Feedback boundary stabilization of 2d fluid-structure interaction systems

Mehdi Badra, Takéo Takahashi

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Mehdi Badra*, Takéo Takahashi †‡

Abstract

We study the feedback stabilization of a system composed by an incompressible viscous fluid and a deformable structure located at the boundary of the fluid domain. We stabilize the position and the velocity of the structure and the velocity of the fluid around a stationary state by means of a Dirichlet control, localized on the exterior boundary of the fluid domain and with values in a finite dimensional space. Our result concerns weak solutions for initial data close to the stationary state. Our method is based on general arguments for stabilization of nonlinear parabolic systems combined with a change of variables to handle the fact that the fluid domain of the stationary state and of the stabilized solution are different. We prove that for initial data close to the stationary state, we can stabilize the position and the velocity of the deformable structure and the velocity of the fluid.

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Key words: feedback stabilization, fluid-structure interaction, Navier-Stokes equations, beam equation.

1 Introduction

We consider the problem of stabilization for a fluid-structure system composed by a viscous incompressible fluid and a deformable structure located at the boundary of the fluid domain. The fluid motion is modeled by the Navier-Stokes system and the structure deformation follows the equation of a "viscous" beam. Such a model is already considered by several authors ([33], [10], etc.). Our aim consists in showing the boundary stabilization of such a system in the 2d case and for weak solutions. The method used here could be adapted for other fluid-structure systems in the case of a fluid modeled by the Navier-Stokes system. In the 3d case or for strong solutions, the stabilization of such fluid-structure systems could be obtained by using the methodology developed in [6] or in [8].

Let us first describe more precisely the system considered in this paper. The domain of reference for the fluid is denoted by $\mathcal{F}_{\text{ref}}$. We assume that it is a smooth domain of $\mathbb{R}^2$ such that its boundary $\partial\mathcal{F}_{\text{ref}}$ contains a flat part Γ_{ref}. We can assume that $\Gamma_{\text{ref}} = (0,1)\times\{0\}$ and we set $\Gamma_0 \stackrel{\text{def}}{=} \partial\mathcal{F}_{\text{ref}}\backslash\overline{\Gamma_{\text{ref}}}$.

On the part Γ_{ref}, we assume that there is a beam that can deform through the action of exterior forces and in particular the force due to the fluid. The fluid boundary is thus moving, Γ_{ref} being transformed into

$$\Gamma_{\text{str}}(\eta(t,\cdot)) \stackrel{\text{def}}{=} \{(s,\eta(t,s)) \ ; \ s\in(0,1)\}, \tag{1.1}$$

whereas Γ_0 remains unchanged. The new domain of fluid $\mathcal{F}(\eta(t,\cdot))$ is the interior of $\Gamma_0 \cup \overline{\Gamma_{\text{str}}(\eta(t))}$. We assume that

$$\eta(t,0) = \eta(t,1) = \partial_s\eta(t,0) = \partial_s\eta(t,1) = 0, \tag{1.2}$$

and that $\Gamma_0 \cap \Gamma_{\text{str}}(\eta(t)) = \emptyset$ so that $\Gamma_0 \cup \overline{\Gamma_{\text{str}}(\eta(t))}$ is a close, simple C^1 curve and this definition makes sense (see Figure 1).

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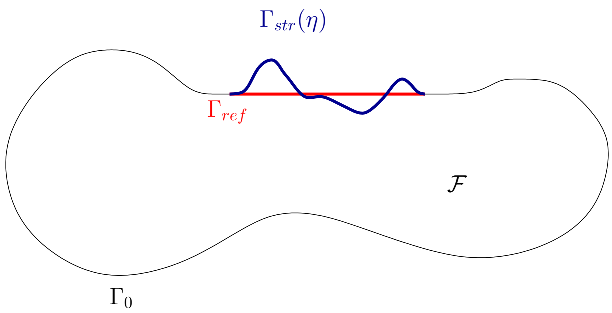

Figure 1: The fluid-plate system

The fluid-structure system that we consider reads as follows

$$\left\{\begin{array}{c}\partial_t v+(v\cdot\nabla)v-\operatorname{div}\mathbb{T}(v,p)=f^S,\quad t>0,\ x\in\mathcal{F}(\eta(t)),\\ \operatorname{div}v=0\quad t>0,\ x\in\mathcal{F}(\eta(t)),\\ v(t,s,\eta(t,s))=(\partial_t\eta)(t,s)e_2\quad t>0,\ s\in(0,1),\\ v=b^S+\Xi(u)\quad t>0,\ x\in\Gamma_0,\\ \partial_{tt}\eta+\alpha\partial_{ssss}\eta-\beta\partial_{ss}\eta-\delta\partial_{tss}\eta=-\mathbb{H}_\eta(v,p)+g^S,\quad t>0,\ s\in(0,1),\\ \eta=\partial_s\eta=0\quad t>0,\ s\in\{0,1\},\end{array}\right. \tag{1.3}$$

with the initial conditions

$$\eta(0)=\eta_1^0\quad\text{and}\quad\partial_t\eta(0)=\eta_2^0\quad\text{in }(0,1),\quad v(0)=v^0\quad\text{in }\mathcal{F}(\eta_1^0). \tag{1.4}$$

Here and in all that follows, (e_1,e_2) is the canonical basis of $\mathbb{R}^2$, in particular $e_2=\begin{bmatrix}0\\1\end{bmatrix}$,

$$\mathbb{T}(v,p)\overset{\text{def}}{=}2\nu D(v)-pI_2,\quad D(v)=\frac{1}{2}\left(\nabla v+(\nabla v)^*\right), \tag{1.5}$$

$$\mathbb{H}_\eta(v,p)\overset{\text{def}}{=}\left\{(1+|\partial_s\eta|^2)^{1/2}\left[\mathbb{T}(v,p)n\right](t,s,\eta(t,s))\cdot e_2\right\}, \tag{1.6}$$

the vector fields n is the unit exterior normal to $\mathcal{F}(\eta(t))$ and in particular, on $\Gamma_{\rm str}(\eta(t))$,

$$n(t,y_1,y_2)=\frac{1}{\sqrt{1+|\partial_s\eta(t,y_1)|^2}}\begin{bmatrix}-\partial_s\eta(t,y_1)\\1\end{bmatrix}. \tag{1.7}$$

