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LEVEL TOTAL CURVATURE INTEGRAL: EULER CHARACTERISTIC AND 2D RANDOM FIELDS

HERMINE BIERMÉ AND AGNÈS DESOLNEUX

ABSTRACT. We introduce the level total curvature function associated with a real valued function f defined on the plane \mathbb{R}^2 as the function that, for any level $t \in \mathbb{R}$, computes the total (signed) curvature of the boundary of the excursion set of f above level t . Thanks to the Gauss-Bonnet theorem, the total curvature is directly related to the Euler Characteristic of the excursion set. We show that the level total curvature function can be explicitly computed in two different frameworks: piecewise constant functions (also called here elementary functions) and smooth (at least C^2) functions. Considering 2D random fields (in particular considering shot noise random fields), we will compute their mean total curvature function, and this will provide new explicit computations of the mean Euler Characteristic of excursion sets, beyond the Gaussian framework.

CONTENTS

1.	Introduction	1
2.	Total curvature and Euler Characteristic of excursion sets	2
2.1.	Total curvature: Definition and properties	2
2.2.	Link with Euler Characteristic	4
2.3.	Link with Euler Integral	5
2.4.	The case of elementary functions	6
2.5.	The case of smooth functions	10
3.	Elementary Shot noise random fields	11
3.1.	Level total curvature of an elementary shot noise random field	11
3.2.	Explicit computations	17
4.	Mean level total curvature of smooth random fields	21
4.1.	A general result for smooth stationary random fields	22
4.2.	Smooth shot noise random fields	29
4.3.	Link with discretization of a smooth Gaussian field	30
	References	34

1. INTRODUCTION

Computing the Euler Characteristic of excursion sets of random fields is a problem that has received much attention. Indeed, in many applications, the Euler Characteristic is a very useful index of the geometry of the field, as explained for instance in the review paper of R. Adler [2], or in the papers of K. Worsley [29] or [30] where applications in astrophysics or in brain imaging are mentioned.

In the framework of Gaussian random field, the Euler Characteristic of excursion sets is well-known and well-studied. For stationary isotropic Gaussian random fields, an explicit formula for any level t may be set for the expectation of the Euler Characteristic, only depending on the variance and the second spectral moment of the field. This is an important result with many statistical applications. In particular, for large levels t , the Euler Characteristic gives a good approximation of the probability that the suprema of the field is greater than t and can therefore be used as a p-value: this is the Euler Characteristic heuristic (see [5] for instance). In a “tour de force”, a Central Limit Theorem has recently been established in [15] that proves the accuracy of the estimation over only one sample path as the size of the observation is growing. There are also some interesting results apart from the Gaussian framework for χ^2 , F and t -fields [29] as well as stable [3] or infinitely divisible random fields [4] for instance. Most of general results rely on strong smoothness regularity assumptions and on conditional distribution densities that are often difficult to evaluate for non-Gaussian fields.

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Now, in this paper, we will be interested in another family of infinitely divisible random fields, that are not Gaussian, namely the *shot noise random fields*. We will work here in dimension $n = 2$. A shot noise random field is defined on \mathbb{R}^2 by

$$\forall x \in \mathbb{R}^2, \quad X(x) = \sum_i g_{m_i}(x - x_i),$$

where the x_i are the points of an homogeneous Poisson point process of intensity λ in \mathbb{R}^2 , and the m_i are “marks”, independent of the Poisson point process. Given a bounded open subset U of \mathbb{R}^2 , we will consider the excursion set of X of level $t \in \mathbb{R}$ in U , defined by

$$E_X(t; U) := \{x \in U \text{ s.t. } X(x) \geq t\}.$$

Despite its “global” definition (the number of connected components minus the number of holes), the Euler Characteristic of an excursion set is in fact a purely local quantity related, by Morse theory, to the number of critical points of X in U , or, by the Gauss-Bonnet theorem, to the total curvature of the boundary of the excursion set. Here, we will extensively use this second equivalence to obtain explicit computations of the mean Euler Characteristic of the excursion sets of a shot noise random field.

More precisely, our contributions are fourfold: (1) We propose a general definition of the *level total curvature integral* of a function, that allows to compute the total curvature (and therefore the Euler Characteristic) of its excursion sets for almost every level. (2) We give explicit computations for *elementary* shot noise random fields, where the functions g_{m_i} are piecewise constant functions (with piecewise smooth discontinuity set). This allows us to generalize results of the literature (about the Boolean model [23], or about “random configurations” [13]). We also give explicit results for isotropic elementary and smooth shot noise random fields. (3) For smooth stationary random fields (not necessarily shot noise ones), we give a new proof of the equivalence of the different definitions of the mean Euler Characteristic of the excursion sets, relying only on the stationarity of the random field. (4) For Gaussian stationary and isotropic random field, we show that its mean total curvature (divided by 2π) when the field is discretized on an hexagonal tiling, converges, when the size of the hexagons goes to 0, to the well-known formula for the Euler Characteristic of its excursion sets.

Let us finally emphasize that we have made here the deliberate choice of not working in the weakest possible functional framework. Our goal is to work with piecewise smooth functions (like the indicator function of a set having a piecewise C^2 boundary for instance). But we believe some of our results can be extended to functions with a weakest regularity. Let us also mention the recent work of R. Lachièze-Rey in [18] and [19] that relates the Euler Characteristic to the three-point joint distribution of the random field. And also the even more recent paper [20] where R. Lachièze-Rey gives formulas for the Euler Characteristic of isotropic shot noise random field that are a.s. Morse functions.

2. TOTAL CURVATURE AND EULER CHARACTERISTIC OF EXCURSION SETS

2.1. Total curvature: Definition and properties. We first recall here some basic facts and definitions about plane curves. Let U be an open bounded set of \mathbb{R}^2 , and let Γ be a continuous oriented simple curve on U of finite length and piecewise C^2 , with a finite number of “corners”. This curve can also be “open”: in this case it can be arc length parametrized by a continuous piecewise C^2 function $\gamma : (0, L) \rightarrow U$ in such a way that there is a finite set of points $c_1, \dots, c_k \in (0, L)$ at which γ is not C^2 (but is continuous). For any $s \in (0, L) \setminus \{c_1, \dots, c_k\}$, the tangent vector at point $x = \gamma(s)$ is given by

$$T_\Gamma(x) = \gamma'(s) \quad \text{with} \quad \|\gamma'(s)\| = 1,$$

where $\|\cdot\|$ is the usual Euclidean norm in \mathbb{R}^2 .

The unit normal vector, denoted by $\nu_\Gamma(x)$, is then defined as the $+\frac{\pi}{2}$ rotation of $T_\Gamma(x)$, also denoted as

$$\nu_\Gamma(x) = T_\Gamma(x)^\perp = \gamma'(s)^\perp,$$

in such a way that $(T_\Gamma(x), \nu_\Gamma(x))$ is a direct orthonormal basis of \mathbb{R}^2 .

The signed curvature $\kappa_\Gamma(x)$ of Γ at $x = \gamma(s)$ is then defined as

$$\kappa_\Gamma(x) = \langle \gamma''(s), \gamma'(s)^\perp \rangle = \langle \gamma''(s), \nu_\Gamma(x) \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the usual Euclidean scalar product on \mathbb{R}^2 .

At the points of discontinuity of the tangent, we assume moreover that $\lim_{s \rightarrow c_i^-} \gamma'(s) = \gamma'(c_i^-)$ and $\lim_{s \rightarrow c_i^+} \gamma'(s) = \gamma'(c_i^+)$, are well defined in the unit sphere S^1 and are not colinear (no cusp), such that we define the *turning angle* at $x = \gamma(c_i)$ as the angle $\alpha_\Gamma(x) \in (-\pi, \pi)$ between the tangent “before” x and the one “after” x , that is

$$\alpha_\Gamma(x) = \text{Arg } \gamma'(c_i^+) - \text{Arg } \gamma'(c_i^-) \quad \in (-\pi, \pi),$$

where Arg is the argument (angle determination) of points on the unit circle S^1 . We write

$$\mathcal{C}_\Gamma = \{\gamma(c_1), \dots, \gamma(c_k)\}$$

the set of these points, and we call them *corners* of the curve Γ .

The curve Γ can also be “closed”: in that case, the function γ is defined on the closed interval $[0, L]$ (meaning, following Do Carmo [14] p.30, that it is the restriction of a piecewise C^2 function on an open interval containing $[0, L]$) and such that all derivatives exist and agree at 0 and L , that is

$$\gamma(0) = \gamma(L), \quad \gamma'(0) = \gamma'(L) \quad \text{and} \quad \gamma''(0) = \gamma''(L).$$

All the above definitions of tangent, normal, curvature and turning angle remain the same.

We will also sometimes just say that Γ is an oriented simple curve piecewise C^2 in U if Γ may be written as a finite disjoint union of $\Gamma_1, \dots, \Gamma_n$ with each Γ_i being a continuous oriented simple curve piecewise C^2 in U with finite length L_i and a finite number of corners. It follows that Γ is of finite length $L = L_1 + \dots + L_n$ and may also be arc length parametrized. It has also a finite number of corners given by $\mathcal{C}_\Gamma = \mathcal{C}_{\Gamma_1} \cup \dots \cup \mathcal{C}_{\Gamma_n}$, with disjoint union. Its curvature is then defined for \mathcal{H}^1 -almost every $x \in \Gamma$ by $\kappa_\Gamma(x) = \kappa_{\Gamma_i}(x)$ if $x \in \Gamma_i$, where \mathcal{H}^1 denotes the one dimensional Hausdorff measure (the length measure) on the curve.

In the following U will denote an open bounded square of \mathbb{R}^2 , of the form $x_0 + (0, T)^2$, for some $x_0 \in \mathbb{R}^2$ and some $T > 0$. Its closure is denoted $\bar{U} = x_0 + [0, T]^2$ and ∂U will denote its boundary.

Definition 1 (Total curvature). *We say that a curve Γ is of finite total curvature in U , if it is an oriented simple curve piecewise C^2 on U whose curvature κ_Γ is integrable on $\Gamma \cap U$, that is $\kappa_\Gamma \in L^1(\Gamma \cap U, \mathcal{H}^1)$. It follows that the total absolute curvature of Γ in U is*

$$\text{TaC}(\Gamma, U) := \int_{\Gamma \cap U} |\kappa_\Gamma(x)| \mathcal{H}^1(dx) + \sum_{x \in \mathcal{C}_\Gamma \cap U} |\alpha_\Gamma(x)| < +\infty.$$

We then define the total curvature of Γ in U as

$$\text{TC}(\Gamma, U) := \int_{\Gamma \cap U} \kappa_\Gamma(x) \mathcal{H}^1(dx) + \sum_{x \in \mathcal{C}_\Gamma \cap U} \alpha_\Gamma(x).$$

with $\int_{\Gamma \cap U} \kappa_\Gamma(x) \mathcal{H}^1(dx) = \int_0^L \kappa_\Gamma(\gamma(s)) ds$, where $\gamma : (0, L) \rightarrow \mathbb{R}^2$ is a parametrization of $\Gamma \cap U$ by arc length.

Notice that the definition of TaC is the same as the one introduced by Milnor in [24]. But here, in this work, we will pay a particular attention to the signed total curvature, and not to its absolute value. Actually, our definition of total curvature is the same as to the one of Santaló in [25], Chapter 7. The total curvature is intrinsic, it doesn't depend on the parametrization of the curve. But it depends on its orientation: if we reverse the orientation of the curve then its total curvature is changed into its opposite.

Examples:

- Consider a positively oriented unit radius circle $\Gamma = \{x \in \mathbb{R}^2; \|x\| = 1\} = \partial D(0, 1)$, where $D(0, 1)$ denotes the disk of center 0 and radius 1. Then Γ is a closed curve. An arc length parametrization of the curve is given by $\gamma : s \in [0, 2\pi] \rightarrow (\cos(s), \sin(s))$ such that $\mathcal{C}_\Gamma = \emptyset$. Now, for $r > 0$, the circle of radius r is given by $r\Gamma$ and admits $\gamma_r : s \in [0, 2\pi r] \rightarrow r\gamma(s/r)$ as an arc length parametrization. It follows that when $r\Gamma \subset U$,

$$\text{TC}(r\Gamma, U) = \text{TaC}(r\Gamma, U) = \int_0^{2\pi r} \frac{1}{r} dt = 2\pi.$$

- A positively oriented boundary of a unit square $\Gamma = \partial(0, 1)^2$ may be parametrized by the piecewise C^2 function γ defined on $[0, 4]$ by $\gamma(s) = (1, s)$ for $s \in [0, 1]$, $\gamma(s) = (2 - s, 1)$ for $s \in [1, 2]$, $\gamma(s) = (0, 3 - s)$ for $s \in [2, 3]$, $\gamma(s) = (s - 3, 0)$ for $s \in [3, 4]$. The curve Γ is closed, it has 4 corners given by $\mathcal{C}_\Gamma = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$. Now, for $r > 0$, as previously, the positively oriented boundary of a square of side r admits $\gamma_r : s \in [0, 4r] \rightarrow r\gamma(s/r)$ as an arc length parametrization and the corners $\mathcal{C}_{r\Gamma}$ are simply given by $r\mathcal{C}_\Gamma$. At the corners we have $\alpha_{r\Gamma}(x) = \pi/2$ and $\kappa_{r\Gamma}(x) = 0$ for points that are not corners, such that when $r\Gamma \subset U$, we also have

$$\text{TC}(r\Gamma, U) = \text{TaC}(r\Gamma, U) = 4 \times \frac{\pi}{2} = 2\pi.$$

Remark: It is a well-known result of differential geometry of plane curves that the total curvature of any continuous simple closed curve is 2π or -2π (depending on the orientation of the curve). This result is sometimes called Hopf's Umlaufsatz, or also the theorem of turning tangents ([14] p.396).

Definition 2 (Level total curvature integral). *Let f be a real-valued function defined on U . For $t \in \mathbb{R}$, we define the excursion set of f for the level t as*

$$E_f(t; U) := \{x \in U \text{ s.t. } f(x) \geq t\}.$$

We assume that for almost every $t \in \mathbb{R}$, the boundary $\partial E_f(t; U)$ is a simple curve piecewise C^2 , positively oriented in such a way that the normals are oriented towards $E_f(t; U)$. We moreover assume that

$$\text{LTaC}_f(U) := \int_{\mathbb{R}} \text{TaC}(\partial E_f(t; U), U) dt < +\infty.$$

Note that this implies that $\partial E_f(t; U)$ is a curve of finite total curvature in U in the sense of Definition 1 for a.e. $t \in \mathbb{R}$. We say that the function f is of finite level total curvature integral (on U). It follows that the function

$$(1) \quad t \mapsto \text{TC}(\partial E_f(t; U), U) := \psi_f(t, U)$$

is in $L^1(\mathbb{R})$. This function, denoted by $\psi_f(\cdot, U)$ is called the level total curvature function of f in U . Then, the level total curvature integral of f is defined for any bounded continuous function h on \mathbb{R} by

$$(2) \quad \text{LTC}_f(h, U) = \int_{\mathbb{R}} h(t) \text{TC}(\partial E_f(t; U), U) dt = \int_{\mathbb{R}} h(t) \psi_f(t, U) dt,$$

and we simply denote $\text{LTC}_f(U)$ for $\text{LTC}_f(1, U)$.

Let us remark that when $t > \max_U f$, then $E_f(t; U) = \emptyset$ and therefore $\psi_f(t, U) = 0$. On the other hand, when $t \leq \min_U f$, then $E_f(t; U) = U$, and thus $\partial E_f(t; U) \cap U = \emptyset$. Therefore we also have $\psi_f(t, U) = 0$. This shows that the level total curvature function is 0 for levels t outside the range of f .

Let us also notice that when H is a C^1 diffeomorphism on \mathbb{R} with bounded derivative $h = H'$, by a simple change of variable, the function $H \circ f$ is also of finite level total curvature integral on U with $\psi_{H \circ f}(t, U) = \psi_f(H^{-1}(t), U)$ and

$$\text{LTC}_{H \circ f}(U) = \text{LTC}_f(h, U).$$

2.2. Link with Euler Characteristic. The link between the level total curvature and Euler Characteristic is given by the Gauss-Bonnet theorem. The precise statement of the theorem is the following (it is taken from the book of Do Carmo [14] p.274).

Theorem 1 (Gauss-Bonnet Theorem). *Let S be an oriented surface, and let $D \subset S$ be a regular region such that its boundary ∂D is formed by n closed, simple and piecewise regular curves $\Gamma_1, \dots, \Gamma_n$. Suppose that each Γ_i is positively oriented and let $\alpha_1, \dots, \alpha_p$ be the set of all turning angles of the curves $\Gamma_1, \dots, \Gamma_n$. Then*

$$\sum_{i=1}^n \int_{\Gamma_i} \kappa_{\Gamma_i}(x) \mathcal{H}^1(dx) + \sum_{i=1}^p \alpha_i + \iint_D K d\sigma = 2\pi\chi(D),$$

where K is the Gaussian curvature of S and $\chi(D)$ is the Euler Characteristic of D .

The intuitive explanation of the Gauss-Bonnet Theorem for subsets of \mathbb{R}^2 is the following. First, we see here the Euler Characteristic as a count of the number of connected components minus the number of holes. Then, for each connected component, the total curvature of its boundary (that is a simple closed curve) is $+2\pi$, whereas for a hole, since it is oriented in an ‘‘inverse way’’, its total curvature is -2π . Therefore adding all these total curvatures, we get 2π times the Euler Characteristic of the set.

A direct consequence of the Gauss-Bonnet Theorem is that if we consider the flat surface $S = x_0 + [0, T]^2$, then its Gaussian curvature is 0. Hence, for $U = x_0 + (0, T)^2$, when considering a regular compact region $D \subset U$, with boundary given by $\Gamma = \partial D \subset U$ of finite total curvature in U , one has

$$2\pi\chi(D) = \text{TC}(\Gamma, U).$$

More generally we obtain the following corollary that gives the link between the Euler Characteristic of the excursion sets and their level total curvature.

Corollary 1. *Let $U = x_0 + (0, T)^2$, and let f be a function defined on an open set containing $\bar{U} = x_0 + [0, T]^2$. We assume that f is of finite level total curvature integral in U (in the sense of Definition 2). Then, for almost every $t \in \mathbb{R}$,*

$$2\pi\chi(E_f(t; U)) = \text{TC}(\partial E_f(t; U), U) + \sum_{j=1}^{k^{(t)}} \beta_j^{(t)} = \psi_f(t, U) + \sum_{j=1}^{k^{(t)}} \beta_j^{(t)},$$

where the $\beta_j^{(t)}$, $1 \leq j \leq k^{(t)}$ are the possible turning angles of $\partial E_f(t; U)$ at points that are on the boundary of U (see Figure 1).

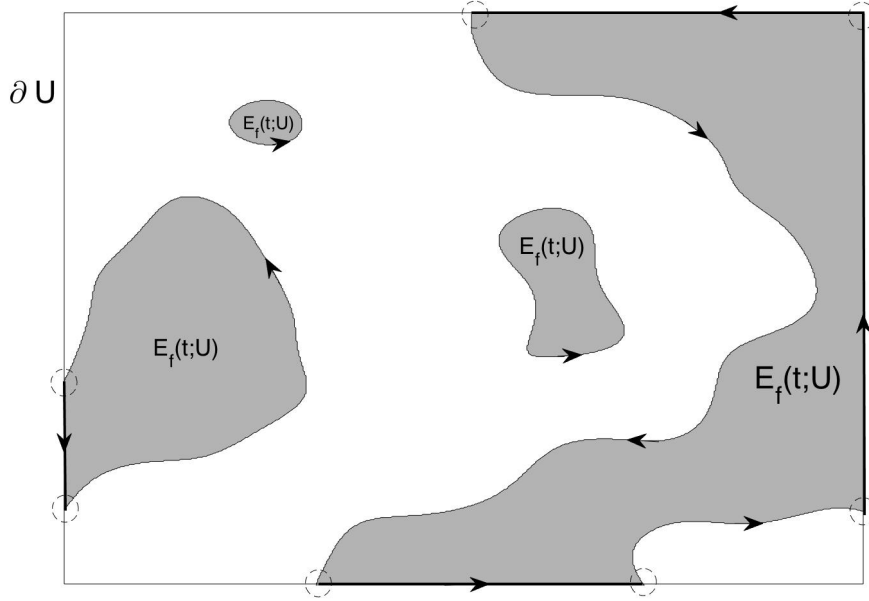


FIGURE 1. By the Gauss-Bonnet theorem, the Euler Characteristic of the excursion set $E_f(t;U)$ (in gray) is equal to the total curvature of its boundary in the open rectangular domain U plus the turning angles at the points where $E_f(t;U)$ meets ∂U and also plus $\pi/2$ for each of the corners of the rectangle U that are in $E_f(t;U)$. All these special points are the ones marked by the small dashed circles on the figure.

Remark: To avoid the problem of the boundary of $U = x_0 + (0, T)^2$, one can consider functions defined on the flat torus $U = \mathbb{T}^2$. Since it has a zero Gaussian curvature everywhere, the Gauss-Bonnet theorem with $S = \mathbb{T}^2$, implies that, for any level t ,

$$2\pi\chi(E_f(t;U)) = \text{TC}(\partial E_f(t;U), U) = \psi_f(t, U).$$

Now, considering a non-periodic domain $U = (0, T)^2 \subset \mathbb{R}^2$, the total curvature of the boundary of an excursion set E , divided by 2π , can be seen as a “modified” Euler Characteristic, in a sense very similar to the one used in the book of Adler and Taylor [6] or in the paper of Estrade and León [15], where critical points in U are only taken into account, and not the ones on the boundary of U . Moreover, we will often consider large domains (that is T goes to infinity), and in that case, the total curvature (in expectation) will grow like T^2 whereas the sum of the turning angles on ∂U will (in expectation also) grow like T .

2.3. Link with Euler Integral. Let us quote that due to the additivity property of the Euler Characteristic

$$\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B),$$

it is natural to set up an integration theory with respect to Euler Characteristic [21, 28]. However, since χ is only finitely additive, a careful choice of integrands must be done. This problem was tackled by defining the class constructible functions [26], then extended by the class of “tame” real-valued functions in [8]. Following this framework, Bobrowski and Borman obtained in [12] the first probabilistic statement about the persistent homology generated by sublevel sets. We briefly recall the definitions used in [12] for comparison with our setting. When f is a real continuous function defined on a compact topological set S , it is said to be a tame function if the homotopy types of $E_f(t; S) = \{x \in S; f(x) \geq t\}$ and $\{x \in S; f(x) \leq t\}$ change only finitely many times as t varies over \mathbb{R} and the Euler Characteristic of each set is always finite. For such a function, a lower and upper Euler integrals are defined by

$$\begin{aligned} \int_S f[d\chi] &= \int_0^{+\infty} (\chi(f \geq t) - \chi(f < -t)) dt \\ \int_S f[d\chi] &= \int_0^{+\infty} (\chi(f > t) - \chi(f \leq t)) dt, \end{aligned}$$

where $\chi(f \geq t) = \chi(E_f(t; S))$, $\chi(f < t) = \chi(S) - \chi(f \geq t)$, etc. In contrast with the level total curvature function $\psi_f(\cdot, U)$ of f in U defined in (1), taking $S = x_0 + [0, T]^2$, for some $x_0 \in \mathbb{R}^2$ and $T > 0$, leads to

$\chi(f \geq t) = \chi(S) = 1$ for any $t \leq \min_S f$ and thus $t \mapsto \chi(f \geq t)$ is not integrable on \mathbb{R} , explaining the above definition of Euler integrals.

However, note that in the periodic case for $S = U = \mathbb{T}^2$ one has $\chi(S) = 0$ and the lower Euler integral just becomes

$$\int_S f [d\chi] = \int_{\mathbb{R}} \chi(f \geq t) dt.$$

And it is therefore equal to our level total curvature integral divided by 2π , in view of the above remark, since

$$\text{LTC}_f(U) = \int_{\mathbb{R}} \psi_f(t, U) dt.$$

We now show how the level total curvature integral can be explicitly computed in different situations. The first situation is the one of sums of piecewise constant functions (also called elementary functions), and the second situation will be the case of smooth (at least C^2) functions.

