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Abstract

This paper deals with the analysis of the propagation of uncertainties in computational linear dynamics for strongly dissipative 3D linear viscoelastic composite structures in the presence of parameter uncertainties as well as model uncertainties. The approach used for modeling uncertainties is the probabilistic nonparametric approach, which consists in replacing the matrices of the nominal reduced-order model with random matrices following a probability distribution obtained using information theory. Special care is taken regarding the stochastic modeling of the random reduced stiffness and damping matrices. These two matrices are statistically dependent through a set of compatibility equations, implied by the causality of the system. This set of equations involves Hilbert transforms of the frequency-dependent part of the two matrices and are used in order to generate statistically independent realizations of each random matrix which satisfy the causality principle for the Monte Carlo stochastic solver.

1 Introduction

The objective of this paper is to present the numerical analysis and the computational aspects of an extension (recently proposed in [1, 2, 3]) of the nonparametric probabilistic approach of uncertainties [4, 5, 6] in computational linear structural dynamics for viscoelastic composite structures. The present work is devoted to the frequency-domain analysis of uncertain viscoelastic structures using the nonparametric probabilistic approach of uncertainty. The approach is formulated for a 3D dissipative composite structure made up of a linear viscoelastic part coupled with an elastic part. In the framework of linear viscoelasticity (see for instance [7, 8, 9]) and in the frequency domain, the generalized damping matrix $[\mathcal{D}(\omega)]$ and the generalized stiffness matrix $[\mathcal{K}(\omega)]$ of the reduced-order computational model depend on frequency ω. The nonparametric probabilistic approach of uncertainties consists in modeling this two frequency-dependent generalized matrices by frequency-dependent random matrices $[\mathbf{D}(\omega)]$ and $[\mathbf{K}(\omega)]$ respectively. However, as these two matrices come from a causal dynamical system, the causality implies two compatibility equations, also known as the Kramers-Kronig relations [10, 11], involving the Hilbert transform [12]. Consequently, the stochastic modeling of random matrices $[\mathbf{D}(\omega)]$ and $[\mathbf{K}(\omega)]$ cannot be constructed as independent random matrices without violating the causality as explained in [1]. In this paper, the compatibility equations are then used directly in order to generate compatible realizations of $[\mathbf{K}(\omega)]$ and $[\mathbf{D}(\omega)]$.

	Upper layer k = 1	Lower layer k = 2
$\nu^{(k)}$	0.27	0.47
$E_\infty^{(k)}$ (GPa)	240	220
$E_1^{(k)}$ (GPa)	126.5	50
$\tau_1^{(k)}$ (s)	7.351 $\times\, 10^{-2}$	1.103 $\times\, 10^{-1}$

Table 1: Parameters of the single-branch generalized Maxwell model

2 Mean reduced-order model

2.1 Description of the numerical model

In order to analyze the influence of viscoelasticity and causality in the propagation of uncertainties, a composite structure is studied in the Low-Frequency (LF) range. The structure that is considered is a thin multi-layered plate of length $L = 1$ m, width $W = 0.3$ m and thickness $H = 0.1$ m, under a nodal load of $F = 1$ N applied in direction the vertical direction $\mathbf{e}_3$ at the point located at (0.5067 m, 0.1565 m, 0.1 m) (see Fig. 1). The three layers are made up of a homogenous elastic medium occupying domain Ω_e that is sandwiched between two homogenous viscoelastic media occupying the domain $\Omega_{ve} = \Omega_1 \cup \Omega_2$ in which the domain Ω_1 is the upper layer and the domain Ω_2 is the lower layer. For the elastic medium, the material is assumed to be isotropic with Young's modulus $E = 210$ GPa, Poisson's ratio $\nu = 0.3$, and a density $\rho = 7,850$ kg/m^3. Its thickness is $h = 4H/5$ in which H is the total thickness of the plate. For the viscoelastic homogenous medium occupying domain Ω_k, with $k = 1, 2$, the material is assumed to be isotropic with a Poisson ratio $\nu^{(k)}$ and a time-dependent viscoelastic coefficient $E^{(k)}(t)$. Let $\widehat{E}^{(k)}(\omega)$ be the Fourier transform of $E^{(k)}(t)$. In the case of a single-branch generalized Maxwell model, we have

$$\widehat{E}_k(\omega) = E_\infty^{(k)} + \frac{E_1^{(k)}(\tau_1^{(k)}\omega)^2}{1+(\tau_1^{(k)}\omega)^2} + i\omega\frac{E_1^{(k)}\tau_1^{(k)}}{1+(\tau_1^{(k)}\omega)^2}\,. \tag{1}$$