The constants α, β and δ are assumed to satisfy

$$\alpha>0,\quad\beta\geqslant 0,\quad\delta>0.$$

Moreover, $f^S:\mathbb{R}^2\to\mathbb{R}^2$, $b^S:\Gamma_0\to\mathbb{R}^2$ and $g^S:(0,1)\to\mathbb{R}$ are time-independent functions corresponding to a stationary state (v^S,p^S,η^S) of the above system:

$$\left\{\begin{array}{c}(v^S\cdot\nabla)v^S-\operatorname{div}\mathbb{T}(v^S,p^S)=f^S,\quad x\in\mathcal{F}(\eta^S),\\ \operatorname{div}v^S=0\quad x\in\mathcal{F}(\eta^S),\\ v^S(t,s,\eta^S(s))=0,\quad s\in(0,1),\\ v^S=b^S\quad x\in\Gamma_0,\\ \alpha\partial_{ssss}\eta^S-\beta\partial_{ss}\eta^S=-\mathbb{H}_{\eta^S}(v^S,p^S)+g^S,\ s\in(0,1),\\ \eta^S=\partial_s\eta^S=0,\ s\in\{0,1\},\end{array}\right. \tag{1.8}$$

where the boundary value b^S is supposed to satisfy $\int_{\Gamma_0}b^S\cdot n\,d\gamma=0$.

Finally, u is a control function that we will search in a feedback form so that the corresponding solution $(v,\eta,\partial_t\eta)$ tends to the stationary solution $(v^S,\eta^S,0)$ as $t\to\infty$. The precise statement of this convergence is given below. Here $\Xi \in \mathcal{L}(\mathbf{L}^2(\Gamma_0))$ is an operator used to localize the action of the control u in a relatively compact subset of Γ_0 and such that $\int_{\Gamma_0} \Xi(u)\cdot n d\gamma = 0$. Precisely, it can be defined by

$$\Xi(u) \stackrel{\text{def}}{=} \rho u - \left(\int_{\Gamma_0} \rho u \cdot n d\gamma\right) \rho n \tag{1.9}$$

where $\rho \in C^2(\Gamma_0)$ is a non zero function compactly supported in Γ_0 such that $\int_{\Gamma_0} \rho d\gamma = 1$.

Before stating in details the main result, let us first rewrite systems (1.3) and (1.8) in a more general way that allows to take into account the incompressibility of the fluid. A formal calculation yields

$$0 = \int_{\mathcal{F}(\eta(t))} \operatorname{div} v \; dx = \int_{\Gamma_{\rm str}(\eta(t))} v \cdot n \; d\gamma = \int_0^1 (\partial_t \eta)(t,s) e_2 \cdot n (1 + |\partial_s \eta(t,s)|^2)^{1/2} \; ds = \frac{d}{dt} \int_0^1 \eta(t,s) \; ds.$$

Consequently, it is natural to work with displacements η with constant mean value along the time. For simplicity, we assume that the mean value of η is zero:

$$\int_0^1 \eta(t,s) \; ds = 0, \tag{1.10}$$

that is

$$\eta(t,\cdot) \in L^2_0(0,1) \stackrel{\text{def}}{=} \left\{ f \in L^2(0,1) \; ; \; \int_0^1 f(s) \; ds = 0 \right\}.$$

With this assumption, we also have $\partial_{tt}\eta(t,\cdot) \in L^2_0(0,1)$ and from the boundary conditions (1.2), we obtain $\partial_{ss}\eta(t,\cdot), \partial_{tss}\eta(t,\cdot) \in L^2_0(0,1)$. Therefore, the equation for η in (1.3) yields the following condition for all $t \geqslant 0$,

$$\int_0^1 \mathbb{H}_\eta(v,p)(t,s) ds = \int_0^1 (g^S(s) - \alpha \partial_{ssss}\eta(t,s)) ds. \tag{1.11}$$

From the definition (1.5)-(1.6) of $\mathbb{H}_\eta(v,p)$ and from (1.7), the above condition can be written as

$$\int_0^1 p(t,s,\eta(t,s)) ds = \int_0^1 \left(-g^S(s) + \alpha \partial_{ssss}\eta(t,s) + 2\nu \left\{ (1 + |\partial_s \eta|^2)^{1/2} \left[D(v) n \right] (t,s,\eta(t,s)) \cdot e_2 \right\} \right) ds.$$

Note that an analogous condition can be imposed on p^S to have $\eta^S \in L^2_0(0,1)$. These conditions imply that, in contrast to the Navier–Stokes system, the pressure is not determined up to a constant in this fluid-structure interaction system. To avoid to deal with this constraint we will use the orthogonal projection

$$M : L^2(0,1) \to L^2_0(0,1). \tag{1.12}$$

Let us introduce the operator $T_\eta : L^2(0,1) \to \mathbf{L}^2(\partial\mathcal{F}(\eta))$ defined by

$$(T_\eta \xi)(x) = \xi(s) e_2 \quad \text{if} \quad x = (s,\eta(s)) \in \Gamma_{\rm str}(\eta) \quad \text{and} \quad (T_\eta \xi)(x) = 0 \quad \text{if} \quad x \in \Gamma_0. \tag{1.13}$$

Let us note that the adjoint $T^*_\eta : \mathbf{L}^2(\partial\mathcal{F}(\eta)) \to L^2(0,1)$ of T_η is given by

$$(T^*_\eta v)(s) = (1 + |\partial_s \eta(s)|^2)^{1/2} v(s,\eta(s)) \cdot e_2. \tag{1.14}$$

