb{E}\left(\mathrm{TC}(\partial E_{X_\Phi}(t),U)\right)dt,$$

where

$$\begin{aligned}
\mathbb{E}(\mathrm{LP}_{X_\Phi}(h,U)) &= \lambda\mathcal{L}(U)\int_{\mathbb{R}^d}\int_{\mathcal{R}_{g_m}}\int_{g_m^-(x)}^{g_m^+(x)}\mathbb{E}(h(X_\Phi(0)+s))ds\mathcal{H}^1(dx)F(dm)\\
\mathbb{E}(\mathrm{LTC}_{X_\Phi}(h,U)) &= \lambda\mathcal{L}(U)\int_{\mathbb{R}^d}\left(R(h,m)+C(h,m)+I(h,m)\right)F(dm),
\end{aligned}$$

where

$$R(h,m)=\int_{\mathcal{R}_{g_m}}\int_{g_m^-(x)}^{g_m^+(x)}\mathbb{E}(h(X_\Phi(0)+s))ds\,\kappa_{g_m}(x)\mathcal{H}^1(dx)\,,\quad C(h,m)=\sum_{x\in\mathcal{C}_{g_m}}\int_{g_m^-(x)}^{g_m^+(x)}\mathbb{E}(h(X_\Phi(0)+s))ds\,\alpha_{g_m}(x),$$

and $I(h,m)=I^{(1)}(h,m)+I^{(2)}(h,m)$, with

$$I^{(1)}(h,m)=\sum_{x\in\mathcal{I}_{g_m}}\left(\int_{{g_m}_-^+(x)}^{g_m^+(x)}\mathbb{E}(h(X_\Phi(0)+s))ds-\int_{g_m^-(x)}^{{g_m}_+^-(x)}\mathbb{E}(h(X_\Phi(0)+s))ds\right)\beta_{g_m}(x),$$

while $I^{(2)}(h,m)$ is equal to

$$\frac{\lambda}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^2}\sum_{z\in\tau_x\mathcal{R}_{g_m}\cap\mathcal{R}_{g_{m'}}}d_{S^1}(\nu_{g_m}(z-x),\nu_{g_{m'}}(z))$$

$$\times \int_{g_m^-(z-x)}^{g_m^+(z-x)} \mathbb{E}\left(h(X_\Phi(0)+s+g_{m'}^+(z)) - h(X_\Phi(0)+s+g_{m'}^-(z))\right) dsdxF(dm').$$

Proof. First note that the results on the perimeter follow from Theorem 2 and Proposition 1 of [11] since in view of (11) and (12) we have $\int_{\mathbb{R}^d} \|g_m\|_{BV(\mathbb{R}^d)} F(dm) < +\infty$. Moreover, according to (10), one has a.s.

$$\mathrm{LTaC}_{X_\Phi}(U) \le \sum_i \mathrm{LTaC}_{g_{m_i}}(\tau_{-x_i}U) + 2\pi \sum_{i,j\neq} \mathcal{H}^0\left(\tau_{x_i}\mathcal{R}_{g_{m_i}} \cap \tau_{x_j}\mathcal{R}_{g_{m_j}} \cap U\right)(\|g_{m_i}\|_\infty + \|g_{m_j}\|_\infty).$$

By Campbell formula,

$$\mathbb{E}\left(\sum_i \mathrm{LTaC}_{g_{m_i}}(\tau_{-x_i}U)\right) = \int_{\mathbb{R}^2\times\mathbb{R}^d} \mathrm{LTaC}_{g_m}(\tau_{-x}U)\lambda dxF(dm).$$

Hence, by Fubini Theorem,

$$\mathbb{E}\left(\sum_i \mathrm{LTaC}_{g_{m_i}}(\tau_{-x_i}U)\right) = \lambda\mathcal{L}(U)\int_{\mathbb{R}^d} \mathrm{LTaC}_{g_m}(\mathbb{R}^2)F(dm) < +\infty.$$

Moreover, by Slivnyak-Mecke formula,

$$\begin{aligned}
&\mathbb{E}\left(\sum_{i,j\neq} \mathcal{H}^0\left(\tau_{x_i}\mathcal{R}_{g_{m_i}} \cap \tau_{x_j}\mathcal{R}_{g_{m_j}} \cap U\right)(\|g_{m_i}\|_\infty + \|g_{m_j}\|_\infty)\right)\\
&= \int_{\mathbb{R}^d\times\mathbb{R}^d}\int_{\mathbb{R}^2\times\mathbb{R}^2} \mathcal{H}^0\left(\tau_x\mathcal{R}_{g_m} \cap \tau_{x'}\mathcal{R}_{g_{m'}} \cap U\right)(\|g_m\|_\infty + \|g_{m'}\|_\infty)\lambda^2 dxdx'F(dm)F(dm')\\
&= \lambda^2\mathcal{L}(U)\int_{\mathbb{R}^d\times\mathbb{R}^d}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \mathbf{I}_{z\in\mathcal{R}_{g_{m'}}\cap\tau_x\mathcal{R}_{g_m}}(\|g_m\|_\infty + \|g_{m'}\|_\infty)\mathcal{H}^0(dz)dxF(dm)F(dm')\\
&= \lambda^2\mathcal{L}(U)\int_{\mathbb{R}^d\times\mathbb{R}^d}\int_{\mathbb{R}^2} (\|g_m\|_\infty + \|g_{m'}\|_\infty)\mathcal{H}^0\left(\mathcal{R}_{g_{m'}} \cap \tau_x\mathcal{R}_{g_m}\right)dxF(dm)F(dm') < +\infty,
\end{aligned}$$

by assumption (22). It follows that $\mathbb{E}(\mathrm{LTaC}_{X_\Phi}(U)) < +\infty$ and therefore $\mathbb{E}\left(\int_{\mathbb{R}} |\mathrm{TC}(\partial E_{X_\Phi}(t), U)|dt\right) < +\infty$. So for a.e. $t\in\mathbb{R}$, the random variable $\mathrm{TC}(\partial E_{X_\Phi}(t), U)$ has a finite expectation and $t \mapsto \mathbb{E}\left(\mathrm{TC}(\partial E_{X_\Phi}(t), U)\right)$, is a function in $L^1(\mathbb{R})$. Note that since $\mathrm{LTC}_{X_\Phi}(U) = \sum_i \mathrm{LTC}_{g_{m_i}}(\tau_{-x_i}U)$ we simply get by Campbell formula and Fubini Theorem that

$$\mathbb{E}(\mathrm{LTC}_{X_\Phi}(U)) = \lambda\mathcal{L}(U)\int_{\mathbb{R}^d} \mathrm{LTC}_{g_m}(\mathbb{R}^2)F(dm).$$

Now, let h be a continuous bounded function with primitive denoted by H. We already know that a.s. $\mathrm{LTC}_{X_\Phi}(h, U) = \int_{\mathbb{R}} h(t)\mathrm{TC}(\partial E_{X_\Phi}(t), U)dt$ may be written as the sum of three terms $R_h + C_h + I_h$ with finite expectation under our assumptions. By Fubini Theorem, it follows that

$$\mathbb{E}(\mathrm{LTC}_{X_\Phi}(h, U)) = \int_{\mathbb{R}} h(t)\mathbb{E}(\mathrm{TC}(\partial E_{X_\Phi}(t), U))dt = \mathbb{E}(R_h) + \mathbb{E}(C_h) + \mathbb{E}(I_h).$$

For the first term we get

$$\begin{aligned}
\mathbb{E}(R_h) &= \mathbb{E}\left(\int_{\mathcal{R}_{X_\Phi}\cap U} [H(X_\Phi^+(z)) - H(X_\Phi^-(x))]\kappa_{X_\Phi}(z)\mathcal{H}^1(dz)\right)\\
&= \mathbb{E}\left(\sum_i \int_{\tau_{x_i}\mathcal{R}_{g_{m_i}}\cap U} [H(X_i(z) + g_{m_i}^+(z-x_i)) - H(X_i(z) + g_{m_i}^-(z-x_i))]\kappa_{g_{m_i}}(z-x_i)\mathcal{H}^1(dz)\right)\\
&= \int_{\mathbb{R}^2\times\mathbb{R}^d}\int_{\tau_x\mathcal{R}_{g_m}\cap U} \mathbb{E}\left([H(X_\Phi(z) + g_m^+(z-x)) - H(X_\Phi(z) + g_m^-(z-x))]\right)\kappa_{g_m}(z-x)\mathcal{H}^1(dz)\lambda dxF(dm),
\end{aligned}$$

by Slivnyak-Mecke formula. Using translation invariance of both $\mathcal{H}^1$ and $\mathcal{L}$ and stationarity of X_Φ we get that

$$\mathbb{E}(R_h) = \lambda\mathcal{L}(U)\int_{\mathbb{R}^d}\int_{\mathcal{R}_{g_m}}\int_{g_m^-(z)}^{g_m^+(z)} \mathbb{E}(h(X_\Phi(0)+s))\,\kappa_{g_m}(z)ds\mathcal{H}^1(dz)F(dm) = \lambda\mathcal{L}(U)\int_{\mathbb{R}^d} R(h,m)F(dm). \quad (23)$$

Similarly, for the second term we get

$$\begin{aligned}
\mathbb{E}(C_h) &= \mathbb{E}\left(\sum_{z\in\mathcal{C}_{X_\Phi}\cap U}[H(X_\Phi^+(z))-H(X_\Phi^-(z))]\alpha_{X_\Phi}(z)\right)\\
&= \mathbb{E}\left(\sum_i \mathbb{1}_{z\in\tau_{x_i}\mathcal{C}_{g_{m_i}}\cap U}[H(X_{\Phi_i}(z)+g_{m_i}^+(z-x_i))-H(X_{\Phi_i}(z)+g_{m_i}^-(z-x_i))]\alpha_{g_{m_i}}(z-x_i)\mathcal{H}^1(dz)\right)\\
&= \lambda\mathcal{L}(U)\int_{\mathbb{R}^d}\sum_{z\in\mathcal{C}_{g_m}}\mathbb{E}\left([H(X_\Phi(0)+g_m^+(z))-H(X_\Phi(0)+g_m^-(z))]\right)\alpha_{g_m}(z)F(dm) \quad (24)\\
&= \lambda\mathcal{L}(U)\int_{\mathbb{R}^d}C(h,m)F(dm), \quad (25)
\end{aligned}$$

with $C(h,m)=\sum_{z\in\mathcal{C}_{g_m}}\int_{g_m^-(z)}^{g_m^+(z)}\mathbb{E}\left(h(X_\Phi(0)+s)\right)ds\,\alpha_{g_m}(z)$. Finally, the last term may be itself decomposed in two terms, say $I_h=I_h^{(1)}+I_h^{(2)}$. With similar computations we get $\mathbb{E}(I_h^{(1)})$ equals to

$$\begin{aligned}
&\lambda\mathcal{L}(U)\int_{\mathbb{R}^d}\sum_{z\in\mathcal{I}_{g_m}}\left(\int_{g_{m_-}^+(z)}^{g_m^+(z)}\mathbb{E}\left(h(X_\Phi(0)+s)\right)ds-\int_{g_m^-(z)}^{g_{m_+}^-(z)}\mathbb{E}\left(h(X_\Phi(0)+s)\right)ds\right)\beta_{g_m}(z)F(dm)\\
&=\lambda\mathcal{L}(U)\int_{\mathbb{R}^d}I^{(1)}(h,m)F(dm). \quad (26)
\end{aligned}$$

Since $\tau_{x_i}\mathcal{R}_{g_{m_i}}\cap\tau_{x_j}\mathcal{R}_{g_{m_j}}=\tau_{x_j}\mathcal{R}_{g_{m_j}}\cap\tau_{x_i}\mathcal{R}_{g_{m_i}}$, writing for $z\in\tau_{x_i}\mathcal{R}_{g_{m_i}}\cap\tau_{x_j}\mathcal{R}_{g_{m_j}}$

$$\Delta_{m_i,m_j}h(t,z-x_i,z-x_j) = \int_{g_{m_j}^-(z-x_j)}^{g_{m_j}^+(z-x_j)}[h(t+g_{m_i}^+(z-x_i)+s)-h(t+g_{m_i}^-(z-x_i)+s)]ds,$$

the term $\mathbb{E}(I_h^{(2)})$ is equal to

$$\begin{aligned}
&\mathbb{E}\left(\frac{1}{2}\sum_{i\neq j}\sum_{z\in\tau_{x_i}\mathcal{R}_{g_{m_i}}\cap\tau_{x_j}\mathcal{R}_{g_{m_j}}\cap U}\Delta_{m_i,m_j}h(X_{\Phi_{ij}}(z),z-x_i,z-x_j)d_{S^1}(\nu_{g_{m_i}}(z-x_i),\nu_{g_{m_j}}(z-x_j))\right)\\
&=\frac{1}{2}\iint\sum_{z\in\tau_x\mathcal{R}_{g_m}\cap\tau_{x'}\mathcal{R}_{g_{m'}}\cap U}\mathbb{E}\left(\Delta_{m,m'}h(X_\Phi(z),z-x,z-x')\right)d_{S^1}(\nu_{g_m}(z-x),\nu_{g_{m'}}(z-x'))\lambda^2dxF(dm)dx'F(dm'),
\end{aligned}$$

by Slivnyak-Mecke formula. By change of variables, translation invariance of $\mathcal{H}^0$ and $\mathcal{L}$, and stationarity of X_Φ, we get

$$\mathbb{E}(I_h^{(2)})=\frac{\lambda^2}{2}\mathcal{L}(U)\int_{\mathbb{R}^d\times\mathbb{R}^d}\int_{\mathbb{R}^2}\sum_{z\in\tau_x\mathcal{R}_{gm}\cap\mathcal{R}_{g_{m'}}}\mathbb{E}\left(\Delta_{m,m'}h(X_\Phi(0),z-x,z)\right)d_{S^1}(\nu_{g_m}(z-x),\nu_{g_{m'}}(z))dxF(dm)F(dm').$$

