



HAL
open science

REPRESENTATIONS OF 3-MANIFOLDS GROUPS IN $\mathrm{PGL}(n, \mathbb{C})$ AND THEIR RESTRICTION TO THE BOUNDARY

Antonin Guilloux

► **To cite this version:**

Antonin Guilloux. REPRESENTATIONS OF 3-MANIFOLDS GROUPS IN $\mathrm{PGL}(n, \mathbb{C})$ AND THEIR RESTRICTION TO THE BOUNDARY. 2016. hal-01370280

HAL Id: hal-01370280

<https://hal.science/hal-01370280>

Preprint submitted on 23 Sep 2016

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

REPRESENTATIONS OF 3-MANIFOLDS GROUPS IN $\mathrm{PGL}(n, \mathbf{C})$ AND THEIR RESTRICTION TO THE BOUNDARY

ANTONIN GUILLOUX

ABSTRACT. Let M be a cusped 3-manifold – e.g. a knot complement – and note ∂M the collection of its peripheral tori. Thurston [Thu79] gave a combinatorial way to produce hyperbolic structures via triangulation and the so-called *gluing equations*. This gives coordinates on the space of representations of $\pi_1(M)$ to $\mathrm{PGL}(2, \mathbf{C})$.

In their paper [NZ85], Neumann and Zagier showed how this coordinates are adapted to describe this space of representations as a totally isotropic subvariety lying inside a space equipped with a 2-form – now called Neumann-Zagier symplectic space. And they related this 2-form with a natural symplectic form on the space of representations of $\pi(\partial M)$ to $\mathrm{PGL}(2, \mathbf{C})$: the Weil-Petersson form.

Subsequent works of Neumann [Neu92] and Kabaya [Kab07] extended the scope of the previous works. We fulfill here, as Garoufalidis-Zickert [GZ13], the generalization of these works to representations to $\mathrm{PGL}(3, \mathbf{C})$.

1. INTRODUCTION

Let M be the 8-knot complement. Thurston [Thu79] explained the following program to construct its hyperbolic structure:

- (1) Triangulate M , here thanks to the Riley’s triangulation.
- (2) Give a set of parameters to each tetrahedra, here cross-ratios, that describe their hyperbolic structure.
- (3) Glue back the tetrahedra, imposing the *gluing equations*. Those insure that the edge will not become singular.
- (4) Add a polynomial condition specifying that the structure is complete, by forcing the peripheral holonomy to be parabolic.

Hence the hyperbolic structure is described by the solution to a polynomial system. Moreover, relaxing the last condition, this parametrize a (Zariski-)open subset of a decorated version of the character variety:

$$\chi_2(M) := \mathrm{Hom}(\pi_1(M), \mathrm{PGL}(2, \mathbf{C})) // \mathrm{PGL}(2, \mathbf{C}).$$

This approach has proven very efficient and is followed in the computer program SnapPy to construct hyperbolic structures on ideally triangulated 3-manifolds.

This program was further developed by Neumann and Zagier in [NZ85]. By a careful analysis of items 2 and 3, they showed that there is a \mathbf{C} -vector space (denoted $\ker(\beta^*) \subset J$ in [Neu92]) carrying an antisymmetric bilinear form ω such that

- the character $\chi_2(M)$, through the parameters, is seen as a subvariety of $\exp(\ker(\beta^*))$ tangent to the kernel of the 2-form ω^1 .
- the symplectic quotient $\mathcal{H}(J)$ of $\ker(\beta^*)$ (the so-called Neumann-Zagier symplectic space) is isomorphic to the cohomology group $H^1(\partial M, \mathbf{C})$ with its Goldman-Weil-Peterson symplectic form (∂M denotes the peripheral torus).

This presentation uses the more precise version given by Neumann [Neu92]. This construction allows to understand the volume of the representations near the holonomy of the hyperbolic structure [NZ85]. It has been used to give a proof of the local rigidity of the holonomy of the hyperbolic structure [Cho04]. Kabaya [Kab07] investigated the case of M being a compact hyperbolic manifold with higher genus boundary.

More recently, several new works revisited Neumann-Zagier strategy and generalized it to understand the character variety:

$$\chi_n(M) := \text{Hom}(\pi_1(M), \text{PGL}(n, \mathbf{C})) // \text{PGL}(n, \mathbf{C}).$$

The reasons of this new interest seems to emanate from two very different fields. First, from a geometric point of view: the construction of representations $\pi_1(M) \rightarrow \text{PU}(2, 1)$, following the initial strategy of Thurston, has been undertaken by Falbel [Fal11] in order to investigate the possibility for M to carry a CR-spherical structure. Using Neumann-Zagier approach, Bergeron, Falbel and the author [BFG12] gave a description of $\chi_3(M)$ similar to the one of $\chi_2(M)$ described above. This leads to a local rigidity result [BFG⁺ar] and actual computations (for $n = 3$) [FKR13]. Those, in turn, leads to construction of geometric structures [DF13]. Another approach is via physical mathematics. I must confess my ignorance and refer to Dimofte and Garofalidis [DG12] for a presentation. This motivated the works of Garofalidis, Goerner, Thurston and Zickert [Zic, GTZ11, GGZ12]. They proposed a set of parameters for the case $\text{PGL}(n, \mathbf{C})$, and generalized partially Neumann-Zagier results for their setting. This also leads to actual computations (mainly when $n = 3$) by the second named author. Dimofte, Gabella and Goncharov [DGG13] also analyzed the problem

¹More precisely, it is the decorated character variety.

for $\mathrm{PGL}(n, \mathbf{C})$ from this point of view, giving a systematic account of a set of coordinates, together with the announcement that they are able to fulfill the Neumann-Zagier strategy. Unfortunately all the proofs are not given in their paper. As mentioned in the abstract, by the very end of the writing of this paper, a prepublication by Garoufalidis and Zickert [GZ13] appeared. Their result and the one discussed in this paper are very similar. However, I still think that the presentations are different enough and that, from a geometrical viewpoint, the approach we follow here allows a better understanding.