2.4. The case of elementary functions. We first introduce some notations. For a point $x \in U$ and a real positive number ρ , we denote by $B_\rho(x)$ the open ball of radius ρ and center x . For an oriented simple piecewise C^2 curve γ with finite length, and a point x on γ , then for ρ small enough, $B_\rho(x) \setminus \gamma$ is made of two connected components. These two “half-balls” are respectively denoted by $B_\rho^+(x, \gamma)$ and $B_\rho^-(x, \gamma)$. The half-ball $B_\rho^+(x, \gamma)$ is the component that is on the side of the normal ν_γ to γ .

Definition 3 (Elementary function). *We say that a function f defined on U is an elementary function on U if f is a piecewise constant function taking a finite number of values (meaning that $f(U)$ is a finite subset of \mathbb{R}), and if the discontinuity set of f in U , denoted by \mathcal{S}_f , can be decomposed as*

$$\mathcal{S}_f = \mathcal{R}_f \cup \mathcal{C}_f \cup \mathcal{I}_f,$$

where (see also Figure 2)

- \mathcal{R}_f is the regular part of the discontinuity set of f : it is the finite and disjoint union of C^2 simple curves having finite length and finite total curvature. More precisely, if $x \in \mathcal{R}_f$, then there exists $\rho > 0$ such that $\mathcal{S}_f \cap B_\rho(x)$ is a simple C^2 oriented curve γ separating the ball $B_\rho(x)$ in two half-balls $B_\rho^+(x, \gamma)$ and $B_\rho^-(x, \gamma)$. Moreover there exist two real numbers $f^+(x) > f^-(x)$ such that $f(y) = f^+(x)$ for all $y \in B_\rho^+(x, \gamma)$ and $f(y) = f^-(x)$ for all $y \in B_\rho^-(x, \gamma)$. We also denote

$$\nu_f(x) = \nu_\gamma(x) \quad \text{and} \quad \kappa_f(x) = \kappa_\gamma(x) \quad \text{with} \quad \kappa_f \in L^1(\mathcal{R}_f, \mathcal{H}^1).$$

- \mathcal{C}_f is the set of corner points: it is a finite set of points (meaning $\mathcal{H}^0(\mathcal{C}_f) < +\infty$, with \mathcal{H}^0 the counting measure) such that if $x \in \mathcal{C}_f$, then there exists $\rho > 0$ such that $\mathcal{S}_f \cap B_\rho(x)$ is a simple piecewise C^2 oriented curve γ having only one corner at x . We write $\alpha_f(x) \in (-\pi, \pi)$ the turning angle of γ at x . As for regular points, γ separates the ball $B_\rho(x)$ in two half-balls $B_\rho^+(x, \gamma)$ and $B_\rho^-(x, \gamma)$, and moreover there exist two real numbers $f^+(x) > f^-(x)$ such that $f(y) = f^+(x)$ for all $y \in B_\rho^+(x, \gamma)$ and $f(y) = f^-(x)$ for all $y \in B_\rho^-(x, \gamma)$. The turning angle at such a corner point is denoted

$$\alpha_f(x) = \alpha_\gamma(x).$$

- \mathcal{I}_f is the set of intersection points: it is a finite set of points (meaning $\mathcal{H}^0(\mathcal{I}_f) < +\infty$) such that for $x \in \mathcal{I}_f$, then there exists $\rho > 0$ such that $\mathcal{S}_f \cap B_\rho(x)$ is the union of two different simple and oriented C^2 curves γ_1 and γ_2 such that $\{x\} = \gamma_1 \cap \gamma_2$ and such that the intersection is non-degenerate (meaning that $\nu_{\gamma_1}(x)$ and $\nu_{\gamma_2}(x)$ are not colinear). Each curve separates the ball in two half-balls, and there exist 4 real numbers $f^-(x) < f_+^-(x)$, $f_+^-(x) < f^+(x)$, such that $f = f^-(x)$ on $B_\rho^-(x, \gamma_1) \cap B_\rho^-(x, \gamma_2)$; $f = f_+^-(x)$ on $B_\rho^-(x, \gamma_1) \cap B_\rho^+(x, \gamma_2)$; $f = f_+^-(x)$ on $B_\rho^+(x, \gamma_1) \cap B_\rho^-(x, \gamma_2)$ and $f = f^+(x)$ on $B_\rho^+(x, \gamma_1) \cap B_\rho^+(x, \gamma_2)$. And we define

$$\begin{aligned} \beta_f(x) &= d_{S^1}(\nu_{\gamma_1}(x), \nu_{\gamma_2}(x)) \\ &= \min(|\text{Arg } \nu_{\gamma_1}(x) - \text{Arg } \nu_{\gamma_2}(x)|, 2\pi - |\text{Arg } \nu_{\gamma_1}(x) - \text{Arg } \nu_{\gamma_2}(x)|) \in (0, \pi), \end{aligned}$$

the geodesic distance between $\nu_{\gamma_1}(x)$ and $\nu_{\gamma_2}(x)$ on S^1 .

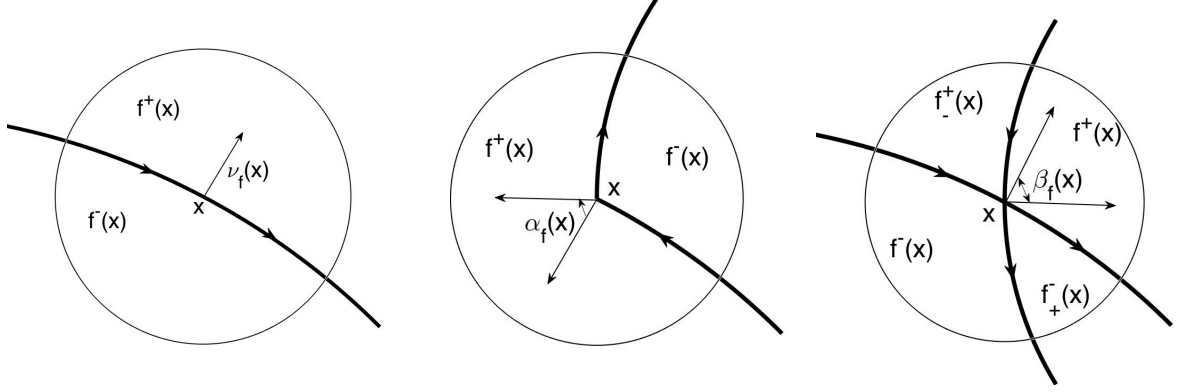


FIGURE 2. The three types of points of the discontinuity set of an elementary function. From left to right: a regular point, a corner point and an intersection point.

Proposition 1. *If f is an elementary function on U , then f is of finite level total curvature integral on U with*

$$(3) \quad \text{LTaC}_f(U) = \int_{\mathcal{R}_f \cap U} [f^+(x) - f^-(x)] |\kappa_f(x)| \mathcal{H}^1(dx) + \sum_{x \in \mathcal{C}_f \cap U} [f^+(x) - f^-(x)] |\alpha_f(x)| \\ + \sum_{x \in \mathcal{I}_f \cap U} [f^+(x) - \max(f_-^+(x), f_-^-(x)) + \min(f_+^-(x), f_+^-(x)) - f^-(x)] \beta_f(x) < +\infty.$$

Moreover, for all $t \in \mathbb{R}$, $\partial E_f(t, U)$ is a curve of finite total curvature in the sense of Definition 1. And if h is a bounded continuous function on \mathbb{R} , and H is a primitive of h (for instance $H(t) = \int_0^t h(u) du$), then the level total curvature integral of f is given by

$$(4) \quad \text{LTC}_f(h, U) = \int_{\mathcal{R}_f} [H(f^+(x)) - H(f^-(x))] \kappa_f(x) \mathcal{H}^1(dx) + \sum_{x \in \mathcal{C}_f} [H(f^+(x)) - H(f^-(x))] \alpha_f(x) \\ + \sum_{x \in \mathcal{I}_f} [H(f^+(x)) + H(f^-(x)) - H(f_-^+(x)) - H(f_+^-(x))] \beta_f(x).$$

In particular, when $h = 1$, we get

$$(5) \quad \text{LTC}_f(U) = \int_{\mathcal{R}_f} [f^+(x) - f^-(x)] \kappa_f(x) \mathcal{H}^1(dx) + \sum_{x \in \mathcal{C}_f} [f^+(x) - f^-(x)] \alpha_f(x) \\ + \sum_{x \in \mathcal{I}_f} [f^+(x) + f^-(x) - f_-^+(x) - f_+^-(x)] \beta_f(x).$$

Proof. We assume that $m = \text{Card}(f(U)) \geq 2$. Otherwise, if $m = 1$, then for all $t \in \mathbb{R}$, $E_f(t; U) = \emptyset$ or U and therefore $\text{TaC}(\partial E_f(t; U), U) = \text{TC}(\partial E_f(t; U), U) = 0$. In the following we denote the values of f in U by $v_1 < \dots < v_m$ and set $v_0 = -\infty$. We first remark that $E_f(t; U) = \emptyset$ for $t > v_m$; $E_f(t; U) = E_f(v_i; U)$ for $v_{i-1} < t \leq v_i$ and $E_f(t; U) = U$ for $t \leq v_1$. The set of discontinuity points is given by $\mathcal{S}_f = \cup_{i=2}^m \partial E_f(v_i; U)$.

Let us compute the total curvature of $\Gamma_i := \partial E_f(v_i; U)$ for each $2 \leq i \leq m$. Since $\Gamma_i \subset \mathcal{S}_f$, we can write

$$\Gamma_i = (\Gamma_i \cap \mathcal{R}_f) \cup (\Gamma_i \cap \mathcal{C}_f) \cup (\Gamma_i \cap \mathcal{I}_f).$$

Note that for $x \in \Gamma_i$, one has $f^+(x) \geq v_i$ and $f^-(x) < v_i$ with $\kappa_{\Gamma_i}(x) = \kappa_f(x)$ for $x \in \Gamma_i \cap \mathcal{R}_f$ and $\alpha_{\Gamma_i}(x) = \alpha_f(x)$ for $x \in \Gamma_i \cap \mathcal{C}_f \subset \mathcal{C}_{\Gamma_i}$.

When $x \in \Gamma_i \cap \mathcal{I}_f$, we have an intersection point of f , and it may become a corner point for Γ_i . Indeed, using the same notations for intersection points as in Definition 3, we may write $x \in \gamma_1 \cap \gamma_2$ with the two simple C^2 curves γ_1 and γ_2 being such that $(\gamma_1 \cup \gamma_2) \cap B_\rho(x) = \mathcal{S}_f \cap B_\rho(x)$. If $v_i \leq \min(f_-^+(x), f_+^-(x))$, then $\Gamma_i \cap B_\rho(x) = (\gamma_1 \cap B_\rho^-(x, \gamma_2)) \cup (\gamma_2 \cap B_\rho^-(x, \gamma_1))$ and $x \in \mathcal{C}_{\Gamma_i}$ with $\alpha_{\Gamma_i}(x) = -\beta_f(x)$. Without loss of generality we may assume that $f_-^+(x) \leq f_+^-(x)$. If $f_-^+(x) < v_i \leq f_+^-(x)$ then $\Gamma_i \cap B_\rho(x) = \gamma_2$ and therefore x is not a corner point of Γ_i . If $v_i > \max(f_-^+(x), f_+^-(x))$, then $\Gamma_i \cap B_\rho(x) = (\gamma_1 \cap B_\rho^+(x, \gamma_2)) \cup (\gamma_2 \cap B_\rho^+(x, \gamma_1))$ and $\alpha_{\Gamma_i}(x) = \beta_f(x)$.

Finally, it follows that

$$\begin{aligned} \text{TaC}(\Gamma_i, U) &= \int_{\mathcal{R}_f \cap \Gamma_i} |\kappa_f(x)| \mathcal{H}^1(dx) + \sum_{x \in \Gamma_i \cap \mathcal{C}_f} |\alpha_f(x)| \\ &\quad + \sum_{x \in \Gamma_i \cap \mathcal{I}_f} \beta_f(x) \left(\mathbf{1}_{v_i > \max(f_+^-(x), f_-^-(x))} + \mathbf{1}_{v_i \leq \min(f_+^-(x), f_-^-(x))} \right), \end{aligned}$$

while

$$\begin{aligned} \text{TC}(\Gamma_i, U) &= \int_{\mathcal{R}_f \cap \Gamma_i} \kappa_f(x) \mathcal{H}^1(dx) + \sum_{x \in \Gamma_i \cap \mathcal{C}_f} \alpha_f(x) \\ &\quad + \sum_{x \in \Gamma_i \cap \mathcal{I}_f} \beta_f(x) \left(\mathbf{1}_{v_i > \max(f_+^-(x), f_-^-(x))} - \mathbf{1}_{v_i \leq \min(f_+^-(x), f_-^-(x))} \right). \end{aligned}$$

Now let us prove that f is of finite level curvature integral. Let h be a non-negative continuous bounded function on \mathbb{R} and H a primitive of h . We have

$$\int_{\mathbb{R}} h(t) \text{TaC}(\partial E_f(t, U)) dt = \sum_{i=2}^m \int_{v_{i-1}}^{v_i} h(t) \text{TaC}(\Gamma_i, U) dt = \sum_{i=2}^m [H(v_i) - H(v_{i-1})] \text{TaC}(\Gamma_i, U).$$

Then, using the above formula for $\text{TaC}(\Gamma_i, U)$, we get the sum of three terms. The first one is given by

$$\sum_{i=2}^m [H(v_i) - H(v_{i-1})] \int_{\mathcal{R}_f \cap \Gamma_i} |\kappa_f(x)| \mathcal{H}^1(dx) = \int_{\mathcal{R}_f} |\kappa_f(x)| \sum_{i=2}^m [H(v_i) - H(v_{i-1})] \mathbf{1}_{\Gamma_i}(x) \mathcal{H}^1(dx).$$

When $x \in \mathcal{S}_f = \cup_{i=2}^m \Gamma_i$, we denote $i(x)$, (resp. $j(x) \geq i(x)$), the minimal (resp. maximal) index $i = 2, \dots, m$ such that $x \in \Gamma_i$ and $f^-(x) := v_{i(x)-1}$, (resp. $f^+(x) := v_{j(x)}$). It follows that

$$\begin{aligned} \sum_{i=2}^m [H(v_i) - H(v_{i-1})] \mathbf{1}_{\Gamma_i}(x) &= \sum_{i=i(x)}^{j(x)} [H(v_i) - H(v_{i-1})] \\ &= H(v_{j(x)}) - H(v_{i(x)-1}) \\ &= H(f^+(x)) - H(f^-(x)). \end{aligned}$$

Therefore the first term is

$$\int_{\mathcal{R}_f} [H(f^+(x)) - H(f^-(x))] |\kappa_f(x)| \mathcal{H}^1(dx).$$

Similarly, the second term is equal to

$$\sum_{i=2}^m [H(v_i) - H(v_{i-1})] \sum_{x \in \Gamma_i \cap \mathcal{C}_f} |\alpha_f(x)| = \sum_{x \in \mathcal{C}_f} [H(f^+(x)) - H(f^-(x))] |\alpha_f(x)|.$$

Finally, the third one is equal to

$$\begin{aligned} \sum_{i=2}^m [H(v_i) - H(v_{i-1})] \sum_{x \in \Gamma_i \cap \mathcal{I}_f} |\beta_f(x)| \left(\mathbf{1}_{v_i > \max(f_+^-(x), f_-^-(x))} + \mathbf{1}_{v_i \leq \min(f_+^-(x), f_-^-(x))} \right) \\ = \sum_{x \in \mathcal{I}_f} \beta_f(x) \left(\sum_{i=l(x)+1}^{j(x)} [H(v_i) - H(v_{i-1})] + \sum_{i=i(x)}^{k(x)} [H(v_i) - H(v_{i-1})] \right), \end{aligned}$$

where we have introduced $k(x)$ and $l(x)$ with $i(x) \leq k(x) \leq l(x) < j(x)$ such that $\min(f_+^-(x), f_-^-(x)) = v_{k(x)}$ and $\max(f_+^-(x), f_-^-(x)) = v_{l(x)}$. It follows that this third term is equal to

$$\begin{aligned} \sum_{x \in \mathcal{I}_f} \beta_f(x) (H(v_{j(x)}) - H(v_{l(x)}) + (H(v_{k(x)}) - H(v_{i(x)-1}))) \\ = \sum_{x \in \mathcal{I}_f} \beta_f(x) (H(f^+(x)) - H(\max(f_+^-(x), f_-^-(x))) + H(\min(f_+^-(x), f_-^-(x))) - H(f^-(x))). \end{aligned}$$

In particular, for $h = 1$, we obtain Formula (4) and the fact that

$$(6) \quad \text{LTaC}_f(U) = \int_{\mathbb{R}} \text{TaC}(\partial E_f(t, U)) dt < +\infty.$$

Therefore, f is of finite level total curvature integral and $t \mapsto \psi_f(t, U) = \text{TC}(\partial E_f(t, U))$ is in $L^1(\mathbb{R})$. Note that the same computations as above for h a bounded continuous function on \mathbb{R} (not necessarily non-negative), gives the result (5) for

$$\text{LTC}_f(h, U) = \int_{\mathbb{R}} h(t) \psi(t, U) dt.$$

Indeed now, the third term is equal to

$$\begin{aligned} & \sum_{i=2}^m [H(v_i) - H(v_{i-1})] \sum_{x \in \Gamma_i \cap \mathcal{I}_f} \beta_f(x) \left(\mathbf{1}_{v_i > \max(f_+^+(x), f_-^-(x))} - \mathbf{1}_{v_i \leq \min(f_+^+(x), f_-^-(x))} \right) \\ &= \sum_{x \in \mathcal{I}_f} \beta_f(x) \left(\sum_{i=l(x)+1}^{j(x)} [H(v_i) - H(v_{i-1})] - \sum_{i=i(x)}^{k(x)} [H(v_i) - H(v_{i-1})] \right) \\ &= \sum_{x \in \mathcal{I}_f} \beta_f(x) (H(f_+^+(x)) + H(f_-^-(x)) - H(f_+^+(x)) - H(f_-^-(x))) \end{aligned}$$

Taking again $h = 1$ we obtain (6). \square

The above proposition gives the formula for the level total curvature of an elementary function on U . Now, since we will be interested in shot-noise random fields that are obtained by summing elementary functions, we need to have also a formula for the level total curvature of a sum of elementary functions on U . This is the aim of the following proposition.

Proposition 2. *Let f, g be elementary functions on U such that their respective discontinuity sets \mathcal{S}_f and \mathcal{S}_g intersect only at a finite number of regular points, that is $\mathcal{S}_f \cap \mathcal{S}_g \subset \mathcal{R}_f \cap \mathcal{R}_g$. We moreover assume that the intersections are non-degenerate, meaning that if $x \in \mathcal{R}_f \cap \mathcal{R}_g$, then $\nu_f(x)$ and $\nu_g(x)$ are not colinear. Then $f + g$ is also an elementary function with*

- $\mathcal{R}_{f+g} = \mathcal{R}_f \Delta \mathcal{R}_g = \mathcal{R}_f \cup \mathcal{R}_g \setminus (\mathcal{R}_f \cap \mathcal{R}_g)$ and
 - if $x \in \mathcal{R}_{f+g} \cap \mathcal{R}_f$, then $\kappa_{f+g}(x) = \kappa_f(x)$ with $(f+g)^+(x) = f^+(x) + g(x)$ and $(f+g)^-(x) = f^-(x) + g(x)$;
 - if $x \in \mathcal{R}_{f+g} \cap \mathcal{R}_g$, then $\kappa_{f+g}(x) = \kappa_g(x)$ with $(f+g)^+(x) = f(x) + g^+(x)$ and $(f+g)^-(x) = f(x) + g^-(x)$;
- $\mathcal{C}_{f+g} = \mathcal{C}_f \cup \mathcal{C}_g$ is a disjoint union and
 - if $x \in \mathcal{C}_f$, then $\alpha_{f+g}(x) = \alpha_f(x)$ with $(f+g)^+(x) = f^+(x) + g(x)$ and $(f+g)^-(x) = f^-(x) + g(x)$,
 - if $x \in \mathcal{C}_g$, then $\alpha_{f+g}(x) = \alpha_g(x)$ with $(f+g)^+(x) = f(x) + g^+(x)$ and $(f+g)^-(x) = f(x) + g^-(x)$;
- $\mathcal{I}_{f+g} = \mathcal{I}_f \cup \mathcal{I}_g \cup (\mathcal{R}_f \cap \mathcal{R}_g)$ is a disjoint union and
 - if $x \in \mathcal{I}_f$, then $\beta_{f+g}(x) = \beta_f(x)$ with $(f+g)^+(x) = f^+(x) + g(x)$, $(f+g)^-(x) = f^-(x) + g(x)$, $(f+g)_+^+(x) = f_+^+(x) + g(x)$ and $(f+g)_-^+(x) = f_-^+(x) + g(x)$
 - if $x \in \mathcal{I}_g$, then $\beta_{f+g}(x) = \beta_g(x)$ with $(f+g)^+(x) = f(x) + g^+(x)$, $(f+g)^-(x) = f(x) + g^-(x)$, $(f+g)_+^-(x) = f(x) + g_+^-(x)$ and $(f+g)_-^-(x) = f(x) + g_-^-(x)$
 - if $x \in \mathcal{R}_f \cap \mathcal{R}_g$ then $\beta_{f+g}(x) = d_{S^1}(\nu_f(x), \nu_g(x)) \in (0, \pi)$ with $(f+g)^+(x) = f^+(x) + g^+(x)$, $(f+g)^-(x) = f^-(x) + g^-(x)$ and $\{(f+g)_+^+(x), (f+g)_-^+(x)\} = \{f^+(x) + g^-(x), f^-(x) + g^+(x)\}$.

Moreover, we have that

$$(7) \quad \text{LTC}_{f+g}(U) = \text{LTC}_f(U) + \text{LTC}_g(U),$$

while

$$\begin{aligned} (8) \quad \text{LTaC}_{f+g}(U) &= \text{LTaC}_f(U) + \text{LTaC}_g(U) \\ &+ \sum_{x \in \mathcal{R}_f \cap \mathcal{R}_g \cap U} \left((f+g)^+(x) - \max((f+g)_+^+(x), (f+g)_+^-(x)) \right. \\ &\quad \left. + \min((f+g)_-^+(x), (f+g)_-^-(x)) - (f+g)^-(x) \right) \beta_{f+g}(x) \\ &\leq \text{LTaC}_f(U) + \text{LTaC}_g(U) + 2\pi \sum_{x \in \mathcal{R}_f \cap \mathcal{R}_g \cap U} [(f^+(x) - f^-(x)) + (g^+(x) - g^-(x))]. \end{aligned}$$

Proof. Since f and g are both piecewise constant on U , then $f + g$ is also piecewise constant on U , and its discontinuity set is $\mathcal{S}_{f+g} \subset \mathcal{S}_f \cup \mathcal{S}_g$. We now need to show that a point in \mathcal{S}_{f+g} is else a regular point, a corner or an intersection point in the sense of Definition 3.

If $x \in \mathcal{S}_f \setminus \mathcal{S}_g$, then we have three cases:

- if $x \in \mathcal{R}_f \setminus \mathcal{S}_g$, then $x \in \mathcal{R}_{f+g}$ with $\kappa_{f+g}(x) = \kappa_f(x)$ and $(f+g)^+(x) = f^+(x) + g(x)$ and $(f+g)^-(x) = f^-(x) + g(x)$.
- if $x \in \mathcal{C}_f \setminus \mathcal{S}_g$, then $x \in \mathcal{C}_{f+g}$ with $\alpha_{f+g}(x) = \alpha_f(x)$ and $(f+g)^+(x) = f^+(x) + g(x)$ and $(f+g)^-(x) = f^-(x) + g(x)$.
- if $x \in \mathcal{I}_f \setminus \mathcal{S}_g$, then $x \in \mathcal{I}_{f+g}$ with $\beta_{f+g}(x) = \beta_f(x)$ and $(f+g)^+(x) = f^+(x) + g(x)$, $(f+g)^-(x) = f^-(x) + g(x)$, $(f+g)_+^-(x) = f_+^-(x) + g(x)$ and $(f+g)_-^+(x) = f_-^+(x) + g(x)$.