Now, let us quote that choosing $h_u(s)=e^{ius}$ for $u,s\in\mathbb{R}$ we can remark that

$$\mathbb{E}(\mathrm{LP}_{X_\Phi}(h_u,\mathbb{R}^2))=\lambda\mathcal{L}(U)\mathbb{E}\left(e^{iuX_\Phi(0)}\right)\int_{\mathbb{R}^d}\mathrm{LP}_{g_m}(h_u,\mathbb{R}^2)F(dm), \quad (27)$$

while (23)+(25)+(26) is equal to

$$\lambda\mathcal{L}(U)\mathbb{E}\left(e^{iuX_\Phi(0)}\right)\int_{\mathbb{R}^d}\mathrm{LTC}_{g_m}(h_u,\mathbb{R}^2)F(dm). \quad (28)$$

Therefore

$$\begin{aligned}
(29)\mathbb{E}(\mathrm{LTC}_{X_\Phi}(h_u,U)) &= \int_{\mathbb{R}}e^{iut}\mathbb{E}(\mathrm{TC}(\partial E_{X_\Phi}(t),U))dt=\lambda\mathcal{L}(U)\mathbb{E}\left(e^{iuX_\Phi(0)}\right)\int_{\mathbb{R}^d}\big[\mathrm{LTC}_{g_m}(h_u,\mathbb{R}^2)+\\
&\frac{\lambda}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^2}\sum_{z\in\tau_x\mathcal{R}_{gm}\cap\mathcal{R}_{g_{m'}}}(e^{iug_{m'}^+(z)}-e^{iug_{m'}{}^-(z)})\int_{g_m^-(z-x)}^{g_m^+(z-x)}e^{ius}d_{S^1}(\nu_{g_m}(z-x),\nu_{g_{m'}}(z))\,dsdxF(dm')\big]\,F(dm)
\end{aligned}$$

□

3.3. **Explicit computations.** In this section, we will give some explicit computations of the mean level total curvature function of elementary shot noise random fields. These results generalize the results of Decreusefond et al. [14] obtained for indicator functions of a square, and also the known results on the Boolean model (that correspond to the excursion set of level $t = 1$). We will also show some numerical simulations.

Let us first recall that for shot noise random fields, the characteristic function of $X_\Phi(0)$ is explicit and given by (see for instance [10])

$$\mathbb{E}(e^{iuX_\Phi(0)}) = \exp\left(\lambda \int_{\mathbb{R}^2\times\mathbb{R}^d} (e^{iug_m(x)} - 1)dxF(dm)\right).$$

In this section we consider D an elementary compact subset of $\mathbb{R}^2$ and regular region so that it has a piecewise smooth boundary given by $\Gamma = \partial D$ a finite union of positively oriented closed simple curves, piecewise C^2 and of finite total curvature, i.e. $\mathrm{TaC}(\Gamma, \mathbb{R}^2) < +\infty$. Note that by Gauss-Bonnet Theorem we have

$$\mathrm{TC}(\Gamma, \mathbb{R}^2) = 2\pi\chi(D).$$

We will focus on the case where the marks are of the form $m = (b, r, \theta) \in \mathbb{R} \times [0, +\infty) \times [0, 2\pi] \subset \mathbb{R}^d$ with $d = 3$, with distribution $F(dm) = F_B(db)F_R(dr)F_\Theta(d\theta)$ and functions g_m given by

$$\forall x \in \mathbb{R}^2, \quad g_m(x) = b\mathbb{1}_{R_\theta rD}(x),$$

where rD is the dilation of D by the factor r, and R_θ denotes the rotation of angle θ. We will denote $b^+ = \max(b, 0)$, $b^- = \min(0, b)$ such that for $x \in \partial R_\theta rD$, we have $g_m^+(x) = b^+$ and $g_{m^-}(x) = b^-$ with $g_m^+(x) - g_{m^-}(x) = b^+ - b^- = |b|$ and $b = (b^+ - b^-)\mathrm{sgn}(b)$.

We denote by B (resp. $B^+ = \max(B, 0)$ and $B^- = \min(B, 0)$), R and Θ independent random variables with distributions F_B, F_R and F_Θ. We will mainly focus on the case where Θ is uniform on $[0, 2\pi]$, that is $F_\Theta(d\theta) = \frac{1}{2\pi}\mathbb{1}_{[0,2\pi]}d\theta$ for random shapes with uniform rotation; or on the case where $\Theta = 0$ a.s., that is $F_\Theta = \delta_0$ corresponding to simpler marks $m = (b, r)$.

Theorem 4. *We assume that*

$$\mathbb{E}(|B|) < +\infty \quad \text{and} \quad \mathbb{E}(R^2) < +\infty.$$

We denote the mean perimeter and the mean area of $R_\theta rD$ by

$$\overline{p} := \int_{\mathbb{R}^2} \mathrm{Per}(R_\theta rD)\, F_R(dr)F_\Theta(d\theta) = \mathrm{Per}(D)\mathbb{E}(R) \quad \text{and} \quad \overline{a} := \int_{\mathbb{R}^2} \mathcal{L}(R_\theta rD)\, F_R(dr)F_\Theta(d\theta) = \mathcal{L}(D)\mathbb{E}(R^2).$$

Then X_Φ satisfies the assumptions of Theorem 2 and 3 and

$$\mathbb{E}(\mathrm{LP}_{X_\Phi}(U)) = \lambda\mathcal{L}(U)\mathbb{E}(|B|)\overline{p} \text{ and } \mathbb{E}(\mathrm{LTC}_{X_\Phi}(U)) = 2\pi\lambda\mathcal{L}(U)\mathbb{E}(B)\chi(D).$$

Moreover, when F_Θ is the uniform law on $[0, 2\pi]$, then, for all $u \neq 0$,

$$\begin{aligned}
\int_{\mathbb{R}} e^{iut}\mathbb{E}(\mathrm{Per}(E_{X_\Phi}(t), U))dt &= \lambda\mathcal{L}(U)\mathbb{E}(e^{iuX_\Phi(0)})\frac{\widehat{F_{B^+}}(u) - \widehat{F_{B^-}}(u)}{iu}\overline{p} \\
\int_{\mathbb{R}} e^{iut}\mathbb{E}(\mathrm{TC}(\partial E_{X_\Phi}(t), U))dt &= \lambda\mathcal{L}(U)\mathbb{E}(e^{iuX_\Phi(0)})\frac{1}{iu}\left(2\pi(\widehat{F_B}(u) - 1)\chi(D) + \frac{\lambda}{2}(\widehat{F_{B^+}}(u) - \widehat{F_{B^-}}(u))^2\overline{p}^2\right),
\end{aligned}$$

where $\widehat{F_B}$, respectively $\widehat{F_{B^+}}$ or $\widehat{F_{B^-}}$, denotes the characteristic function of B, respectively B^+ or B^-. In the case where $B = 1 (= B^+)$ a.s. we deduce that $\forall k \in \mathbb{N}$, $\forall t \in (k, k + 1]$,

(30)

$$\mathbb{E}(\mathrm{Per}(E_{X_\Phi}(t), U)) = \lambda\mathcal{L}(U)e^{-\lambda\overline{a}}\frac{(\lambda\overline{a})^k}{k!} \text{ and } \mathbb{E}(\mathrm{TC}(\partial E_{X_\Phi}(t), U)) = 2\pi\lambda\mathcal{L}(U)e^{-\lambda\overline{a}}\frac{(\lambda\overline{a})^k}{k!}\left(\chi(D) - \frac{\lambda}{4\pi}\overline{p}^2 + \frac{\overline{p}^2}{4\pi\overline{a}}k\right).$$

Proof. Since $\mathcal{H}^1(R_\theta r\Gamma) = r\mathcal{H}^1(\Gamma)$ and $\mathrm{TaC}(R_\theta r\Gamma, \mathbb{R}^2) = \mathrm{TaC}(\Gamma, \mathbb{R}^2)$, the g_m's are elementary functions with for $F(dm)$ almost every $m = (b, r, \theta)$, writing $L = \mathcal{H}^1(\Gamma) = \mathrm{Per}(D)$, we have

$$V_{g_m}(\mathbb{R}^2) = |b|rL \quad \text{and} \quad \mathrm{LTaC}(g_m, \mathbb{R}^2) = |b|\,\mathrm{TaC}(\Gamma, \mathbb{R}^2).$$

Let us remark that Equation (11) becomes

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^2} |g_m(x)|dxF(dm) = \mathbb{E}(|B|)\,\overline{a} < +\infty.$$

Equation (12) is easily checked since

$$\int_{\mathbb{R}^d} V_{g_m}(\mathbb{R}^2)F(dm) = \mathbb{E}(|B|)\mathbb{E}(R)L < +\infty \text{ and } \int_{\mathbb{R}^d} \mathrm{LTaC}_{g_m}(\mathbb{R}^2)F(dm) = \mathbb{E}(|B|)\,\mathrm{TaC}(\Gamma, \mathbb{R}^2) < +\infty.$$

Assumption (13) also follows from the fact that $\overline{a} < +\infty$ and $\mathbb{E}(|B|) < +\infty$. For (15), let us remark that $\mathcal{I}_{g_m} = \emptyset$ and $\mathcal{C}_{g_m} = R_\theta r \mathcal{C}_\Gamma$ such that $\mathcal{H}^0\left(\mathcal{S}_{g_m} \setminus \mathcal{R}_{g_m}\right) = \mathcal{H}^0\left(\mathcal{C}_\Gamma\right)$, ensuring (15), by assumption on Γ.

Now, in order to check (16) and (17) we need a kind of *kinematic formula*. For f a measurable non-negative function on $[-\pi,\pi)^2$, periodically extended on $\mathbb{R}^2$, and Γ_1, Γ_2 two simple closed curves, we can compute the integral

$$I_{\Gamma_1,\Gamma_2}(f) := \int_{\mathbb{R}^2} \sum_{z \in \Gamma_1 \cap \tau_x \Gamma_2} f(\operatorname{Arg} \nu_{\Gamma_1}(z), \operatorname{Arg} \nu_{\Gamma_2}(z-x))\, dx\,.$$

Using computations analogous to the kinematic formula of Santaló [29], Chapter 7 (where he considers random translations and rotations), we have that if Γ_1 and Γ_2 are two simple closed curves, then

$$I_{\Gamma_1,\Gamma_2}(f) = \int_0^{L_1}\int_0^{L_2} f(\theta_1(s_1)+\frac{\pi}{2}, \theta_2(s_2)+\frac{\pi}{2}) |\sin(\theta_1(s_1)-\theta_2(s_2))|\, ds_1\, ds_2, \tag{31}$$

where the curve Γ_1 (resp. Γ_2) of length $L_1 = \mathcal{H}^1(\Gamma_1)$ (resp. $L_2 = \mathcal{H}^1(\Gamma_2)$), is parametrized by $s_1 \mapsto \gamma_1(s_1)$ (resp. by $s_2 \mapsto \gamma_2(s_2)$) where s_1 (resp. s_2) is arc length, and $\theta_1(s_1) = \operatorname{Arg}\gamma_1'(s_1)$ (resp. $\theta_2(s_2) = \operatorname{Arg}\gamma_2'(s_2)$). Heuristically, this formula can be obtained using the change of variable $x = F(s_1,s_2) = \gamma_1(s_1) - \gamma_2(s_2)$, for which the Jacobian is

$$|\det(dF)| = |\sin(\theta_1(s_1)-\theta_2(s_2))|,$$

and noticing that the intersection point of Γ_1 and $\tau_x\Gamma_2$ is then $z = \gamma_1(s_1) = x + \gamma_2(s_2)$.

In particular, taking $f = 1$, it follows that $I_{\Gamma_1,\Gamma_2}(1) \le L_1 L_2$. Note also that we moreover have the exact formula

$$\int_0^{2\pi}\int_0^{2\pi} I_{R_\theta\Gamma_1, R'_\theta\Gamma_2}(1) d\theta d\theta' = 2\pi \times \int_0^{2\pi} |\sin(\theta)| d\theta L_1 L_2 = 2\pi \times 4 L_1 L_2,$$

according to a generalization of Poincaré's Formula (see [29] for instance). Using the fact that Γ is a finite disjoint union of closed curves we obtain that $I_{R_\theta r\Gamma, R'_\theta r'\Gamma}(1) \le L^2 r r'$, since the length of $R_\theta r\Gamma$, resp. $R'_\theta r'\Gamma$, is rL, resp. $r'L$, with $L = \mathcal{H}^1(\Gamma)$ the length of Γ and

$$\int_0^{2\pi}\int_0^{2\pi} I_{R_\theta r\Gamma, R'_\theta r'\Gamma}(1) d\theta d\theta' = 2\pi \times 4L^2 r r'.$$

It follows that

$$\begin{aligned}\int_{\mathbb{R}^d\times\mathbb{R}^d}\int_{\mathbb{R}^2} \mathcal{H}^0\left(\mathcal{R}_{g_{m'}} \cap \tau_x \mathcal{R}_{g_m}\right) dx F(dm) F(dm') &= \int_{\mathbb{R}_+^2}\int_{[0,2\pi]^2} I_{R_\theta r\Gamma, R_{\theta'} r'\Gamma}(1) F_\Theta(d\theta) F_\Theta(d\theta') F_R(dr) F_R(dr') \\ &\le 4L^2\left(\int_{\mathbb{R}^+} r F_r(dr)\right)^2 = 4\overline{p}^2,\end{aligned}$$

with $\overline{p} = L\,\mathbb{E}(R)$ the mean perimeter, proving (16).