We also set

$$\mathcal{H}_S \stackrel{\text{def}}{=} L^2_0(0,1), \tag{1.15}$$

$$\mathcal{D}(A_1) \stackrel{\text{def}}{=} H^4(0,1) \cap H^2_0(0,1) \cap L^2_0(0,1), \quad A_1 \xi \stackrel{\text{def}}{=} \alpha M \partial_{ssss}\xi - \beta \partial_{ss}\xi, \tag{1.16}$$

and

$$\mathcal{D}(A_2) \stackrel{\text{def}}{=} H^2(0,1) \cap H^1_0(0,1) \cap L^2_0(0,1), \quad A_2 \xi \stackrel{\text{def}}{=} -\delta M \partial_{ss}\xi. \tag{1.17}$$

The properties of these operators are described in Section 3. In particular their square roots $A_1^{1/2}$ and $A_2^{1/2}$ are well defined. We also denote by $(\cdot,\cdot)_{\mathcal{H}_S}$ the usual scalar product of $L^2(0,1)$.

With the above notation, systems (1.3) and (1.8) rewrite:

$$\left\{ \begin{array}{c} \partial_t v + (v\cdot\nabla) v - \operatorname{div} \mathbb{T}(v,p) = f^S, \quad t>0, \; x \in \mathcal{F}(\eta(t)), \\ \operatorname{div} v = 0 \quad t > 0, \; x \in \mathcal{F}(\eta(t)), \\ v = T_{\eta(t)} \partial_t \eta + \mathbf{1}_{\Gamma_0}(b^S + \Xi(u)) \quad t>0, \; x \in \partial\mathcal{F}(\eta(t)), \\ \partial_{tt}\eta + A_1 \eta + A_2 \partial_t \eta = -M T^*_{\eta(t)} \mathbb{T}(v,p) n + M g^S, \quad t>0, \; s \in (0,1), \end{array} \right. \tag{1.18}$$

and

$$
\begin{cases}
(v^S\cdot\nabla)v^S-\operatorname{div}\mathbb{T}(v^S,p^S)=f^S, & x\in\mathcal{F}(\eta^S),\\
\operatorname{div}v^S=0 & x\in\mathcal{F}(\eta^S),\\
v^S=\mathbf{1}_{\Gamma_0}b^S & x\in\partial\mathcal{F}(\eta^S),\\
A_1\eta^S=-MT^*_{\eta^S}\mathbb{T}(v^S,p^S)n+Mg^S, & s\in(0,1),
\end{cases}
\tag{1.19}
$$

In above settings $\mathbf{1}_{\Gamma_0}$ denotes the characteristic function of Γ_0.

Our aim is to use the control u in (1.3) in order to "reach" the above stationary state. More precisely, we impose a feedback law depending on the difference between $(v(t,\cdot),\eta(t,\cdot),\partial_t\eta(t,\cdot))$ and $(v^S,\eta^S,0)$ in order to obtain that the difference between these states goes to 0 exponentially.

Such results of stabilization are classical for the classical Navier-Stokes system (without any structure), see for instance, [23], [38], [9], [37], [2], [5], etc. Note that for this problem there is a difference between the dimension 2 and the dimension 3: due to the nonlinearity in the Navier-Stokes system, and to the method developed (stabilization of the linearized system, fixed point), in dimension 2 one can take initial data in L^2 (or H^s, $s<1/2$), whereas in dimension 3, one needs to take the initial data in H^1 (or at least H^s, $s\geqslant 1/2$). As a consequence, in dimension 3, we have to impose compatibility condition at $t=0$ between the initial condition and the feedback control u (see [4, 3, 2] for details). Several techniques have been considered to overcome this difficulty: [34], [2], [5], etc. For instance in [5], the solution consists in assuming that the control u satisfies an evolution equation with another feedback control. We are thus reduced to stabilize a system coupling the fluid velocity and the control u. In dimension 2, the method allows to consider classical feedback operators for weak solutions. This is done for the Navier-Stokes system in [38]. Note that in dimension 2, the stabilization of strong solutions leads to the same problem of compatibility conditions.

For the fluid-structure interaction systems, there are few results of stabilization. A first result was obtained in [36] for the system considered in this article. The target velocity v^S is zero, the control is acting in the whole structure and the author works with strong solutions (initial data in H^1 for the fluid velocity). The case of a deformable structure immersed in a fluid is considered in [19], [18]. For the case of a rigid body, a 1d simplified model is treated in [7] whereas the 2d and 3d case are considered in [6]. In this last paper, we work with a notion of weak solutions in order to deal with the 2d case without the problem of the compatibility conditions. However, in [6] we need that the initial and the final position of the structure are equal.

The main novelty of this work is to prove stabilizability result for *weak* solutions of a fluid-structure system. Moreover, we consider a nontrivial target velocity v^S to be stabilized. We work in the 2d case only and the method for the stabilization follows the same idea as the papers quoted above. One important difficulty that we need to deal with is that there is no proof in the literature for the existence of *weak* solutions of a fluid-structure system with a Banach fixed point. In order to do this here, a first step consists in performing a change of variables to work on a cylindrical domain (see Section 2). Such an approach is already considered for *strong* solutions and there exists changes of variables that allow to keep the divergence free conditions and the form of the boundary conditions. We don't employ such a change of variables on the unknowns but on the test functions. This leads to transform our system in a cylindrical domain with non homogeneous divergence conditions and non homogeneous boundary conditions. We can overcome the corresponding difficulty by using a framework developed in [39] for the Navier-Stokes system. All this work could be adapted to other fluid-structure systems such as the case of rigid bodies moving into a viscous incompressible fluid as in [6]. Indeed, the presence here of the deformable structure that follows a beam equation leads to lots of technical difficulties (see the three sections in the appendix).