The viscoelastic coefficients used in the simulations are listed in Table 1.

2.1.1 Mean computational model

The finite element mesh of the structure is constituted of 8-nodes 3D finite elements (see Fig. 1). The total number of degrees of freedom is 468402. Applying the standard finite element method (see for instance [13, 14]) in the framework of linear viscoelasticity (see for instance [7, 8, 9]) and in the frequency domain, yields the following mean computational model

$$(-\omega^2[\mathbb{M}] + i\omega[\mathbb{D}(\omega)] + [\mathbb{K}_0] + [\mathbb{K}(\omega)])\,\widehat{\mathbb{u}}(\omega) = \widehat{\mathbb{f}}(\omega)\,, \tag{2}$$

where $[\mathbb{K}_0]$ is a symmetric real matrix that is positive definite and frequency independent and where the symmetric positive real matrix $[\mathbb{K}(\omega)]$ is such that

$$[\mathbb{K}(\omega)] = \frac{2\,\omega}{\pi}\,p.v\int_0^{+\infty}\frac{[\mathbb{D}(\omega u)]}{1-u^2}\,du\,. \tag{3}$$

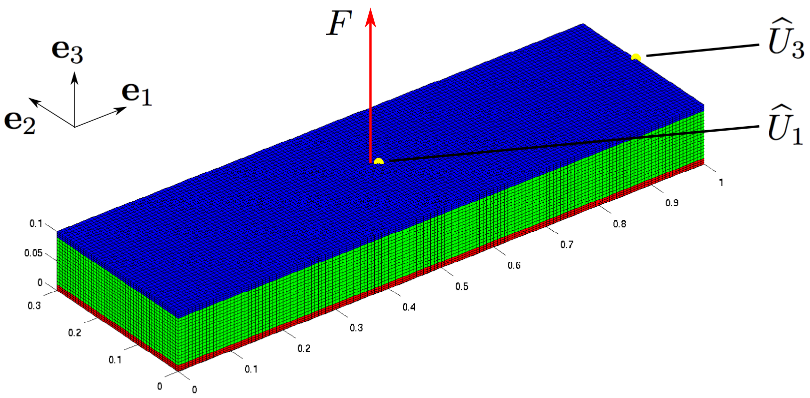

Figure 1: Finite element mesh of the plate with force applied $F = 1$ N and observation dofs $\widehat{U}_1$ and $\widehat{U}_3$.

2.2 Mean reduced-order model

The mean reduced-order computational model is constructed by using the reduced basis represented by the rectangular real matrix $[\Phi_N]$ whose N columns are to the first N modes associated with the first N positive eigenvalues $0 < \omega_1^2 \leq \ldots \leq \omega_N^2$ of the underlying undamped mechanical system for which the mass matrix is $[\mathbb{M}]$ and the stiffness matrix is $[\mathbb{K}_0]$. The reduced-order computational model is written as

$$(-\omega^2[\mathcal{M}] + i\omega\,[\mathcal{D}(\omega)] + [\mathcal{K}_0] + [\mathcal{K}(\omega)])\,\widehat{\mathbf{q}}(\omega) = \widehat{\mathbf{f}}(\omega)\,, \tag{4}$$

in which the full $(N \times N)$ real matrices $[\mathcal{D}(\omega)]$ and $[\mathcal{K}(\omega)]$ are symmetric positive, and are linked by the following equation,

$$[\mathcal{K}(\omega)] = \frac{2\,\omega}{\pi}\; p.v\!\int_0^{+\infty} \frac{1}{1-u^2}[\mathcal{D}(\omega u)]\,du \quad , \quad \omega \geq 0\,. \tag{5}$$