The same symmetric formulas hold when $x \in \mathcal{S}_g \setminus \mathcal{S}_f$. Now, when $x \in \mathcal{S}_f \cap \mathcal{S}_g = \mathcal{R}_f \cap \mathcal{R}_g$, since we made the hypothesis that the intersection points are non-degenerate, then x becomes an intersection point of $f+g$, that is $x \in \mathcal{I}_{f+g}$ and moreover $\beta_{f+g}(x) = d_{S^1}(\nu_f(x), \nu_g(x))$, $(f+g)^+(x) = f^+(x) + g^+(x)$, $(f+g)^-(x) = f^-(x) + g^-(x)$, and $\{(f+g)_+^-(x), (f+g)_-^+(x)\} = \{f^+(x) + g^-(x), f^-(x) + g^+(x)\}$.

Finally, having identified the discontinuity set of $f+g$, since we assume that $\mathcal{H}^0(\mathcal{R}_f \cap \mathcal{R}_g) < +\infty$ we deduce that

$$\begin{aligned} \int_{\mathcal{R}_{f+g}} |\kappa_{f+g}(x)| \mathcal{H}^1(dx) &= \int_{\mathcal{R}_f} |\kappa_f(x)| \mathcal{H}^1(dx) + \int_{\mathcal{R}_g} |\kappa_g(x)| \mathcal{H}^1(dx) < +\infty, \\ \mathcal{H}^0(\mathcal{C}_{f+g}) &= \mathcal{H}^0(\mathcal{C}_{f+g}) + \mathcal{H}^0(\mathcal{C}_{f+g}) < +\infty \\ \text{and } \mathcal{H}^0(\mathcal{I}_{f+g}) &= \mathcal{H}^0(\mathcal{I}_f) + \mathcal{H}^0(\mathcal{I}_g) + \mathcal{H}^0(\mathcal{R}_f \cap \mathcal{R}_g) < +\infty. \end{aligned}$$

This finishes to prove that $f+g$ is an elementary function. We also notice in particular that when $x \in \mathcal{R}_f \cap \mathcal{R}_g$, one has $(f+g)^+(x) + (f+g)^-(x) = (f+g)_+^-(x) + (f+g)_-^+(x) = f^+(x) + g^+(x) + f^-(x) + g^-(x)$, such that applying the result of the previous proposition (Equation (5)) we can obtain the formula for the level total curvature of $f+g$. In particular, taking $h = 1$ we get (7). \square

Remark: Formula (7) says that the total curvature of a sum of two elementary functions is the sum of their total curvature. This result is quite striking, but we have to underline that it does not hold in general for two non elementary functions. This will appear clearly in the next section where we will consider smooth functions f and where the formula for the level total curvature integral is obviously non linear in f .

2.5. The case of smooth functions. In the previous section, we have computed the level total curvature of elementary functions, that are piecewise constant functions (that are in particular not continuous). Here, we now consider the case of smooth (at least C^2) functions. The question of the link between these two cases will be discussed in Section 4.3 where we will consider a ‘‘discretization’’ of a smooth Gaussian random field. But here, at the moment, there is no randomness. In the sequel, for f a C^2 function we denote by ∇f its gradient vector and by D^2f its Hessian matrix.

Proposition 3. *Let f be a smooth (at least C^2) function on an open set containing \bar{U} . Then f is of finite level total curvature integral on U with*

$$\text{LTaC}_f(U) \leq \int_U \|D^2f(x)\|_2 dx,$$

where $\|\cdot\|_2$ is the matrix norm subordinated to the Euclidean norm. Moreover, for h a bounded continuous function on \mathbb{R} , the level total curvature integral of f is given by

$$\text{LTC}_f(h, U) = \int_{\mathbb{R}} h(t) \psi_f(t, U) dt = - \int_U h(f(x)) \frac{D^2f(x) \cdot (\nabla f(x)^\perp, \nabla f(x)^\perp)}{\|\nabla f(x)\|^2} dx.$$

Proof. Since f is a C^2 function on an open set containing \bar{U} , by Morse-Sard theorem (see [17] p.69 for instance), the set of critical values of f has measure 0 in \mathbb{R} . Let us also recall the co-area formula for Lipschitz mappings (see [16] p.117 for instance), for any \mathcal{L} -integrable function g ,

$$\int_U g(x) \|\nabla f\|(x) dx = \int_{\mathbb{R}} \int_{\partial E_f(t; U) \cap U} g(x) \mathcal{H}^1(dx) dt.$$

Taking $g = 1$, we get

$$\int_U \|\nabla f\|(x) dx = \int_{\mathbb{R}} \mathcal{H}^1(\partial E_f(t; U) \cap U) dt,$$

where $\partial E_f(t; U) \cap U = \{x \in U; f(x) = t\}$, by continuity of f . In other words, combining this with Morse-Sard theorem we have that for almost every $t \in \mathbb{R}$, $\mathcal{H}^1(\partial E_f(t; U) \cap U) < +\infty$ and for all points $x \in U$ such that $f(x) = t$ then $\nabla f(x) \neq 0$. Let t be such a non-critical value. Now, for a curve γ given by an implicit form $f(\gamma(s)) = t$, we have $\gamma'(s)^\perp = \nabla f(\gamma(s)) / \|\nabla f(\gamma(s))\|$ and therefore the curvature at $x = \gamma(s)$ is given by

$$(9) \quad \kappa_f(x) = - \frac{D^2f(x) \cdot (\nabla f^\perp(x), \nabla f^\perp(x))}{\|\nabla f(x)\|^3},$$

where if $A = (a_{ij})_{1 \leq i, j \leq 2}$ is a 2×2 symmetric matrix and $y = (y_1, y_2) \in \mathbb{R}^2$, we use the notation

$$A.(y, y) = {}^t y A y = a_{11}y_1^2 + a_{22}y_2^2 + 2a_{12}y_1y_2.$$

And thus, we get

$$\begin{aligned} \text{TaC}(\partial E_f(t; U), U) &= \int_{\partial E_f(t; U) \cap U} |\kappa_f(x)| \mathcal{H}^1(dx) \\ \text{and } \psi_f(t, U) = \text{TC}(\partial E_f(t; U), U) &= \int_{\partial E_f(t; U) \cap U} \kappa_f(x) \mathcal{H}^1(dx). \end{aligned}$$

Now, using the co-area formula with $g = |\kappa_f|$,

$$\begin{aligned} \text{LTaC}_f(U) &= \int_{\mathbb{R}} \int_{\partial E_f(t; U) \cap U} |\kappa_f(x)| \mathcal{H}^1(dx) = \int_U |\kappa_f(x)| \|\nabla f(x)\| dx \\ &\leq \int_U \|D^2 f(x)\|_2 dx, \end{aligned}$$

in view of (9). It follows that f is of finite level total curvature integral on U and, for h a bounded continuous function on \mathbb{R} , using again the co-area formula but now with $g = (h \circ f) \kappa_f$,

$$\begin{aligned} \text{LTC}_f(h) &= \int_{\mathbb{R}} h(t) \psi_f(t, U) dt = \int_{\mathbb{R}} \int_{\partial E_f(t; U) \cap U} h(f(x)) \kappa_f(x) \mathcal{H}^1(dx) dt \\ &= \int_U h(f(x)) \kappa_f(x) \|\nabla f(x)\| dx \\ &= - \int_U h(f(x)) \frac{D^2 f(x) \cdot (\nabla f(x)^\perp, \nabla f(x)^\perp)}{\|\nabla f(x)\|^2} dx, \end{aligned}$$

in view of (9). □

3. ELEMENTARY SHOT NOISE RANDOM FIELDS

3.1. Level total curvature of an elementary shot noise random field. We consider here a shot-noise random field defined on \mathbb{R}^2 by

$$\forall x \in \mathbb{R}^2, \quad X_\Phi(x) = \sum_{i \in I} g_{m_i}(x - x_i),$$

where $\Phi = \{(x_i, m_i)\}_{i \in I}$ is a Poisson point process on $\mathbb{R}^2 \times \mathbb{R}^d$, defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, of intensity $\lambda \mathcal{L} \times F$, with $\lambda > 0$ real, \mathcal{L} the Lebesgue measure on \mathbb{R}^2 and F a probability measure on \mathbb{R}^d . Note that equivalently, we may define Φ as an independently *marked Poisson point process* where $\{x_i\}_i$ is an homogeneous Poisson point process of intensity λ and the m_i are ‘‘marks’’, following a law $F(dm)$ on \mathbb{R}^d (with $d \geq 1$) and independent of the Poisson point process $\{x_i\}_i$. Let $g : \mathbb{R}^2 \times \mathbb{R}^d \rightarrow \mathbb{R}$ be a measurable function such that the functions $g_m := g(\cdot, m)$ satisfy

$$(10) \quad \int_{\mathbb{R}^2 \times \mathbb{R}^d} |g_m(x)| dx F(dm) < +\infty.$$

Then, the random field X_Φ is well defined as an almost surely locally integrable function on \mathbb{R}^2 (see [10]). Note that moreover, the random field X_Φ is stationary. We will first give formulas for the level total curvature integral of X_Φ on an open bounded set U in the case where the g_m are elementary functions on \mathbb{R}^2 , then we will compute the Fourier transform of the function $t \mapsto \mathbb{E}(\psi_{X_\Phi}(t, U))$. Finally we will give explicit results in the case of weighted indicator functions of random set, obtained from a deterministic compact set D with a boundary given by a finite union of closed curves with finite total curvatures, by random rotation and dilation. Specific computations for disks with $D = D(0, 1)$ and squares with $D = [0, 1]^2$ are linked with some recent results on Boolean models.

Throughout the rest of this section we also assume that the g_m are elementary functions on \mathbb{R}^2 , with compact support and such that

$$(11) \quad \int_{\mathbb{R}^d} \text{LTaC}_{g_m}(\mathbb{R}^2) F(dm) < +\infty,$$

where $\text{LTaC}_{g_m}(\mathbb{R}^2)$ is defined by (4) choosing $U = \mathbb{R}^2$. Note that this implies that for F -almost every $m \in \mathbb{R}^d$, g_m is of finite level total curvature integral and we may define $\psi_{g_m}(\cdot, \mathbb{R}^2)$, the level total curvature function of g_m in \mathbb{R}^2 as a function in $L^1(\mathbb{R})$. For F -almost every m , g_m is assumed to have a compact support, that can

be included in a square $[-T_m, T_m]^2$ with $T_m \in \mathbb{R}_+$, and its maximal value $\|g_m\|_\infty = \max_{[-T_m, T_m]^2} |g_m|$ is finite.

We will assume moreover that

$$(12) \quad \int_{\mathbb{R}^d} T_m^2 F(dm) < +\infty \quad \text{and} \quad \int_{\mathbb{R}^d} \|g_m\|_\infty F(dm) < +\infty.$$

Note that the first assumption of (12) implies that there is only a finite random number of g_m , denoted by $N(U)$, contributing to the values of X_Φ on the bounded set $U = (0, T)^2$, for $T > 0$. Actually, it is clear that

$$N(U) \leq \#\{i; \bar{U} \cap (x_i + [-T_{m_i}, T_{m_i}]^2) \neq \emptyset\} \leq \#\{i; \|x_i\|_\infty \leq 2T_{m_i} + T\},$$

with $\|x\|_\infty = \max(|x_1|, |x_2|)$, for $x \in \mathbb{R}^2$. It follows that $\mathbb{E}(N(U)) \leq \lambda \int_{\mathbb{R}^d} (2T_m + T)^2 F(dm)$. Since F is a finite measure, under (12), we get

$$(13) \quad \mathbb{E}(N(U)) < +\infty.$$

In the following we will use the notation τ_x to denote the translation of x in \mathbb{R}^2 (i.e. $\tau_x y = y + x$ for all $y \in \mathbb{R}^2$). We will also denote $\Phi_i = \Phi \setminus \{(x_i, m_i)\}$ for $i \in I$, $\Phi_{ij} = \Phi \setminus \{(x_i, m_i), (x_j, m_j)\}$ for $i \neq j$ in I , and their associated shot noise random fields

$$\forall i, \quad X_{\Phi_i}(x) = \sum_{k; k \neq i} g_{m_k}(x - x_k) \quad \text{and} \quad \forall j \neq i, \quad X_{\Phi_{ij}}(x) = \sum_{k; k \neq i, k \neq j} g_{m_k}(x - x_k).$$

Theorem 2. *Assume that for F -almost every $m \in \mathbb{R}^d$, the function g_m is an elementary function on \mathbb{R}^2 (in the sense of Definition 3) satisfying (10), (11) and (12), and such that*

$$(14) \quad \int_{\mathbb{R}^d} \mathcal{H}^0(\mathcal{S}_{g_m} \setminus \mathcal{R}_{g_m}) F(dm) < +\infty$$

$$(15) \quad \int_{\mathbb{R}^d \times \mathbb{R}^d} \int_{\mathbb{R}^2} \mathcal{H}^0(\mathcal{R}_{g_{m'}} \cap \tau_x \mathcal{R}_{g_m}) dx F(dm) F(dm') < +\infty$$

$$(16) \quad \int_{\mathbb{R}^d \times \mathbb{R}^d} \int_{\mathbb{R}^2} \mathcal{H}^0(\{y \in \mathcal{R}_{g_{m'}} \cap \tau_x \mathcal{R}_{g_m}; \nu_{g_{m'}}(y) = \pm \nu_{g_m}(y - x)\}) dx F(dm) F(dm') = 0.$$

Then, almost surely, for all $T > 0$ and $U = (0, T)^2$, X_Φ is an elementary function on U and its discontinuity set on U is given by $\mathcal{S}_{X_\Phi} \cap U$ where $\mathcal{S}_{X_\Phi} = \mathcal{R}_{X_\Phi} \cup \mathcal{C}_{X_\Phi} \cup \mathcal{I}_{X_\Phi}$, with

- $\mathcal{R}_{X_\Phi} = \left(\bigcup_i \tau_{x_i} \mathcal{R}_{g_{m_i}} \right) \setminus \left(\bigcup_{i, j \neq i} \tau_{x_i} \mathcal{R}_{g_{m_i}} \cap \tau_{x_j} \mathcal{R}_{g_{m_j}} \right)$, and if $x \in \mathcal{R}_{X_\Phi} \cap U$, there exists a unique i such that $x \in \tau_{x_i} \mathcal{R}_{g_{m_i}}$ with $\kappa_{X_\Phi}(x) = \kappa_{g_{m_i}}(x - x_i)$ and

$$X_\Phi^+(x) = g_{m_i}^+(x - x_i) + X_{\Phi_i}(x) \quad \text{and} \quad X_\Phi^-(x) = g_{m_i}^-(x - x_i) + X_{\Phi_i}(x);$$

- $\mathcal{C}_{X_\Phi} = \bigcup_i \tau_{x_i} \mathcal{C}_{g_{m_i}}$, and if $x \in \mathcal{C}_{X_\Phi} \cap U$, there exists a unique i such that $x \in \tau_{x_i} \mathcal{C}_{g_{m_i}}$ with $\alpha_{X_\Phi}(x) = \alpha_{g_{m_i}}(x - x_i)$ and

$$X_\Phi^+(x) = g_{m_i}^+(x - x_i) + X_{\Phi_i}(x) \quad \text{and} \quad X_\Phi^-(x) = g_{m_i}^-(x - x_i) + X_{\Phi_i}(x);$$

- $\mathcal{I}_{X_\Phi} = \left(\bigcup_i \tau_{x_i} \mathcal{I}_{g_{m_i}} \right) \cup \left(\bigcup_{i, j \neq i} \tau_{x_i} \mathcal{R}_{g_{m_i}} \cap \tau_{x_j} \mathcal{R}_{g_{m_j}} \right)$ and if $x \in \mathcal{I}_{X_\Phi} \cap U$, only two situations occur
 - there exists a unique i such that $x \in \tau_{x_i} \mathcal{I}_{g_{m_i}}$, with $\beta_{X_\Phi}(x) = \beta_{g_{m_i}}(x - x_i)$ and

$$\begin{aligned} X_\Phi^+(x) &= g_{m_i}^+(x - x_i) + X_{\Phi_i}(x), & X_\Phi^-(x) &= g_{m_i}^-(x - x_i) + X_{\Phi_i}(x), \\ X_{\Phi_+}^-(x) &= g_{m_i}^-(x - x_i) + X_{\Phi_i}(x), & X_{\Phi_-}^+(x) &= g_{m_i}^+(x - x_i) + X_{\Phi_i}(x); \end{aligned}$$

- or there exists a unique pair $\{i, j\}$ with $i \neq j$ such that $x \in \tau_{x_i} \mathcal{R}_{g_{m_i}} \cap \tau_{x_j} \mathcal{R}_{g_{m_j}}$ with

$$\beta_{X_\Phi}(x) = d_{S^1}(\nu_{g_{m_i}}(x - x_i), \nu_{g_{m_j}}(x - x_j)) \in (0, \pi)$$

$$X_\Phi^+(x) = g_{m_i}^+(x - x_i) + g_{m_j}^+(x - x_j) + X_{\Phi_{ij}}(x), \quad X_\Phi^-(x) = g_{m_i}^-(x - x_i) + g_{m_j}^-(x - x_j) + X_{\Phi_{ij}}(x),$$

$$\{X_{\Phi_-}^+(x), X_{\Phi_+}^-(x)\} = \{g_{m_i}^+(x - x_i) + g_{m_j}^-(x - x_j) + X_{\Phi_{ij}}(x), g_{m_i}^-(x - x_i) + g_{m_j}^+(x - x_j) + X_{\Phi_{ij}}(x)\}.$$

In particular, a.s.

$$\begin{aligned}
\text{LTC}_{X_\Phi}(U) &= \sum_i \text{LTC}_{g_{m_i}}(\tau_{-x_i}U) \\
&= \sum_i \int_{U \cap \tau_{x_i} \mathcal{R}_{g_{m_i}}} [g_{m_i}^+(x-x_i) - g_{m_i}^-(x-x_i)] \kappa_{g_{m_i}}(x-x_i) \mathcal{H}^1(dx) \\
&\quad + \sum_i \sum_{x \in U \cap \tau_{x_i} \mathcal{C}_{g_{m_i}}} [g_{m_i}^+(x-x_i) - g_{m_i}^-(x-x_i)] \alpha_{g_{m_i}}(x-x_i) \\
&\quad + \sum_i \sum_{x \in U \cap \tau_{x_i} \mathcal{I}_{g_{m_i}}} [g_{m_i}^+(x-x_i) + g_{m_i}^-(x-x_i) - g_{m_i}^+(x-x_i) - g_{m_i}^-(x-x_i)] \beta_{g_{m_i}}(x-x_i).
\end{aligned}$$

Proof. Since it is sufficient to prove the result for all $U = (0, T)^2$ with $T \in \mathbb{Q}$, we only have to prove it holds almost surely on some fixed $U = (0, T)^2$, with $T > 0$.

Let us first remark that when A_Φ is a finite set of points of \mathbb{R}^2 depending on the marked Poisson point process $\Phi = \{(x_i, m_i)\}$, as soon as $\mathbb{E}(\mathcal{H}^0(A_\Phi)) < +\infty$, one has,

$$\bigcup_j A_{\Phi_j} \cap \tau_{x_j} \mathcal{S}_{g_{m_j}} = \emptyset \quad \text{a.s.}$$

This follows from the fact that, by Slivnyak-Mecke formula (see [7] Theorem 1.4.5),

$$\begin{aligned}
\mathbb{E} \left(\mathcal{H}^0 \left(\bigcup_j A_{\Phi_j} \cap \tau_{x_j} \mathcal{S}_{g_{m_j}} \right) \right) &\leq \lambda \int_{\mathbb{R}^2 \times \mathbb{R}^d} \mathbb{E}(\mathcal{H}^0(A_\Phi \cap \tau_x \mathcal{S}_{g_m})) dx F(dm) \\
&\leq \lambda \mathbb{E}(\mathcal{H}^0(A_\Phi)) \int_{\mathbb{R}^d} \mathcal{L}(\mathcal{S}_{g_m}) F(dm) = 0,
\end{aligned}$$

since $\mathcal{L}(\mathcal{S}_{g_m}) = 0$, using Fubini Theorem and translation invariance of both \mathcal{H}^0 and \mathcal{L} .

Our first assumption (14) implies that

$$(17) \quad \bigcup_{i,j \neq} \tau_{x_i} (\mathcal{S}_{g_{m_i}} \setminus \mathcal{R}_{g_{m_i}}) \cap \tau_{x_j} \mathcal{S}_{g_{m_j}} \cap U = \emptyset \quad \text{a.s.}$$

Actually, taking $A_\Phi = \bigcup_i \tau_{x_i} (\mathcal{S}_{g_{m_i}} \setminus \mathcal{R}_{g_{m_i}}) \cap U$, Campbell formula (see [7] Theorem 1.4.3) ensures that

$$\begin{aligned}
\mathbb{E}(\mathcal{H}^0(A_\Phi)) &\leq \lambda \int_{\mathbb{R}^2 \times \mathbb{R}^d} \mathcal{H}^0(\tau_x (\mathcal{S}_{g_m} \setminus \mathcal{R}_{g_m}) \cap U) dx F(dm) \\
&\leq \lambda \mathcal{L}(U) \int_{\mathbb{R}^d} \mathcal{H}^0(\mathcal{S}_{g_m} \setminus \mathcal{R}_{g_m}) F(dm) < +\infty.
\end{aligned}$$

Then, (17) follows from the preceding remark since $\bigcup_{i,j \neq} \tau_{x_i} (\mathcal{S}_{g_{m_i}} \setminus \mathcal{R}_{g_{m_i}}) \cap \tau_{x_j} \mathcal{S}_{g_{m_j}} \cap U = \bigcup_j A_{\Phi_j} \cap \tau_{x_j} \mathcal{S}_{g_{m_j}}$.

The second assumption (15) will ensure both that

$$(18) \quad \mathcal{H}^0 \left(\bigcup_{i,j \neq} \tau_{x_i} \mathcal{R}_{g_{m_i}} \cap \tau_{x_j} \mathcal{R}_{g_{m_j}} \cap U \right) < \infty \quad \text{a.s.},$$

and

$$(19) \quad \bigcup_{i,j,k \neq} \tau_{x_i} \mathcal{R}_{g_{m_i}} \cap \tau_{x_j} \mathcal{R}_{g_{m_j}} \cap \tau_{x_k} \mathcal{S}_{g_{m_k}} \cap U = \emptyset \quad \text{a.s.}$$

Here we set $A_\Phi = \bigcup_{i,j \neq} \tau_{x_i} \mathcal{R}_{g_{m_i}} \cap \tau_{x_j} \mathcal{R}_{g_{m_j}} \cap U$. Using again Slivnyak-Mecke formula and Campbell formula we obtain that

$$\begin{aligned}
\mathbb{E}(\mathcal{H}^0(A_\Phi)) &\leq \lambda^2 \int_{\mathbb{R}^2 \times \mathbb{R}^d} \int_{\mathbb{R}^2 \times \mathbb{R}^d} \mathcal{H}^0(\tau_x \mathcal{R}_{g_m} \cap \tau_{x'} \mathcal{R}_{g_{m'}} \cap U) dx F(dm) dx' F(dm') \\
&\leq \lambda^2 \mathcal{L}(U) \int_{\mathbb{R}^d \times \mathbb{R}^d} \int_{\mathbb{R}^2} \mathcal{H}^0(\mathcal{R}_{g_{m'}} \cap \tau_x \mathcal{R}_{g_m}) dx F(dm) F(dm') < +\infty.
\end{aligned}$$

It follows that $\mathcal{H}^0(A_\Phi) < +\infty$ a.s. and $\bigcup_k A_{\Phi_k} \cap \tau_{x_k} \mathcal{S}_{g_{m_k}} = \emptyset$ a.s.

Finally and similarly, the last assumption ensures that

$$(20) \quad \left\{ y \in \bigcup_{i,j \neq} \tau_{x_i} \mathcal{R}_{g_{m_i}} \cap \tau_{x_j} \mathcal{R}_{g_{m_j}} \cap U; \nu_{g_{m_i}}(y - x_i) = \pm \nu_{g_{m_j}}(y - x_j) \right\} = \emptyset \quad \text{a.s.}$$

This follows from the fact that the expected \mathcal{H}^0 measure of this set is zero, according to Slivnyak-Mecke formula and Campbell formula, Fubini Theorem and translation invariance.

Now let us consider the random variable $N(U)$ counting the number of functions g_{m_i} contributing to the values of X_Φ on U and recall that under the assumption that $\int_{\mathbb{R}^d} T_m^2 F(dm) < +\infty$, $N(U)$ is a.s. finite, as a consequence of (13).

We now will prove the result of Theorem 2 by induction on the value of $N(U)$ once we have fixed an almost sure realization ensuring the previous configuration.