Let us also mention that several works have been devoted to the study of the fluid–plate system. The model was proposed in [33]. The existence of weak solutions was proved in [16] and the existence of strong solutions was obtained in [10] and in [29]. For these two results, as in our case, the damping term, that is $-\delta\partial_{tss}\eta$, is essential. In [24], the author manages to prove the existence of weak solutions (in 2d or in 3d) for this problem in the case without damping ($\delta=0$).

Let us write the weak formulations for both systems (1.18) and (1.19). Assume

$$
\varphi\in C^1_0([0,\infty);\mathbf{C}^1(\overline{\mathcal{F}(\eta(t))})),\quad \zeta\in C^1_0([0,\infty);C^2([0,1])\cap L^2_0(0,1))
$$

(see Section 2.1 for the precise definition of such a functional space) satisfy

$$
\operatorname{div}\varphi=0\quad\text{in }\mathcal{F}(\eta(t)),\tag{1.20}
$$

$$
\varphi=T_\eta\zeta\quad\text{on }\partial\mathcal{F}(\eta(t)),\tag{1.21}
$$

$$\zeta(t,0) = \zeta(t,1) = \partial_s\zeta(t,0) = \partial_s\zeta(t,1) = 0. \tag{1.22}$$

The weak formulation for (1.18) is

$$\begin{aligned}
&-\int_{\mathcal{F}(\eta_1^0)} v^0 \cdot \varphi(0,\cdot)\, dx - \int_{\mathbb{R}_+}\int_{\mathcal{F}(\eta(t))} v\cdot(\partial_t\varphi + (v\cdot\nabla)\varphi)\ dx\ dt + 2\nu\int_{\mathbb{R}_+}\int_{\mathcal{F}(\eta(t))} D(v) : D(\varphi)\ dx\ dt \\
&\qquad - \left(\eta_2^0, \partial_t\zeta(0,\cdot)\right)_{\mathcal{H}_{\mathcal{S}}} - \int_{\mathbb{R}_+}\left(\partial_t\eta, \partial_t\zeta\right)_{\mathcal{H}_{\mathcal{S}}}\ dt + \int_{\mathbb{R}_+}\left(A_1^{1/2}\eta, A_1^{1/2}\zeta\right)_{\mathcal{H}_{\mathcal{S}}}\ dt \\
&- \left(A_2^{1/2}\eta_1^0, A_2^{1/2}\zeta\right)_{\mathcal{H}_{\mathcal{S}}} - \int_{\mathbb{R}_+}\left(A_2^{1/2}\eta, A_2^{1/2}\partial_t\zeta\right)_{\mathcal{H}_{\mathcal{S}}}\ dt = \int_{\mathbb{R}_+}\int_{\mathcal{F}(\eta(t))} f^S\cdot\varphi\ dx\ dt + \int_{\mathbb{R}_+}\left(g^S, \zeta\right)_{\mathcal{H}_{\mathcal{S}}}\ dt.
\end{aligned} \tag{1.23}$$

Assume $\varphi^S \in C^1(\overline{\mathcal{F}(\eta^S)})$ and $\zeta^S \in C^2([0,1]) \cap L_0^2(0,1)$ satisfy

$$\operatorname{div}\varphi^S = 0 \quad \text{in } \mathcal{F}(\eta^S), \tag{1.24}$$

$$\varphi^S = T_{\eta^S}\zeta^S \quad \text{on } \partial\mathcal{F}(\eta^S), \tag{1.25}$$

$$\zeta^S(0) = \zeta^S(1) = \partial_s\zeta^S(0) = \partial_s\zeta^S(1) = 0. \tag{1.26}$$

The weak formulation for (1.19) is

$$\begin{aligned}
-\int_{\mathcal{F}(\eta^S)} v^S\cdot(v^S\cdot\nabla)\varphi^S\ dy + 2\nu\int_{\mathcal{F}(\eta^S)} D(v^S) : D(\varphi^S)\ dy + \int_{\mathbb{R}_+}\left(A_1^{1/2}\eta^S, A_1^{1/2}\zeta^S\right)_{\mathcal{H}_{\mathcal{S}}} \\
= \int_{\mathcal{F}(\eta^S)} f^S\cdot\varphi^S\ dy + \left(g^S, \zeta\right)_{\mathcal{H}_{\mathcal{S}}}.
\end{aligned} \tag{1.27}$$

One of the difficulties, that is classical in fluid-structure interaction problems, is coming from the fact that the solutions and the test functions of (1.18) and of (1.19) are not written in the same spatial domains. To overcome this issue, we transform the system (1.18) by using a change of variables X such that $X(t, \mathcal{F}(\eta^S)) = \mathcal{F}(\eta(t,\cdot))$ and $X(t, \Gamma_{\mathrm{str}}(\eta^S)) = \Gamma_{\mathrm{str}}(\eta(t,\cdot))$, and also such that $X(t,\cdot) = \mathrm{Id}$ on Γ_0 for all t. More precisely, here we use a change of variables depending on time through a dependance on $\eta(t)$ only: we choose the particular form $X(t,\cdot) = X_{\eta(t)}$ for all $t \geqslant 0$ where, for any deformation $\eta \in H_0^2(0,1)$, $X_\eta : \mathbb{R}^2 \to \mathbb{R}^2$ satisfies $X_\eta(\mathcal{F}(\eta^S)) = \mathcal{F}(\eta)$ and

$$X_\eta(s, \eta^S(s)) = (s, \eta(s)) \quad s \in (0,1), \quad X_\eta(\cdot) = \mathrm{Id} \ \text{ on } \Gamma_0. \tag{1.28}$$

The construction of X_η is given in Section 2.2. Note that we will obtain that $X_{\eta(t)}$ is a C^1-diffeomorphism of $\mathbb{R}^2$ into itself for all $t \geqslant 0$ by assuming that η is a continuous bounded function in time with values in $C^1([0,1])$ and is close to η^S.