Moreover, for $f(\theta_1,\theta_2) = \mathbb{1}_{\theta_1 \equiv \theta_2} + \mathbb{1}_{\theta_1 \equiv \theta_2 + \pi}$, where $\equiv$ stands for equality modulo 2π, we clearly have

$$I_{\Gamma_1,\Gamma_2}(f) = 0,$$

in view of (31). Since Γ is a finite disjoint union of closed curves, it follows that

$$\begin{aligned}&\int_{\mathbb{R}^d\times\mathbb{R}^d}\int_{\mathbb{R}^2} \mathcal{H}^0\left(\left\{y \in \mathcal{R}_{g_{m'}} \cap \tau_x\mathcal{R}_{g_m}; \nu_{g_{m'}}(y) = \pm\nu_{g_m}(y-x)\right\}\right) dx F(dm) F(dm') \\ = &\int_{\mathbb{R}_+^2}\int_{[0,2\pi]^2} I_{R_\theta r\Gamma, R_{\theta'} r'\Gamma}(f) F_\Theta(d\theta) F_\Theta(d\theta') F_R(dr) F_R(dr') = 0,\end{aligned}$$

so that (17) holds. Therefore we get the statement of Theorem 2. Moreover, we also get

$$\begin{aligned}&\int_{\mathbb{R}^d\times\mathbb{R}^d}\int_{\mathbb{R}^2} (\|g_m\|_\infty + \|g_{m'}\|_\infty) \mathcal{H}^0\left(\mathcal{R}_{g_{m'}} \cap \tau_x \mathcal{R}_{g_m}\right) dx F(dm) F(dm') \\ \le\ & 4\overline{p}^2 \int_{\mathbb{R}\times\mathbb{R}} (|b| + |b'|) F_B(db) F_B(db') = 8\overline{p}^2 \mathbb{E}(|B|) < +\infty,\end{aligned}$$

and (22) is also satisfied so that Theorem 3 holds.

Note that, we simply have

$$\mathbb{E}(V_{X_\Phi}(U)) = \lambda \mathcal{L}(U) \int_{\mathbb{R}^d} V_{g_m}(\mathbb{R}^2) F(dm) = \lambda \mathcal{L}(U) \mathbb{E}(|B|) \mathbb{E}(R) \mathcal{H}^1(\Gamma),$$

and

$$\mathbb{E}(\mathrm{LTC}_{X_\Phi}(U)) = \lambda\mathcal{L}(U)\int_{\mathbb{R}^d}\mathrm{LTC}_{g_m}(\mathbb{R}^2)F(dm) = \lambda\mathcal{L}(U)\mathbb{E}(B)\mathrm{TC}(\Gamma,\mathbb{R}^2),$$

using the fact that $\mathrm{LTC}_{g_m}(\mathbb{R}^2) = (b^+ - b^-)\times\mathrm{sgn}(b)\times\mathrm{TC}(\Gamma,\mathbb{R}^2)$. Since $\mathrm{TC}(\Gamma,\mathbb{R}^2) = 2\pi\chi(D)$, we obtain the first general statement. For $u\neq 0$, we can explicitly compute the characteristic function of $X_\Phi(0)$, given by

$$\mathbb{E}(e^{iuX_\Phi(0)}) = \exp\left(\lambda\int_{\mathbb{R}^d\times\mathbb{R}^2}(e^{iug_m(x)}-1)F(dm)dx\right) = e^{\lambda\overline{a}\left(\widehat{F}_B(u)-1\right)},$$

where $\widehat{F}_B(u) = \mathbb{E}(e^{iuB}) = \int_{\mathbb{R}} e^{iub}F_B(db)$ is the characteristic function of B. In particular for $B=1$ a.s. $X_\Phi(0)$ is a Poisson random variable of parameter $\lambda\overline{a}$. We also have, writing h_u as previoulsy,

$$\begin{aligned}\mathrm{LP}_{g_m}(h_u,\mathbb{R}^2) &= \int_{b^-}^{b+} e^{iut}\mathcal{H}^1(R_\theta r\Gamma)dt = r\mathcal{H}^1(\Gamma)\frac{e^{iub^+}-e^{iub^-}}{iu}\\ \mathrm{LTC}_{g_m}(h_u,\mathbb{R}^2) &= \int_{b^-}^{b^+} e^{iut}\mathrm{sgn}(b)\mathrm{TC}(R_\theta r\Gamma,\mathbb{R}^2)dt = \mathrm{TC}(\Gamma,\mathbb{R}^2)\frac{e^{iub^+}-e^{iub^-}}{iu}\mathrm{sgn}(b),\end{aligned}$$

so that

$$\int_{\mathbb{R}^d}\mathrm{LP}_{g_m}(h_u,\mathbb{R}^2)F(dm) = \mathbb{E}(R)\mathcal{H}^1(\Gamma)\frac{\widehat{F_{B^+}}(u)-\widehat{F_{B^-}}(u)}{iu}$$

and

$$\int_{\mathbb{R}^d}\mathrm{LTC}_{g_m}(h_u,\mathbb{R}^2)F(dm) = \mathrm{TC}(\Gamma,\mathbb{R}^2)\frac{\widehat{F_B}(u)-1}{iu}.$$

Therefore

$$\mathbb{E}(\mathrm{LP}_{X_\Phi}(h_u,U)) = \int_{\mathbb{R}} e^{iut}\mathbb{E}(\mathcal{H}^1(\partial E_{X_\Phi}(t,U))dt = \lambda\mathcal{L}(U)\mathbb{E}(e^{iuX_\Phi(0)})\mathbb{E}(R)\mathcal{H}^1(\Gamma)\frac{\widehat{F_{B^+}}(u)-\widehat{F_{B^-}}(u)}{iu}.$$

Finally, let us remark that for $f(\theta_1,\theta_2) = \min(|\theta_1-\theta_2|, 2\pi-|\theta_1-\theta_2|)$ (distance between two angles) and $f_\pi(\theta_1,\theta_2) = f(\pi+\theta_1,\theta_2)$, we get

$$\begin{aligned}&\int_{\mathbb{R}^d}\int_{\mathbb{R}^2\times\mathbb{R}^d}(e^{iug^+_{m'}(z)}-e^{iug^-_{m'}(z)})\left(\int_{g^-_m(z-x)}^{g^+_m(z-x)}e^{ius}ds\right)\sum_{z\in\tau_x\mathcal{R}_{g_m}\cap\mathcal{R}_{g_{m'}}} d_{S^1}(\nu_{g_m}(z-x),\nu_{g_{m'}}(z))\,dx\,F(dm)F(dm')\\ =\;& \frac{(\int_0^{+\infty}(e^{iub}-1)F_B(db))^2+(\int_{-\infty}^0(1-e^{iub})F_B(db))^2}{iu}\int_{\mathbb{R}_+^2}\int_{[0,2\pi]^2} I_{R_\theta r\Gamma,R_{\theta'}r'\Gamma}(f)F_\Theta(d\theta)F_\Theta(d\theta')F_R(dr)F_R(dr')\\ +\;& 2\frac{(\int_0^{+\infty}(e^{iub}-1)F_B(db))(\int_{-\infty}^0(1-e^{iub'})F_B(db'))}{iu}\int_{\mathbb{R}_+^2}\int_{[0,2\pi]^2} I_{R_\theta r\Gamma,R_{\theta'}r'\Gamma}(f_\pi)F_\Theta(d\theta)F_\Theta(d\theta')F_R(dr)F_R(dr')\end{aligned}$$

This last expression may not be simple to compute. However, assuming from now on that $F_\Theta(d\theta) = \frac{1}{2\pi}\mathbf{I}_{[0,2\pi]}d\theta$, we obtain by (31), for two simple closed curves Γ_1,Γ_2,

$$\begin{aligned}&\int_{[0,2\pi]^2} I_{R_\theta\Gamma_1,R_{\theta'}\Gamma_2}(f)F_\Theta(d\theta)F_\Theta(d\theta')\\ =\;& \frac{1}{2\pi}\int_0^{2\pi}\int_0^{L_1}\int_0^{L_2}\min(|\theta_1(s_1)-\theta_2(s_2)-\theta|, 2\pi-|\theta_1(s_1)-\theta_2(s_2)-\theta|)|\sin(\theta_1(s_1)-\theta_2(s_2)-\theta)|ds_1ds_2d\theta\\ =\;& L_1L_2.\end{aligned}$$

Similarly we have

$$\int_{[0,2\pi]^2} I_{R_\theta\Gamma_1,R_{\theta'}\Gamma_2}(f_\pi)F_\Theta(d\theta)F_\Theta(d\theta') = L_1L_2.$$

Since Γ is a finite disjoint union of such closed curves, we get

$$\int_{[0,2\pi]^2} I_{R_\theta r\Gamma,R_{\theta'}r'\Gamma}(f)F_\Theta(d\theta)F_\Theta(d\theta') = \int_{[0,2\pi]^2} I_{R_\theta r\Gamma,R_{\theta'}r'\Gamma}(f_\pi)F_\Theta(d\theta)F_\Theta(d\theta') = L^2rr'.$$

It follows that for uniform rotations, we have

$$\mathbb{E}(\mathrm{LTC}_{X_\Phi}(h_u,U)) = \lambda\mathcal{L}(U)\mathbb{E}(e^{iuX_\Phi(0)})\frac{1}{iu}\left((\widehat{F_B}(u)-1)\mathrm{TC}(\Gamma,\mathbb{R}^2)+\frac{\lambda}{2}(\widehat{F_{B^+}}(u)-\widehat{F_{B^-}}(u))^2\overline{p}^2\right).$$

When moreover $B=1$ a.s., $B^+=1$ and $B^-=0$ a.s., we can deduce an exact formula for the mean level total curvature function of X_Φ. Actually, in this case $X_\Phi(0)$ follows a Poisson law of parameter $\lambda\overline{a}$

and $\frac{\widehat{F_{B^+}}(u)-\widehat{F_{B^-}}(u)}{iu} = \frac{\widehat{F_B}(u)-1}{iu}$ is the characteristic function of a uniform random variable Z on $[0,1]$, while $\widehat{F_B}(u)\frac{\widehat{F_B}(u)-1}{iu}$ is the characteristic function of $Z+1$. Then, considering Z independent from $X_\Phi(0)$ we recognize

$$\mathbb{E}(\mathrm{LP}_{X_\Phi}(h_u,U)) = \lambda\mathcal{L}(U)\mathbb{E}\left(e^{iu[X_\Phi(0)+Z]}\right)\mathbb{E}(R)\mathcal{H}^1(\Gamma), \text{ and}$$

$$\mathbb{E}(\mathrm{LTC}_{X_\Phi}(h_u,U)) = \lambda\mathcal{L}(U)\left[\mathbb{E}\left(e^{iu[X_\Phi(0)+Z]}\right)\left(\mathrm{TC}(\Gamma,\mathbb{R}^2)-\frac{\lambda}{2}\overline{p}^2\right)+\mathbb{E}\left(e^{iu[X_\Phi(0)+Z+1]}\right)\frac{\lambda}{2}\overline{p}^2\right].$$

Note also that since here the random field X_Φ has integer values, then $\forall k\in\mathbb{N}$,

$$\forall t\in(k,k+1],\quad \{x\in U; X_\Phi(x)\geq t\} = \{x\in U; X_\Phi(x)\geq k+1\},$$

and therefore $\mathbb{E}(\mathrm{Per}(E_{X_\Phi}(t),U)) = \mathbb{E}(\mathrm{Per}(E_{X_\Phi}(k+1),U))$ and $\mathbb{E}(\mathrm{TC}(\partial E_{X_\Phi}(t),U)) = \mathbb{E}(\mathrm{TC}(\partial E_{X_\Phi}(k+1),U))$. Hence we may conclude that $\forall k\in\mathbb{N}$, $\forall t\in(k,k+1]$

(32)

$$\mathbb{E}(\mathrm{Per}(E_{X_\Phi}(t),U)) = \lambda\mathcal{L}(U)e^{-\lambda\overline{a}}\frac{(\lambda\overline{a})^k}{k!}\overline{p} \quad \text{and } \mathbb{E}(\mathrm{TC}(\partial E_{X_\Phi}(t,U)) = \lambda\mathcal{L}(U)e^{-\lambda\overline{a}}\frac{(\lambda\overline{a})^k}{k!}\left(\mathrm{TC}(\Gamma,\mathbb{R}^2)-\frac{\lambda}{2}\overline{p}^2+\frac{\overline{p}^2}{2\overline{a}}k\right).$$

Note that, thanks to Gauss-Bonnet Theorem, we have $\mathrm{TC}(\Gamma,\mathbb{R}^2) = 2\pi\chi(D)$, so that we may rewrite this as in (30).

