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# REPRESENTATIONS OF 3-MANIFOLDS GROUPS IN $\mathrm{PGL}(n, \mathbf{C})$ AND THEIR RESTRICTION TO THE BOUNDARY

ANTONIN GUILLOUX

ABSTRACT. Let  $M$  be a cusped 3-manifold – e.g. a knot complement – and note  $\partial M$  the collection of its peripheral tori. Thurston [Thu79] gave a combinatorial way to produce hyperbolic structures via triangulation and the so-called *gluing equations*. This gives coordinates on the space of representations of  $\pi_1(M)$  to  $\mathrm{PGL}(2, \mathbf{C})$ .

In their paper [NZ85], Neumann and Zagier showed how this coordinates are adapted to describe this space of representations as a totally isotropic subvariety lying inside a space equipped with a 2-form – now called Neumann-Zagier symplectic space. And they related this 2-form with a natural symplectic form on the space of representations of  $\pi(\partial M)$  to  $\mathrm{PGL}(2, \mathbf{C})$ : the Weil-Petersson form.

Subsequent works of Neumann [Neu92] and Kabaya [Kab07] extended the scope of the previous works. We fulfill here, as Garoufalidis-Zickert [GZ13], the generalization of these works to representations to  $\mathrm{PGL}(3, \mathbf{C})$ .

## 1. INTRODUCTION

Let  $M$  be the 8-knot complement. Thurston [Thu79] explained the following program to construct its hyperbolic structure:

- (1) Triangulate  $M$ , here thanks to the Riley's triangulation.
- (2) Give a set of parameters to each tetrahedra, here cross-ratios, that describe their hyperbolic structure.
- (3) Glue back the tetrahedra, imposing the *gluing equations*. Those insure that the edge will not become singular.
- (4) Add a polynomial condition specifying that the structure is complete, by forcing the peripheral holonomy to be parabolic.

Hence the hyperbolic structure is described by the solution to a polynomial system. Moreover, relaxing the last condition, this parametrize a (Zariski-)open subset of a decorated version of the character variety:

$$\chi_2(M) := \mathrm{Hom}(\pi_1(M), \mathrm{PGL}(2, \mathbf{C})) // \mathrm{PGL}(2, \mathbf{C}).$$

This approach has proven very efficient and is followed in the computer program SnapPy to construct hyperbolic structures on ideally triangulated 3-manifolds.

This program was further developed by Neumann and Zagier in [NZ85]. By a careful analysis of items 2 and 3, they showed that there is a  $\mathbf{C}$ -vector space (denoted  $\ker(\beta^*) \subset J$  in [Neu92]) carrying an antisymmetric bilinear form  $\omega$  such that

- the character  $\chi_2(M)$ , through the parameters, is seen as a subvariety of  $\exp(\ker(\beta^*))$  tangent to the kernel of the 2-form  $\omega^1$ .
- the symplectic quotient  $\mathcal{H}(J)$  of  $\ker(\beta^*)$  (the so-called Neumann-Zagier symplectic space) is isomorphic to the cohomology group  $H^1(\partial M, \mathbf{C})$  with its Goldman-Weil-Peterson symplectic form ( $\partial M$  denotes the peripheral torus).

This presentation uses the more precise version given by Neumann [Neu92]. This construction allows to understand the volume of the representations near the holonomy of the hyperbolic structure [NZ85]. It has been used to give a proof of the local rigidity of the holonomy of the hyperbolic structure [Cho04]. Kabaya [Kab07] investigated the case of  $M$  being a compact hyperbolic manifold with higher genus boundary.

More recently, several new works revisited Neumann-Zagier strategy and generalized it to understand the character variety:

$$\chi_n(M) := \text{Hom}(\pi_1(M), \text{PGL}(n, \mathbf{C})) // \text{PGL}(n, \mathbf{C}).$$

The reasons of this new interest seems to emanate from two very different fields. First, from a geometric point of view: the construction of representations  $\pi_1(M) \rightarrow \text{PU}(2, 1)$ , following the initial strategy of Thurston, has been undertaken by Falbel [Fal11] in order to investigate the possibility for  $M$  to carry a CR-spherical structure. Using Neumann-Zagier approach, Bergeron, Falbel and the author [BFG12] gave a description of  $\chi_3(M)$  similar to the one of  $\chi_2(M)$  described above. This leads to a local rigidity result [BFG<sup>+</sup>ar] and actual computations (for  $n = 3$ ) [FKR13]. Those, in turn, leads to construction of geometric structures [DF13]. Another approach is via physical mathematics. I must confess my ignorance and refer to Dimofte and Garofalidis [DG12] for a presentation. This motivated the works of Garofalidis, Goerner, Thurston and Zickert [Zic, GTZ11, GGZ12]. They proposed a set of parameters for the case  $\text{PGL}(n, \mathbf{C})$ , and generalized partially Neumann-Zagier results for their setting. This also leads to actual computations (mainly when  $n = 3$ ) by the second named author. Dimofte, Gabella and Goncharov [DGG13] also analyzed the problem

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<sup>1</sup>More precisely, it is the decorated character variety.

for  $\mathrm{PGL}(n, \mathbf{C})$  from this point of view, giving a systematic account of a set of coordinates, together with the announcement that they are able to fulfill the Neumann-Zagier strategy. Unfortunately all the proofs are not given in their paper. As mentioned in the abstract, by the very end of the writing of this paper, a prepublication by Garoufalidis and Zickert [GZ13] appeared. Their result and the one discussed in this paper are very similar. However, I still think that the presentations are different enough and that, from a geometrical viewpoint, the approach we follow here allows a better understanding.