For $N(U) = 0$ there is nothing to prove since $X_\Phi = 0$ on U in this case. Let us assume the result holds when $N(U) = n \geq 0$ and let us prove it for $N(U) = n + 1$. We can assume that there exists (x_i, m_i) such that $\tau_{-x_i} g_{m_i}$ contributes to the values of X on U and write $X_\Phi = X_{\Phi_i} + \tau_{-x_i} g_{m_i}$. The number of functions contributing to X_{Φ_i} is given by $N(U) - 1$ so we can use our induction to state that X_{Φ_i} is an elementary function on U with discontinuity set $\mathcal{S}_{X_{\Phi_i}} \cap U$ where

$$\begin{aligned} \mathcal{S}_{X_{\Phi_i}} &= \bigcup_{j:j \neq i} \tau_{x_j} \mathcal{S}_{g_{m_j}}, \quad \text{with} \quad \mathcal{R}_{X_{\Phi_i}} = \left(\bigcup_{j:j \neq i} \tau_{x_j} \mathcal{R}_{g_{m_j}} \right) \setminus \left(\bigcup_{j,k \neq i, j \neq i, k \neq i} \tau_{x_j} \mathcal{R}_{g_{m_j}} \cap \tau_{x_k} \mathcal{R}_{g_{m_k}} \right), \\ \mathcal{C}_{X_{\Phi_i}} &= \bigcup_{j:j \neq i} \tau_{x_j} \mathcal{C}_{g_{m_j}} \quad \text{and} \quad \mathcal{I}_{X_{\Phi_i}} = \left(\bigcup_{j:j \neq i} \tau_{x_j} \mathcal{I}_{g_{m_j}} \right) \cup \left(\bigcup_{j,k \neq i, j \neq i, k \neq i} \tau_{x_j} \mathcal{R}_{g_{m_j}} \cap \tau_{x_k} \mathcal{R}_{g_{m_k}} \right). \end{aligned}$$

Then the discontinuity points are given by $\mathcal{S}_{X_\Phi} \cap U$ with $\mathcal{S}_{X_\Phi} \subset \mathcal{S}_{X_{\Phi_i}} \cup \tau_{x_i} \mathcal{S}_{g_{m_i}} = \bigcup_j \tau_{x_j} \mathcal{S}_{g_{m_j}}$ with

$$\mathcal{S}_{X_{\Phi_i}} \cap \tau_{x_i} (\mathcal{S}_{g_{m_i}} \setminus \mathcal{R}_{g_{m_i}}) \cap U \subset \bigcup_{j:j \neq i} \tau_{x_j} \mathcal{S}_{g_{m_j}} \cap \tau_{x_i} (\mathcal{S}_{g_{m_i}} \setminus \mathcal{R}_{g_{m_i}}) \cap U = \emptyset$$

by (17). Moreover,

$$\mathcal{S}_{X_{\Phi_i}} \setminus \mathcal{R}_{X_{\Phi_i}} \subset \bigcup_{j:j \neq i} \tau_{x_j} (\mathcal{S}_{g_{m_j}} \setminus \mathcal{R}_{g_{m_j}}) \cup \bigcup_{j,k \neq i, j \neq i, k \neq i} \tau_{x_j} \mathcal{R}_{g_{m_j}} \cap \tau_{x_k} \mathcal{R}_{g_{m_k}},$$

with again $\bigcup_{j:j \neq i} \tau_{x_j} (\mathcal{S}_{g_{m_j}} \setminus \mathcal{R}_{g_{m_j}}) \cap \tau_{x_i} \mathcal{S}_{g_{m_i}} \cap U = \emptyset$ by (17) and $\bigcup_{j,k \neq i, j \neq i, k \neq i} \tau_{x_j} \mathcal{R}_{g_{m_j}} \cap \tau_{x_k} \mathcal{R}_{g_{m_k}} \cap \tau_{x_i} \mathcal{S}_{g_{m_i}} \cap U = \emptyset$ by (19). Therefore, we may conclude that

$$\mathcal{S}_{X_{\Phi_i}} \cap \tau_{x_i} \mathcal{S}_{g_{m_i}} \cap U \subset \mathcal{R}_{X_{\Phi_i}} \cap \tau_{x_i} \mathcal{R}_{g_{m_i}} \cap U,$$

with $\mathcal{H}^0(\mathcal{R}_{X_{\Phi_i}} \cap \tau_{x_i} \mathcal{R}_{g_{m_i}} \cap U) < +\infty$ by (18). Finally let us quote that the intersections are non-degenerate in view of (20). Hence, according to Proposition 2, X_Φ is an elementary function on U . Moreover, if $x \in \mathcal{R}_{X_{\Phi_i}} \setminus \tau_{x_i} \mathcal{S}_{g_{m_i}}$ we get $X_\Phi^\pm(x) = X_{\Phi_i}^\pm(x) + g_{m_i}(x - x_i)$ and by induction there exists a unique $j \neq i$ such that $x \in \tau_{x_j} \mathcal{R}_{g_{m_j}}$ so that $X_\Phi^\pm(x) = X_{\Phi_{ij}}^\pm(x) + g_{m_j}^\pm(x - x_j) + g_{m_i}(x - x_i) = X_{\Phi_j}^\pm(x) + g_{m_j}^\pm(x - x_j)$ and $\kappa_{X_\Phi}(x) = \kappa_{g_{m_j}}(x - x_j)$. Similarly, if $x \in \tau_{x_i} \mathcal{R}_{g_{m_i}} \setminus \mathcal{S}_{X_{\Phi_i}}$ we get $X_\Phi^\pm(x) = X_{\Phi_i}^\pm(x) + g_{m_i}^\pm(x - x_i)$ and $\kappa_{X_\Phi}(x) = \kappa_{g_{m_i}}(x - x_i)$. In the same way, by induction and using the fact that \mathcal{C}_{X_Φ} is the disjoint union of $\tau_{x_j} \mathcal{C}_{g_{m_j}}$ we obtain that if $x \in \mathcal{C}_{X_\Phi}$, there exists a unique i such that $x \in \tau_{x_i} \mathcal{C}_{g_{m_i}}$ and $\alpha_{X_\Phi}(x) = \alpha_{g_{m_i}}(x - x_i)$ with $X_\Phi^\pm(x) = g_{m_i}^\pm(x - x_i) + X_{\Phi_i}(x)$. Finally \mathcal{I}_{X_Φ} is the disjoint union of $\mathcal{I}_{X_{\Phi_i}}, \tau_{x_i} \mathcal{I}_{g_{m_i}}$ and $\mathcal{R}_{X_{\Phi_i}} \cap \tau_{x_i} \mathcal{R}_{g_{m_i}}$. By induction, $\mathcal{I}_{X_{\Phi_i}} = \bigcup_{j:j \neq i} \tau_{x_j} \mathcal{I}_{g_{m_j}} \cup \bigcup_{j,k \neq i, j \neq i, k \neq i} \tau_{x_j} \mathcal{R}_{g_{m_j}} \cap \tau_{x_k} \mathcal{R}_{g_{m_k}}$ and $\mathcal{R}_{X_{\Phi_i}} \cap \tau_{x_i} \mathcal{R}_{g_{m_i}} = \bigcup_{j:j \neq i} \tau_{x_j} \mathcal{R}_{g_{m_j}} \cap \tau_{x_i} \mathcal{R}_{g_{m_i}}$ where unions are all disjoint. Hence, grouping the terms we get the result. \square

Theorem 3. *Under the hypothesis of Theorem 2, assuming moreover that*

$$(21) \quad \int_{\mathbb{R}^d \times \mathbb{R}^d} \int_{\mathbb{R}^2} (\|g_m\|_\infty + \|g_{m'}\|_\infty) \mathcal{H}^0(\mathcal{R}_{g_{m'}} \cap \tau_x \mathcal{R}_{g_m}) dx F(dm) F(dm') < +\infty,$$

then $(\omega, t) \mapsto \psi_{X_\Phi(\omega)}(t, U) \in L^1(\Omega \times \mathbb{R})$ for $U = (0, T)^2$, with $T > 0$. Let us denote for a.e. $t \in \mathbb{R}$,

$$\Psi_{X_\Phi}(t, U) = \mathbb{E}(\psi_{X_\Phi}(t, U)) = \mathbb{E}(\text{TC}(\partial E_{X_\Phi}(t, U))).$$

Then $\Psi_{X_\Phi}(\cdot, U)$ is an integrable function on \mathbb{R} and its Fourier transform is given for all $u \in \mathbb{R}$, $u \neq 0$ by

$$\begin{aligned}\widehat{\Psi_{X_\Phi}}(u, U) &= \int_{\mathbb{R}} e^{iut} \Psi_{X_\Phi}(t, U) dt = \mathbb{E}(\text{LTC}_{X_\Phi}(e^{iu\cdot}, U)) \\ &= \lambda \mathcal{L}(U) \mathbb{E}\left(e^{iuX_\Phi(0)}\right) \left(\int_{\mathbb{R}^d} \widehat{\psi_{g_m}}(u, \mathbb{R}^2) F(dm) \right. \\ &\quad \left. + \frac{\lambda}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^2 \times \mathbb{R}^d} \sum_{z \in \tau_x \mathcal{R}_{g_m} \cap \mathcal{R}_{g_{m'}}} d_{S^1}(\nu_{g_m}(z-x), \nu_{g_{m'}}(z)) \Delta_{m, m'}^u(z-x, z) dx F(dm) F(dm') \right),\end{aligned}$$

$$\text{where } \Delta_{m, m'}^u(y, y') = \frac{(e^{iug_m^+(y)} - e^{iug_m^-(y)})(e^{iug_{m'}^+(y')} - e^{iug_{m'}^-(y')})}{iu}, \quad \forall y \in \mathcal{R}_{g_m} \text{ and } y' \in \mathcal{R}_{g_{m'}},$$

and

$$\begin{aligned}\widehat{\psi_{g_m}}(u, \mathbb{R}^2) &= \text{LTC}_{g_m}(e^{iu\cdot}, \mathbb{R}^2) = \int_{\mathcal{R}_{g_m}} \frac{e^{iug_m^+(z)} - e^{iug_m^-(z)}}{iu} \kappa_{g_m}(z) \mathcal{H}^1(dz) \\ &\quad + \sum_{z \in \mathcal{C}_{g_m}} \frac{e^{iug_m^+(z)} - e^{iug_m^-(z)}}{iu} \alpha_{g_m}(z) + \sum_{z \in \mathcal{I}_{g_m}} \frac{e^{iug_m^+(z)} + e^{iug_m^-(z)} - e^{iug_m^+(z)} - e^{iug_m^-(z)}}{iu} \beta_{g_m}(z).\end{aligned}$$

And for $u = 0$, we have

$$\widehat{\Psi_{X_\Phi}}(0, U) = \int_{\mathbb{R}} \Psi_{X_\Phi}(t, U) dt = \mathbb{E}(\text{LTC}_{X_\Phi}(U)) = \lambda \mathcal{L}(U) \int_{\mathbb{R}^d} \text{LTC}_{g_m}(\mathbb{R}^2) F(dm)$$

with

$$\begin{aligned}\text{LTC}_{g_m}(\mathbb{R}^2) &= \widehat{\psi_{g_m}}(0, \mathbb{R}^2) \\ &= \int_{\mathcal{R}_{g_m}} [g_m^+(z) - g_m^-(z)] \kappa_{g_m}(z) \mathcal{H}^1(dz) + \sum_{z \in \mathcal{C}_{g_m}} [g_m^+(z) - g_m^-(z)] \alpha_{g_m}(z) \\ &\quad + \sum_{z \in \mathcal{I}_{g_m}} [g_m^+(z) + g_m^-(z) - g_m^+(z) - g_m^-(z)] \beta_{g_m}(z).\end{aligned}$$

Proof. First note that in view of (8), one has a.s.

$$\text{LTaC}_{X_\Phi}(U) \leq \sum_i \text{LTaC}_{g_{m_i}}(\tau_{-x_i} U) + 2\pi \sum_{i, j \neq} \mathcal{H}^0(\tau_{x_i} \mathcal{R}_{g_{m_i}} \cap \tau_{x_j} \mathcal{R}_{g_{m_j}} \cap U) (\|g_{m_i}\|_\infty + \|g_{m_j}\|_\infty).$$

By Campbell formula,

$$\mathbb{E}\left(\sum_i \text{LTaC}_{g_{m_i}}(\tau_{-x_i} U)\right) = \int_{\mathbb{R}^2 \times \mathbb{R}^d} \text{LTaC}_{g_m}(\tau_{-x} U) \lambda dx F(dm).$$

Hence, by Fubini Theorem,

$$\mathbb{E}\left(\sum_i \text{LTaC}_{g_{m_i}}(\tau_{-x_i} U)\right) = \lambda \mathcal{L}(U) \int_{\mathbb{R}^d} \text{LTaC}_{g_m}(\mathbb{R}^2) F(dm) < +\infty.$$

Moreover, by Slivnyak-Mecke formula,

$$\begin{aligned}&\mathbb{E}\left(\sum_{i, j \neq} \mathcal{H}^0(\tau_{x_i} \mathcal{R}_{g_{m_i}} \cap \tau_{x_j} \mathcal{R}_{g_{m_j}} \cap U) (\|g_{m_i}\|_\infty + \|g_{m_j}\|_\infty)\right) \\ &= \int_{\mathbb{R}^d \times \mathbb{R}^d} \int_{\mathbb{R}^2 \times \mathbb{R}^2} \mathcal{H}^0(\tau_x \mathcal{R}_{g_m} \cap \tau_{x'} \mathcal{R}_{g_{m'}} \cap U) (\|g_m\|_\infty + \|g_{m'}\|_\infty) \lambda^2 dx dx' F(dm) F(dm') \\ &= \lambda^2 \mathcal{L}(U) \int_{\mathbb{R}^d \times \mathbb{R}^d} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \mathbf{1}_{z \in \mathcal{R}_{g_{m'}} \cap \tau_x \mathcal{R}_{g_m}} (\|g_m\|_\infty + \|g_{m'}\|_\infty) \mathcal{H}^0(dz) dx F(dm) F(dm') \\ &= \lambda^2 \mathcal{L}(U) \int_{\mathbb{R}^d \times \mathbb{R}^d} \int_{\mathbb{R}^2} (\|g_m\|_\infty + \|g_{m'}\|_\infty) \mathcal{H}^0(\mathcal{R}_{g_{m'}} \cap \tau_x \mathcal{R}_{g_m}) dx F(dm) F(dm') < +\infty,\end{aligned}$$

by assumption (21). It follows that $\mathbb{E}(\text{LTaC}_{X_\Phi}(U)) < +\infty$ and therefore $\mathbb{E}(\int_{\mathbb{R}} |\psi_{X_\Phi}(t, U)| dt) < +\infty$. So for a.e. $t \in \mathbb{R}$ we can define $\Psi_{X_\Phi}(t, U) = \mathbb{E}(\psi_{X_\Phi}(t, U))$, which is a function in $L^1(\mathbb{R})$. Note that since $\text{LTC}_{X_\Phi}(U) = \sum_i \text{LTC}_{g_{m_i}}(\tau_{-x_i} U)$ we simply get by Campbell formula and Fubini Theorem that

$$\widehat{\Psi_{X_\Phi}}(0, U) = \int_{\mathbb{R}} \Psi_{X_\Phi}(t, U) dt = \lambda \mathcal{L}(U) \int_{\mathbb{R}^d} \text{LTC}_{g_m}(\mathbb{R}^2) F(dm).$$

Now, let $u \neq 0$ and consider the continuous bounded function $j_u(t) = e^{iut}$ with primitive $J_u(t) = \frac{e^{iut} - 1}{iu}$, for $t \in \mathbb{R}$. We already know that a.s. $\text{LTC}_{X_\Phi}(j_u, U) = \int_{\mathbb{R}} j_u(t) \psi_{X_\Phi}(t, U) dt$ may be written as the sum of three terms $R_u + C_u + I_u$ with finite expectation under our assumptions. By Fubini Theorem, it follows that

$$\widehat{\Psi_{X_\Phi}}(u, U) = \mathbb{E} \left(\int_{\mathbb{R}} j_u(t) \psi_{X_\Phi}(t, U) dt \right) = \mathbb{E}(R_u) + \mathbb{E}(C_u) + \mathbb{E}(I_u).$$

For the first term we get

$$\begin{aligned} \mathbb{E}(R_u) &= \mathbb{E} \left(\int_{\mathcal{R}_{X_\Phi} \cap U} \frac{e^{iuX_\Phi^+(z)} - e^{iuX_\Phi^-(x)}}{iz} \kappa_{X_\Phi}(z) \mathcal{H}^1(dz) \right) \\ &= \mathbb{E} \left(\sum_i \int_{\tau_{x_i} \mathcal{R}_{g_{m_i}} \cap U} e^{iuX_{\Phi_i}(z)} \frac{e^{iug_{m_i}^+(z-x_i)} - e^{iug_{m_i}^-(z-x_i)}}{iu} \kappa_{g_{m_i}}(z-x_i) \mathcal{H}^1(dz) \right) \\ &= \int_{\mathbb{R}^2 \times \mathbb{R}^d} \int_{\tau_x \mathcal{R}_{g_m} \cap U} \mathbb{E} \left(e^{iuX_\Phi(z)} \frac{e^{iug_m^+(z-x)} - e^{iug_m^-(z-x)}}{iu} \kappa_{g_m}(z-x) \mathcal{H}^1(dz) \right) \lambda dx F(dm), \end{aligned}$$

by Slivnyak-Mecke formula. Using translation invariance of both \mathcal{H}^1 and \mathcal{L} and stationarity of X_Φ we get that

$$(22) \quad \mathbb{E}(R_u) = \lambda \mathcal{L}(U) \mathbb{E} \left(e^{iuX_\Phi(0)} \int_{\mathbb{R}^d} \int_{\mathcal{R}_{g_m}} \frac{e^{iug_m^+(z)} - e^{iug_m^-(z)}}{iu} \kappa_{g_m}(z) \mathcal{H}^1(dz) F(dm) \right).$$

Similarly, for the second term we get

$$\begin{aligned} \mathbb{E}(C_u) &= \mathbb{E} \left(\sum_{z \in \mathcal{C}_{X_\Phi} \cap U} \frac{e^{iuX_\Phi^+(z)} - e^{iuX_\Phi^-(z)}}{iu} \alpha_{X_\Phi}(z) \right) \\ &= \mathbb{E} \left(\sum_i \mathbf{1}_{z \in \tau_{x_i} \mathcal{C}_{g_{m_i}} \cap U} e^{iuX_{\Phi_i}(z)} \frac{e^{iug_{m_i}^+(z-x_i)} - e^{iug_{m_i}^-(z-x_i)}}{iu} \alpha_{g_{m_i}}(z-x_i) \mathcal{H}^1(dz) \right) \\ (23) \quad &= \lambda \mathcal{L}(U) \mathbb{E} \left(e^{iuX_\Phi(0)} \int_{\mathbb{R}^d} \sum_{z \in \mathcal{C}_{g_m}} \frac{e^{iug_m^+(z)} - e^{iug_m^-(z)}}{iu} \alpha_{g_m}(z) F(dm) \right), \end{aligned}$$

Finally, the last term may be itself decomposed in two terms, say $I_u = I_u^{(1)} + I_u^{(2)}$. With similar computations we get

$$(24) \quad \mathbb{E}(I_u^{(1)}) = \lambda \mathcal{L}(U) \mathbb{E} \left(e^{iuX_\Phi(0)} \int_{\mathbb{R}^d} \sum_{z \in \mathcal{I}_{g_m}} \frac{e^{iug_m^+(z)} + e^{iug_m^-(z)} - e^{iug_m^+(z)} - e^{iug_m^-(z)}}{iu} \beta_{g_m}(z) F(dm) \right),$$

while, by Slivnyak-Mecke formula, since $\tau_{x_i} \mathcal{R}_{g_{m_i}} \cap \tau_{x_j} \mathcal{R}_{g_{m_j}} = \tau_{x_j} \mathcal{R}_{g_{m_j}} \cap \tau_{x_i} \mathcal{R}_{g_{m_i}}$,

$$\begin{aligned} &\mathbb{E}(I_u^{(2)}) \\ &= \mathbb{E} \left(\frac{1}{2} \sum_{i \neq j} \sum_{z \in \tau_{x_i} \mathcal{R}_{g_{m_i}} \cap \tau_{x_j} \mathcal{R}_{g_{m_j}} \cap U} e^{iuX_{\Phi_{ij}}(z)} \Delta_{m_i, m_j}^u(z-x_i, z-x_j) d_{S^1}(\nu_{g_{m_i}}(z-x_i), \nu_{g_{m_j}}(z-x_j)) \right) \\ &= \frac{1}{2} \iint \sum_{z \in \tau_x \mathcal{R}_{g_m} \cap \tau_{x'} \mathcal{R}_{g_{m'}} \cap U} \mathbb{E} \left(e^{iuX_\Phi(z)} \right) \Delta_{m, m'}^u(z-x, z-x') d_{S^1}(\nu_{g_m}(z-x), \nu_{g_{m'}}(z-x')) \lambda^2 dx F(dm) dx' F(dm'), \end{aligned}$$

By change of variables, translation invariance of \mathcal{H}^0 and \mathcal{L} , and stationarity of X_Φ , we get

$$\mathbb{E}(I_u^{(2)}) = \frac{\lambda^2}{2} \mathcal{L}(U) \mathbb{E} \left(e^{iuX_\Phi(0)} \int_{\mathbb{R}^d \times \mathbb{R}^d} \int_{\mathbb{R}^2} \sum_{z \in \tau_x \mathcal{R}_{g_m} \cap \tau_{x'} \mathcal{R}_{g_{m'}}} \Delta_{m, m'}^u(z-x, z) d_{S^1}(\nu_{g_m}(z-x), \nu_{g_{m'}}(z)) dx F(dm) F(dm') \right).$$

Now, (22)+(23)+(24) is equal to

$$(25) \quad \lambda \mathcal{L}(U) \mathbb{E} \left(e^{iuX_\Phi(0)} \right) \int_{\mathbb{R}^d} \widehat{\psi}_{g_m}(u, \mathbb{R}^2) F(dm),$$

which concludes the proof. \square

3.2. Explicit computations. In this section, we will give some explicit computations of the mean level total curvature function of elementary shot noise random fields. These results generalize the results of Decreusefond et. al. [13] obtained for indicator functions of a square, and also the known results on the Boolean model (that correspond to the excursion set of level $t = 1$). We will also show some numerical simulations.

Let us first recall that for shot noise random fields, the characteristic function of $X_\Phi(0)$ is explicit and given by (see for instance [9])

$$\mathbb{E}(e^{iuX_\Phi(0)}) = \exp \left(\lambda \int_{\mathbb{R}^2 \times \mathbb{R}^d} (e^{iug_m(x)} - 1) dx F(dm) \right).$$

In this section we consider D a compact subset of \mathbb{R}^2 having a piecewise smooth boundary given by $\Gamma = \partial D$ a finite union of positively oriented closed simple curves, piecewise C^2 and of finite total curvature, i.e. $\text{TaC}(\Gamma, \mathbb{R}^2) < +\infty$. Note that by Gauss-Bonnet Theorem we have

$$\text{TC}(\Gamma, \mathbb{R}^2) = 2\pi\chi(D).$$

We will focus on the case where the marks are of the form $m = (b, r, \theta) \in [0, +\infty)^2 \times [0, 2\pi] \subset \mathbb{R}^d$ with $d = 3$, with distribution $F(dm) = F_B(db)F_R(dr)F_\Theta(d\theta)$ and functions g_m given by

$$\forall x \in \mathbb{R}^2, \quad g_m(x) = b \mathbf{1}_{R_\theta r D}(x),$$

where rD is the dilation of D by the factor r , and R_θ denotes the rotation of angle θ . We denote by B , R and Θ independent random variables with distributions F_B , F_R (with support in \mathbb{R}_+) and F_Θ (with support in $[0, 2\pi]$). We will mainly focus on the case where Θ is uniform on $[0, 2\pi]$, that is $F_\Theta(d\theta) = \frac{1}{2\pi} \mathbf{1}_{[0, 2\pi]} d\theta$ for random shapes with uniform rotation; or on the case where $\Theta = 0$ a.s., that is $F_\Theta = \delta_0$ corresponding to simpler marks $m = (b, r)$.