□

Let us remark that Formula (30) only involves the Euler Characteristic, the mean perimeter and the mean area of the shapes. When $k=0$, we find the formula of the mean Euler Characteristic density of a rotation invariant Boolean model as obtained by Mecke and Wagner in [26] and by Mecke in [25] stating that

$$\lim_{r\to+\infty}\frac{\mathbb{E}(\chi(\{x\in rU; X(x)\geq 1\}))}{\pi\mathcal{L}(rU)} = e^{-\lambda m_0(D)}\left(\lambda m_2(D)-\lambda^2 m_1(D)^2\right),$$

with $m_0(D)=\overline{a}$, $m_1(D)=\overline{p}/2\pi$ and $m_2(D)=\chi(D)/\pi$. Actually, following the framework of [31] for Boolean models, we can define some volume densities for excursion sets as, for all $k\in\mathbb{N}$ and $t\in(k,k+1]$,

$$\begin{aligned}
\overline{\mathcal{L}(E_X(t))} &:= \mathbb{P}(X(0)\geq k+1) = 1-e^{-(\lambda\overline{a})}\sum_{l=0}^{k}\frac{(\lambda\overline{a})^l}{l!}\\
\overline{\mathrm{Per}(E_X(t))} &= e^{-(\lambda\overline{a})}\frac{(\lambda\overline{a})^k}{k!}(\lambda\overline{p})\\
\overline{\chi(E_X(t))} &= e^{-(\lambda\overline{a})}\frac{(\lambda\overline{a})^k}{k!}\left(\lambda\chi(D)-\frac{1}{4\pi}(\lambda\overline{p})^2+\frac{\overline{p}^2}{4\pi\overline{a}}k\right),
\end{aligned}$$

recovering the results of Boolean model for $k=0$ in dimension 2 (see p.389 of [31]). The typical behavior of $\overline{\chi(E_X(k))}$, as a function of $k\in\mathbb{N}\setminus\{0\}$, is the following:

- It starts, when k is small, by being negative. This is explained by the fact that $\{X_\Phi\geq k\}$ is essentially made of one big connected component with many small holes in it. In particular the minimum value is achieved for an integer denoted k_-. The explicit value of k_- can be computed from Equation (32). The formula is not very nice, but it has a simple asymptotic behavior when λ is large, since then we have
$$k_- = \lambda\overline{a}-\sqrt{\lambda\overline{a}}+O(1).$$
- Then, after k_-, the mean level total curvature $\mathbb{E}(\mathrm{TC}(\partial E_{X_\Phi}(k,U))$ increases and it crosses 0 in the interval that contains k_0 where
$$k_0 = \lambda\overline{a}-\frac{2\pi\overline{a}}{\overline{p}^2}\chi(D) = \lambda\overline{a}+O(1).$$
For this level, there are as many connected components as holes.
- After k_0, the mean level total curvature is positive and it increases till a value k_+ and afterwards it decreases and goes to 0 as k goes to infinity. As for k_- the value of k_+ is explicit, and its asymptotic behavior when λ is large is
$$k_+ = \lambda\overline{a}+\sqrt{\lambda\overline{a}}+O(1).$$

Example 1: Random disks
We assume here that $D = D(0,1)$ is a disk of radius 1, and that $B = 1$ a.s. In this case we have $\mathrm{TC}(\partial D) = 2\pi\chi(D) = 2\pi$, $\overline{p} = 2\pi\mathbb{E}(R)$ and $\overline{a} = \pi\mathbb{E}(R^2)$. Note also that since $R_\theta rD = rD$ for all θ, whatever F_Θ is, the shot noise field has the same law than one with marks given by $\tilde{m} = (b, r) \in [0, +\infty)^2 \subset \mathbb{R}^2$, with distribution $G(d\tilde{m}) = F_B(db)F_R(dr)$. An example of such a random field with comparisons between the theoretical value of $\mathbb{E}(\mathrm{TC}(\partial E_{X_\phi}(t), U))/2\pi$ of Euler Characteristic and an empirical estimate on a large domain are shown on Figure 4. The caption of the figure gives the practical and technical details of the simulation.

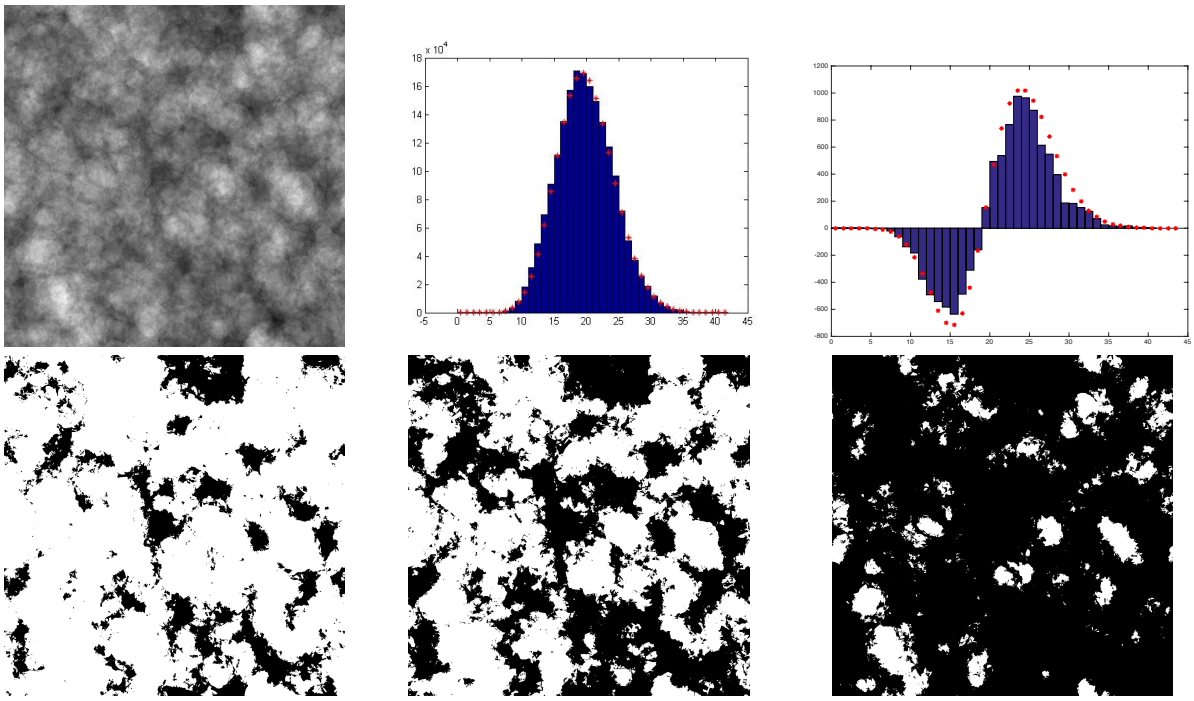

FIGURE 4. Shot noise random field with indicator functions of random disks. This sample has been obtained using Matlab, with a domain of size 2000×2000 pixels, a Poisson point process of intensity $\lambda = 0.001$, and random disks of radius $R = 50$ or $R = 100$ (each with probability 0.5). Top middle and right figure: empirical Perimeter and Euler Characteristic as a function of the level t (computed thanks to the Matlab functions `bwperim` and `bweuler`), compared with the theoretical values (red stars) of Equation (30). Bottom line: Three excursion sets corresponding respectively from left to right to the level $t = 15$, $t = 19$ (that is the "critical level" where the Euler Characteristic turns from negative to positive) and $t = 25$.

Let us quote that we can also compute the mean level total curvature for a non isotropic shape. This is the case of squares for instance, as developed in the following example.
Example 2: Random squares
We assume here that D is a square of side length 1 and $\Theta = 0$ a.s. with $F(dm) = F_B(db)F_R(dr)\delta_0(d\theta)$ or equivalently that marks are given by $\tilde{m} = (b, r) \in \mathbb{R} \times [0, +\infty) \subset \mathbb{R}^2$, with distribution $G(d\tilde{m}) = F_B(db)F_R(dr)$. In this case, $\Gamma = \partial D$ is made of four line segments, with $\mathrm{TC}(\Gamma, \mathbb{R}^2) = 2\pi\chi(D) = 2\pi$, $\overline{p} = 4\mathbb{E}(R)$ and $\overline{a} = \mathbb{E}(R^2)$. On the boundary of a square, the curvature is 0, and it has four corner points with a turning angle equal to $\pi/2$. Hence for all h continuous bounded function m we have $R(h, m) = 0$, $I^{(1)}(h, m) = 0$, and $C(h, m) = 4 \times \mathrm{sgn}(b)\frac{\pi}{2}\int_{b^-}^{b^+} \mathbb{E}(h(X_\Phi(0) + s))ds$. Now, according to the kinematic formula (31), the only remaining terms are for $\theta_1(s_1) = \theta_2(s_2) \pm \frac{\pi}{2}$ for which

$$f(\theta_1(s_1), \theta_2(s_2)) = \min(|\theta_1(s_1) - \theta_2(s_2)|, 2\pi - |\theta_1(s_1) - \theta_2(s_2)|) = \frac{\pi}{2} = f_\pi(\theta_1(s_1), \theta_2(s_2)).$$

It follows that

$$I_{\Gamma_1,\Gamma_2}(f) = \frac{\pi}{2} \times 8r_1 \times r_2.$$

Therefore, for $u \neq 0$ and $h_u(t) = e^{iut}$

$$\frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^2\times\mathbb{R}^d}\sum_{z\in\tau_x\mathcal{R}_{g_m}\cap\mathcal{R}_{g_{m'}}}(e^{iug^+_{m'}(z)}-e^{iug^-_{m'}(z)})\left(\int_{g^-_m(z-x)}^{g^+_m(z-x)}e^{ius}ds\right)d_{S^1}(\nu_{g_m}(z-x),\nu_{g_{m'}}(z))\,dx\,F(dm)F(dm')$$

$$=\frac{(\widehat{F_{B^+}}(u)-\widehat{F_{B^-}}(u))^2}{iu}\int_{\mathbb{R}^2_+}I_{r\Gamma,r'\Gamma}(k)F_R(dr)F_R(dr')=2\pi\frac{(\widehat{F_{B^+}}(u)-\widehat{F_{B^-}}(u))^2}{iu}\frac{\overline{p}^2}{16}.$$

It follows that we get in this case

$$\int_{\mathbb{R}}e^{iut}\mathbb{E}(\mathrm{TC}(\partial E_{X_\Phi}(t),U))dt=\lambda\mathcal{L}(U)\mathbb{E}(e^{iuX_\Phi(0)})\frac{1}{iu}\left((\widehat{F_B}(u)-1)\mathrm{TC}(\Gamma,\mathbb{R}^2)+2\pi\lambda(\widehat{F_{B^+}}(u)-\widehat{F_{B^-}}(u))^2\frac{\overline{p}^2}{16}\right).$$

For $B=1$ a.s., inverting as previously, we obtain

$$\forall k\in\mathbb{N},\forall t\in(k,k+1],\quad \frac{1}{2\pi}\mathbb{E}(\mathrm{TC}(\partial E_{X_\Phi}(t),U))=\lambda\mathcal{L}(U)e^{-\lambda\overline{a}}\frac{(\lambda\overline{a})^k}{k!}\left(1-\frac{\lambda}{16}\overline{p}^2+\frac{\overline{p}^2}{16\overline{a}}k\right). \tag{33}$$

It is illustrated on Figure 5. This formula generalizes one of the results of Decreusefond et al. [14]. Actually, considering the Boolean model made of squares of constant size $R=2\varepsilon$ a.s. for some $\varepsilon>0$, we get for $k=0$, and $a>0$,

$$\forall t\in(0,1],\quad \frac{1}{2\pi}\mathbb{E}(\mathrm{TC}(\partial E_{X_\Phi}(t),(0,a)^2))=\lambda a^2e^{-\lambda(2\varepsilon)^2}\left(1-\lambda(2\varepsilon)^2\right),$$

that corresponds to the mean Euler Characteristic of the Boolean model in dimension 2, considered in the torus of size $a>0$ in Theorem 11 of [14].