3 Stochastic reduced-order computational model

For N fixed, the nonparametric probabilistic approach of uncertainties consists in substituting in Eqs. (4) and (5), the deterministic matrices $[\mathcal{M}]$, $[\mathcal{D}(\omega)]$, $[\mathcal{K}_0]$, and $[\mathcal{K}(\omega)]$, by the $(N \times N)$ real random matrices $[\mathbf{M}]$, $[\mathbf{D}(\omega)]$, $[\mathbf{K}_0]$, and $[\mathbf{K}(\omega)]$ respectively, in preserving the positive-definiteness property of $[\mathbf{M}]$, $[\mathcal{D}(\omega)]$, $[\mathbf{K}_0]$, and the positiveness property of $[\mathbf{K}(\omega)]$. Consequently $\widehat{\mathbf{q}}(\omega)$ becomes the random vectors $\widehat{\mathbf{Q}}(\omega)$ such that

$$(-\omega^2[\mathbf{M}] + i\omega\,[\mathbf{D}(\omega)] + [\mathbf{K}_0] + [\mathbf{K}(\omega)])\,\widehat{\mathbf{Q}}(\omega) = \widehat{\mathbf{f}}(\omega)\,, \tag{6}$$

$$[\mathbf{K}(\omega)] = \frac{2\,\omega}{\pi}\; p.v\!\int_0^{+\infty} \frac{1}{1-u^2}[\mathbf{D}(\omega u)]\,du \quad , \quad \omega \geq 0\,. \tag{7}$$

Furthermore, Eq. (7) means that the probabilistic model of random matrix $[\mathbf{K}(\omega)]$ is completely defined by the probabilistic model of random matrix $[\mathbf{D}(\omega)]$ and consequently, the two random matrices are not statistically independent such that the probabilistic model for random matrix $[\mathbf{K}(\omega)]$ allows satisfying almost-surely the causality principle and will be referred as the probabilistic model with almost-sure causality. Consequently, only the probabilistic models of random matrices $[\mathbf{M}]$, $[\mathbf{K}_0]$ and $[\mathbf{D}(\omega)]$ have to be constructed. In the framework of the nonparametric probabilistic approach of uncertainties, these random matrices are statistically independent and they constructed as explained in [1, 4, 5]. Their level of uncertainty is respectively controlled by parameters δ_M, δ_K, $\delta_M < (\frac{N+1}{N+5})^{1/2}$.

On an other hand, if the causality principle was not taken into account for the construction of the stochastic model of random matrix $[\mathbf{K}(\omega)]$, then the stochastic model that would be constructed would be causal in average but would not be almost-surely causal. Such a model would be erroneous from the point of view of

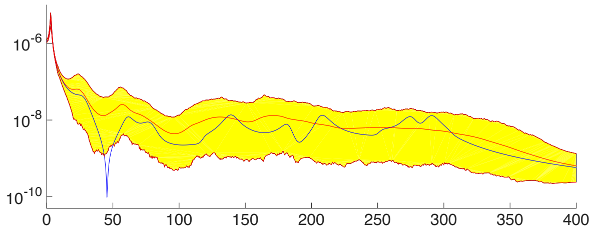

Figure 2: Graph of $\nu \mapsto |\widehat{U}_1(2\pi\nu)|$ for the probabilistic model with almost-sure causality: 95% confidence region (yellow region) and mean value (red line). Graph of $\nu \mapsto |\widehat{u}_1(2\pi\nu)|$ (blue line). Horizontal axis: frequency ν in Hz.