Theorem 4. *We assume that*

$$\mathbb{E}(|B|) = \mathbb{E}(B) < +\infty \quad \text{and} \quad \mathbb{E}(R^2) < +\infty.$$

Then X_Φ satisfies assumptions of Theorem 2 and 3 and

$$\widehat{\Psi}_{X_\Phi}(0, U) = \int_{\mathbb{R}} \mathbb{E}(\psi_{X_\Phi}(t, U)) dt = \mathbb{E}(\text{LTC}_{X_\Phi}(U)) = 2\pi\lambda\mathcal{L}(U)\mathbb{E}(B)\chi(D).$$

We denote the mean perimeter and the mean area of D by

$$\bar{p} := \int_{\mathbb{R}^2} \mathcal{H}^1(\partial R_\theta r D) F_R(dr) F_\Theta(d\theta) = \mathcal{H}^1(\partial D) \mathbb{E}(R) \quad \text{and} \quad \bar{a} := \int_{\mathbb{R}^2} \mathcal{L}(R_\theta r D) F_R(dr) F_\Theta(d\theta) = \mathcal{L}(D) \mathbb{E}(R^2).$$

When moreover, F_Θ is the uniform law on $[0, 2\pi]$, then, for all $u \neq 0$,

$$\widehat{\Psi}_{X_\Phi}(u, U) = \lambda \mathcal{L}(U) \mathbb{E}(e^{iuX_\Phi(0)}) \frac{\widehat{F}_B(u) - 1}{iu} \left(2\pi\chi(D) + \frac{\lambda}{2} (\widehat{F}_B(u) - 1) \bar{p}^2 \right).$$

In the case where $B = 1$ a.s. we deduce that

$$(26) \quad \forall k \in \mathbb{N}, \forall t \in (k, k+1], \quad \mathbb{E}(\psi_{X_\Phi}(t, U)) = 2\pi\lambda\mathcal{L}(U) e^{-\lambda\bar{a}} \frac{(\lambda\bar{a})^k}{k!} \left(\chi(D) - \frac{\lambda}{4\pi} \bar{p}^2 + \frac{\bar{p}^2}{4\pi\bar{a}} k \right).$$

Proof. Since $B \geq 0$ a.s. and $\text{TaC}(R_\theta r \Gamma, \mathbb{R}^2) = \text{TaC}(\Gamma, \mathbb{R}^2)$, the g_m 's are elementary functions with for $F(dm)$ almost every $m = (b, r, \theta)$,

$$\text{LTaC}(g_m, \mathbb{R}^2) = |b| \text{TaC}(\Gamma, \mathbb{R}^2).$$

Let us remark that Equation (10) becomes

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^2} |g_m(x)| dx F(dm) = \mathbb{E}(|B|) \bar{a} < +\infty.$$

Equation (11) is easily checked since

$$\int_{\mathbb{R}^d} \text{LTaC}_{g_m}(\mathbb{R}^2) F(dm) = \mathbb{E}(|B|) \text{TaC}(\Gamma, \mathbb{R}^2) < +\infty.$$

Assumption (12) also follows from the fact that $\bar{a} < +\infty$ and $\mathbb{E}(|B|) < +\infty$. For (14), let us remark that $\mathcal{I}_{g_m} = \emptyset$ and $\mathcal{C}_{g_m} = R_\theta r \mathcal{C}_\Gamma$ such that $\mathcal{H}^0(\mathcal{S}_{g_m} \setminus \mathcal{R}_{g_m}) = \mathcal{H}^0(\mathcal{C}_\Gamma)$, ensuring (14), by assumption on Γ .

Now, in order to check (15) and (16) we need a kind of *kinematic formula*. For f a measurable non-negative function on $[-\pi, \pi]^2$, periodically extended on \mathbb{R}^2 , and Γ_1, Γ_2 two simple closed curves, we can compute the integral

$$I_{\Gamma_1, \Gamma_2}(f) := \int_{\mathbb{R}^2} \sum_{z \in \Gamma_1 \cap \tau_x \Gamma_2} f(\text{Arg } \nu_{\Gamma_1}(z), \text{Arg } \nu_{\Gamma_2}(z - x)) dx.$$

Using computations analogous to the kinematic formula of Santaló [25], Chapter 7 (where he considers random translations and rotations), we have that if Γ_1 and Γ_2 are two simple closed curves, then

$$(27) \quad I_{\Gamma_1, \Gamma_2}(f) = \int_0^{L_1} \int_0^{L_2} f(\theta_1(s_1) + \frac{\pi}{2}, \theta_2(s_2) + \frac{\pi}{2}) |\sin(\theta_1(s_1) - \theta_2(s_2))| ds_1 ds_2,$$

where the curve Γ_1 (resp. Γ_2) of length $L_1 = \mathcal{H}^1(\Gamma_1)$ (resp. $L_2 = \mathcal{H}^1(\Gamma_2)$), is parametrized by $s_1 \mapsto \gamma_1(s_1)$ (resp. by $s_2 \mapsto \gamma_2(s_2)$) where s_1 (resp. s_2) is arc length, and $\theta_1(s_1) = \text{Arg } \gamma_1'(s_1)$ (resp. $\theta_2(s_2) = \text{Arg } \gamma_2'(s_2)$). Heuristically, this formula can be obtained using the change of variable $x = F(s_1, s_2) = \gamma_1(s_1) - \gamma_2(s_2)$, for which the Jacobian is

$$|\det(dF)| = |\sin(\theta_1(s_1) - \theta_2(s_2))|,$$

and noticing that the intersection point of Γ_1 and $\tau_x \Gamma_2$ is then $z = \gamma_1(s_1) = x + \gamma_2(s_2)$.

In particular, taking $f = 1$, it follows that $I_{\Gamma_1, \Gamma_2}(1) \leq L_1 L_2$. Note also that we moreover have the exact formula

$$\int_0^{2\pi} \int_0^{2\pi} I_{R_\theta \Gamma_1, R'_\theta \Gamma_2}(1) d\theta d\theta' = 2\pi \times \int_0^{2\pi} |\sin(\theta)| d\theta L_1 L_2 = 2\pi \times 4L_1 L_2,$$

according to a generalization of Poincaré's Formula (see [25] for instance). Using the fact that Γ is a finite disjoint union of closed curves we obtain that $I_{R_\theta r \Gamma, R'_\theta r' \Gamma}(1) \leq L^2 r r'$, since the length of $R_\theta r \Gamma$, resp. $R'_\theta r' \Gamma$, is rL , resp. $r'L$, with $L = \mathcal{H}^1(\Gamma)$ the length of Γ and

$$\int_0^{2\pi} \int_0^{2\pi} I_{R_\theta r \Gamma, R'_\theta r' \Gamma}(1) d\theta d\theta' = 2\pi \times 4L^2 r r'.$$

It follows that

$$\begin{aligned} \int_{\mathbb{R}^d \times \mathbb{R}^d} \int_{\mathbb{R}^2} \mathcal{H}^0(\mathcal{R}_{g_{m'}} \cap \tau_x \mathcal{R}_{g_m}) dx F(dm) F(dm') &= \int_{\mathbb{R}_+^2} \int_{[0, 2\pi]^2} I_{R_\theta r \Gamma, R'_\theta r' \Gamma}(1) F_\Theta(d\theta) F_\Theta(d\theta') F_R(dr) F_R(dr') \\ &\leq 4L^2 \left(\int_{\mathbb{R}_+} r F_r(dr) \right)^2 = 4\bar{p}^2, \end{aligned}$$

with $\bar{p} = L \mathbb{E}(R)$ the mean perimeter, proving (15).

Moreover, for $f(\theta_1, \theta_2) = \mathbf{1}_{\theta_1 \equiv \theta_2} + \mathbf{1}_{\theta_1 \equiv \theta_2 + \pi}$, where \equiv stands for equality modulo 2π , we clearly have

$$I_{\Gamma_1, \Gamma_2}(f) = 0,$$

in view of (27). Since Γ is a finite disjoint union of closed curves, it follows that

$$\begin{aligned} &\int_{\mathbb{R}^d \times \mathbb{R}^d} \int_{\mathbb{R}^2} \mathcal{H}^0(\{y \in \mathcal{R}_{g_{m'}} \cap \tau_x \mathcal{R}_{g_m}; \nu_{g_{m'}}(y) = \pm \nu_{g_m}(y - x)\}) dx F(dm) F(dm') \\ &= \int_{\mathbb{R}_+^2} \int_{[0, 2\pi]^2} I_{R_\theta r \Gamma, R'_\theta r' \Gamma}(f) F_\Theta(d\theta) F_\Theta(d\theta') F_R(dr) F_R(dr') = 0, \end{aligned}$$

so that (16) holds.

Therefore we get the statement of Theorem 2.

Moreover, we also get

$$\begin{aligned} &\int_{\mathbb{R}^d \times \mathbb{R}^d} \int_{\mathbb{R}^2} (\|g_m\|_\infty + \|g_{m'}\|_\infty) \mathcal{H}^0(\mathcal{R}_{g_{m'}} \cap \tau_x \mathcal{R}_{g_m}) dx F(dm) F(dm') \\ &\leq 4\bar{p}^2 \int_{\mathbb{R} \times \mathbb{R}} (|b| + |b'|) F_B(db) F_B(db') = 8\bar{p}^2 \mathbb{E}(|B|) < +\infty, \end{aligned}$$

and (21) is also satisfied so that Theorem 3 holds.

Note that, for $u = 0$, we simply have

$$\widehat{\Psi}_{X_\Phi}(0, U) = \lambda \mathcal{L}(U) \int_{\mathbb{R}^d} \text{LTC}_{g_m}(\mathbb{R}^2) F(dm) = \lambda \mathcal{L}(U) \mathbb{E}(B) \text{TC}(\Gamma, \mathbb{R}^2).$$

Since $\text{TC}(\Gamma, \mathbb{R}^2) = 2\pi\chi(D)$, we obtain the first general statement. For $u \neq 0$, we can explicitly compute the characteristic function of $X_\Phi(0)$, given by

$$\mathbb{E}(e^{iuX_\Phi(0)}) = \exp\left(\lambda \int_{\mathbb{R}^d \times \mathbb{R}^2} (e^{iug_m(x)} - 1)F(dm)dx\right) = e^{\lambda\bar{a}(\widehat{F}_B(u)-1)},$$

where $\widehat{F}_B(u) = \mathbb{E}(e^{iuB}) = \int_{\mathbb{R}} e^{iub}F_B(db)$ is the characteristic function of B . In particular for $B = 1$ a.s. $X_\Phi(0)$ is a Poisson random variable of parameter $\lambda\bar{a}$. We also have

$$\begin{aligned} \widehat{\psi}_{g_m}(u, \mathbb{R}^2) &= \int_{\mathcal{R}_{g_m}} \frac{e^{iug_m^+(z)} - e^{iug_m^-(z)}}{iu} \kappa_{g_m}(z) \mathcal{H}^1(dz) + \sum_{z \in \mathcal{C}_{g_m}} \frac{e^{iug_m^+(z)} - e^{iug_m^-(z)}}{iu} \alpha_{g_m}(z) \\ &= \frac{e^{iub} - 1}{iu} \text{TC}(\mathcal{R}_{\theta r}\Gamma, \mathbb{R}^2) = \frac{e^{iub} - 1}{iu} \text{TC}(\Gamma, \mathbb{R}^2), \end{aligned}$$

so that

$$\int_{\mathbb{R}^d} \widehat{\psi}_{g_m}(u, \mathbb{R}^2) F(dm) = \frac{\widehat{F}_B(u) - 1}{iu} \text{TC}(\Gamma, \mathbb{R}^2).$$

Finally, let us remark that for $f(\theta_1, \theta_2) = \min(|\theta_1 - \theta_2|, 2\pi - |\theta_1 - \theta_2|)$ (distance between two angles), we get

$$\begin{aligned} &\int_{\mathbb{R}^d} \int_{\mathbb{R}^2 \times \mathbb{R}^d} \sum_{z \in \tau_x \mathcal{R}_{g_m} \cap \mathcal{R}_{g_{m'}}} d_{S^1}(\nu_{g_m}(z-x), \nu_{g_{m'}}(z)) \Delta_{m, m'}^u(z-x, z) dx F(dm) F(dm') \\ &= \frac{(\widehat{F}_B(u) - 1)^2}{iu} \int_{\mathbb{R}_+^2} \int_{[0, 2\pi]^2} I_{R_{\theta r}\Gamma, R_{\theta' r'}\Gamma}(f) F_\Theta(d\theta) F_\Theta(d\theta') F_R(dr) F_R(dr'). \end{aligned}$$

This last expression may not be simple to compute. However, assuming from now on that $F_\Theta(d\theta) = \frac{1}{2\pi} \mathbf{1}_{[0, 2\pi]} d\theta$, we obtain by (27), for two simple closed curves Γ_1, Γ_2 ,

$$\begin{aligned} &\int_{[0, 2\pi]^2} I_{R_{\theta}\Gamma_1, R_{\theta'}\Gamma_2}(f) F_\Theta(d\theta) F_\Theta(d\theta') \\ &= \frac{1}{2\pi} \int_0^{2\pi} \int_0^{L_1} \int_0^{L_2} \min(|\theta_1(s_1) - \theta_2(s_2) - \theta|, 2\pi - |\theta_1(s_1) - \theta_2(s_2) - \theta|) \sin(\theta_1(s_1) - \theta_2(s_2) - \theta) ds_1 ds_2 d\theta \\ &= L_1 L_2. \end{aligned}$$

Since Γ is a finite disjoint union of such closed curves, we get

$$\int_{[0, 2\pi]^2} I_{R_{\theta r}\Gamma, R_{\theta' r'}\Gamma}(f) F_\Theta(d\theta) F_\Theta(d\theta') = L^2 r r'.$$

It follows that for uniform rotations, we have

$$\widehat{\Psi}_{X_\Phi}(u, U) = \lambda \mathcal{L}(U) \mathbb{E}(e^{iuX_\Phi(0)}) \frac{\widehat{F}_B(u) - 1}{iu} \left(\text{TC}(\Gamma, \mathbb{R}^2) + \frac{\lambda}{2} (\widehat{F}_B(u) - 1) \bar{p}^2 \right).$$

When moreover $B = 1$ a.s., we can deduce an exact formula for the mean level total curvature function of X_Φ . Actually, in this case $X_\Phi(0)$ follows a Poisson law of parameter $\lambda\bar{a}$ and $\frac{\widehat{F}_B(u)-1}{iu}$ is the characteristic function of a uniform random variable Z on $[0, 1]$, while $\widehat{F}_B(u) \frac{\widehat{F}_B(u)-1}{iu}$ is the characteristic function of $Z + 1$. Then, considering Z independent from $X_\Phi(0)$ we recognize

$$\widehat{\Psi}_{X_\Phi}(u, U) = \lambda \mathcal{L}(U) \left[\mathbb{E}\left(e^{iu[X_\Phi(0)+Z]}\right) \left(\text{TC}(\Gamma, \mathbb{R}^2) - \frac{\lambda}{2} \bar{p}^2 \right) + \mathbb{E}\left(e^{iu[X_\Phi(0)+Z+1]}\right) \frac{\lambda}{2} \bar{p}^2 \right].$$

Note also that since here the random field X_Φ has integer values, then $\forall k \in \mathbb{N}$,

$$\forall t \in (k, k+1], \quad \{x \in U; X_\Phi(x) \geq t\} = \{x \in U; X_\Phi(x) \geq k+1\},$$

$$\text{and therefore } \mathbb{E}(\psi_{X_\Phi}(t, U)) = \mathbb{E}(\psi_{X_\Phi}(k+1, U)).$$

Hence we may conclude that

$$(28) \quad \forall k \in \mathbb{N}, \forall t \in (k, k+1], \quad \mathbb{E}(\psi_{X_\Phi}(t, U)) = \lambda \mathcal{L}(U) e^{-\lambda\bar{a}} \frac{(\lambda\bar{a})^k}{k!} \left(\text{TC}(\Gamma, \mathbb{R}^2) - \frac{\lambda}{2} \bar{p}^2 + \frac{\bar{p}^2}{2\bar{a}} k \right)$$

Note that, thanks to Gauss-Bonnet Theorem, we have $\text{TC}(\Gamma, \mathbb{R}^2) = 2\pi\chi(D)$, so that we may rewrite this as in (26). □

Let us remark that formula (26) only involves the Euler Characteristic, the mean perimeter and the mean area of the shapes. When $k = 0$, we find the formula of the mean Euler Characteristic of a rotation invariant Boolean model as obtained by Mecke and Wagner in [23] and by Mecke in [22] stating that

$$\frac{\mathbb{E}(\chi(\{x \in U; X(x) \geq 1\}))}{\pi \mathcal{L}(U)} = e^{-\lambda m_0(D)} (\lambda m_2(D) - \lambda^2 m_1(D)^2),$$

with $m_0(D) = \bar{a}$, $m_1(D) = \bar{p}/2\pi$ and $m_2(D) = \chi(D)/\pi$.

The typical behavior of $\mathbb{E}(\psi_X(k, U))$, as a function of $k \in \mathbb{N}$, is the following:

- It starts, when k is small, by being negative. This is explained by the fact that $\{X_\Phi \geq k\}$ is essentially made of one big connected component with many small holes in it. In particular the minimum value is achieved for an integer denoted k_- . The explicit value of k_- can be computed from Equation (28). The formula is not very nice, but it has a simple asymptotic behavior when λ is large, since then we have

$$k_- = \lambda \bar{a} - \sqrt{\lambda \bar{a}} + O(1).$$

- Then, after k_- , the mean level total curvature $\mathbb{E}(\psi_{X_\Phi}(k, U))$ increases and it crosses 0 in the interval that contains k_0 where

$$k_0 = \lambda \bar{a} - \frac{2\pi \bar{a}}{\bar{p}^2} \chi(D) = \lambda \bar{a} + O(1).$$

For this level, there are as many connected components as holes.

- After k_0 , the mean level total curvature is positive and it increases till a value k_+ and afterwards it decreases and goes to 0 as k goes to infinity. As for k_- the value of k_+ is explicit, and its asymptotic behavior when λ is large is

$$k_+ = \lambda \bar{a} + \sqrt{\lambda \bar{a}} + O(1).$$

Example 1: Random disks

We assume here that $D = D(0, 1)$ is a disk of radius 1, and that $B = 1$ a.s.. In this case we have $\text{TC}(\partial D) = 2\pi \chi(D) = 2\pi$, $\bar{p} = 2\pi \mathbb{E}(R)$ and $\bar{a} = \pi \mathbb{E}(R^2)$. Note also that since $R_\theta r D = r D$ for all θ , whatever F_Θ is, the shot noise field has the same law than one with marks given by $\tilde{m} = (b, r) \in [0, +\infty)^2 \subset \mathbb{R}^2$, with distribution $G(d\tilde{m}) = F_B(db)F_R(dr)$. Then, for all θ, θ' , we may compute, $I_{R_\theta r \Gamma, R_{\theta'} r' \Gamma}(f) = I_{r \Gamma, r' \Gamma}(f)$ as

$$r r' \int_0^{2\pi} \int_0^{2\pi} |\theta_1 - \theta_2| |\sin(\theta_1 - \theta_2)| d\theta_1 d\theta_2 = 2\pi r \times 2\pi r'.$$

Hence,

$$(29) \quad \forall k \in \mathbb{N}, \forall t \in (k, k+1], \quad \frac{1}{2\pi} \mathbb{E}(\psi_{X_\Phi}(t, U)) = \lambda \mathcal{L}(U) e^{-\lambda \bar{a}} \frac{(\lambda \bar{a})^k}{k!} \left(1 - \pi \lambda \mathbb{E}(R)^2 + \frac{\mathbb{E}(R)^2}{\mathbb{E}(R^2)} k \right).$$

An example of such a random field with comparisons between the theoretical value of $\mathbb{E}(\psi_{X_\Phi}(t, U))/2\pi$, corresponding to mean Euler characteristic of excursion sets, and an empirical estimate on a large domain are shown on Figure 3. The caption of the figure gives the practical and technical details of the simulation.

Let us quote that we can also compute the mean level total curvature for a non isotropic shape. This is the case of squares for instance, as developed in the following example.

Example 2: Random squares

We assume here that D is a square of side length 1 and $\Theta = 0$ a.s. or equivalently that marks are given by $\tilde{m} = (b, r) \in [0, +\infty)^2 \subset \mathbb{R}^2$, with distribution $G(d\tilde{m}) = F_B(db)F_R(dr)$. In this case, $\Gamma = \partial D$ is made of four line segments, with $\text{TC}(\Gamma, \mathbb{R}^2) = 2\pi \chi(D) = 2\pi$, $\bar{p} = 4\mathbb{E}(R)$ and $\bar{a} = \mathbb{E}(R^2)$. On the boundary of a square, the curvature is 0, and it has four corner points with a turning angle equal to $\pi/2$. Now, according to the kinematic formula (27), the only remaining terms are for $\theta_1(s_1) = \theta_2(s_2) \pm \frac{\pi}{2}$ for which $f(\theta_1(s_1), \theta_2(s_2)) = \min(|\theta_1(s_1) - \theta_2(s_2)|, 2\pi - |\theta_1(s_1) - \theta_2(s_2)|) = \frac{\pi}{2}$. It follows that

$$I_{\Gamma_1, \Gamma_2}(f) = \frac{\pi}{2} \times 8r_1 \times r_2.$$

Therefore

$$\begin{aligned} & \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^2 \times \mathbb{R}^d} \sum_{z \in \tau_x \mathcal{R}_{g_m} \cap \mathcal{R}_{g_{m'}}} d_{S^1}(\nu_{g_m}(z-x), \nu_{g_{m'}}(z)) \Delta_{m, m'}^u(z-x, z) dx F(dm) F(dm') \\ &= \frac{(\widehat{F}_B(u) - 1)^2}{iu} \int_{\mathbb{R}_+^2} I_{r \Gamma, r' \Gamma}(k) F_R(dr) F_R(dr') = 2\pi \frac{(\widehat{F}_B(u) - 1)^2 \bar{p}^2}{iu} \frac{1}{16}. \end{aligned}$$

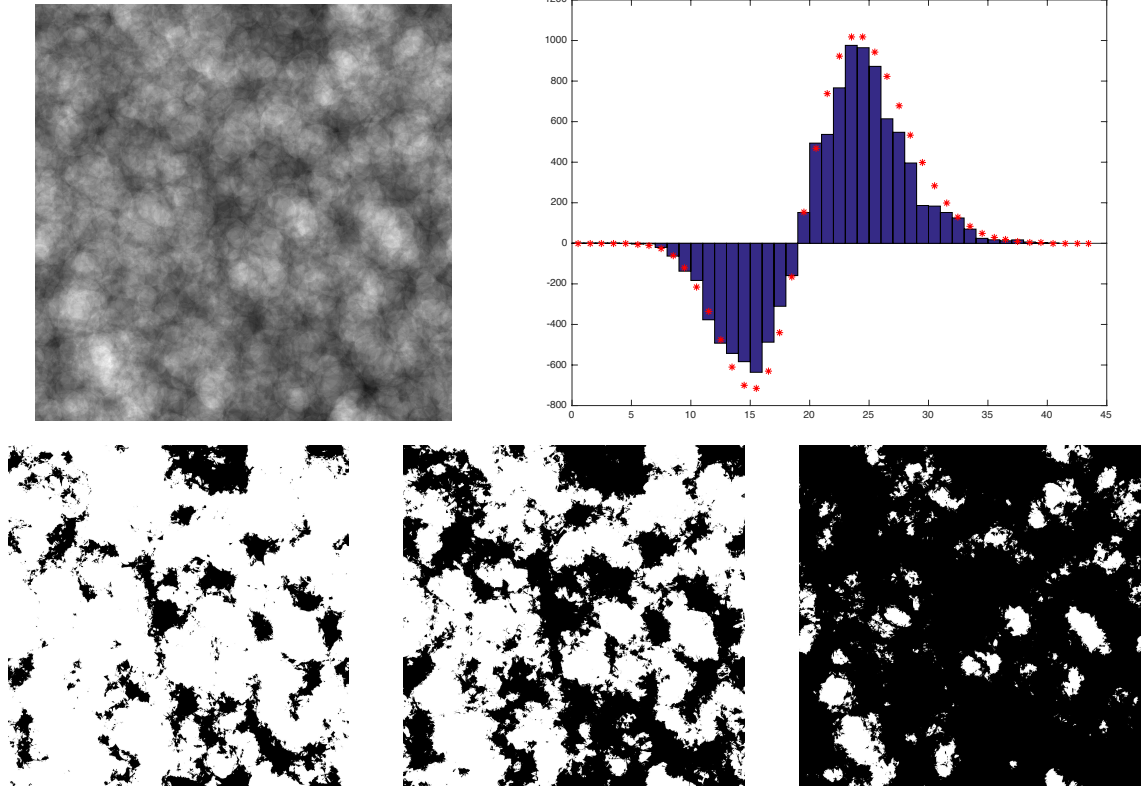


FIGURE 3. Shot noise random field with indicator functions of random disks. This sample has been obtained using Matlab, with a domain of size 2000×2000 pixels, a Poisson point process of intensity $\lambda = 0.001$, and random disks of radius $R = 50$ or $R = 100$ (each with probability 0.5). Top right figure: empirical Euler Characteristic as a function of the level t (computed thanks to the Matlab function `bweuler`), compared with the theoretical value (red stars) of Equation (29). Bottom line: Three excursion sets corresponding respectively from left to right to the level $t = 15$, $t = 19$ (that is the “critical level” where the Euler Characteristic turns from negative to positive) and $t = 25$.