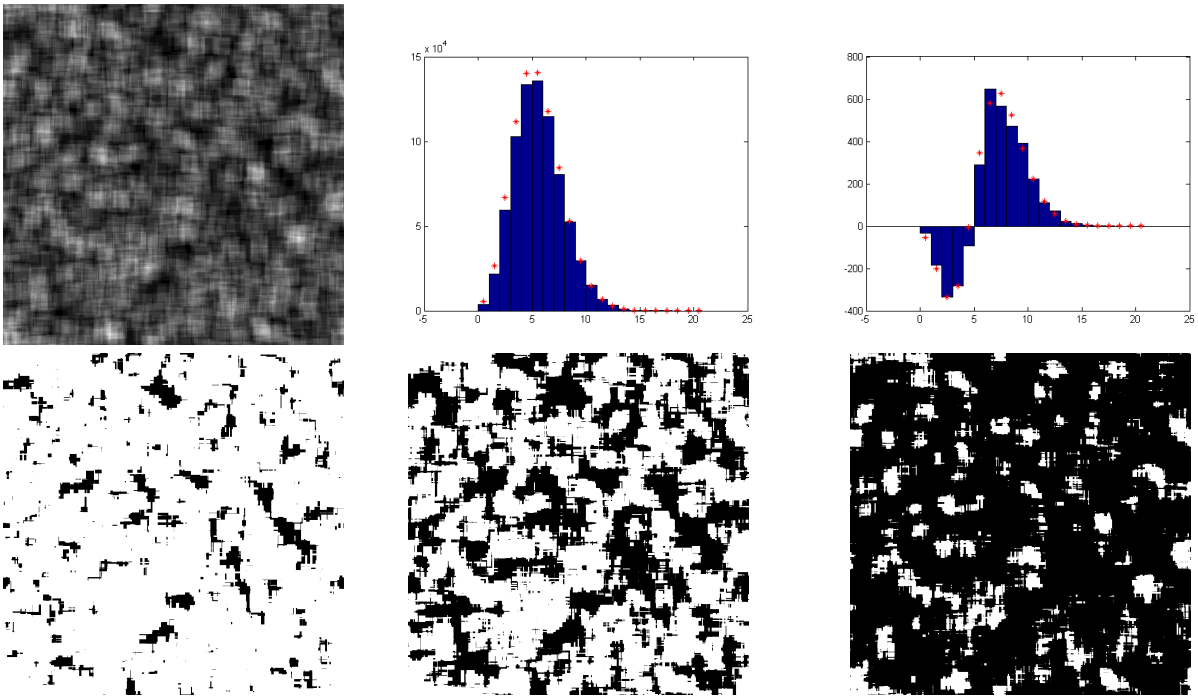

FIGURE 5. Shot noise random field with indicator functions of random squares. This sample has been on a domain of size 2000×2000 pixels, a Poisson point process of intensity $\lambda = 0.005$, and random squares of fixed side length $R = 100$. Top middle and right figures: empirical Perimeter and Euler Characteristic as a function of the level t, compared with the theoretical value (red stars) of Equation (33). Bottom line: Three excursion sets corresponding respectively from left to right to the level $t = 3$, $t = 5$ (that is the "critical level" where the Euler Characteristic turns from negative to positive) and $t = 8$.

4. Level integrals of smooth functions and random fields

4.1. **The case of smooth functions.** In the previous section, we have computed the level total curvature of elementary functions, that are piecewise constant functions (that are in particular not continuous). Here, we now consider the case of smooth (at least C^2) functions. The question of the link between these two cases will be discussed in a future paper.

In the sequel, for f a C^2 function we denote by ∇f its gradient vector and by $D^2 f$ its Hessian matrix.

Proposition 3. *Let f be a smooth (at least C^2) function on an open set containing $\overline{U}$. Then f is of special bounded variation and of finite level total curvature integral on U with*

$$V_f(U) = \int_U \|\nabla f(x)\|\, dx \quad \text{and} \quad \mathrm{LTaC}_f(U) \leq \int_U \|D^2 f(x)\|_2\, dx,$$

where $\|\cdot\|_2$ is the matrix norm subordinated to the Euclidean norm. Moreover, for h a bounded continuous function on $\mathbb{R}$, the level perimeter and total curvature integrals of f are given by

$$\mathrm{LP}_f(h,U) = \int_U h(f(x))\|\nabla f(x)\|\, dx \quad \text{and}$$

$$\mathrm{LTC}_f(h,U) = -\int_U h(f(x)) \frac{D^2 f(x).(\nabla f(x)^\perp, \nabla f(x)^\perp)}{\|\nabla f(x)\|^2}\, \mathbf{1}_{\|\nabla f(x)\|>0}\, dx,$$

where if $A = (a_{ij})_{1\leq i,j\leq 2}$ is a 2×2 symmetric matrix and $y = (y_1, y_2) \in \mathbb{R}^2$, we use the notation

$$A.(y,y) = {}^t yAy = a_{11}y_1^2 + a_{22}y_2^2 + 2a_{12}y_1y_2.$$

Proof. Note that since f is C^1 on an open set containing $\overline{U}$ we clearly have that $f \in SBV(U)$ and thus $E_f(t)$ is of finite perimeter in U for a.e. $t \in \mathbb{R}$ by co-area formula (see Theorem 3.40 [7]). Moreover, since f is actually C^2 on an open set containing $\overline{U}$, by Morse-Sard theorem (see [20] p.69 for instance), the set of critical values of f has measure 0 in $\mathbb{R}$. Hence for a.e $t \in \mathbb{R}$, for all points $x \in U$ such that $f(x) = t$ then $\nabla f(x) \neq 0$. Let t be such a non-critical value. It follows that $\partial E_f(t) \cap U = \{x \in U; f(x) = t\}$, and we clearly have $S_{\mathbf{1}_{E_f(t)}} = \partial E_f(t) \cap U$ is a regular curve with finite length such that for any $x \in \partial E_f(t) \cap U$ one can find a parametrization γ given by an implicit form $f(\gamma(s)) = t$, with $x = \gamma(s)$ and normal vector $\gamma'(s)^\perp = \nabla f(x)/||\nabla f(x)||$ and curvature given by

$$\kappa_f(x) = -\frac{D^2 f(x).(\nabla f^\perp(x), \nabla f^\perp(x))}{||\nabla f(x)||^3}. \tag{34}$$

Therefore

$$\mathrm{TaC}(\partial E_f(t), U) = \int_{\partial E_f(t)\cap U} |\kappa_f(x)|\mathcal{H}^1(dx) < +\infty \text{ and } \quad \mathrm{TC}(\partial E_f(t), U) = \int_{\partial E_f(t)\cap U} \kappa_f(x)\mathcal{H}^1(dx).$$

The fact that $\partial E_f(t) \cap U$ is given by a finite union of simple connected curves comes from compactness of $\{x \in \overline{U}; f(x) = t\}$ in which it is included. Hence we have already proven that for a.e. $t \in \mathbb{R}$, the set $E_f(t)$ is an elementary set of U. Now let us prove that $t \mapsto \mathrm{TaC}(\partial E_f(t), U)$ is integrable on $\mathbb{R}$. Let us define $|\kappa_f|$ as a measurable non-negative function on U by setting $|\kappa_f(x)| = +\infty$ if $\nabla f(x) = 0$ and $|\kappa_f(x)| = \left|\frac{D^2 f(x).(\nabla f^\perp(x), \nabla f^\perp(x))}{||\nabla f(x)||^3}\right|$, otherwise. Now, let us recall the co-area formula for Lipschitz mappings (see [18] p.117 for instance): for any non-negative measurable or $\mathcal{L}$-integrable function g, the function $t \mapsto \int_{\partial E_f(t)\cap U} g(x)\mathcal{H}^1(dx)$ is measurable and

$$\int_U g(x)\|\nabla f\|(x)\, dx = \int_{\mathbb{R}} \int_{\partial E_f(t)\cap U} g(x)\mathcal{H}^1(dx)\, dt.$$

Taking $g = 1$, we recover the co-area formula:

$$\int_U \|\nabla f\|(x) dx = \int_{\mathbb{R}} \mathcal{H}^1(\partial E_f(t) \cap U)\, dt = V_f(U),$$

while for $g = |\kappa_f|$,

$$\begin{aligned} \mathrm{LTaC}_f(U) = \int_{\mathbb{R}} \int_{\partial E_f(t)\cap U} |\kappa_f(x)|\mathcal{H}^1(dx) &= \int_U |\kappa_f(x)|\|\nabla f(x)\| dx \\ &\leq \int_U \|D^2 f(x)\|_2 dx, \end{aligned}$$

in view of (34). This concludes the proof that f is of special bounded variation and finite level total curvature integral on U and, for h a bounded continuous function on $\mathbb{R}$, using again twice the co-area formula but now with $\max(g,0)$ and $-\min(g,0)$ for $g = h\circ f$ or $g = (h\circ f)\,\kappa_f$, and substracting we get

$$\begin{aligned}\mathrm{LP}_f(h) &= \int_{\mathbb{R}} h(x)\|\nabla f(x)\|dx\\ \mathrm{LTC}_f(h) &= \int_U h(f(x))\kappa_f(x)||\nabla f(x)||\,dx = -\int_U h(f(x))\frac{D^2f(x).(\nabla f(x)^\perp,\nabla f(x)^\perp)}{\|\nabla f(x)\|^2}\,\mathbf{1}_{\|\nabla f(x)\|>0}dx,\end{aligned}$$

in view of (34).

□

4.2. **A general result for smooth stationary random fields.** In this section we consider a smooth stationary random field X defined on $\mathbb{R}^2$. Let us introduce some notations for the derivatives of X. A point $x\in\mathbb{R}^2$ is defined by its two coordinates $x=(x_1,x_2)$ and we denote for $i,j=1,2$

$$X_i := \frac{\partial X}{\partial x_i} \quad\text{and}\quad X_{ij} := \frac{\partial^2 X}{\partial x_i\partial x_j}.$$

With these notations it follows that $\nabla X = \begin{pmatrix} X_1\\ X_2\end{pmatrix}$ and $D^2X = \begin{pmatrix} X_{11} & X_{12}\\ X_{12} & X_{22}\end{pmatrix}$. Note that since X is stationary, for any $x\in\mathbb{R}^2$,

$$(X(x),\nabla X(x),D^2X(x)) \overset{d}{=} (X(0),\nabla X(0),D^2X(0)).$$

When X, ∇X and D^2X have also finite second order moment, $X(x)$ and $\nabla X(x)$ are not correlated, as well as $\nabla X(x)$ and $D^2X(x)$ (see [1] p.31 for instance). This is very useful for Gaussian fields since it implies that $\nabla X(x)$ is independent from $(X(x),D^2X(x))$.

Using the result of Section 4.1 and the stationarity of X, we have the following formula.

Theorem 5. *Let X be a stationary C^2 random field on $\mathbb{R}^2$, such that $X(0), X_i(0), X_{ii}(0)$ and $X_{12}(0)$ have finite expectations for $i=1,2$. Then, $X\in SBV(U)$ with $DX=\nabla X\mathcal{L}$ and for a.e. $t\in\mathbb{R}$, the random variables $\mathrm{Per}(E_X(t),U)$ and $\mathrm{TC}(\partial E_X(t),U)$ have finite expectation such that for all h bounded continuous function on $\mathbb{R}$, one has*

$$\mathbb{E}(\mathrm{LP}_X(h,U)) = \int_{\mathbb{R}} h(t)\mathbb{E}(\mathrm{Per}(E_X(t),U))\,dt \quad \text{and} \quad \mathbb{E}(\mathrm{LTC}_X(h,U)) = \int_{\mathbb{R}} h(t)\mathbb{E}(\mathrm{TC}(\partial E_X(t),U))\,dt,$$

with

$$\begin{aligned}\mathbb{E}(\mathrm{LP}_X(h,U)) &= \mathcal{L}(U)\mathbb{E}\left(h(X(0))\|\nabla X(0)\|\right)\\ \mathbb{E}(\mathrm{LTC}_X(h,U)) &= -\mathcal{L}(U)\mathbb{E}\left(h(X(0))\frac{D^2X(0).(\nabla X(0)^\perp,\nabla X(0)^\perp)}{\|\nabla X(0)\|^2}\mathbf{1}_{\|\nabla X(0)\|>0}\right)\end{aligned}$$

It follows that when the field X is isotropic, i.e. $X\circ A \overset{fdd}{=} X$ for all orthogonal matrix A, then the above formulas reduce to, $\forall i=1,2$,

$$\begin{aligned}\mathbb{E}(\mathrm{LP}_X(h,U)) &= \frac{\pi}{2}\mathcal{L}(U)\mathbb{E}\left(h(X(0))|X_i(0)|\right) \quad \text{and}\\ \mathbb{E}(\mathrm{LTC}_X(h,U)) &= -\mathcal{L}(U)\left(\mathbb{E}\left(h(X(0))X_{ii}(0)\mathbf{1}_{\|\nabla X(0)\|>0}\right) - 4\mathbb{E}(h(X(0))\frac{X_1(0)X_2(0)}{\|\nabla X(0)\|^2}X_{12}(0)\mathbf{1}_{\|\nabla X(0)\|>0})\right).\end{aligned}$$

Assuming moreover that the field X can be written as $X=X^{(1)}+X^{(2)}$, with $X^{(1)}$, $X^{(2)}$ iid isotropic and symmetric ($X^{(i)}\overset{fdd}{=}-X^{(i)}$), the previous formula reduced to

$$\mathbb{E}(\mathrm{LTC}_X(h,U)) = -\mathcal{L}(U)\mathbb{E}\left(h(X(0))X_{ii}(0)\mathbf{1}_{\|\nabla X(0)\|>0}\right).$$

In particular, taking $h=1$ we have $V_X(U)=\frac{\pi}{2}\mathcal{L}(U)\mathbb{E}(|X_i(0)|)$ and

$$\mathbb{E}(\mathrm{LTC}_X(U)) = -\mathcal{L}(U)\mathbb{E}\left(X_{11}(0)\mathbf{1}_{\|\nabla X(0)\|>0}\right) = -\mathcal{L}(U)\mathbb{E}\left(X_{22}(0)\mathbf{1}_{\|\nabla X(0)\|>0}\right) = 0.$$

It follows that when $X(0)$ admits a density $p_{X(0)}$, we get for almost every $t\in\mathbb{R}$, $\forall i=1,2$,

$$\begin{aligned}\mathbb{E}(\mathrm{Per}(E_X(t),U)) &= \frac{\pi}{2}\mathcal{L}(U)\mathbb{E}\left(|X_i(0)||X(0)=t\right)p_{X(0)}(t)\\ \mathbb{E}(\mathrm{TC}(\partial E_X(t),U)) &= -\mathcal{L}(U)\mathbb{E}\left(X_{ii}(0)\mathbf{1}_{\|\nabla X(0)\|>0}|X(0)=t\right)p_{X(0)}(t).\end{aligned}$$

Proof. According to Proposition 3, since X is a.s C^2, it is a.s. of special bounded variation and of finite level curvature integral on U with

$$V_X(U) = \int_U \|\nabla X(x)\| dx \quad \text{and} \quad \mathrm{LTaC}_X(U) \leq \int_U \|D^2X(x)\|_2 \, dx \quad \text{a.s..}$$

Since X is stationary, the finite expectation assumption implies that $V_X(U)$ and $\mathrm{LTaC}_X(U)$ are non-negative random variables with finite expectation. By Fubini's Theorem, this implies that $(t,\omega) \mapsto \mathrm{Per}(E_X(t),U) \in L^1(\mathbb{R}\times\Omega)$ and $(t,\omega) \mapsto \mathrm{TaC}(\partial E_X(t),U) \in L^1(\mathbb{R}\times\Omega)$ so that we also have $(t,\omega) \mapsto \mathrm{TC}(\partial E_X(t),U) \in L^1(\mathbb{R}\times\Omega)$. Moreover, a.s., for any h a bounded continuous function on $\mathbb{R}$,

$$\mathrm{LP}_X(h,U) = \int_{\mathbb{R}} h(t)\mathrm{Per}(E_X(t),U)\, dt \quad \text{and} \quad \mathrm{LTC}_X(h,U) = \int_{\mathbb{R}} h(t)\mathrm{TC}(\partial E_X(t),U)\, dt, \text{ and}$$

$$\mathrm{LP}_X(h,U) = \int_U h(X(x))\|\nabla X(x)\| dx \text{ and } \mathrm{LTC}_X(h,U) = -\int_U h(X(x)) \frac{D^2X(x).(\nabla X(x)^\perp, \nabla X(x)^\perp)}{\|\nabla X(x)\|^2} \mathbf{1}_{\|\nabla X(x)\|>0} dx.$$

Hence, Fubini's Theorem yields the first result for expectation, while the second one is a straightforward consequence from the stationarity of X.