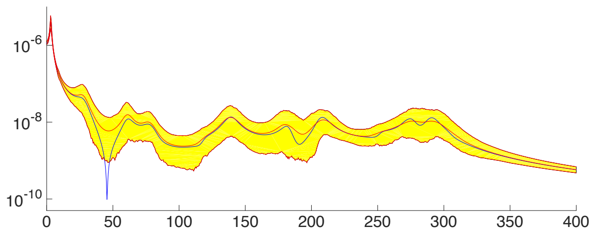

Figure 3: Graph of $\nu \mapsto |\widehat{U}_1(2\pi\nu)|$ for the probabilistic model with causality in mean: 95% confidence region (yellow region) and mean value (red line). Graph of $\nu \mapsto |\widehat{u}_1(2\pi\nu)|$ (blue line). Horizontal axis: frequency ν in Hz.

the theory of physically realizable systems. In the following, such an erroneous stochastic construction will be referred as the probabilistic model with a causality in mean. Such a model can be constructed by rewriting Eq. (6) as

$$([\mathbf{M}] + i\omega[\mathbf{D}(\omega)] + [\widetilde{\mathbf{K}}(\omega)])\,\widehat{\mathbf{Q}}(\omega) = \widehat{\mathbf{f}}(\omega)\,, \tag{8}$$

where $[\widetilde{\mathbf{K}}(\omega)] = [\mathbf{K}_0] + [\mathbf{K}(\omega)]$. The random matrices $[\mathbf{M}]$, $[\mathbf{K}(\omega)]$ and $[\mathbf{D}(\omega)]$ are statistically independent and are constructed as explained in [1, 4, 5]. Their level of uncertainty is respectively controlled by parameters δ_M, δ_K, $\delta_M < (\frac{N+1}{N+5})^{1/2}$.

4 Results

Hereinafter, the two random models with almost-sure causality (see Eqs. (6) and (7)) and with the causality in mean (see Eq. (8)) are compared. It is assumed that $[\mathbf{M}]$ remains deterministic ($\delta_M = 0$), and that $\delta_K = 0.15$ and $\delta_D = 0.7$. Let $\widehat{U}_k(\omega) = \{\widehat{\mathbb{U}}(\omega)\}_k$ be the k-th component of $\widehat{\mathbb{U}}(\omega) = [\Phi_N]\,\widehat{\mathbf{Q}}(\omega)$. Let $\widehat{u}_k(\omega) = \{\widehat{\mathbb{u}}(\omega)\}_k$ be the k-th component of the response calculated with computational model. The numbering of degrees of freedom is such that, for $k = 1$, $\widehat{U}_k(\omega)$ is related to the degree of freedom in direction $\mathbf{e}_1$ of the node located, respectively, at (0.5067 m, 0.1630 m, 0.1 m) (see Fig. 1). The confidence region of the stochastic response $\omega \mapsto |\widehat{U}_1(\omega)|$ with a probability level $p_c = 0.95$, its statistical mean value, and the deterministic response $\omega \mapsto |\widehat{u}_1(\omega)|$, are displayed, for the random model with almost-sure causality, in Fig. 2, and are displayed, for the random model with causality in mean, in Fig. 3 In Figs. 2 and 3, it can be seen that the confidence region is not the same for the probabilistic model with almost-sure causality and for the probabilistic model with causality in mean. The mean values are also different. It is important to note that in some cases, it can be seen that the values given by the computational model are outside of the 95% confidence interval for some frequencies, revealing that the computational model is not robust with respect to uncertainties for these frequencies. It can be seen that, for the probabilistic model with causality in mean, the width of the confidence region is smaller than for the probabilistic model with almost-sure causality. In addition, while the mean value of $\widehat{U}_1(\omega)$ is very close of $\widehat{u}_1(\omega)$, it can be seen that it is not the case for the probabilistic model with almost-surely causality.

5 Conclusions

In the framework of the nonparametric probabilistic approach of uncertainties, a new stochastic modeling has been proposed for taking into account uncertainties in the computational models of linear viscoelastic dynamical structures. This method is based on the construction of a compatibility equation that allows for satisfying the causality principle for the stochastic dynamical system in order to obtain compatible realizations of the random stiffness matrix and the random damping matrix at each frequency point of analysis. A numerical example has been presented for analyzing the propagation of uncertainties in a computational model of a composite viscoelastic structure. The results obtained show that it is very important to construct a probabilistic model which satisfies the causality principle.

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