It follows that we get in this case

$$\widehat{\Psi}_{X_\Phi}(u, U) = \lambda \mathcal{L}(U) \mathbb{E}(e^{iuX_\Phi(0)}) \frac{\widehat{F}_B(u) - 1}{iu} \left(\text{TC}(\Gamma, \mathbb{R}^2) + 2\pi\lambda(\widehat{F}_B(u) - 1) \frac{\bar{p}^2}{16} \right).$$

For $B = 1$ a.s., inverting as previously, we obtain

$$(30) \quad \forall k \in \mathbb{N}, \forall t \in (k, k+1], \quad \frac{1}{2\pi} \mathbb{E}(\psi_{X_\Phi}(t, U)) = \lambda \mathcal{L}(U) e^{-\lambda \bar{a}} \frac{(\lambda \bar{a})^k}{k!} \left(1 - \frac{\lambda}{16} \bar{p}^2 + \frac{\bar{p}^2}{16 \bar{a}} k \right).$$

It is illustrated on Figure 4.

This formula generalizes one of the results of Decreusefond et. al. [13]. Actually, considering the Boolean model made of squares of constant size $R = 2\varepsilon$ a.s. for some $\varepsilon > 0$, we get for $k = 0$, and $a > 0$,

$$\forall t \in (0, 1], \quad \frac{1}{2\pi} \mathbb{E}(\psi_{X_\Phi}(t, [0, a]^2)) = \lambda a^2 e^{-\lambda(2\varepsilon)^2} (1 - \lambda(2\varepsilon)^2),$$

that corresponds to the mean Euler Characteristic of the Boolean model in dimension 2, considered in the torus of size $a > 0$ in Theorem 11 of [13].

4. MEAN LEVEL TOTAL CURVATURE OF SMOOTH RANDOM FIELDS

In this section we consider a smooth stationary random field X defined on \mathbb{R}^2 . Let us introduce some notations for the derivatives of X . A point $x \in \mathbb{R}^2$ is defined by its two coordinates $x = (x_1, x_2)$ and we denote for $i, j = 1, 2$

$$X_i := \frac{\partial X}{\partial x_i} \quad \text{and} \quad X_{ij} := \frac{\partial^2 X}{\partial x_i \partial x_j}.$$

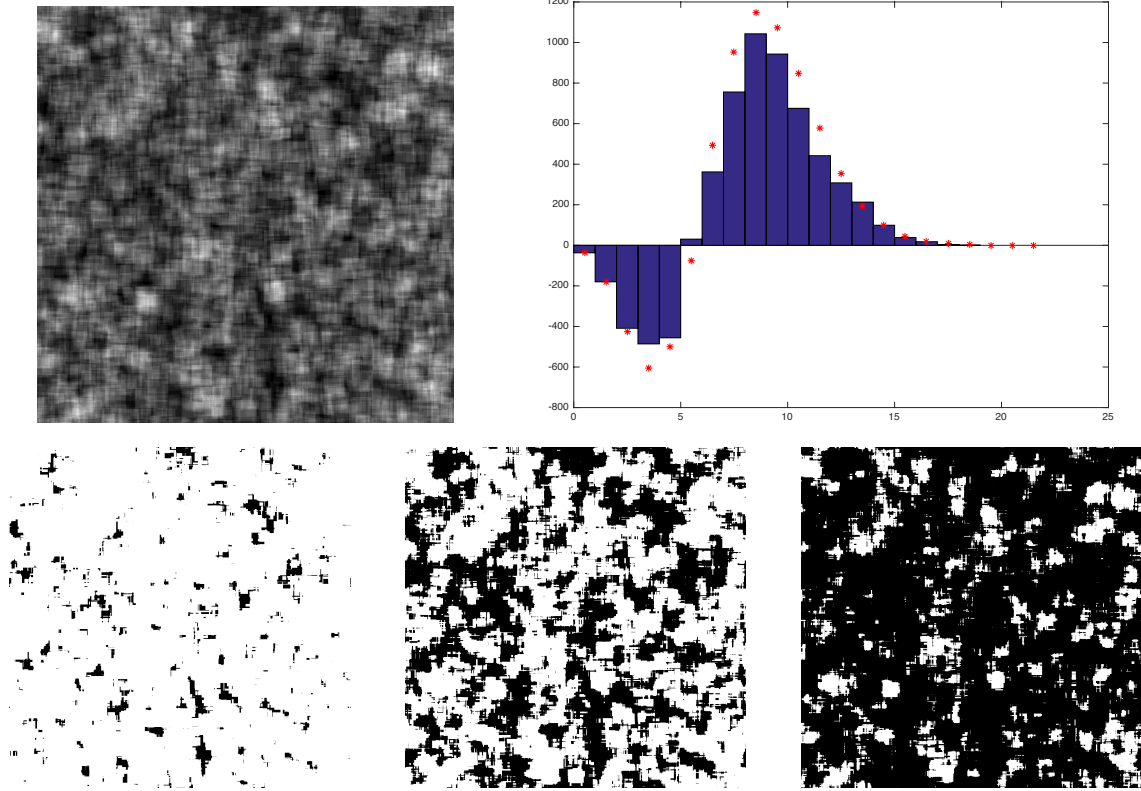


FIGURE 4. Shot noise random field with indicator functions of random squares. This sample has been on a domain of size 2000×2000 pixels, a Poisson point process of intensity $\lambda = 0.001$, and random squares of side length $R = 50$ or $R = 100$ (each with probability 0.5). Top right figure: empirical Euler Characteristic as a function of the level t , compared with the theoretical value (red stars) of Equation (30). Bottom line: Three excursion sets corresponding respectively from left to right to the level $t = 4$, $t = 6$ (that is the “critical level” where the Euler Characteristic turns from negative to positive) and $t = 9$.

With these notations it follows that $\nabla X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$ and $D^2 X = \begin{pmatrix} X_{11} & X_{12} \\ X_{12} & X_{22} \end{pmatrix}$. Note that since X is stationary, for any $x \in \mathbb{R}^2$,

$$(X(x), \nabla X(x), D^2 X(x)) \stackrel{d}{=} (X(0), \nabla X(0), D^2 X(0)).$$

When X , ∇X and $D^2 X$ have also finite second order moment, $X(x)$ and $\nabla X(x)$ are not correlated, as well as $\nabla X(x)$ and $D^2 X(x)$ (see [1] p.31 for instance). This is very useful for Gaussian fields since it implies that $\nabla X(x)$ is independent from $(X(x), D^2 X(x))$.

4.1. A general result for smooth stationary random fields. Using the result of Section 2.5 and the stationarity of X , we have the following formula.

Theorem 5. *Let X be a stationary C^2 random field on \mathbb{R}^2 , such that $D^2 X$ has a finite expectation. Then, $(\omega, t) \mapsto \psi_{X(\omega)}(t, U) \in L^1(\Omega \times \mathbb{R})$. Let us denote $\Psi_X(t, U) = \mathbb{E}(\psi_X(t, U))$ for a.e. $t \in \mathbb{R}$, then for any h bounded continuous function on \mathbb{R} , one has*

$$\int_{\mathbb{R}} h(t) \Psi_X(t, U) dt = -\mathcal{L}(U) \mathbb{E} \left(h(X(0)) \frac{D^2 X(0) \cdot (\nabla X(0)^\perp, \nabla X(0)^\perp)}{\|\nabla X(0)\|^2} \right).$$

It follows that when the field X is isotropic, i.e. $X \circ A \stackrel{fdd}{=} X$ for all orthogonal matrix A , then the above formula reduces to

$$\int_{\mathbb{R}} h(t) \Psi_X(t, U) dt = -\mathcal{L}(U) \mathbb{E}(h(X(0)) X_{ii}(0)), \quad \forall i = 1, 2.$$

In particular, taking $h = 1$ we have

$$\mathbb{E}(\text{LTC}_X(U)) = -\mathcal{L}(U) \mathbb{E}(X_{11}(0)) = -\mathcal{L}(U) \mathbb{E}(X_{22}(0)) = 0.$$

Assuming moreover that $X(0)$ admits a density $p_{X(0)}$ we get for almost every $t \in \mathbb{R}$

$$\Psi_X(t, U) = -\mathcal{L}(U)\mathbb{E}(X_{ii}(0)|X(0) = t)p_{X(0)}(t), \quad \forall i = 1, 2.$$

Proof. Let h be a bounded continuous function on \mathbb{R} . According to Proposition 3, since X is a.s. C^2 , we obtain

$$\text{LTaC}_X(U) \leq \int_U \|D^2X(x)\|_2 dx \quad \text{a.s.}$$

and

$$\text{LTC}_X(h, U) = \int_{\mathbb{R}} h(t)\psi_X(t, U) dt = - \int_U h(X(x)) \frac{D^2X(x) \cdot (\nabla X(x)^\perp, \nabla X(x)^\perp)}{\|\nabla X(x)\|^2} dx \quad \text{a.s.}$$

Note that $\|D^2X(x)\|_2 \leq |X_{11}(x)| + |X_{22}(x)| + 2|X_{12}(x)|$ so that

$$\int_{\mathbb{R}} \mathbb{E}(|\psi_X(t, U)|) dt \leq \mathbb{E}(\text{LTaC}_X(U)) < +\infty,$$

since $D^2X(x)$ has finite expectation. Hence $(\omega, t) \mapsto \psi_{X(\omega)}(t, U) \in L^1(\Omega \times \mathbb{R})$ and using Fubini Theorem we obtain for $\Psi_X(t, U) = \mathbb{E}(\psi_X(t, U))$,

$$\begin{aligned} \int_{\mathbb{R}} h(t)\Psi_X(t, U) dt &= \mathbb{E}(\text{LTC}_X(h, U)) \\ &= - \int_U \mathbb{E} \left(h(X(x)) \frac{D^2X(x) \cdot (\nabla X(x)^\perp, \nabla X(x)^\perp)}{\|\nabla X(x)\|^2} \right) dx \\ &= -\mathcal{L}(U)\mathbb{E} \left(h(X(0)) \frac{D^2X(0) \cdot (\nabla X(0)^\perp, \nabla X(0)^\perp)}{\|\nabla X(0)\|^2} \right), \end{aligned}$$

by stationarity of X .

Under the assumption that the field is isotropic, we can exploit further on this formula. First let us recall that by Taylor formula, since X is a.s. C^2 we have a.s. for all $x, z \in \mathbb{R}^2$,

$$X(x+z) = X(x) + \langle \nabla X(x), z \rangle + \frac{1}{2} D^2X(x) \cdot (z, z) + o_{\|z\| \rightarrow 0}(\|z\|^2).$$

In particular we obtain that, for any orthogonal matrix A ,

$$\nabla(X \circ A)(x) = {}^tA \nabla X(Ax) \text{ and } D^2(X \circ A)(x) = {}^tA(D^2X)(Ax)A.$$

Writing (e_1, e_2) the canonical basis of \mathbb{R}^2 , we also have for $i = 1, 2$,

$$\begin{aligned} X_{ii}(x) &= \lim_{\varepsilon \rightarrow 0} \frac{X(x + \varepsilon e_i) + X(x - \varepsilon e_i) - 2X(x)}{\varepsilon^2}, \\ X_{12}(x) &= \lim_{\varepsilon \rightarrow 0} \frac{X(x + \varepsilon(e_1 + e_2)) + X(x - \varepsilon(e_1 + e_2)) - X(x + \varepsilon(e_1 - e_2)) - X(x - \varepsilon(e_1 - e_2))}{2\varepsilon^2} \end{aligned}$$

Since $X \circ A \stackrel{fdd}{=} X$, we deduce that

$$(X(x), \nabla X(x), D^2X(x)) \stackrel{d}{=} (X(Ax), {}^tA \nabla X(Ax), {}^tA(D^2X)(Ax)A),$$

and specifying to $x = 0$, it follows that

$$(31) \quad (X(0), \nabla X(0), D^2X(0)) \stackrel{d}{=} (X(0), {}^tA \nabla X(0), {}^tA(D^2X)(0)A).$$

Let us introduce the random variable Θ with values in $2\pi\mathbb{T}$ identified with $[0, 2\pi)$ such that

$$\nabla X(0) = \begin{pmatrix} X_1(0) \\ X_2(0) \end{pmatrix} = \|\nabla X(0)\| \begin{pmatrix} \cos \Theta \\ \sin \Theta \end{pmatrix}.$$

For any $\theta \in [0, 2\pi)$ let us consider a rotation matrix $R_\theta = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$ and a reflexion matrix $S_\theta = \begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{pmatrix}$. Let us quote that $R_{-\theta} \nabla X(0) = {}^tR_\theta \nabla X(0) = \|\nabla X(0)\| \begin{pmatrix} \cos(\Theta - \theta) \\ \sin(\Theta - \theta) \end{pmatrix}$, $S_\theta \nabla X(0) = {}^tS_\theta \nabla X(0) = \|\nabla X(0)\| \begin{pmatrix} \cos(\theta - \Theta) \\ \sin(\theta - \Theta) \end{pmatrix}$. From (31) with $A = R_\theta$ and $A = S_\theta$ we have that

$$\begin{aligned} (X(0), \|\nabla X(0)\|, \Theta, D^2X(0)) &\stackrel{d}{=} (X(0), \|\nabla X(0)\|, \Theta - \theta, {}^tR_\theta D^2X(0) R_\theta) \\ &\stackrel{d}{=} (X(0), \|\nabla X(0)\|, \theta - \Theta, {}^tS_\theta D^2X(0) S_\theta) \end{aligned}$$

It follows that

$$\begin{aligned} (X(0), \|\nabla X(0)\|, \Theta, {}^tR_\theta D^2 X(0) R_\theta) &\stackrel{d}{=} (X(0), \|\nabla X(0)\|, \Theta + \theta, D^2 X(0)) \\ (X(0), \|\nabla X(0)\|, \Theta, {}^tS_\theta D^2 X(0) S_\theta) &\stackrel{d}{=} (X(0), \|\nabla X(0)\|, \theta - \Theta, D^2 X(0)) \end{aligned}$$

But ${}^tR_\theta D^2(X)(0)R_\theta \cdot (e_i, e_i) = {}^tS_\theta D^2(X)(0)S_\theta \cdot (e_i, e_i)$ and ${}^tR_\theta D^2(X)(0)R_\theta \cdot (e_1, e_2) = -{}^tS_\theta D^2(X)(0)S_\theta \cdot (e_1, e_2)$, for $i = 1, 2$. Hence we can deduce that

$$(X(0), \|\nabla X(0)\|, \Theta + \theta, X_{11}(0), X_{22}(0), X_{12}(0)) \stackrel{d}{=} (X(0), \|\nabla X(0)\|, \theta - \Theta, X_{11}(0), X_{22}(0), -X_{12}(0)),$$

from which we have for all $\theta \in [0, 2\pi)$,

$$(32) \quad (X(0), \|\nabla X(0)\|, \Theta, X_{11}(0), X_{22}(0), X_{12}(0)) \stackrel{d}{=} (X(0), \|\nabla X(0)\|, 2\theta - \Theta, X_{11}(0), X_{22}(0), -X_{12}(0)).$$

Now,

$$\mathbb{E} \left(h(X(0)) \frac{D^2 X(0) \cdot (\nabla X(0)^\perp, \nabla X(0)^\perp)}{\|\nabla X(0)\|^2} \right) = \mathbb{E} (h(X(0)) (X_{11}(0) \sin^2 \Theta + X_{22}(0) \cos^2 \Theta - 2X_{12}(0) \sin \Theta \cos \Theta))$$

Choosing $\theta = \frac{\pi}{4}$ in (32) we get on the one hand

$$\mathbb{E}(h(X(0))[X_{11}(0) \sin^2 \Theta + X_{22} \cos^2 \Theta]) = \mathbb{E}(h(X(0))[X_{11}(0) \cos^2 \Theta + X_{22} \sin^2 \Theta]).$$

Therefore, adding the two side,

$$\mathbb{E}(h(X(0))[X_{11}(0) \sin^2 \Theta + X_{22} \cos^2 \Theta]) = \frac{1}{2} \mathbb{E}(h(X(0))[X_{11}(0) + X_{22}]).$$

On the other hand

$$\mathbb{E}(h(X(0))X_{12}(0) \sin \Theta \cos \Theta) = \mathbb{E}(h(X(0))[-X_{12}(0)] \cos \Theta \sin \Theta),$$

implying that $\mathbb{E}(h(X(0))X_{12}(0) \sin \Theta \cos \Theta) = 0$. Finally, considering S_θ with $\theta = \frac{\pi}{2}$, we have $(X(0), X_{11}(0)) \stackrel{d}{=} (X(0), X_{22}(0))$, that concludes the proof for the formula. Recall that since X is stationary, all its derivatives are centered so that $\mathbb{E}(\text{LTC}_X(U)) = 0$, by taking $h = 1$ in the above formula.

When, moreover $X(0)$ admits a density $p_{X(0)}$, we can further write

$$\mathbb{E}(h(X(0))X_{ii}(0)) = \int_{\mathbb{R}} h(t) \mathbb{E}(X_{ii}(0)|X(0) = t) p_{X(0)}(t) dt,$$

such that for any continuous bounded function h

$$\int_{\mathbb{R}} h(t) \Psi_X(t, U) dt = \int_{\mathbb{R}} h(t) (-\mathcal{L}(U) \mathbb{E}(X_{ii}(0)|X(0) = t) p_{X(0)}(t)) dt,$$

implying that, for almost every $t \in \mathbb{R}$,

$$\Psi_X(t, U) = -\mathcal{L}(U) \mathbb{E}(X_{ii}(0)|X(0) = t) p_{X(0)}(t).$$

□

Remark. Let us quote that using twice (32), we also get that for any θ, θ' ,

$$\begin{aligned} (X(0), \|\nabla X(0)\|, \Theta + \theta', X_{11}(0), X_{22}(0), X_{12}(0)) &\stackrel{d}{=} (X(0), \|\nabla X(0)\|, 2\theta - \Theta + \theta', X_{11}(0), X_{22}(0), -X_{12}(0)) \\ &\stackrel{d}{=} (X(0), \|\nabla X(0)\|, 2(\theta + \theta'/2) - \Theta, X_{11}(0), X_{22}(0), -X_{12}(0)) \\ &\stackrel{d}{=} (X(0), \|\nabla X(0)\|, \Theta, X_{11}(0), X_{22}(0), X_{12}(0)). \end{aligned}$$

Hence we may deduce that the conditional law of Θ knowing $(X(0), \|\nabla X(0)\|, X_{11}(0), X_{22}(0), X_{12}(0))$ is uniform on $[0, 2\pi]$ (see [11] for instance) and therefore Θ is also independent from $(X(0), \|\nabla X(0)\|, X_{11}(0), X_{22}(0), X_{12}(0))$. Finally, let us mention that for Gaussian random field, invariance under all orthogonal matrices is a direct consequence from invariance under all rotation matrices. This follows from the fact that its covariance function must be radial.

Remark. As we were finishing this paper, we found a similar recent result (see Corollary 2.3 of [20]) that should rewrite in our stationary setting, under additional assumption on X , as

$$\mathbb{E} \left(\int_{\mathbb{R}} h(t) \chi(E_X(t, U)) dt \right) = -\mathcal{L}(U) \mathbb{E} \left(h(X(0)) \left[\sum_{i=1}^2 \mathbf{1}_{\nabla X(0) \in Q_i} X_{ii}(0) \right] + h'(X(0)) \left[\sum_{i=1}^2 \mathbf{1}_{\nabla X(0) \in Q_i} X_i(0)^2 \right] \right),$$

for $h : \mathbb{R} \rightarrow \mathbb{R}$ a C^1 function with compact support, $Q_1 = \{x = (x_1, x_2); x_2 < x_1 < 0\}$ and $Q_2 = \{x = (x_1, x_2); x_1 < x_2 < 0\}$. Under the assumption that X is also isotropic, according to Remark 2.5 of [20],

$$\mathbb{E} (h'(X(0)) \mathbf{1}_{\nabla X(0) \in Q_i} X_i(0)^2) = \frac{\pi - 2}{16\pi} \mathbb{E} (h'(X(0)) \|\nabla X(0)\|^2).$$

Note that by stationarity

$$\begin{aligned} \mathbb{E} (h'(X(0)) X_i(0)^2) &= \int_0^1 \mathbb{E} (h'(X(te_i)) X_i(te_i)^2) dt = \mathbb{E} \left(\int_0^1 h'(X(te_i)) X_i(te_i)^2 dt \right) \\ &= -\mathbb{E} \left(\int_0^1 h(X(te_i)) X_{ii}(te_i)^2 dt \right) = -\mathbb{E} (h(X(0)) X_{ii}(0)), \end{aligned}$$

integrating by parts and using $\mathbb{E} (h(X(e_i)) X_i(e_i)) = \mathbb{E} (h(X(0)) X_i(0))$. Moreover, by (32),

$$\mathbb{E} (h(X(0)) \mathbf{1}_{\nabla X(0) \in Q_i} X_{ii}(0)) = \mathbb{E} \left(h(X(0)) \mathbf{1}_{\Theta \in \pi + \frac{\pi}{4}(i-1, i)} X_{ii}(0) \right) = \frac{1}{8} \mathbb{E} (h(X(0)) X_{ii}(0)).$$

It follows that

$$\mathbb{E} \left(\int_{\mathbb{R}} h(t) \chi(E_X(t, U)) dt \right) = -\mathcal{L}(U) \sum_{i=1}^2 \left(\frac{1}{8} - \frac{\pi - 2}{8\pi} \right) \mathbb{E} (h(X(0)) X_{ii}(0)) = \frac{1}{2\pi} \int_{\mathbb{R}} h(t) \Psi_X(t, U) dt,$$

by Theorem 5.

Example. Let X be a stationary isotropic Gaussian random field. Then,

$$\mathbb{E} (X_{11}(0) | X(0)) = \frac{\text{Cov}(X(0), X_{11}(0))}{\text{Var}(X(0))} X(0) = \frac{\partial_1^2 \rho_X(0)}{\rho_X(0)} X(0) = \frac{-\lambda_2}{\sigma^2} X(0),$$

where $\rho_X(x) = \text{Cov}(X(x), X(0))$, $\sigma^2 = \rho_X(0)$ and λ_2 denotes the second spectral moment. Hence, we get in this case that, for almost every $t \in \mathbb{R}$,

$$\Psi_X(t, U) = \mathcal{L}(U) \frac{\lambda_2}{\sigma^2} t \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{t^2}{2\sigma^2}}.$$

Let us emphasize that this is exactly the formula obtained for $2\pi \mathbb{E}(\chi(E_X(t, U)))$, stated for all $t \in \mathbb{R}$, under additional assumptions on X (see (3.2.8) of [2] for instance).

Examples of such stationary isotropic random fields with comparisons between the theoretical values of $\mathbb{E}(\psi_X(t, U))/2\pi$ (corresponding to the mean Euler characteristic of excursion sets) and an empirical estimate on the square of fixed size $[0, 1]$ are shown on Figure 5 and 6 (with $\sigma^2 = 1$ and $\lambda_2 = 2T^2$). The captions of the figures give the practical and technical details on simulations. Note that in view of the covariance functions a scaling relation may be set between T and the size of the square.