Under the assumption that the field is isotropic, we can exploit further on this formula. First let us recall that by Taylor formula, since X is a.s. C^2 we have a.s. for all $x, z \in \mathbb{R}^2$,

$$X(x+z) = X(x) + \langle \nabla X(x), z\rangle + \frac{1}{2} D^2X(x).(z,z) + o_{\|z\|\to 0}(\|z\|^2).$$

In particular we obtain that, for any orthogonal matrix A,

$$\nabla(X\circ A)(x) = {}^tA\nabla X(Ax) \text{ and } D^2(X\circ A)(x) = {}^tA(D^2X)(Ax)A.$$

Writing (e_1,e_2) the canonical basis of $\mathbb{R}^2$, we also have for $i=1,2$,

$$\begin{aligned} X_{ii}(x) &= \lim_{\varepsilon\to 0} \frac{X(x+\varepsilon e_i) + X(x-\varepsilon e_i) - 2X(x)}{\varepsilon^2}, \\ X_{12}(x) &= \lim_{\varepsilon\to 0} \frac{X(x+\varepsilon(e_1+e_2)) + X(x-\varepsilon(e_1+e_2)) - X(x+\varepsilon(e_1-e_2)) - X(x-\varepsilon(e_1-e_2))}{2\varepsilon^2} \end{aligned}$$

Since $X\circ A \overset{fdd}{=} X$, we deduce that

$$(X(x), \nabla X(x), D^2X(x)) \overset{d}{=} (X(Ax), {}^tA\nabla X(Ax), {}^tA(D^2X)(Ax)A),$$

and specifying to $x=0$, it follows that

$$(X(0), \nabla X(0), D^2X(0)) \overset{d}{=} (X(0), {}^tA\nabla X(0), {}^tA(D^2X)(0)A). \tag{35}$$

Hence $(X(0),\nabla X(0)) \overset{d}{=} (X(0), {}^tA\nabla X(0))$ and for any $\theta\in[0,2\pi)$, denoting $u(\theta) = (\cos\theta, \sin\theta) \in S^1$, one has $(X(0), X_i(0)) \overset{d}{=} (X(0), \langle u(\theta), \nabla X(0)\rangle)$, according to orthogonal invariance (see Proposition 4.8 of [12]). Moreover $\|\nabla X(0)\| = \frac{1}{4}\int_0^{2\pi} |\langle u(\theta), \nabla X(0)\rangle| d\theta$ so that we may deduce that

$$\mathbb{E}\left(h(X(0))\|\nabla X(0)\|\right) = \frac{1}{4}\int_0^{2\pi} \mathbb{E}\left(h(X(0))|\langle u(\theta),\nabla X(0)\rangle|\right) d\theta = \frac{\pi}{2}\mathbb{E}\left(h(X(0))|X_i(0)|\right),$$

and the result for $\mathbb{E}(\mathrm{LP}_X(h,U))$ follows. Now, let us introduce the random variable Θ with values in $2\pi\mathbb{T}$ (identified with $[0,2\pi)$), such that

$$\nabla X(0) = \begin{pmatrix} X_1(0) \\ X_2(0) \end{pmatrix} = \|\nabla X(0)\| \begin{pmatrix} \cos\Theta \\ \sin\Theta \end{pmatrix}.$$

For any $\theta\in[0,2\pi)$ let us consider a rotation matrix $R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ and a reflexion matrix $S_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$. Let us quote that $R_{-\theta}\nabla X(0) = {}^tR_\theta\nabla X(0) = \|\nabla X(0)\| \begin{pmatrix} \cos(\Theta-\theta) \\ \sin(\Theta-\theta) \end{pmatrix}$, $S_\theta\nabla X(0) = {}^tS_\theta\nabla X(0) = \|\nabla X(0)\| \begin{pmatrix} \cos(\theta-\Theta) \\ \sin(\theta-\Theta) \end{pmatrix}$. From (35) with $A=S_\theta$ for $\theta=\pi/2$ we have that

$$(X(0), \|\nabla X(0)\|, \Theta, X_{11}(0), X_{22}(0), X_{12}(0)) \overset{d}{=} (X(0), \|\nabla X(0)\|, \pi/2-\Theta, X_{22}(0), X_{11}(0), X_{12}(0)) \tag{36}$$

But

$$\begin{aligned}&\mathbb{E}\left(h(X(0))\frac{D^2X(0).(\nabla X(0)^\perp,\nabla X(0)^\perp)}{\|\nabla X(0)\|^2}\mathbf{1}_{\|\nabla X(0)\|>0}\right)\\ =\;&\mathbb{E}\left(h(X(0))\left(X_{11}(0)\sin^2\Theta+X_{22}(0)\cos^2\Theta-2X_{12}(0)\sin\Theta\cos\Theta\right)\mathbf{1}_{\|\nabla X(0)\|>0}\right)\end{aligned}$$

Note that

$$\mathbb{E}\left(h(X(0))X_{11}(0)\sin^2\Theta\mathbf{1}_{\|\nabla X(0)\|>0}\right)=\mathbb{E}\left(h(X(0))X_{11}(0)\mathbf{1}_{\|\nabla X(0)\|>0}\right)-\mathbb{E}\left(h(X(0))X_{11}(0)\cos^2\Theta\mathbf{1}_{\|\nabla X(0)\|>0}\right),$$

so that

$$\begin{aligned}&\mathbb{E}\left(h(X(0))[X_{11}(0)\sin^2\Theta+X_{22}(0)\cos^2\Theta]\mathbf{1}_{\|\nabla X(0)\|>0}\right)\\ =\;&\mathbb{E}\left(h(X(0))X_{11}(0)\mathbf{1}_{\|\nabla X(0)\|>0}\right)\big)+\mathbb{E}\left(h(X(0))[X_{22}(0)-X_{11}(0)]\cos^2\Theta\mathbf{1}_{\|\nabla X(0)\|>0}\right).\end{aligned}$$

From (36),

$$\mathbb{E}\left(h(X(0))[X_{22}(0)-X_{11}(0)]\cos^2\Theta\mathbf{1}_{\|\nabla X(0)\|>0}\right)=-\mathbb{E}\left(h(X(0))[X_{22}(0)-X_{11}(0)]\sin^2\Theta\mathbf{1}_{\|\nabla X(0)\|>0}\right),$$

and hence

$$\mathbb{E}\left(h(X(0))[X_{22}(0)-X_{11}(0)]\cos^2\Theta\mathbf{1}_{\|\nabla X(0)\|>0}\right)=\mathbb{E}\left(h(X(0))\frac{X_{22}(0)-X_{11}(0)}{2}\cos(2\Theta)\mathbf{1}_{\|\nabla X(0)\|>0}\right).$$

Now, from (35) with $A=R_\theta$ for $\theta=\pi/4$,

$$(X(0),\|\nabla X(0)\|,\Theta,X_{12}(0))\stackrel{d}{=}(X(0),\|\nabla X(0)\|,\Theta-\pi/4,\frac{X_{22}(0)-X_{11}(0)}{2}),$$

and it follows that

$$(X(0),\|\nabla X(0)\|,\Theta,\frac{X_{22}(0)-X_{11}(0)}{2})\stackrel{d}{=}(X(0),\|\nabla X(0)\|,\Theta+\pi/4,X_{12}(0)). \tag{37}$$

Therefore,

$$\mathbb{E}\left(h(X(0))[X_{22}(0)-X_{11}(0)]\cos^2\Theta\mathbf{1}_{\|\nabla X(0)\|>0}\right)=-\mathbb{E}\left(h(X(0))\sin(2\Theta)X_{12}(0)\right).$$

The general result follows, by remarking that in view of (36)

$$\mathbb{E}\left(h(X(0))X_{11}(0)\mathbf{1}_{\|\nabla X(0)\|>0}\right)=\mathbb{E}\left(h(X(0))X_{22}(0)\mathbf{1}_{\|\nabla X(0)\|>0}\right).$$

Now, let us assume that the field X can be written as $X=X^{(1)}+X^{(2)}$, with $X^{(1)}$, $X^{(2)}$ iid isotropic and symmetric ($X^{(i)}\stackrel{fdd}{=}-X^{(i)}$). Let us quote that X is itself symmetric and therefore

$$\mathbb{E}\left(h(X(0))\sin(2\Theta)X_{12}(0)\mathbf{1}_{\|\nabla X(0)\|>0}\right)=-\mathbb{E}\left(h(-X(0))\sin(2\Theta)X_{12}(0)\mathbf{1}_{\|\nabla X(0)\|>0}\right).$$

Hence this term vanishes when h is an even function. So let us assume that h is odd. From (35) with $A=S_\theta$ for $\theta=-\pi/2$ we have that

$$(X^{(2)}(0),X_1^{(2)}(0),X_2^{(2)}(0),X_{11}^{(2)}(0),X_{22}^{(2)}(0),X_{12}^{(2)}(0))\stackrel{d}{=}(X^{(2)}(0),-X_2^{(2)}(0),-X_1^{(2)}(0),X_{11}^{(2)}(0),X_{22}^{(2)}(0),X_{12}^{(2)}(0)) \tag{38}$$

Hence by independence between $X^{(1)}$ and $X^{(2)}$

$$(X(0),X_1(0),X_2(0),X_{12}^{(i)}(0))\stackrel{d}{=}(X^{(1)}(0)+X^{(2)}(0),X_1^{(1)}(0)-X_2^{(2)}(0),X_2^{(1)}(0)-X_1^{(2)}(0),X_{12}^{(i)}(0)). \tag{39}$$

By symmetry, we also have $X=X^{(1)}-X^{(2)}$ with $X^{(2)}$ satisfying (36) so that

$$(X(0),X_1(0),X_2(0),X_{12}^{(i)}(0))\stackrel{d}{=}(X^{(1)}(0)-X^{(2)}(0),X_1^{(1)}(0)-X_2^{(2)}(0),X_2^{(1)}(0)-X_1^{(2)}(0),X_{12}^{(i)}(0)). \tag{40}$$

Using the fact that $(X^{(1)},X^{(2)})\stackrel{fdd}{=}(X^{(2)},X^{(1)})$, (39) becomes

$$\begin{aligned}(X(0),X_1(0),X_2(0),X_{12}^{(i)}(0))&\stackrel{d}{=}(X^{(2)}(0)+X^{(1)}(0),X_1^{(2)}(0)-X_2^{(1)}(0),X_2^{(2)}(0)-X_1^{(1)}(0),X_{12}^{(j)}(0))\\ &\stackrel{d}{=}(X(0),-X_2(0),-X_1(0),X_{12}^{(j)}(0))\end{aligned} \tag{41}$$

where $\{i,j\}=\{1,2\}$, while (40) becomes

$$\begin{aligned}(X(0),X_1(0),X_2(0),X_{12}^{(i)}(0))&\stackrel{d}{=}(X^{(2)}(0)-X^{(1)}(0),X_1^{(2)}(0)-X_2^{(1)}(0),X_2^{(2)}(0)-X_1^{(1)}(0),X_{12}^{(j)}(0))\\ &\stackrel{d}{=}(-X(0),-X_2(0),-X_1(0),X_{12}^{(j)}(0))\end{aligned} \tag{42}$$

From (41) and (42) we can deduce that

$$(X(0),X_1(0),X_2(0),X_{12}^{(j)}(0))\stackrel{d}{=}(-X(0),X_1(0),X_2(0),X_{12}^{(j)}(0)).$$

therefore, since h is odd

$$\mathbb{E}\left(h(X(0))\sin(2\Theta)X_{12}^{(i)}(0)\mathbf{1}_{\|\nabla X(0)\|>0}\right)=0.$$

Hence we conclude that

$$\mathbb{E}\left(h(X(0))\sin(2\Theta)X_{12}(0)\mathbf{1}_{\|\nabla X(0)\|>0}\right)=\sum_{i=1}^{2}\mathbb{E}\left(h(X(0))\sin(2\Theta)X_{12}^{(i)}(0)\mathbf{1}_{\|\nabla X(0)\|>0}\right)=0.$$

Note also that we have $\mathbb{E}(\mathrm{LTC}_X(U))=0$, by taking $h=1$ and using symmetry in the above formula.