An application of our theorem is a variational formula for the volume of a representation; this is thoroughly discussed in [DGG13, section 4.4]. We present another, more geometric, application: we prove the local rigidity result generalizing [Cho04, BFG<sup>+</sup>ar].

This paper links the work of [DGG13] with [BFG12] to complete Neumann-Zagier program in the case of  $\mathrm{PGL}(n, \mathbf{C})$ . My feeling is that the coordinates given in [DGG13] are very well adapted to understand of the "lagrangian part" of the strategy of Neumann-Zagier – i.e. describe the analog of the vector space  $\ker(\beta^*) \subset J$  with its form  $\omega$  such that  $\chi_n(M)$  is tangent to its kernel in  $\exp(\ker(\beta^*))$  – and define the volume of those representations. But, in order to understand the "symplectic isomorphism part", a direct generalization of [BFG12] seems suitable.

After this rather long introduction, let me warn the reader that this paper heavily relies on three sources:

- Fock and Goncharov combinatorics described in [FG06],
- Dimofte, Gabella and Goncharov work in [DGG13],
- Bergeron, Falbel and G. work in [BFG12] (and through it to the original Neumann-Zagier strategy [NZ85, Neu92]).

It is not an easy task to give a understandable and reasonable account of those works. So I rather choosed to give precise references to them. This makes this paper absolutely not self-contained.

## 2. TRIANGULATION, FLAGS, AFFINE FLAGS AND THEIR CONFIGURATIONS

**2.1. Triangulated manifold.** We will consider in this paper triangles and tetrahedra. Those will always be oriented: an orientation is an ordering of the vertices up to even permutations. Note that the faces of a tetrahedron inherits an orientation.

An abstract triangulation is defined as a pair  $\mathcal{T} = ((T_\nu)_{\nu=1, \dots, N}, \Phi)$  where  $(T_\nu)_{\nu=1, \dots, N}$  is a finite family of tetrahedra and  $\Phi$  is a matching of the faces of the  $T_\nu$ 's reversing the orientation. For any tetrahedron  $T$ ,

we define  $\text{Trunc}(T)$  as the tetrahedron truncated at each vertex. The topological space obtained from  $\text{Trunc}(T_\mu)$  after matching the faces will be denoted by  $K_{\mathcal{T}}$ .

A *triangulation* of an oriented compact 3-manifold  $M$  with boundary is an abstract triangulation  $\mathcal{T}$  together with an oriented homeomorphism  $M \simeq K_{\mathcal{T}}$ .

Remark that a knot complement is homeomorphic to the interior of such a triangulated manifold [BFG12, Section 1.2]. And a theorem of Luo-Schleimer-Tillman [LST08] states that, up to passing to a finite cover, any complete cusped hyperbolic 3-manifold may be seen as the interior of a compact triangulated manifold.

From now on, we fix a triangulation  $\mathcal{T}$  of a compact manifold  $M$  with boundary  $\partial M$ . We moreover add some combinatorial hypothesis on the triangulation: we assume that the link of any vertex is a disc, a torus or an annulus – [BFG12, Section 5.1] and [DGG13, Section 2.1]. Thus the boundary  $\partial M$  decomposes as a union of hexagons lying in the boundary of the complex  $K_{\mathcal{T}}$  and discs, tori and annuli lying in the links of the vertices. The latter are naturally triangulated by the traces of the tetrahedra.

**2.2. Flags, Affine Flags.** As in the work of Fock and Goncharov [FG06], the main technical tool will be the flags, affine flags, and their configuration.

Let  $V = \mathbf{C}^n$ , with its natural basis  $(e_1, \dots, e_n)$ . All our flags will be complete: they are defined as "a line in a plane in a 3-dim plane... in a hyperplane".

More precisely, consider the exterior powers of  $V$  and their projectivizations, for  $m = 1$  to  $n - 1$ :

$$\Lambda^m V \text{ and } \mathbb{P}(\Lambda^m V).$$

Note that  $\Lambda^1 V \simeq V$  and  $\Lambda^{n-1} V \simeq V^*$ , the dual of  $V$ . We fix *once for all* the isomorphism  $\Lambda^n(V) \simeq \mathbf{C}$  by assigning 1 to the element  $e_1 \wedge \dots \wedge e_n$ .

The space of flags in  $V$  is a subset of  $\prod_1^{n-1} \mathbb{P}(\Lambda^m V)$ . To describe it, recall that  $G$  acts on each exterior power of  $V$ , hence diagonally on the product. Moreover the standard flag  $F_{\text{st}}$  is defined by:

$$F_{\text{st}} = ([e_1], [e_1 \wedge e_2], \dots, [e_1 \wedge \dots \wedge e_{n-1}]).$$

Then the flag variety is the orbit of  $F_{\text{st}}$

$$\mathcal{Fl} := \text{PGL}(n, \mathbf{C}) \cdot F_{\text{st}} \subset \prod_1^{n-1} \mathbb{P}(\Lambda^m V).$$