Actually, in the review paper of Adler [2], several “equivalent” definitions of the Euler Characteristic of an excursion set are given. The equivalence of the different definitions involves deep results of differential geometry, mainly the Gauss-Bonnet theorem and the Morse theory. We summarize here three definitions of the mean Euler Characteristic of an excursion set $E_X(t; U)$, not taking into account the boundary of $U = (0, 1)^2$. In the following, we assume that $(X(0), \nabla X(0), D^2 X(0))$ have a joint density of probability on $\mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^3$, where here the Hessian matrix $D^2 X(0)$ is seen as the vector $(X_{11}(0), X_{22}(0), X_{12}(0))$ in \mathbb{R}^3 . We denote by $p : (t, y_1, y_2, a, b, c) \in \mathbb{R}^6 \mapsto p(t, y_1, y_2, a, b, c)$ this joint probability density. The additional hypothesis needed for all the following formulas to be well-defined will be given in the statement of Theorem 6.

- (1) Using the recursive formula that consider intersections with horizontal lines, a first way to define the mean Euler Characteristic of the excursion set of level t is

$$\chi_1(t) = -\mathbb{E}(X_2(0) X_{11}(0) \mathbf{1}_{X_2(0) > 0} | (X, X_1) = (t, 0)) p_{X, X_1}(t, 0),$$

where p_{X, X_1} is the density of $(X(0), X_1(0))$. We can rewrite this as

$$\chi_1(t) = - \int a y_2 \mathbf{1}_{y_2 > 0} p(t, 0, y_2, a, b, c) dy_2 da db dc.$$

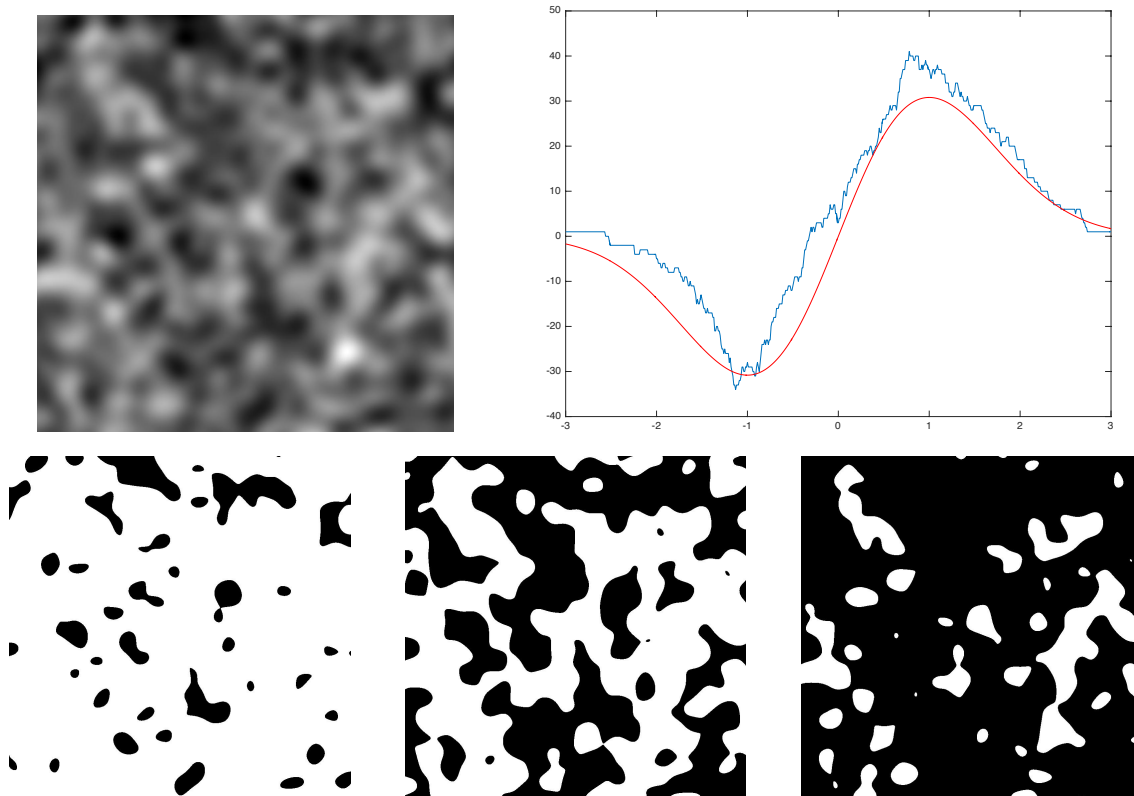


FIGURE 5. Gaussian random field with covariance $\rho(x) = e^{-T^2\|x\|^2}$ for $T = 10$. This sample has been obtained using Matlab, with a discretized domain of size $2^{10} \times 2^{10}$ pixels, using circulant embedding matrix. Top right figure: empirical Euler Characteristic as a function of the level t (computed thanks to the Matlab function `bweuler`), compared with the theoretical value (red one) of Equation (35). Bottom line: Three excursion sets corresponding respectively from left to right to the level $t = -1$, $t = 0$ (that is the “critical level” where the Euler Characteristic turns from negative to positive), and $t = 1$.

This formula amounts to count the mean number of points that are such that $X(x) = t$ and such that the set $E_X(t; U)$ is locally above the horizontal line passing through x .

- (2) The second definition is a consequence of Morse Theorem: it is the signed sum (according the index of the Hessian matrix) of the number of critical points of X (i.e. points such that $\nabla X(x) = 0$) and that are above the level t . That gives the second formula, introducing $p_{\nabla X}$ the density of $\nabla X(0)$,

$$\chi_2(t) = \mathbb{E}(\det(D^2 X(0)) \mathbf{1}_{X(0) \geq t} | \nabla X(0) = 0) p_{\nabla X}(0) = \int (ab - c^2) \mathbf{1}_{s \geq t} p(s, 0, 0, a, b, c) ds da db dc.$$

- (3) The third formula comes from the Gauss-Bonnet theorem, that we have recalled in Section 2.2. Since we don’t consider boundary terms, the third definition for the mean Euler Characteristic of the excursion set becomes

$$\chi_3(t) = -\frac{1}{2\pi} \int \frac{ay_2^2 + by_1^2 - 2cy_1y_2}{y_1^2 + y_2^2} p(t, y_1, y_2, a, b, c) dy_1 dy_2 da db dc.$$

With our notations for the mean level total curvature, this third definition is also

$$\chi_3(t) = \frac{1}{2\pi} \mathbb{E}(\psi_X(t, U)).$$

We now show that these three definitions are equal. This is a known fact, but the interesting point is that the proof only uses the stationarity of X and doesn’t need any result of differential geometry!

Theorem 6. *Let X be a stationary C^2 random field on \mathbb{R}^2 , such that X , ∇X and $D^2 X$ have finite second order moment. We moreover assume that*

- (A1) *The joint probability density p of $(X(0), \nabla X(0), D^2 X(0))$ exists and is continuous on \mathbb{R}^6 ,*

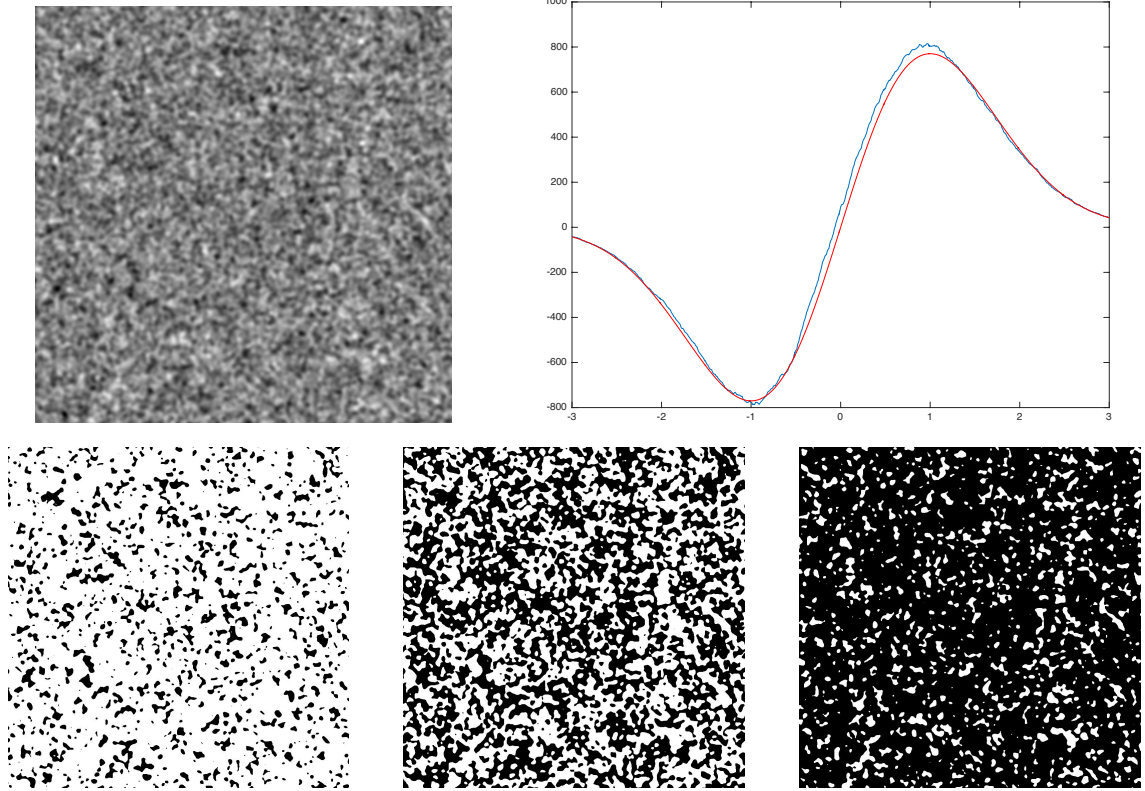


FIGURE 6. Gaussian random field with covariance $\rho(x) = e^{-T^2\|x\|^2}$ for $T = 100$. This sample has been obtained using Matlab, with a discretized domain of size $2^{10} \times 2^{10}$ pixels, using circulant embedding matrix. Top right figure: empirical Euler Characteristic as a function of the level t (computed thanks to the Matlab function `bweuler`), compared with the theoretical value (red one) of Equation (35). Bottom line: Three excursion sets corresponding respectively from left to right to the level $t = -1$, $t = 0$ (that is the “critical level” where the Euler Characteristic turns from negative to positive), and $t = 1$.

(A2) There exists a constant $C > 0$ such that for all $i, j = 1$ or 2 , and for all $t, y_1, y_2 \in \mathbb{R}$,

$$\mathbb{E}(X_{ij}(0)^2 | (X(0), X_1(0), X_2(0)) = (t, y_1, y_2)) p_{X, X_1, X_2}(t, y_1, y_2) \leq C$$

and $\mathbb{E}(X_i(0)^2 | (X(0), X_j(0)) = (t, y_j)) p_{X, X_j}(t, y_j) \leq C.$

Then, the stationarity of X implies that

$$\forall t \in \mathbb{R}, \quad \chi_1(t) = \chi_2(t) = \chi_3(t).$$

Proof. Let us first notice that thanks to the hypothesis on X , the three functions χ_1 , χ_2 and χ_3 are well-defined and continuous on \mathbb{R} .

In the following we will denote by ϕ the Gaussian function on \mathbb{R} , given by

$$\forall s \in \mathbb{R}, \quad \phi(s) = \frac{1}{\sqrt{2\pi}} e^{-s^2/2}.$$

And we will denote by Φ its primitive given by $\Phi(s) = \int_{-\infty}^s \phi(v) dv$. (This notation Φ is also the same as the notation of the Poisson Point Process, but we believe the context is clear and there is no danger of confusion). The two functions ϕ and Φ will be used, with a rescaling of parameter $\varepsilon > 0$ to approximate respectively the Dirac distribution at 0 and the indicator function of \mathbb{R}_+ .

Let h be a bounded continuous function on \mathbb{R} and let H denote a primitive of h . We consider the three following integrals: I_1 , I_2 and I_3 , given by $I_i = \int h(t) \chi_i(t) dt$, and we will show that they are equal (we drop the point notation (0) in the following for sake of conciseness). Since the χ_i are continuous, this will imply that they are equal on \mathbb{R} .

We start, thanks to the hypothesis (A1) and (A2), by rewriting I_1 as a limit:

$$\begin{aligned} I_1 &= \int h(t)\psi_1(t) dt = - \int h(t)ay_2\mathbf{1}_{y_2>0} p(t, 0, y_2, a, b, c) dy_2 da db dc dt \\ &= - \lim_{\varepsilon \rightarrow 0} \mathbb{E} \left[h(X)X_2X_{11}\Phi\left(\frac{X_2}{\varepsilon}\right)\frac{1}{\varepsilon}\phi\left(\frac{X_1}{\varepsilon}\right) \right] \end{aligned}$$

In the same way we can rewrite I_2 as a limit:

$$\begin{aligned} I_2 &:= \int h(t)\psi_2(t) dt = \int h(t)(ab - c^2)\mathbf{1}_{s \geq t} p(s, 0, 0, a, b, c) ds da db dc dt \\ &= \lim_{\varepsilon \rightarrow 0} \mathbb{E} \left[H(X)(X_{11}X_{22} - X_{12}^2)\frac{1}{\varepsilon^2}\phi\left(\frac{X_1}{\varepsilon}\right)\phi\left(\frac{X_2}{\varepsilon}\right) \right] \end{aligned}$$

And we recall that I_3 is given by:

$$\begin{aligned} I_3 &:= \int h(t)\psi_3(t) dt = -\frac{1}{2\pi} \int \frac{ay_2^2 + by_1^2 - 2cy_1y_2}{y_1^2 + y_2^2} p(t, y_1, y_2, a, b, c) dy_1 dy_2 da db dc dt \\ &= -\frac{1}{2\pi} \mathbb{E} \left[h(X) \frac{X_{11}X_2^2 + X_{22}X_1^2 - 2X_{12}X_1X_2}{X_1^2 + X_2^2} \right]. \end{aligned}$$

For $\varepsilon > 0$, let us denote

$$I_1^\varepsilon = \mathbb{E} \left[h(X)X_2X_{11}\Phi\left(\frac{X_2}{\varepsilon}\right)\frac{1}{\varepsilon}\phi\left(\frac{X_1}{\varepsilon}\right) \right] \quad \text{and} \quad I_2^\varepsilon = \mathbb{E} \left[H(X)(X_{11}X_{22} - X_{12}^2)\frac{1}{\varepsilon^2}\phi\left(\frac{X_1}{\varepsilon}\right)\phi\left(\frac{X_2}{\varepsilon}\right) \right].$$

We first notice that we can also write

$$I_2^\varepsilon = \mathbb{E} \left[H(X)\frac{\partial}{\partial x_1}\phi\left(\frac{X_1}{\varepsilon}\right)\frac{\partial}{\partial x_2}\Phi\left(\frac{X_2}{\varepsilon}\right) \right] - \mathbb{E} \left[H(X)\frac{\partial}{\partial x_2}\Phi\left(\frac{X_1}{\varepsilon}\right)\frac{\partial}{\partial x_1}\phi\left(\frac{X_2}{\varepsilon}\right) \right].$$

Then, by stationarity of X , we have that

$$0 = \frac{\partial}{\partial x_2} \mathbb{E} \left[H(X)\frac{\partial}{\partial x_1}\Phi\left(\frac{X_1}{\varepsilon}\right)\Phi\left(\frac{X_2}{\varepsilon}\right) \right] \quad \text{and} \quad 0 = \frac{\partial}{\partial x_1} \mathbb{E} \left[H(X)\frac{\partial}{\partial x_2}\Phi\left(\frac{X_1}{\varepsilon}\right)\Phi\left(\frac{X_2}{\varepsilon}\right) \right].$$

Computing these two partial derivatives, and considering their difference, some terms cancel and we get

$$\begin{aligned} I_2^\varepsilon &= \mathbb{E} \left[X_1h(X)\frac{\partial}{\partial x_2}\Phi\left(\frac{X_1}{\varepsilon}\right)\Phi\left(\frac{X_2}{\varepsilon}\right) \right] - \mathbb{E} \left[X_2h(X)\frac{\partial}{\partial x_1}\Phi\left(\frac{X_1}{\varepsilon}\right)\Phi\left(\frac{X_2}{\varepsilon}\right) \right] \\ &= \mathbb{E} \left[X_1h(X)X_{12}\frac{1}{\varepsilon}\phi\left(\frac{X_1}{\varepsilon}\right)\Phi\left(\frac{X_2}{\varepsilon}\right) \right] + I_1^\varepsilon. \end{aligned}$$

Now, as ε goes to 0 the first term of the right-hand side goes to 0, and thus

$$I_1 = I_2.$$

In the definition of $\chi_1(t)$, the idea was to consider horizontal sections of the excursion set. But it is natural to consider also the sections in any other direction. For $\alpha \in [0, 2\pi)$, we first introduce the partial derivatives in the direction α and $\alpha^\perp = \alpha + \frac{\pi}{2}$ by:

$$\frac{\partial}{\partial e_{\alpha 1}} = \cos \alpha \frac{\partial}{\partial x_1} + \sin \alpha \frac{\partial}{\partial x_2} \quad \text{and} \quad \frac{\partial}{\partial e_{\alpha 2}} = -\sin \alpha \frac{\partial}{\partial x_1} + \cos \alpha \frac{\partial}{\partial x_2}.$$

Then, the equivalent of I_1^ε in the direction α is given by

$$I_1^\varepsilon(\alpha) = -\mathbb{E} \left[h(X)\frac{\partial X}{\partial e_{\alpha 2}}\frac{\partial^2 X}{\partial e_{\alpha 1}^2}\Phi\left(\frac{\partial X}{\partial e_{\alpha 2}}\right)\frac{1}{\varepsilon}\phi\left(\frac{1}{\varepsilon}\frac{\partial X}{\partial e_{\alpha 1}}\right) \right].$$

The same computations as above with the partial derivatives in the directions α and α^\perp show that

$$I_1(\alpha) = \lim_{\varepsilon \rightarrow 0} \mathbb{E} \left[\det(D_\alpha^2 X) H(X)\frac{1}{\varepsilon^2}\phi\left(\frac{1}{\varepsilon}\frac{\partial X}{\partial e_{\alpha 1}}\right)\phi\left(\frac{1}{\varepsilon}\frac{\partial X}{\partial e_{\alpha 2}}\right) \right] = I_2.$$

Indeed, computing the partial derivatives in the directions α et α^\perp amounts to apply a rotation of angle α to the gradient of X , and this doesn't change the norms, nor the determinant of the Hessian.

Therefore we also have

$$I_2 = \frac{1}{2\pi} \int_0^{2\pi} I_1(\alpha) d\alpha.$$

To end the proof, we show that the integral on the right-hand side is equal to I_3 . Indeed, we have

$$I_1(\alpha) = - \int y \mathbf{1}_{y>0} h(t) (a \cos^2 \alpha + b \sin^2 \alpha + 2c \sin \alpha \cos \alpha) p(t, -y \sin \alpha, y \cos \alpha, a, b, c) dt dy da db dc.$$

Then, integrating on α , and making the polar change of variable $y_1 = -y \sin \alpha$ and $y_2 = y \cos \alpha$, we get

$$\int_0^{2\pi} I_1(\alpha) d\alpha = \int h(t) \frac{ay_2^2 + by_1^2 - 2cy_1y_2}{y_1^2 + y_2^2} p(t, y_1, y_2, a, b, c) dt dy_1 dy_2 da db dc = 2\pi I_3.$$

□

4.2. Smooth shot noise random fields. As in Section 3, we consider here a shot noise random field defined on \mathbb{R}^2 by

$$\forall x \in \mathbb{R}^2, \quad X_\Phi(x) = \sum_{i \in I} g_{m_i}(x - x_i),$$

where $\Phi = \{(x_i, m_i)\}_{i \in I}$ is a Poisson point process on $\mathbb{R}^2 \times \mathbb{R}^d$ of intensity $\lambda \mathcal{L} \times F$, with $\lambda > 0$ real, \mathcal{L} the Lebesgue measure on \mathbb{R}^2 and F a probability measure on \mathbb{R}^d . In order to get explicit formulas we have to make an assumption of isotropy. Moreover, since smooth shot noise random fields do not always admit a probability density (we have discussed this through several examples in our first paper [9]) we have to work with their characteristic functions.

Theorem 7. *Let assume that $g : \mathbb{R}^2 \times \mathbb{R}^d \rightarrow \mathbb{R}$ is a measurable function such that for F -almost every $m \in \mathbb{R}^d$ the functions $g_m := g(\cdot, m)$ are C^3 on \mathbb{R}^2 satisfying*

$$(33) \quad \int_{\mathbb{R}^2 \times \mathbb{R}^d} |D^{\mathbf{j}} g_m(x)| dx F(dm) < +\infty,$$

for all $\mathbf{j} = (j_1, j_2) \in \mathbb{N}^2$ with $|\mathbf{j}| = j_1 + j_2 \leq 3$ and where $D^{\mathbf{j}} g_m = \frac{\partial^{|\mathbf{j}|} g_m}{\partial x_1^{j_1} \partial x_2^{j_2}}$. Then X_Φ is a.s. a stationary C^2 field such that $D^2 X$ has finite expectation, ensuring the assumptions of Theorem 5.

We assume moreover that $m = (\tilde{m}, \theta) \in \mathbb{R}^{d-1} \times [0, 2\pi] \subset \mathbb{R}^d$ and $F(dm) = G(d\tilde{m})F_\Theta(d\theta)$, with F_Θ the uniform law on $[0, 2\pi]$, and

$$g(\cdot, m) = g(\cdot, (\tilde{m}, \theta)) = \tilde{g}(R_\theta \cdot, \tilde{m}),$$

for F -a.e. m , with $\tilde{g}(S_0 \cdot, \tilde{m}) = \tilde{g}(\cdot, \tilde{m})$, recalling that $S_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $R_\theta = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$. Then X_Φ is isotropic and the Fourier transform of $\Psi_{X_\Phi}(\cdot, U)$ is given for all $u \in \mathbb{R}$ by, for all $j = 1, 2$,

$$\widehat{\Psi_{X_\Phi}}(u, U) = -\lambda \mathcal{L}(U) \mathbb{E} \left(e^{iu X_\Phi(0)} \right) \left(\int_{\mathbb{R}^d} \int_{\mathbb{R}^2} \partial_j^2 g_m(x) e^{iug_m(x)} dx F(dm) \right),$$

where the notation ∂_j^2 stands for $\frac{\partial^2}{\partial x_j^2}$.

Remark: Note that when $g(A \cdot, m) = g(\cdot, m)$, for all orthogonal matrix A , X_Φ has the same law as the shot noise random field given with marks $\tilde{m} \in \mathbb{R}^{d-1}$ of law $G(d\tilde{m})$.

Proof. Following similar arguments as in Proposition 3 of [9], (33) will ensure that X_Φ is a.s. a stationary C^2 field such that $D^2 X$ has finite expectation, and we can differentiate under the sum. In particular, for all $j = 1, 2$,

$$\partial_j^2 X_\Phi(x) = \sum_{i \in I} \partial_j^2 g_{m_i}(x - x_i).$$

Hence the general formula of Theorem 5 is valid for X_Φ .

Under the additional assumption on the kernel, we can prove isotropy. Actually, for any $k \geq 1$, $u_1, \dots, u_k \in \mathbb{R}$ and $y_1, \dots, y_k \in \mathbb{R}^2$, one has

$$\mathbb{E} \left(e^{i \sum_{j=1}^k u_j X_\Phi(y_j)} \right) = \exp \left(\lambda \int_{\mathbb{R}^d} \int_{\mathbb{R}^2} (e^{iu \sum_{j=1}^k u_j g_m(y_j - x)} - 1) dx F(dm) \right).$$

Hence, for any orthogonal matrix A , by the change of variables $x = Ay$,

$$\mathbb{E} \left(e^{i \sum_{j=1}^k u_j X_\Phi(Ay_j)} \right) = \exp \left(\lambda \int_{\mathbb{R}^d} \int_{\mathbb{R}^2} (e^{iu \sum_{j=1}^k u_j g_m(A(y_j - y))} - 1) dy F(dm) \right)$$

Note that there exists $\theta_0 \in [0, 2\pi]$ such that $A = S_{\theta_0}$ or $A = R_{\theta_0}$ but for $m = (\tilde{m}, \theta)$,

$$g_m \circ S_{\theta_0} = \tilde{g}_{\tilde{m}} \circ R_\theta S_{\theta_0} = \tilde{g}_{\tilde{m}} \circ S_0 R_\theta S_{\theta_0} = \tilde{g}_{\tilde{m}} \circ R_{-\theta - \theta_0}.$$

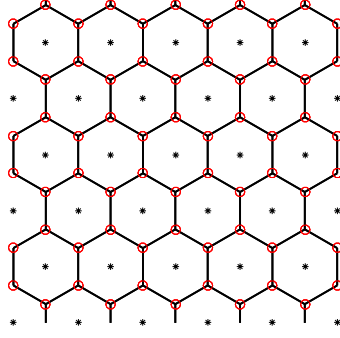


FIGURE 7. Hexagonal tiling restricted to a square domain $(0, T)^2$. The centers (set \mathcal{C}_ε) of the hexagons are the black stars, and the vertices (set \mathcal{R}_ε) are the points marked by a red circle. The distance between two neighbouring centers is $\sqrt{3}\varepsilon$ and the side length of the hexagons is ε .