Moreover, when $X(0)$ admits a density $p_{X(0)}$, we can further write

$$\begin{aligned}\mathbb{E}(h(X(0))|X_i(0)|) &= \int_{\mathbb{R}} h(t)\mathbb{E}(|X_i(0)||X(0)=t)p_{X(0)}(t)dt,\\ \mathbb{E}(h(X(0))X_{ii}(0)\mathbf{1}_{\|\nabla X(0)\|>0}) &= \int_{\mathbb{R}} h(t)\mathbb{E}(X_{ii}(0)\mathbf{1}_{\|\nabla X(0)\|>0}|X(0)=t)p_{X(0)}(t)dt,\end{aligned}$$

such that for any continuous bounded function h,

$$\begin{aligned}\int_{\mathbb{R}} h(t)\mathbb{E}(\mathrm{TC}(\partial E_X(t,U)))\,dt &= \int_{\mathbb{R}} h(t)\mathbb{E}(|X_i(0)||X(0)=t)p_{X(0)}(t)dt \quad \text{and}\\ \int_{\mathbb{R}} h(t)\mathbb{E}(\mathrm{TC}(\partial E_X(t,U)))\,dt &= \int_{\mathbb{R}} h(t)\left(-\mathcal{L}(U)\mathbb{E}(X_{ii}(0)\mathbf{1}_{\|\nabla X(0)\|>0}|X(0)=t)p_{X(0)}(t)\right)dt,\end{aligned}$$

implying the stated equalities for almost every $t\in\mathbb{R}$. □

Remark 1: Let us quote that our assumption on symmetry is always satisfied for symmetric isotropic infinitely divisible random fields, in particular centered isotropic Gaussian fields. Let us quote also that, for Gaussian random field, the conditional law of Θ with respect to $(X(0),\|\nabla X(0)\|,D^2X(0))$ is actually uniform. Moreover, invariance under all orthogonal matrices is a direct consequence from invariance under all rotation matrices. This follows from the fact that its covariance function must be radial.

Remark 2: As we were finishing this paper, we found a similar recent result (see Corollary 2.3 of [23]) that should rewrite in our stationary setting, under additional assumption on X, as

$$\mathbb{E}\left(\int_{\mathbb{R}} h(t)\overline{\chi(E_X(t))}\,dt\right)=-\mathbb{E}\left(h(X(0))\left[\sum_{i=1}^{2}\mathbb{1}_{\nabla X(0)\in Q_i}X_{ii}(0)\right]+h'(X(0))\left[\sum_{i=1}^{2}\mathbb{1}_{\nabla X(0)\in Q_i}X_i(0)^2\right]\right),$$

for $h:\mathbb{R}\to\mathbb{R}$ a C^1 function with compact support, $Q_1=\{x=(x_1,x_2);x_2<x_1<0\}$ and $Q_2=\{x=(x_1,x_2);x_1<x_2<0\}$, and where $\overline{\chi(E_X(t))}$ stands for an Euler Characteristic density. Under the assumption that X is also isotropic, according to Remark 2.5 of [23],

$$\mathbb{E}\left(h'(X(0))\mathbb{1}_{\nabla X(0)\in Q_i}X_i(0)^2\right)=\frac{\pi-2}{16\pi}\mathbb{E}\left(h'(X(0))\|\nabla X(0)\|^2\right).$$

Note that by stationarity

$$\begin{aligned}\mathbb{E}\left(h'(X(0))X_i(0)^2\right) &= \int_0^1\mathbb{E}\left(h'(X(te_i))X_i(te_i)^2\right)dt=\mathbb{E}\left(\int_0^1 h'(X(te_i))X_i(te_i)^2dt\right)\\ &= -\mathbb{E}\left(\int_0^1 h(X(te_i))X_{ii}(te_i)^2dt\right)=-\mathbb{E}\left(h(X(0))X_{ii}(0)\right),\end{aligned}$$

integrating by parts and using $\mathbb{E}\left(h(X(te_i))X_i(te_i)\right)=\mathbb{E}\left(h(X(0))X_i(0)\right)$ for all $t\in[0,1]$.

Moreover, by (37) we clearly have

$$\mathbb{E}\left(h(X(0))\mathbb{1}_{\nabla X(0)\in Q_i}[X_{22}(0)-X_{11}(0)]\right)=2\mathbb{E}\left(h(X(0))\mathbb{1}_{R_{-\pi/4}\nabla X(0)\in Q_i}X_{12}\right),$$

that vanishes for odd h, using our additionnal divisibility and symmetry assumptions. Hence we may deduce that

$$\mathbb{E}\left(h(X(0))\mathbb{1}_{\nabla X(0)\in Q_i}X_{22}(0)\right)=\mathbb{E}\left(h(X(0))\mathbb{1}_{\nabla X(0)\in Q_i}X_{11}(0)\right),$$

and using rotations of angle $\pi/2$, we get

$$\mathbb{E}\left(h(X(0))\mathbb{1}_{\nabla X(0)\in Q_i}X_{ii}(0)\right)=\mathbb{E}\left(h(X(0))\mathbb{1}_{\Theta\in\pi+\frac{\pi}{4}(i-1,i)}X_{ii}(0)\right)=\frac{1}{8}\mathbb{E}\left(h(X(0))X_{ii}(0)\right).$$

It follows that

$$\mathbb{E}\left(\int_{\mathbb{R}} h(t)\overline{\chi(E_X(t))}\,dt\right) = -\sum_{i=1}^{2}\left(\frac{1}{8}-\frac{\pi-2}{8\pi}\right)\mathbb{E}\left(h(X(0))X_{ii}(0)\right) = \frac{1}{2\pi\mathcal{L}(U)}\int_{\mathbb{R}} h(t)\mathbb{E}(\mathrm{TC}(\partial E_X(t),U))\,dt,$$

by Theorem 5, since it is also assumed that $\|\nabla X(0)\| > 0$ a.s. .

Example. Let X be a stationary centered C^2 isotropic Gaussian random field. Then, $X_1(0)$ being independent from $X(0)$, we get $\mathbb{E}\left(|X_1(0)||X(0)\right) = \mathbb{E}\left(|X_1(0)|\right) = \sqrt{\frac{2\lambda_2}{\pi}}$, where $\lambda_2 = \mathrm{Var}(X_1(0)) > 0$ denotes the second spectral moment. Moreover,

$$\mathbb{E}\left(X_{11}(0)|X(0)\right) = \frac{\mathrm{Cov}(X(0),X_{11}(0))}{\mathrm{Var}(X(0))}X(0) = \frac{\partial_1^2\rho_X(0)}{\rho_X(0)}X(0) = \frac{-\lambda_2}{\sigma^2}X(0),$$

where $\rho_X(x) = \mathrm{Cov}(X(x),X(0))$ and $\sigma^2 = \rho_X(0)$. Hence, since $\mathbb{P}(\|\nabla X(0)\| = 0) = 0$, we get in this case that, for almost every $t \in \mathbb{R}$,

$$\mathbb{E}(\mathrm{Per}(E_X(t),U)) = \mathcal{L}(U)\sqrt{\frac{\pi\lambda_2}{2}}\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{t^2}{2\sigma^2}} \tag{43}$$

$$\mathbb{E}(\mathrm{TC}(\partial E_X(t),U)) = \mathcal{L}(U)\frac{\lambda_2}{\sigma^2}t\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{t^2}{2\sigma^2}} \tag{44}$$

(45)

Let us emphasize that this last expression is exactly the formula obtained for $2\pi\mathbb{E}\left(\chi\left(E_X(t)\cap\overline{U}\right)\right)$, stated for all $t \in \mathbb{R}$, under additional assumptions on X (see (3.2.8) of [2] for instance), where χ denotes the DT (Differential Topology) Characteristic of the set.
Examples of such stationary isotropic random fields with comparisons between the theoretical values of $\mathbb{E}(\mathrm{TC}(\partial E_X(t),U))/(2\pi\mathcal{L}(U))$ (corresponding to the mean Euler Characteristic density of excursion sets) and an empirical estimate of Euler Characteristic on the square of fixed size $[0,1]$ are shown on Figures 6 and 7 (with $\sigma^2 = 1$ and $\lambda_2 = 2T^2$). The captions of the figures give the practical and technical details of the simulations. Note that in view of the covariance functions, a scaling relation may be set between T and the size of the square, explaining the convergence without boundary effects as T increases.

4.3. **Smooth shot noise random fields.** As in Section 3.2, we consider here a shot noise random field defined on $\mathbb{R}^2$ by

$$\forall x \in \mathbb{R}^2, \quad X_\Phi(x) = \sum_{i\in I} g_{m_i}(x-x_i),$$

where $\Phi = \{(x_i,m_i)\}_{i\in I}$ is a Poisson point process on $\mathbb{R}^2\times\mathbb{R}^d$ of intensity $\lambda\mathcal{L}\times F$, with $\lambda > 0$ real, $\mathcal{L}$ the Lebesgue measure on $\mathbb{R}^2$ and F a probability measure on $\mathbb{R}^d$. In order to get explicit formulas we have to make an assumption of isotropy. Moreover, since smooth shot noise random fields do not always admit a probability density (we have discussed this through several examples in our first paper [10]) we have to work with their characteristic functions.

Theorem 6. *Let assume that $g : \mathbb{R}^2\times\mathbb{R}^d \to \mathbb{R}$ is a measurable function such that for F-almost every $m \in \mathbb{R}^d$ the functions $g_m := g(\cdot,m)$ are C^3 on $\mathbb{R}^2$ satisfying*

$$\int_{\mathbb{R}^2\times\mathbb{R}^d} |D^{\mathbf{j}}g_m(x)|\,dx\,F(dm) < +\infty, \tag{46}$$

for all $\mathbf{j} = (j_1,j_2) \in \mathbb{N}^2$ with $|\mathbf{j}| = j_1+j_2 \le 3$ and where $D^{\mathbf{j}}g_m = \frac{\partial^{|\mathbf{j}|}g_m}{\partial x_1^{j_1}\partial x_2^{j_2}}$. Then X_Φ is a.s. a stationary C^2 field such that D^2X has finite expectation, ensuring the assumptions of Theorem 5.
We assume moreover that $m = (\tilde{m},\theta) \in \mathbb{R}^{d-1}\times[0,2\pi] \subset \mathbb{R}^d$ and $F(dm) = G(d\tilde{m})F_\Theta(d\theta)$, with F_Θ the uniform law on $[0,2\pi]$, and

$$g(\cdot\,,m) = g(\cdot\,,(\tilde{m},\theta)) = \tilde{g}(R_\theta\cdot\,,\tilde{m}),$$

for F-a.e. m, with $\tilde{g}(S_0\cdot\,,\tilde{m}) = \tilde{g}(\cdot\,,\tilde{m})$, recalling that $S_0 = \begin{pmatrix}1&0\\0&-1\end{pmatrix}$ and $R_\theta = \begin{pmatrix}\cos\theta&-\sin\theta\\\sin\theta&\cos\theta\end{pmatrix}$. Then X_Φ is isotropic. When moreover X_Φ is symmetric, the Fourier transform of the mean perimeter and of the

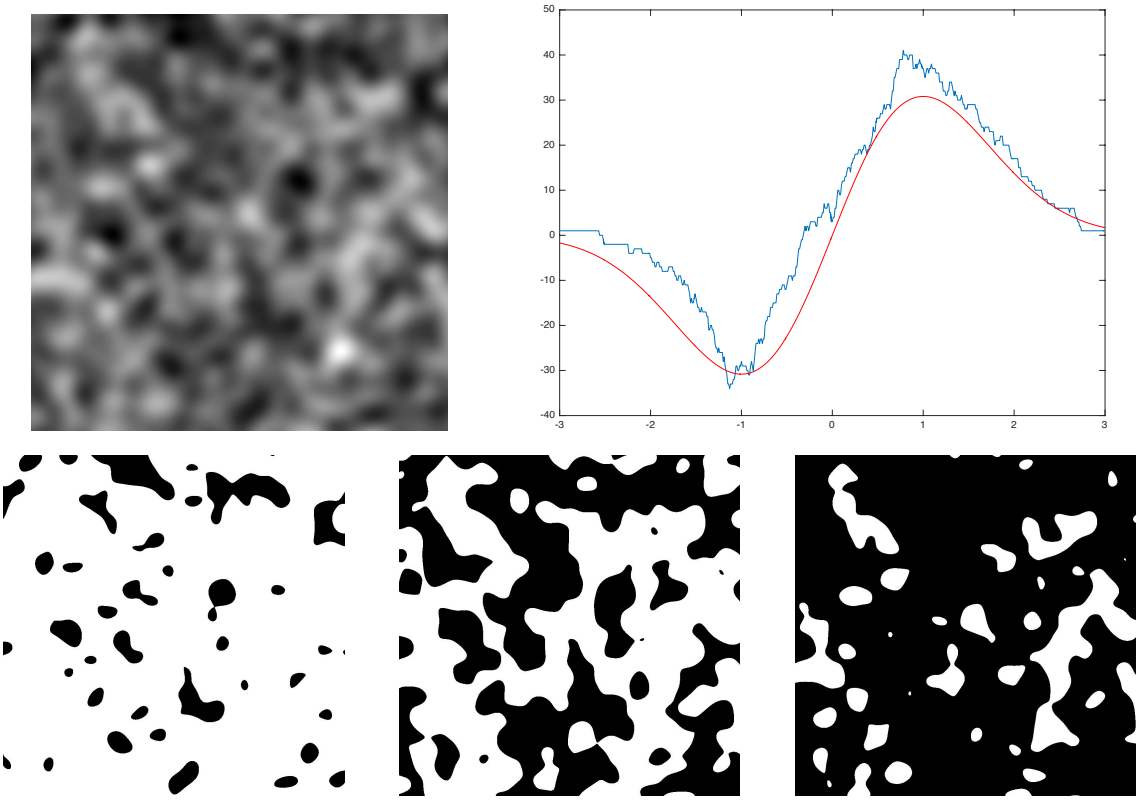

FIGURE 6. Gaussian random field with covariance $\rho(x) = e^{-T^2\|x\|^2}$ for $T = 10$. This sample has been obtained using Matlab, with a discretized domain of size $2^{10} \times 2^{10}$ pixels, using circulant embedding matrix. Top right figure: empirical Euler Characteristic as a function of the level t (computed thanks to the Matlab function `bweuler`), compared with the theoretical value (red one) of Equation (44). Bottom line: Three excursion sets corresponding respectively from left to right to the level $t = -1$, $t = 0$ (that is the "critical level" where the Euler Characteristic turns from negative to positive), and $t = 1$.