The change of variables also allows us to describe the feedback law satisfied by u in (1.3). We take u under the form

$$u(t,x) = \sum_{j=1}^{N_\sigma} F_j(v(t,.), \eta(t,.), \partial_t\eta(t,.))v_j(x) \quad t > 0,\ x \in \Gamma_0, \tag{1.29}$$

with

$$\begin{aligned}
F_j(v, \eta_1, \eta_2) = \int_{\mathcal{F}(\eta^S)}\left(\nabla X_{\eta_1}(y)^* v(X_{\eta_1}(y)) - v^S(y)\right)\cdot\varphi_j(y)\ dy \\
+ \left(A_1^{1/2}(\eta_1 - \eta^S), A_1^{1/2}\zeta_j^1\right)_{\mathcal{H}_{\mathcal{S}}} + \left(\eta_2, \zeta_j^2\right)_{\mathcal{H}_{\mathcal{S}}}
\end{aligned} \tag{1.30}$$

and

$$N_\sigma \in \mathbb{N}, \quad v_j \in \mathbf{H}^2(\Gamma_0), \quad [\varphi_j, \zeta_j^1, \zeta_j^2] \in \mathbf{L}^2(\mathcal{F}(\eta^S)) \times (H_0^2(0,1) \cap \mathcal{H}_S) \times \mathcal{H}_S, \quad j \in \{1,\ldots,N_\sigma\}. \tag{1.31}$$

Now, let us give the definition of a weak solution for our problem.

Definition 1.1. *The pair (v,η) is a weak solution of (1.18) if it satisfies the following properties*

1. *it satisfies the regularity*

$$\left\{\begin{array}{l}\eta \in C([0,+\infty); H_0^2(0,1)\cap L_0^2(0,1))\cap L^2_{\mathrm{loc}}(0,+\infty; H^3(0,1)),\\ \partial_t\eta \in C([0,+\infty); L_0^2(0,1))\cap L^2_{\mathrm{loc}}(0,+\infty; H_0^1(0,1)),\\ v\in C([0,+\infty); \mathbf{L}^2(\mathcal{F}(\eta(t))))\cap L^2_{\mathrm{loc}}(0,+\infty; \mathbf{H}^1(\mathcal{F}(\eta(t))));\end{array}\right. \tag{1.32}$$

2. *there exists a family $\{X(t,\cdot)\}_{t\geqslant 0}$ of C^1-diffeomorphisms that transforms $\mathcal{F}$ onto $\mathcal{F}(\eta(t,\cdot))$ and such that both X and X^{-1} belong to $C_b(\mathbf{C}^1(\overline{\mathcal{F}}))$.*
3. *we have $v = b^S + \Xi(u)$ on $(0,+\infty)\times\Gamma_0$, with u obtained through* (1.29)–(1.31)*;*
4. *relation* (1.23) *holds for all $(\varphi,\zeta)\in C_0^1([0,\infty); C^1(\overline{\mathcal{F}(\eta(t))})\times(C^2([0,1])\cap L_0^2(0,1)))$ satisfying* (1.20)-(1.22)*.*

We refer to Section 2.1 for the precise definition of the functional spaces used above.

Theorem 1.2. *Assume*

$$\eta^S \in C^3([0,1]) \quad \text{and} \quad \mathcal{F}(\eta^S) \text{ is of class } C^{1,1}, \tag{1.33}$$

and

$$f^S\in \mathbf{W}^{2,\infty}(\mathbb{R}^2), \quad v^S\in\mathbf{W}^{2,\infty}(\mathcal{F}(\eta^S)). \tag{1.34}$$

Then for all $\sigma>0$, there exist $N_\sigma\in\mathbb{N}$, $\mu>0$, $C>0$, $v_j\in\mathbf{H}^2(\Gamma_0)$, $j=1,\dots,N_\sigma$ such that if

$$\|v^0\circ X_{\eta^0} - v^S\|_{\mathbf{L}^2(\mathcal{F}(\eta^S))} + \|\eta_1^0-\eta^S\|_{H^2(0,1)} + \|\eta_2^0\|_{L^2(0,1)} \leqslant \mu$$

then there exists a weak solution (v,η) of (1.18), (1.29)-(1.31) *(in the sense of Definition 1.1) and:*

$$\begin{aligned}&\|v\circ X_{\eta(t)} - v^S\|_{\mathbf{L}^2(\mathcal{F}(\eta^S))} + \|\eta(t)-\eta^S\|_{H^2(0,1)} + \|\partial_t\eta(t)\|_{L^2(0,1)}\\ &\qquad\qquad \leqslant Ce^{-\sigma t}\left(\|v^0\circ X_{\eta_1^0} - v^S\|_{\mathbf{L}^2(\mathcal{F}(\eta^S))} + \|\eta_1^0-\eta^S\|_{H^2(0,1)} + \|\eta_2^0\|_{L^2(0,1)}\right).\end{aligned}$$

Remark 1.3. *Note that assumption $\mathcal{F}(\eta^S)$ of class $C^{1,1}$ allows us to freely use H^2-regularity results for the Laplace equation and for the Stokes equations. It is also a natural assumption because, even if the reference domain $\mathcal{F}_{\mathrm{ref}}$ and η^S are regular, the boundary conditions $\eta^S=\partial_s\eta^S=0$ on $\{0,1\}$ do not guarantee a class of regularity for $\mathcal{F}(\eta^S)$ better than $C^{1,1}$.*

Remark 1.4. *The family $\left[\varphi_j,\zeta_j^1,\zeta_j^2\right]$, $j=1,\dots,N_\sigma$ can be obtained for instance from $\left[\varphi_j,\zeta_j^1,\zeta_j^2\right]=Lv_j$ where L is finite rank linear operator on $\mathbf{L}^2(\mathcal{F}(\eta^S))\times\left(H_0^2(0,1)\cap\mathcal{H}_S\right)\times\mathcal{H}_S$ independent of j which can be computed from the solution of a finite dimensional Riccati equation, see [5, 8] or [37] for details.*