Since, F_Θ is the uniform law, we may assume without loss of generality that

$$\begin{aligned} \mathbb{E} \left(e^{i \sum_{j=1}^k u_j X_\Phi(Ay_j)} \right) &= \exp \left(\lambda \int_{\mathbb{R}^d} \int_{\mathbb{R}^2} (e^{iu \sum_{j=1}^k u_j g_m(R_{\theta_0}(y_j - y))} - 1) dy F(dm) \right) \\ &= \exp \left(\lambda \int_{\mathbb{R}^d} \int_{\mathbb{R}^2} (e^{iu \sum_{j=1}^k u_j g_m(y_j - y)} - 1) dy F(dm) \right) = \mathbb{E} \left(e^{i \sum_{j=1}^k u_j X_\Phi(y_j)} \right) \end{aligned}$$

using the fact that $R_\theta R_{\theta_0} = R_{\theta_0 + \theta}$ and F_Θ uniform.

It follows that by Theorem 5, for any h bounded continuous function on \mathbb{R} , one has

$$\int_{\mathbb{R}} h(t) \Psi_{X_\Phi}(t, U) dt = -\mathcal{L}(U) \mathbb{E} (h(X_\Phi(0)) \partial_j^2 X_\Phi(0)), \quad \forall j = 1, 2.$$

Taking $h = e^{iu \cdot}$ for $u \in \mathbb{R}$ we obtain

$$\widehat{\Psi}_{X_\Phi}(u, U) = -\mathcal{L}(U) \mathbb{E} \left(e^{iu X_\Phi(0)} \partial_j^2 X_\Phi(0) \right) = i\mathcal{L}(U) \frac{\partial}{\partial v} \varphi_{X_\Phi, \partial_j^2 X_\Phi}(u, 0),$$

where $\varphi_{X_\Phi, \partial_j^2 X_\Phi}$ is the characteristic function of $(X_\Phi(0), \partial_j^2 X_\Phi(0))$ given by, for $(u, v) \in \mathbb{R}^2$

$$\begin{aligned} \varphi_{X_\Phi, \partial_j^2 X_\Phi}(u, v) &= \mathbb{E} \left(e^{i[u X_\Phi(0) + v \partial_j^2 X_\Phi(0)]} \right) \\ &= \exp \left(\lambda \int_{\mathbb{R}^d} \int_{\mathbb{R}^2} (e^{iug_m(x) + iv \partial_j^2 g_m(x)} - 1) dx F(dm) \right). \end{aligned}$$

Then,

$$\widehat{\Psi}_{X_\Phi}(u, U) = -\mathcal{L}(U) \lambda \left(\int_{\mathbb{R}^d} \int_{\mathbb{R}^2} \partial_j^2 g_m(x) e^{iug_m(x)} dx F(dm) \right) \varphi_{X_\Phi}(u),$$

with $\varphi_{X_\Phi} = \varphi_{X_\Phi, \partial_j^2 X_\Phi}(\cdot, 0)$ the characteristic function of $X_\Phi(0)$. \square

Let us also mention that in the recent paper [20] such a formula is also proposed in a similar isotropic framework in Theorem 3.3.

4.3. Link with discretization of a smooth Gaussian field. In the previous sections of this paper, we have considered two types of functions: the elementary (piecewise constant) ones and the smooth ones. Now, given a regular tiling of the plane, one may consider a discretization of a smooth function f by taking it constant on each tile. This is a classical setting when doing numerical simulations where functions are seen as images discretized on pixels. But using pixels (that are small squares) is not very convenient since at each point of the dual grid, we have to order the 4 neighbouring values to be able to compute the level total curvature. In the following, we will consider a more convenient tiling given by regular hexagons (since here there will be only 3 neighbouring values).

Let us first introduce some notations for the tiling with hexagons. Let $\varepsilon > 0$ and for $\theta \in \mathbb{R}$, let us denote by e_θ the unit vector of coordinates $(\cos \theta, \sin \theta)$. Let us consider a regular tiling with hexagons of “size” ε where the set of the centers of the hexagons is given by

$$\mathcal{C}_\varepsilon = \{k_1\sqrt{3}\varepsilon e_0 + k_2\sqrt{3}\varepsilon e_{\pi/3}; k_1, k_2 \in \mathbb{Z}\} = \left\{ \left(\left(k_1 + \frac{1}{2}k_2 \right) \sqrt{3}\varepsilon, \frac{3}{2}k_2\varepsilon \right); k_1, k_2 \in \mathbb{Z} \right\}$$

The distance between the centers of two neighbouring hexagons is $\sqrt{3}\varepsilon$, the side length of the hexagons is ε and the area of each hexagon is $\frac{3\sqrt{3}}{2}\varepsilon^2$. The vertices of the hexagons is the set of points \mathcal{R}_ε given by

$$\mathcal{R}_\varepsilon = \mathcal{C}_\varepsilon + \left\{ \varepsilon e_{\frac{\pi}{6} + n\frac{\pi}{3}}; 0 \leq n \leq 5 \right\}.$$

On Figure 7, we show such a tiling with regular hexagons. The points of \mathcal{C}_ε are plotted with black stars and the points of \mathcal{R}_ε are the vertices of the hexagons marked by small red circles. For $z \in \mathcal{C}_\varepsilon$ we will denote by $\mathcal{D}(z, \varepsilon)$ the (open) hexagon of center z and size ε . Notice that the distance between a vertex $x \in \mathcal{R}_\varepsilon$ and the centers of its three neighbouring hexagons is equal to ε .

Let f be a function defined on \mathbb{R}^2 , and let us consider a tiling with regular hexagons of size $\varepsilon > 0$. We then consider a discretized version f_ε of f defined by

$$(34) \quad \forall x \in \mathbb{R}^2, \quad f_\varepsilon(x) = \sum_{z \in \mathcal{C}_\varepsilon} f(z) \mathbf{1}_{\mathcal{D}(z, \varepsilon)}(x).$$

The function f_ε is piecewise constant, and the set of its discontinuity points is given by the hexagons edges, such that $\kappa_{f_\varepsilon} = 0$. Each vertex $x \in \mathcal{R}_\varepsilon$ separates three values of f_ε and therefore can not be seen as a corner point or as an intersection point, as defined in Definition 3. However we can still compute the level total curvature of f_ε following the same line as for elementary functions. If we denote for each vertex $x \in \mathcal{V}_\varepsilon$, its three ordered neighbouring values by $f_\varepsilon^{(1)}(x) \leq f_\varepsilon^{(2)}(x) \leq f_\varepsilon^{(3)}(x)$, we obtain the following proposition.

Proposition 4. *Let $U \subset \mathbb{R}^2$ be open and bounded, and let f be defined on U . Then for all $\varepsilon > 0$, the function f_ε is of finite level total curvature integral with*

$$\text{LTaC}_{f_\varepsilon}(U) = \frac{\pi}{3} \sum_{x \in \mathcal{V}_\varepsilon \cap U} [f_\varepsilon^{(3)}(x) - f_\varepsilon^{(1)}(x)].$$

Moreover, for h a bounded continuous function on \mathbb{R} , and H a primitive of h , the level total curvature integral of f_ε is given by

$$\text{LTC}_{f_\varepsilon}(h, U) = \frac{\pi}{3} \sum_{x \in \mathcal{V}_\varepsilon \cap U} [H(f_\varepsilon^{(3)}(x)) + H(f_\varepsilon^{(1)}(x)) - 2H(f_\varepsilon^{(2)}(x))].$$

In particular,

$$\text{LTC}_{f_\varepsilon}(U) = \frac{\pi}{3} \sum_{x \in \mathcal{V}_\varepsilon \cap U} [f_\varepsilon^{(3)}(x) + f_\varepsilon^{(1)}(x) - 2f_\varepsilon^{(2)}(x)].$$

When f is a smooth function, we may hope that $\text{LTC}_{f_\varepsilon}(h, U)$ will converge, as ε goes to 0 to some limit. Now, because of the ordering of the three neighbouring values of each vertex, such a result is difficult to obtain in general. Now, if we consider that f is in fact a smooth random field X , assuming moreover Gaussianity and isotropy, one can compute the level total curvature in expectation, and find its limit as ε goes to 0. This is the aim of the following theorem.

Theorem 8. *Let X be a Gaussian stationary isotropic random field on \mathbb{R}^2 admitting a finite spectral moment of order 2: $\lambda_2 = -\partial_1^2 \rho_X(0) = -\partial_2^2 \rho_X(0)$, where $\rho_X(x) = \text{Cov}(X(x), X(0))$ for $x \in \mathbb{R}^2$. Let $T > 0$ and $U = (0, T)^2$. For any $\varepsilon > 0$, let us denote X_ε the discretized field defined by (34). Then $(\omega, t) \mapsto \psi_{X_\varepsilon(\omega)}(t, U) \in L^1(\Omega \times \mathbb{R})$. Let us denote*

$$\Psi_{X_\varepsilon}(t, U) = \mathbb{E}(\psi_{X_\varepsilon}(t, U)) = \mathbb{E}(\text{TC}(\partial E_{X_\varepsilon}(t, U) \cap U)).$$

Then, for a.e. $t \in \mathbb{R}$,

$$(35) \quad \Psi_{X_\varepsilon}(t, U) \xrightarrow{\varepsilon \rightarrow 0} \Psi_X(t, U) = \mathcal{L}(U) \lambda_2 \sigma^{-2} \frac{t}{\sigma \sqrt{2\pi}} e^{-\frac{t^2}{2\sigma^2}},$$

with $\sigma^2 = \rho_X(0) = \text{Var}X(0)$.

Remark. Let us recall that according to Theorem 3.2.2 of [2], when X satisfies strong additional properties, for all level $t \in \mathbb{R}$, the mean Euler Characteristic of excursion set is given by

$$\mathbb{E}(\chi(E_X(t, U))) = \frac{1}{2\pi} \mathcal{L}(U) \lambda_2 \sigma^{-2} \frac{t}{\sigma \sqrt{2\pi}} e^{-\frac{t^2}{2\sigma^2}}.$$

See also Figure 5 and 6 for illustration.

Proof. We first assume that X is a centered stationary isotropic Gaussian random field, with unit variance ($\sigma^2 = 1$) and with covariance function given by

$$\forall t, s \in \mathbb{R}^2, \quad \rho_X(t) = \text{Cov}(X(t+s), X(s)) = \rho(\|t\|^2),$$

for $\rho : \mathbb{R} \rightarrow \mathbb{R}$, a C^2 function on \mathbb{R} and note that $\lambda_2 = -2\rho'(0)$. Let $\varepsilon > 0$ and consider X_ε defined by (34) with X instead of f . For $T > 0$ we will consider the level total curvature of X_ε on $U = (0, T)^2$. Note that, by stationary and isotropy, for any $x \in \mathcal{V}_\varepsilon$,

$$(X_\varepsilon^{(1)}(x), X_\varepsilon^{(2)}(x), X_\varepsilon^{(3)}(x)) \stackrel{d}{=} \left(X^{(1)}\left(\varepsilon \frac{\sqrt{3}}{3} e\right), X^{(2)}\left(\varepsilon \frac{\sqrt{3}}{3} R e\right), X^{(3)}\left(\varepsilon \frac{\sqrt{3}}{3} R^2 e\right) \right),$$

where $e \in S^1$ is a unit vector of \mathbb{R}^2 and R is the rotation of angle $\frac{2\pi}{3}$. Our assumptions imply that the 3-dimensional random vector $(X_\varepsilon^1, X_\varepsilon^2, X_\varepsilon^3) := (X(\varepsilon \frac{\sqrt{3}}{3} e), X(\varepsilon \frac{\sqrt{3}}{3} R e), X(\varepsilon \frac{\sqrt{3}}{3} R^2 e))$ is a centered Gaussian vector with covariance matrix given by

$$\begin{pmatrix} 1 & \rho(\varepsilon^2) & \rho(\varepsilon^2) \\ \rho(\varepsilon^2) & 1 & \rho(\varepsilon^2) \\ \rho(\varepsilon^2) & \rho(\varepsilon^2) & 1 \end{pmatrix}.$$

Hence, it is a correlated Gaussian vector with equal correlation and hence exchangeable. Since $\rho(0) = \sigma^2 = 1$, we may choose ε in such a way that $\rho(\varepsilon^2) > 0$. In this case, denoting as usual ϕ the standard normal density and Φ its cumulative distribution, marginal densities of order statistics are given for $1 \leq i \leq 3$ by: (see Corollary 6.1.1 of [27])

$$g_{(i)}(x) = \int_{\mathbb{R}} \frac{1}{\sqrt{1-\rho(\varepsilon^2)}} f_{(i)}\left(\frac{x + \sqrt{\rho(\varepsilon^2)}z}{\sqrt{1-\rho(\varepsilon^2)}}\right) \phi(z) dz,$$

with

$$f_{(1)}(y) = 3\Phi(-y)^2\phi(y), \quad f_{(2)}(y) = 6\Phi(y)\Phi(-y)\phi(y), \quad \text{and} \quad f_{(3)}(y) = 3\Phi(y)^2\phi(y),$$

corresponding to the marginal densities of the ordered statistics of an i.i.d. standard Gaussian vector. It follows that the random variable $\text{LTaC}_{X_\varepsilon}(U)$ has finite expectation and we can define the $L^1(\mathbb{R})$ function

$$\Psi_{X_\varepsilon}(t, U) = \mathbb{E}(\psi_{X_\varepsilon}(t, U)) = \mathbb{E}(\text{TC}(\partial E_{X_\varepsilon}(t, U) \cap U)).$$

Moreover, by Fubini Theorem, for any function h continuous and bounded, the random variable $\text{LTC}_{X_\varepsilon}(h, U)$ is integrable and

$$\int_{\mathbb{R}} h(t) \Psi_{X_\varepsilon}(t, U) dt = \mathbb{E}(\text{LTC}_{X_\varepsilon}(h, U)) = \frac{\pi}{3} \#(\mathcal{V}_\varepsilon \cap U) \mathbb{E}\left(H(X_\varepsilon^{(3)}) + H(X_\varepsilon^{(1)}) - 2H(X_\varepsilon^{(2)})\right),$$

with

$$\begin{aligned} \mathbb{E}\left(H(X_\varepsilon^{(3)}) + H(X_\varepsilon^{(1)}) - 2H(X_\varepsilon^{(2)})\right) &= \int_{\mathbb{R}} H(x)[g_{(3)}(x) + g_{(1)}(x) - 2g_{(2)}(x)] dx \\ &= 3 \int_{\mathbb{R}} \int_{\mathbb{R}} H(y \sqrt{1-\rho(\varepsilon^2)} - \sqrt{\rho(\varepsilon^2)}z) [\Phi(y)^2 + \Phi(-y)^2 - 4\Phi(y)\Phi(-y)] \phi(y)\phi(z) dy dz, \end{aligned}$$

after the change of variables $y = \frac{x + \sqrt{\rho(\varepsilon^2)}z}{\sqrt{1-\rho(\varepsilon^2)}}$. Note that, integrating by parts, we have

$$\int_{\mathbb{R}} \Phi(y)\Phi(-y)\phi(y) dy = \frac{1}{2} \int_{\mathbb{R}} \Phi(y)^2\phi(-y) dy.$$

Hence by parity of ϕ ,

$$(36) \quad \int_{\mathbb{R}} [\Phi(y)^2 + \Phi(-y)^2 - 4\Phi(y)\Phi(-y)] \phi(y) dy = 0.$$

Moreover, by odd parity we get

$$(37) \quad \int_{\mathbb{R}} [\Phi(y)^2 + \Phi(-y)^2 - 4\Phi(y)\Phi(-y)] y \phi(y) dy = 0.$$

It follows that for $h = 1$,

$$\mathbb{E}(\text{LTC}_{X_\varepsilon}(U)) = \int_{\mathbb{R}} \Psi_{X_\varepsilon}(t, U) dt = 0.$$

Now let $u \neq 0$ and consider the functions $t \mapsto j_u(t) = e^{iut}$ and $J_u(t) = \frac{e^{iut}-1}{iu}$. Then, denoting by $\widehat{\phi}$ the Fourier transform of ϕ , we have

$$\begin{aligned} \mathbb{E} \left(J_u(X_\varepsilon^{(3)}) + J_u(X_\varepsilon^{(1)}) - 2J_u(X_\varepsilon^{(2)}) \right) \\ = 3 \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{e^{iu[y\sqrt{1-\rho(\varepsilon^2)} - \sqrt{\rho(\varepsilon^2)}z]} - 1}{iu} [\Phi(y)^2 + \Phi(-y)^2 - 4\Phi(y)\Phi(-y)] \phi(y)\phi(z) dy dz \\ = 3\widehat{\phi}(u\sqrt{\rho(\varepsilon^2)}) \int_{\mathbb{R}} \frac{e^{iu[y\sqrt{1-\rho(\varepsilon^2)}]} - 1}{iu} [\Phi(y)^2 + \Phi(-y)^2 - 4\Phi(y)\Phi(-y)] \phi(y) dy, \end{aligned}$$

using (36). By Taylor-Lagrange and assumption on ρ , assuming that $\varepsilon < 1$, we get that

$$\left| e^{iu[y\sqrt{1-\rho(\varepsilon^2)}]} - 1 - iu[y\sqrt{1-\rho(\varepsilon^2)}] + \frac{(uy)^2}{2} \int_0^{\varepsilon^2} (-\rho'(s)) ds \right| \leq \frac{|uy|^3}{3!} \max_{[0,1]} |\rho'|^{3/2} \varepsilon^3,$$

Using (36) and (37), it follows that

$$\begin{aligned} \left| \int_{\mathbb{R}} \frac{e^{iu[y\sqrt{1-\rho(\varepsilon^2)}]} - 1}{iu} [\Phi(y)^2 + \Phi(-y)^2 - 4\Phi(y)\Phi(-y)] \phi(y) dy \right. \\ \left. - \frac{iu}{2} \left(\int_0^{\varepsilon^2} (-\rho'(s)) ds \right) \int_{\mathbb{R}} y^2 [\Phi(y)^2 + \Phi(-y)^2 - 4\Phi(y)\Phi(-y)] \phi(y) dy \right| \\ \leq u^2 \max_{[0,1]} |\rho'|^{3/2} \varepsilon^3 \int_{\mathbb{R}} |y|^3 \phi(y) dy. \end{aligned}$$

But, denoting by H_k the Hermite polynomial of order k defined by $H_k = (-1)^k \phi^{(k)} \phi^{-1}$, since $H_2(y) = y^2 - 1$, by (36) we get

$$\begin{aligned} \int_{\mathbb{R}} y^2 [\Phi(y)^2 + \Phi(-y)^2 - 4\Phi(y)\Phi(-y)] \phi(y) dy &= \int_{\mathbb{R}} H_2(y) [\Phi(y)^2 + \Phi(-y)^2 - 4\Phi(y)\Phi(-y)] \phi(y) dy \\ &= \int_{\mathbb{R}} [\Phi(y)^2 + \Phi(-y)^2 - 4\Phi(y)\Phi(-y)] \phi''(y) dy \\ &= - \int_{\mathbb{R}} [6\Phi(y) - 6\Phi(-y)] \phi(y) \phi'(y) dy \\ &= 12 \int_{\mathbb{R}} y \phi(y)^2 \Phi(y) dy, \end{aligned}$$

by integration by parts. But

$$12 \int_{\mathbb{R}} y \phi(y)^2 \Phi(y) dy = \frac{6}{\pi} \int_{\mathbb{R}} y e^{-y^2} \Phi(y) dy = \frac{3}{\pi} \int_{\mathbb{R}} e^{-y^2} \phi(y) dy = \frac{3}{\sqrt{3}\pi},$$

integrating again by parts and recognizing the density of a centered Gaussian variable with variance $1/3$. Hence,

$$\mathbb{E} \left(J_u(X_\varepsilon^{(3)}) + J_u(X_\varepsilon^{(1)}) - 2J_u(X_\varepsilon^{(2)}) \right) = \frac{3^2 iu}{2\sqrt{3}\pi} \left(\int_0^{\varepsilon^2} (-\rho'(s)) ds \right) \widehat{\phi}(u\sqrt{\rho(\varepsilon^2)}) + o(\varepsilon^2).$$

Now since $U = (0, T)^2$, the number of vertices in U is about two times the number of hexagons centers, such that $\#(\mathcal{V}_\varepsilon \cap U) = 2T^2 \frac{2}{\sqrt{3}} \varepsilon^{-2} + o(\varepsilon^{-2})$. Therefore, since $\mathcal{L}(U) = T^2$,

$$\mathbb{E}(\text{LTC}_{X_\varepsilon}(j_u, U)) = \frac{\pi}{3} 2\mathcal{L}(U) \frac{2}{\sqrt{3}} \frac{3^2 iu}{2\sqrt{3}\pi} (-\rho'(0)) \widehat{\phi}(u) + o(1) = \mathcal{L}(U) (-2\rho'(0)) \widehat{H}_1 \widehat{\phi}(u) + o(1).$$

It follows that we have the pointwise convergence

$$\widehat{\Psi_{X_\varepsilon}(\cdot, U)} \xrightarrow{\varepsilon \rightarrow 0} \mathcal{L}(U) (-2\rho'(0)) \widehat{H}_1 \widehat{\phi}(u).$$

But we can bound $\widehat{\Psi_{X_\varepsilon}(\cdot, U)}$ uniformly on ε by a function in $L^1(\mathbb{R})$ such that by inverse Fourier transform, they are also pointwise convergent according to Lebesgue Theorem. Fourier transform injectivity in $L^1(\mathbb{R})$ gives the conclusion for a.e. $t \in \mathbb{R}$, once we have noticed that $\lambda_2 = -2\rho'(0)$.

For the general case, since $\sigma^2 = \text{Var}(X(x))$, the Gaussian random field X/σ is itself unit variance with covariance ρ_X/σ^2 . The assumption on spectral moment, due to isotropy, implies that ρ_X is C^2 on \mathbb{R}^2 and satisfies $\partial_1 \rho_X(0) = \partial_2 \rho_X(0) = \partial_1 \partial_2 \rho_X(0) = 0$, while $\partial_1^2 \rho_X(0) = \partial_2^2 \rho_X(0) = \lambda_2$. It follows that, for $e \in S^1$, the function $\rho(t) = \frac{1}{\sigma^2} \rho_X(\sqrt{|t|}e)$ is C^1 on \mathbb{R} and satisfies $\text{Cov}(X(x)/\sigma, X(0)/\sigma) = \rho(\|x\|^2)$ with $\rho'(0) = -\frac{\lambda_2}{\sigma^2}$. But

$$\Psi_{X_\varepsilon}(t, U) = \mathbb{E}(\text{TC}(\partial E_{X_\varepsilon}(t, U) \cap U)) = \mathbb{E}(\text{TC}(\partial E_{X_\varepsilon/\sigma}(t/\sigma, U) \cap U)).$$

By the previous proof we know that

$$\mathbb{E}(\text{TC}(\partial E_{X_\varepsilon/\sigma}(t/\sigma, U) \cap U)) \xrightarrow{\varepsilon \rightarrow 0} \mathcal{L}(U)(-2\rho'(0))(H_1\phi)(t/\sigma),$$

and conclude for the proof since $\mathcal{L}(U)(-2\rho'(0))(H_1\phi)(t/\sigma) = \mathcal{L}(U) \frac{\lambda_2}{\sigma^2} \frac{t}{\sigma} e^{-\frac{t^2}{2\sigma^2}}$. \square

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