An application of our theorem is a variational formula for the volume of a representation; this is thoroughly discussed in [DGG13, section 4.4]. We present another, more geometric, application: we prove the local rigidity result generalizing [Cho04, BFG⁺ar].

This paper links the work of [DGG13] with [BFG12] to complete Neumann-Zagier program in the case of $\mathrm{PGL}(n, \mathbf{C})$. My feeling is that the coordinates given in [DGG13] are very well adapted to understand of the "lagrangian part" of the strategy of Neumann-Zagier – i.e. describe the analog of the vector space $\ker(\beta^*) \subset J$ with its form ω such that $\chi_n(M)$ is tangent to its kernel in $\exp(\ker(\beta^*))$ – and define the volume of those representations. But, in order to understand the "symplectic isomorphism part", a direct generalization of [BFG12] seems suitable.

After this rather long introduction, let me warn the reader that this paper heavily relies on three sources:

- Fock and Goncharov combinatorics described in [FG06],
- Dimofte, Gabella and Goncharov work in [DGG13],
- Bergeron, Falbel and G. work in [BFG12] (and through it to the original Neumann-Zagier strategy [NZ85, Neu92]).

It is not an easy task to give a understandable and reasonable account of those works. So I rather choosed to give precise references to them. This makes this paper absolutely not self-contained.

2. TRIANGULATION, FLAGS, AFFINE FLAGS AND THEIR CONFIGURATIONS

2.1. Triangulated manifold. We will consider in this paper triangles and tetrahedra. Those will always be oriented: an orientation is an ordering of the vertices up to even permutations. Note that the faces of a tetrahedron inherits an orientation.

An abstract triangulation is defined as a pair $\mathcal{T} = ((T_\nu)_{\nu=1, \dots, N}, \Phi)$ where $(T_\nu)_{\nu=1, \dots, N}$ is a finite family of tetrahedra and Φ is a matching of the faces of the T_ν 's reversing the orientation. For any tetrahedron T ,

we define $\text{Trunc}(T)$ as the tetrahedron truncated at each vertex. The topological space obtained from $\text{Trunc}(T_\mu)$ after matching the faces will be denoted by $K_{\mathcal{T}}$.

A *triangulation* of an oriented compact 3-manifold M with boundary is an abstract triangulation \mathcal{T} together with an oriented homeomorphism $M \simeq K_{\mathcal{T}}$.

Remark that a knot complement is homeomorphic to the interior of such a triangulated manifold [BFG12, Section 1.2]. And a theorem of Luo-Schleimer-Tillman [LST08] states that, up to passing to a finite cover, any complete cusped hyperbolic 3-manifold may be seen as the interior of a compact triangulated manifold.

From now on, we fix a triangulation \mathcal{T} of a compact manifold M with boundary ∂M . We moreover add some combinatorial hypothesis on the triangulation: we assume that the link of any vertex is a disc, a torus or an annulus – [BFG12, Section 5.1] and [DGG13, Section 2.1]. Thus the boundary ∂M decomposes as a union of hexagons lying in the boundary of the complex $K_{\mathcal{T}}$ and discs, tori and annuli lying in the links of the vertices. The latter are naturally triangulated by the traces of the tetrahedra.

2.2. Flags, Affine Flags. As in the work of Fock and Goncharov [FG06], the main technical tool will be the flags, affine flags, and their configuration.

Let $V = \mathbf{C}^n$, with its natural basis (e_1, \dots, e_n) . All our flags will be complete: they are defined as "a line in a plane in a 3-dim plane... in a hyperplane".

More precisely, consider the exterior powers of V and their projectivizations, for $m = 1$ to $n - 1$:

$$\Lambda^m V \text{ and } \mathbb{P}(\Lambda^m V).$$

Note that $\Lambda^1 V \simeq V$ and $\Lambda^{n-1} V \simeq V^*$, the dual of V . We fix *once for all* the isomorphism $\Lambda^n(V) \simeq \mathbf{C}$ by assigning 1 to the element $e_1 \wedge \dots \wedge e_n$.

The space of flags in V is a subset of $\prod_1^{n-1} \mathbb{P}(\Lambda^m V)$. To describe it, recall that G acts on each exterior power of V , hence diagonally on the product. Moreover the standard flag F_{st} is defined by:

$$F_{\text{st}} = ([e_1], [e_1 \wedge e_2], \dots, [e_1 \wedge \dots \wedge e_{n-1}]).$$

Then the flag variety is the orbit of F_{st}

$$\mathcal{Fl} := \text{PGL}(n, \mathbf{C}) \cdot F_{\text{st}} \subset \prod_1^{n-1} \mathbb{P}(\Lambda^m V).$$