Remark 1.5. *The uniqueness of the controlled weak solution (in the sense of Definition 1.1) is not proved in Theorem 1.2. Since the proof relies on a Banach fixed point argument it is indeed true that the solution is unique within a class of stable solutions sufficiently close to the stationary state. But uniqueness is not obtained in the classical energy space defined by* (1.32)*. The uniqueness of weak solution is not an easy issue, even under the hypothesis of small initial data. It must be the subject of further investigations.*

Remark 1.6. *Using the method developed here, we can obtain the same result for other fluid-structure systems. For instance, we could obtain the stabilization of weak solutions for the case where the structure is a rigid body (see [6]). One could also consider the case of a deformable structure in the case where the equation of deformation is approximated by a finite dimensional method: see [20], [28], [14]. For these cases, the fixed point and the estimates are simpler than here. The case of a deformable structure modeled by the Lame equation or by the wave equation with an adequate damping can also obtained directly from our work. For other damping laws or without damping, even the well-posedness is not always done and the corresponding stabilization problems have to be studied differently. Let us quote some references on the well-posedness of such systems: [15], [12], [13], [30], [40], etc.*

The outline of the paper is as follows. In Section 2, we construct the change of variables and we rewrite the system in a fixed domain. We then obtain the system satisfied by the difference between the controlled solution and the stationary state. By linearizing this system, we obtain in Section 3 the coupled system (3.1)–(3.4) that couples an Oseen's type system with a beam type system with dissipation. With this dissipation, we prove that the semigroup associated with system (3.1)–(3.4) is analytic. That allows us to use the general theory developed in [5, 8] to deduce in Section 4 the feedback stabilization of our linear system, first in the homogeneous case and then in the non homogeneous case (and in particular with terms corresponding to the non null divergence condition and non null boundary condition). In Section 5, we use a fixed point procedure to obtain the stabilization of the nonlinear system and thus to prove the main result. In the appendix, we postpone technical proofs to the three sections: Section A is devoted to the change of variables, Section B to the linearization, and Section C to some estimates for the fixed point.

2 Notation and change of variables

2.1 Notation

The classical Lebesgue and Sobolev spaces are written L^α, H^k and we denote by C_b the continuous and bounded maps. We use the bold notation for the spaces of vector fields: $\mathbf{L}^\alpha = (L^\alpha)^2$, $\mathbf{H}^k = (H^k)^2$ etc. For a Hilbert space $\mathcal{X}$ and $0 < T \leqslant +\infty$, $L^p(0,T;\mathcal{X})$ and $H^s(0,T;\mathcal{X})$, $p \in [1,\infty]$, $s \geqslant 0$, are usual vector-valued Lebesgue and Sobolev spaces and in the case $T = +\infty$, we use the shorter expressions $L^p(\mathcal{X}) \stackrel{\text{def}}{=} L^p(0,+\infty;\mathcal{X})$ and $H^s(\mathcal{X}) \stackrel{\text{def}}{=} H^s(0,+\infty;\mathcal{X})$. We denote by $L^2_{\text{loc}}(0,T;\mathcal{X})$ (resp. $H^s_{\text{loc}}(0,T;\mathcal{X})$) the set of functions belonging to $L^2(0,T;\mathcal{X})$ (resp. $H^s(0,T;\mathcal{X})$) for all $T > 0$. For two Hilbert spaces $\mathcal{X}$, $\mathcal{Y}$ we write $W(\mathcal{X},\mathcal{Y}) \stackrel{\text{def}}{=} L^2(\mathcal{X}) \cap H^1(\mathcal{Y})$.

If $\mathcal{Z}$ is a vector-valued function space of the time variable $t \geqslant 0$, for $\sigma > 0$ we use the subscript σ in $\mathcal{Z}_\sigma$ to denote the space

$$\mathcal{Z}_\sigma \stackrel{\text{def}}{=} \left\{ \mathbf{Z} \in \mathcal{Z} \ ; \ t \mapsto e^{\sigma t}\mathbf{Z}(t) \in \mathcal{Z} \right\}. \tag{2.1}$$

For instance,

$$W_\sigma(\mathcal{X},\mathcal{Y}) \stackrel{\text{def}}{=} \left\{ \mathbf{Z} \in L^2(\mathcal{X}) \cap H^1(\mathcal{Y}) \ ; \ t \mapsto e^{\sigma t}\mathbf{Z}(t) \in W(\mathcal{X},\mathcal{Y}) \right\}.$$

We use the notation $(\mathcal{X})'$, or simply $\mathcal{X}'$, for the dual space of $\mathcal{X}$, we use the notation $\mathcal{L}(\mathcal{X},\mathcal{Y})$ for the bounded linear maps from $\mathcal{X}$ into $\mathcal{Y}$ and the notation $\mathcal{X} \hookrightarrow \mathcal{Y}$ for the continuous embedding of $\mathcal{X}$ into $\mathcal{Y}$. Moreover, $[\mathcal{X},\mathcal{Y}]_\theta$ denotes the complex interpolation space of index $\theta \in (0,1)$. If $\mathcal{X} \hookrightarrow \mathcal{Y}$ the following continuous embeddings hold for all $\theta \in (0,1)$ and $s \in (1/2, 1]$:

$$L^2(\mathcal{X}) \cap H^s(\mathcal{Y}) \hookrightarrow C_b([\mathcal{X},\mathcal{Y}]_{1/(2s)}) \quad \text{and} \quad L^2(\mathcal{X}) \cap H^s(\mathcal{Y}) \hookrightarrow H^{\theta s}([\mathcal{X},\mathcal{Y}]_\theta). \tag{2.2}$$

The first above embedding is an easy consequence of the fact that $[\mathcal{X},\mathcal{Y}]_{1/(2s)}$ is the trace space of $L^2(\mathcal{X}) \cap H^s(\mathcal{Y})$, see e.g. [25]. The second one comes from the equality $[L^2(\mathcal{X}), H^s(\mathcal{Y})]_\theta = H^{\theta s}([\mathcal{X},\mathcal{Y}]_\theta)$ (see Theorem 5.1 and (6.8) in [27]) combined with the embedding $L^2(\mathcal{X}) \cap H^s(\mathcal{Y}) \hookrightarrow [L^2(\mathcal{X}), H^s(\mathcal{Y})]_\theta$.