mean total curvature are given for all $u \in \mathbb{R}$ *by, for all* $j = 1, 2$,

$$\begin{aligned}\mathbb{E}(\mathrm{LP}_{X_\Phi}(h_u, U)) &= -\frac{1}{2}\int_0^{+\infty} \frac{1}{v}\frac{\partial \varphi_{X_\Phi(0),\partial_j X_\Phi(0)}(u,v)}{\partial v} dv \\ \mathbb{E}(\mathrm{LTC}_{X_\Phi}(h_u, U)) &= -\lambda \mathcal{L}(U)\mathbb{E}\left(e^{iuX_\Phi(0)}\right)\left(\int_{\mathbb{R}^d}\int_{\mathbb{R}^2} \partial_j^2 g_m(x) e^{iug_m(x)}\, dx\, F(dm)\right),\end{aligned}$$

where $h_u(t) = e^{iut}$,

$$\varphi_{X_\Phi(0),\partial_j X_\Phi(0)}(u,v) = \mathbb{E}\left(e^{iuX_\Phi(0)+iv\partial_j X_\Phi(0)}\right) = \exp\left(\lambda \int_{\mathbb{R}^d\times\mathbb{R}^2} [e^{i[ug_m(x)+v\partial_j g_m(x)]} - 1]F(dm)dx\right),$$

and the notation ∂_j, *respectively* ∂_j^2, *stands for* $\frac{\partial}{\partial x_j}$, *respectively* $\frac{\partial^2}{\partial x_j^2}$.

Remark: Note that when $g(A\cdot, m) = g(\cdot, m)$, for all orthogonal matrix A, X_Φ has the same law as the shot noise random field given with marks $\tilde{m} \in \mathbb{R}^{d-1}$ of law $G(d\tilde{m})$.

Proof. Following similar arguments as in Proposition 3 of [10], (46) will ensure that X_Φ is a.s. a stationary C^2 field such that X, ∇X and D^2X have finite expectations, and we can differentiate under the sum. In particular, for all $j = 1, 2$,

$$\partial_j X_\Phi(x) = \sum_{i\in I} \partial_j g_{m_i}(x - x_i) \text{ and } \partial_j^2 X_\Phi(x) = \sum_{i\in I} \partial_j^2 g_{m_i}(x - x_i).$$

Hence the general formula of Theorem 5 is valid for X_Φ.

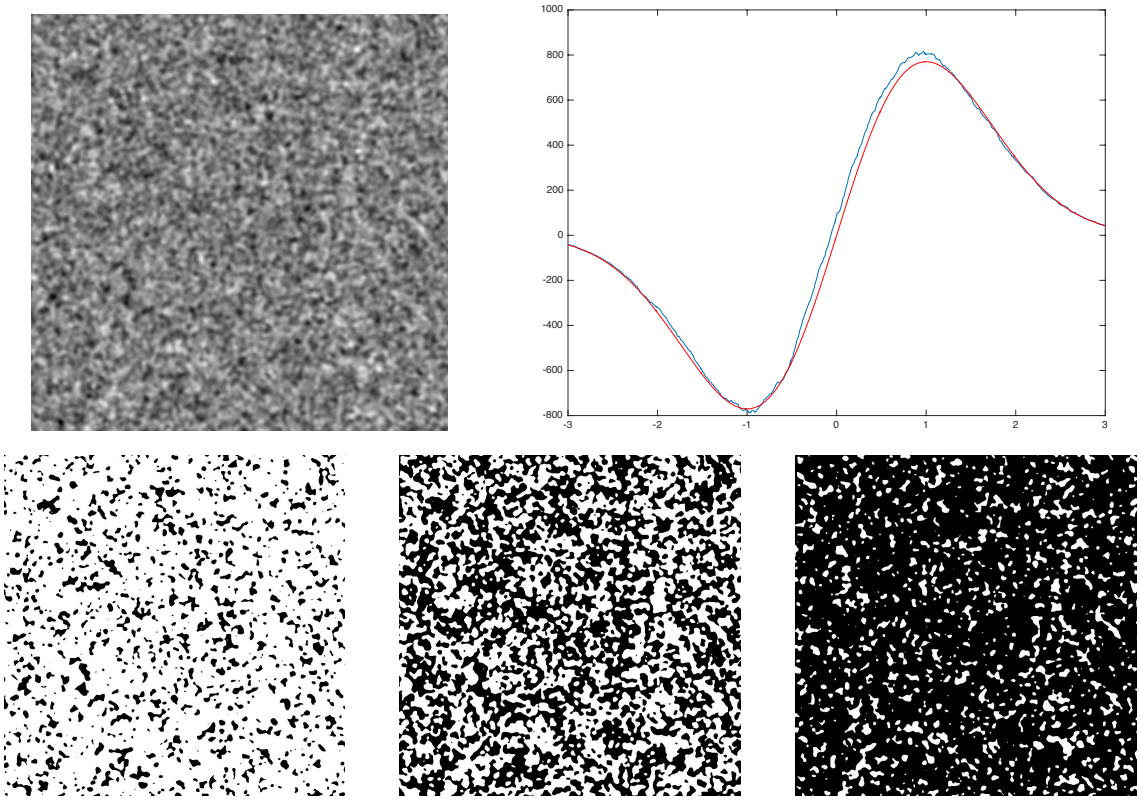

FIGURE 7. Gaussian random field with covariance $\rho(x) = e^{-T^2\|x\|^2}$ for $T = 100$. This sample has been obtained using Matlab, with a discretized domain of size $2^{10} \times 2^{10}$ pixels, using circulant embedding matrix. Top right figure: empirical Euler Characteristic as a function of the level t (computed thanks to the Matlab function `bweuler`), compared with the theoretical value (red one) of Equation (44). Bottom line: Three excursion sets corresponding respectively from left to right to the level $t = -1$, $t = 0$ (that is the "critical level" where the Euler Characteristic turns from negative to positive), and $t = 1$.

Under the additional assumption on the kernel, we can prove isotropy. Actually, for any $k \geq 1$, $u_1, \cdots, u_k \in \mathbb{R}$ and $y_1, \cdots, y_k \in \mathbb{R}^2$, one has

$$\mathbb{E}\left(e^{i\sum_{j=1}^k u_j X_\Phi(y_j)}\right) = \exp\left(\lambda \int_{\mathbb{R}^d}\int_{\mathbb{R}^2} (e^{i\sum_{j=1}^k u_j g_m(y_j - x)} - 1)\, dx\, F(dm)\right).$$

Hence, for any orthogonal matrix A, by the change of variables $x = Ay$,

$$\mathbb{E}\left(e^{i\sum_{j=1}^k u_j X_\Phi(Ay_j)}\right) = \exp\left(\lambda \int_{\mathbb{R}^d}\int_{\mathbb{R}^2} (e^{i\sum_{j=1}^k u_j g_m(A(y_j - y))} - 1)\, dy\, F(dm)\right)$$

Note that there exists $\theta_0 \in [0, 2\pi)$ such that $A = S_{\theta_0}$ or $A = R_{\theta_0}$ but for $m = (\tilde{m}, \theta)$,

$$g_m \circ S_{\theta_0} = \tilde{g}_{\tilde{m}} \circ R_\theta S_{\theta_0} = \tilde{g}_{\tilde{m}} \circ S_0 R_\theta S_{\theta_0} = \tilde{g}_{\tilde{m}} \circ R_{-\theta-\theta_0}.$$

Since F_Θ is the uniform law, we may assume without loss of generality that

$$\begin{aligned}\mathbb{E}\left(e^{i\sum_{j=1}^k u_j X_\Phi(Ay_j)}\right) &= \exp\left(\lambda \int_{\mathbb{R}^d}\int_{\mathbb{R}^2} (e^{iu\sum_{j=1}^k u_j g_m(R_{\theta_0}(y_j - y))} - 1)\, dy\, F(dm)\right) \\ &= \exp\left(\lambda \int_{\mathbb{R}^d}\int_{\mathbb{R}^2} (e^{iu\sum_{j=1}^k u_j g_m(y_j - y)} - 1)\, dy\, F(dm)\right) = \mathbb{E}\left(e^{i\sum_{j=1}^k u_j X_\Phi(y_j)}\right)\end{aligned}$$

using the fact that $R_\theta R_{\theta_0} = R_{\theta_0+\theta}$ and F_Θ uniform.

It follows that by Theorem 5, for any h bounded continuous function on $\mathbb{R}$, one has for all $j = 1, 2$,

$$\mathbb{E}(\mathrm{LP}_{X_\Phi}(h, U)) = \frac{\pi}{2}\mathcal{L}(U)\mathbb{E}(h(X_\Phi(0))|\partial_j X_\Phi(0)|)$$

and assuming moreover that X_Φ is symmetric,

$$\mathbb{E}(\mathrm{LTC}_{X_\Phi}(h,U) = -\mathcal{L}(U)\mathbb{E}\left(h(X_\Phi(0))\partial_j^2 X_\Phi(0)\right).$$

Taking $h_u = e^{iu\cdot}$ for $u \in \mathbb{R}$ we obtain

$$\begin{aligned}\mathbb{E}(\mathrm{LP}_{X_\Phi}(h_u,U)) &= \frac{\pi}{2}\mathcal{L}(U)\mathbb{E}(e^{iuX_\Phi(0)}|\partial_j X_\Phi(0)|)\\ &= \frac{\pi}{2}\mathcal{L}(U)\times -\frac{1}{\pi}\int_0^{+\infty}\frac{1}{v}\frac{\partial\varphi_{X_\Phi(0),\partial_j X_\Phi(0)}(u,v)}{\partial v}dv,\end{aligned}$$

according to Proposition 2 of [11], using the fact that $\mathbb{E}(|\partial_j X_\Phi(0)|) < +\infty$ and $\varphi_{X_\Phi(0),\partial_j X_\Phi(0)}(u,v) = \varphi_{X_\Phi(0),\partial_j X_\Phi(0)}(u,-v)$ by isotropy. Moreover,

$$\mathbb{E}(\mathrm{LTC}_{X_\Phi}(h_u,U)) = -\mathcal{L}(U)\mathbb{E}\left(e^{iuX_\Phi(0)}\partial_j^2 X_\Phi(0)\right) = i\mathcal{L}(U)\frac{\partial\varphi_{X_\Phi,\partial_j^2 X_\Phi}}{\partial v}(u,0),$$

where $\varphi_{X_\Phi,\partial_j^2 X_\Phi}$ is the characteristic function of $(X_\Phi(0),\partial_j^2 X_\Phi(0))$ given by, for $(u,v)\in\mathbb{R}^2$

$$\begin{aligned}\varphi_{X_\Phi,\partial_j^2 X_\Phi}(u,v) &= \mathbb{E}\left(e^{i[uX_\Phi(0)+v\partial_j^2 X_\Phi(0)]}\right)\\ &= \exp\left(\lambda\int_{\mathbb{R}^d}\int_{\mathbb{R}^2}(e^{iug_m(x)+iv\partial_j^2 g_m(x)}-1)\,dx\,F(dm)\right).\end{aligned}$$

Then,

$$\mathbb{E}(\mathrm{LTC}_{X_\Phi}(h_u,U)) = -\mathcal{L}(U)\lambda\left(\int_{\mathbb{R}^d}\int_{\mathbb{R}^2}\partial_j^2 g_m(x)e^{iug_m(x)}\,dx\,F(dm)\right)\varphi_{X_\Phi}(u),$$

with $\varphi_{X_\Phi} = \varphi_{X_\Phi,\partial_j^2 X_\Phi}(\cdot,0)$ the characteristic function of $X_\Phi(0)$. □

Let us also mention that in the recent paper [23] such a formula is also proposed in a similar isotropic framework in Theorem 3.3.

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