In order to simplify the notation, we write in what follows

$$\mathcal{F} \stackrel{\text{def}}{=} \mathcal{F}(\eta^S), \quad \Gamma_{\text{str}} \stackrel{\text{def}}{=} \Gamma_{\text{str}}(\eta^S), \tag{2.3}$$

and we introduce spaces of free divergence functions in $\mathcal{F}$ as well as the corresponding trace spaces on $\partial\mathcal{F}$:

$$\mathbf{V}^0_n(\mathcal{F}) \stackrel{\text{def}}{=} \left\{ f \in \mathbf{L}^2(\mathcal{F}) \ ; \ \operatorname{div} f = 0 \quad \text{in} \quad \mathcal{F} \quad \text{and} \quad f \cdot n = 0 \quad \text{on} \quad \partial\mathcal{F} \right\},$$

$$\mathbf{V}^s_0(\mathcal{F}) \stackrel{\text{def}}{=} \{ f \in \mathbf{H}^s(\mathcal{F}) \ ; \ \operatorname{div} f = 0 \quad \text{in} \quad \mathcal{F} \quad \text{and} \quad f = 0 \quad \text{on} \quad \partial\mathcal{F} \}, \quad (s > 1/2),$$

$$\mathbf{V}^s(\partial\mathcal{F}) \stackrel{\text{def}}{=} \left\{ w \in \mathbf{H}^s(\partial\mathcal{F}) \ ; \ \langle w \cdot n, 1 \rangle_{H^{-\frac{1}{2}}(\partial\mathcal{F}), H^{\frac{1}{2}}(\partial\mathcal{F})} = 0 \right\} \quad (s \geqslant -1/2).$$

We also use functional spaces of type $L^2(0,\infty;\mathbf{H}^1(\mathcal{F}(\eta(t))))$. Such a space is defined through a family $\{X(t,\cdot)\}_{t\geqslant 0}$ of C^1-diffeomorphisms that transforms $\mathcal{F}$ onto $\mathcal{F}(\eta(t,\cdot))$ and such that both X and X^{-1} belong to $C_b(\mathbf{C}^1(\overline{\mathcal{F}}))$. We say that $v \in L^2(0,\infty;\mathbf{H}^1(\mathcal{F}(\eta(t))))$ if $v \circ X \in L^2(\mathbf{H}^1(\mathcal{F}))$. It can be seen that the above definition of $L^2(0,\infty;\mathbf{H}^1(\mathcal{F}(\eta(t))))$ is independent of the choice of X. For instance, if $\eta \in C_b(C^1([0,1]))$ one can choose the family of change of variables $\{X_{\eta(t)}\}_{t\geqslant 0}$ introduced in Section 2.2 below. Other spaces of functions defined on a non cylindrical domain of $\mathbb{R}^3$ are defined similarly: $C_b([0,\infty);\mathbf{L}^2(\mathcal{F}(\eta(t))))$, $C^1([0,\infty);\mathbf{C}^1(\overline{\mathcal{F}(\eta(t))}))$, etc.

In what follows, $C > 0$ denotes a generic constant that may change from line to line and which is independent on the other terms of the relation where it is used.

2.2 Construction of the change of variables

We transform the system (1.18) written in the non cylindrical domain

$$\bigcup_{t>0} \{t\} \times \mathcal{F}(\eta(t))$$

onto the domain

$$(0, \infty) \times \mathcal{F}.$$

We recall that, here and in what follows, we use the simplified notation $\mathcal{F} \stackrel{\text{def}}{=} \mathcal{F}(\eta^S)$ and $\Gamma_{\text{str}} \stackrel{\text{def}}{=} \Gamma_{\text{str}}(\eta^S)$. Since $\eta^S \in C^3([0,1])$, see (1.33), we can extend η^S to a function in $C^3(\mathbb{R})$.

We consider the set

$$\mathcal{V}_\alpha = \left\{ (y_1, y_2) \in \mathbb{R}^2 \ ; \ y_1 \in (0,1), \ y_2 \in (\eta^S(y_1) - \alpha, \eta^S(y_1)) \right\}. \tag{2.4}$$

We suppose that $\alpha > 0$ is small enough in order that $\mathcal{V}_\alpha \subset \mathcal{F}$. Note that $\partial \mathcal{V}_\alpha \cap \partial \mathcal{F} = \Gamma_{\text{str}}$. We consider $\widehat{\theta} \in C_c^\infty(\mathbb{R})$ such that $\widehat{\theta} \equiv 1$ in $(-\alpha/2, \alpha/2)$ and $\widehat{\theta} \equiv 0$ in $\mathbb{R} \backslash (-\alpha, \alpha)$, and for $(y_1, y_2) \in \mathbb{R}^2$ we define $\theta(y_1, y_2) \stackrel{\text{def}}{=} \widehat{\theta}(y_2 - \eta^S(y_1))$. Then $\theta \in C^3(\mathbb{R}^2)$ and for a given function $\eta \in H_0^2(0,1)$ we define the change of variables:

$$X_\eta : \mathbb{R}^2 \to \mathbb{R}^2, \quad y \mapsto \begin{cases} y + \theta(y)(\eta(y_1) - \eta^S(y_1))e_2 & \text{if } y_1 \in (0,1), \\ y & \text{if } y_1 \notin (0,1). \end{cases}$$

We can check that $X_\eta = \text{Id}$ in $\mathcal{F} \backslash \mathcal{V}_\alpha$ and that (1.28) holds true, and in particular

$$X_\eta(\partial \mathcal{F}) = \partial \mathcal{F}(\eta). \tag{2.5}$$

Note that $\eta \in H_0^2(0,1)$ and $\eta^S \in H_0^2(0,1)$ imply that the extension of $\eta - \eta^S$ by zero outside $(0,1)$ belongs to $H^2(\mathbb{R})$ and is supported in $[0,1]$. As a consequence, $X_\eta \in \mathbf{H}^2(\mathbb{R}^2)$. From the continuous embedding $H_0^2(0,1) \hookrightarrow C^1([0,1])$ we also deduce that $X_\eta \in \mathbf{C}^1(\mathbb{R}^2)$. Moreover, we can check that for $(y_1, y_2) \in \mathbb{R}^2$,

$$\det(\nabla X_\eta(y_1, y_2)) = 1 + \widehat{\theta}'(y_2 - \eta^S(y_1))(\eta(y_1) - \eta^S(y_1)), \tag{2.6}$$

and that the mapping X_η is a C^1-diffeomorphism of $\mathbb{R}^2$ onto itself if we assume that

$$\|\widehat{\theta}'\|_{L^\infty(\mathbb{R})} \|\eta - \eta^S\|_{L^\infty(0,1)} < 1. \tag{2.7}$$

In that case, from (2.5), we deduce that X_η is a C^1-diffeomorphism of $\mathcal{F}$ onto $\mathcal{F}(\eta)$. We denote by Y_η the inverse of X_η.

In what follows, we will construct a solution $t \mapsto \eta(t)$ in $C_b(H_0^2(0,1))$ such that

$$\|\eta - \eta^S\|_{L^\infty(L^\infty(0,1))} \leqslant c_0 \tag{2.8}$$

for $c_0 \in (0, 1/\|\widehat{\theta}'\|_{L^\infty(\mathbb{R})})$. This will guarantee that for all $t \geqslant 0$, $X_{\eta(t)}$ is a C^1-diffeomorphism of $\mathcal{F}$ onto $\mathcal{F}(\eta(t))$ and that $X_{\eta(\cdot)} \in C_b(\mathbf{C}^1(\mathcal{F}))$ and $Y_{\eta(\cdot)} \in C_b(\mathbf{C}^1(\mathcal{F}))$. In what follows, we will use the simplified notation

$$\forall (t,y) \in \mathbb{R}^+ \times \mathcal{F}, \quad X(t,y) \stackrel{\text{def}}{=} X_{\eta(t)}(y) \quad \text{and} \quad Y(t,y) \stackrel{\text{def}}{=} Y_{\eta(t)}(y).$$

From their definitions we observe that

$$\forall t \geqslant 0, \quad X(t, \cdot) = \text{Id} \quad \text{in } \overline{\mathcal{F}} \backslash \mathcal{V}_\alpha,$$

and

$$\forall t \geqslant 0, \quad \nabla X(t, \cdot) = I_2 \quad \text{in } \overline{\mathcal{F}} \backslash \mathcal{V}_\alpha.$$

In the sequel, we use the above relation on Γ_0.

Finally, for a 2×2 matrix M we denote by $\text{Cof}(M)$ the cofactor matrix and we recall the classical relations

$$\det(M) = M \,\text{Cof}(M)^* = \text{Cof}(M)^* M. \tag{2.9}$$

We remark that (2.7), (2.6) imply $\det(\nabla X) > 0$ and using (2.9) we deduce that for v, φ in $L^2(\mathbf{L}^2(\mathcal{F}(\eta(t))))$,

$$\int_{\mathcal{F}(\eta(t))} v(t,x) \cdot \varphi(t,x) dx = \int_{\mathcal{F}} \widetilde{v}(t,y) \cdot \widetilde{\varphi}(t,y) dy, \tag{2.10}$$

where $\widetilde{v}(t,y) = \nabla X(t,y)^* v(t, X(t,y))$ and $\widetilde{\varphi}(t,y) = \text{Cof}(\nabla X(t,y))^* \varphi(t, X(t,y))$. It is such an observation that motivates the change of variables that are introduced in the next section (see (2.11) and (2.15) below).

2.3 Rewriting the system (1.3) in a fixed domain

In this section we assume (2.8). Our change of variables is defined by

$$\widetilde{v}(t,y) \stackrel{\text{def}}{=} \nabla X(t,y)^* v(t,X(t,y)). \tag{2.11}$$

Remark 2.1. *We could have used the change of variables $\widetilde{v}(t,y) \stackrel{def}{=} v(t,X(t,y))$ or, as in [41] or [14], $\widetilde{v}(t,y) \stackrel{def}{=} \mathrm{Cof}(\nabla X(t,y))^* v(t,X(t,y))$. The advantage of the latter choice is that it preserves the divergence free condition. Here we use this formula to transform the test function φ, see* (2.15) *below.*

We have the following results (the technical proof is postponed in Section A.1).

Lemma 2.2. *With the notation* (2.11)*, we have*

$$\det(\nabla X)(\operatorname{div} v)\circ X = \operatorname{div}(K\widetilde{v}), \tag{2.12}$$

with

$$K \stackrel{def}{=} \det(\nabla X)[\nabla Y](X)[\nabla Y]^*(X). \tag{2.13}$$

Moreover the equation

$$v = (\partial_t \eta) e_2 \quad on\ \Gamma_{\mathrm{str}}(\eta(t)),$$

is equivalent to

$$\widetilde{v} = (\nabla X)^* \left[\mathrm{Cof}(\nabla X)\right]^{-*} (\partial_t \eta) e_2 \quad on \quad \Gamma_{\mathrm{str}}. \tag{2.14}$$

For the test function in (1.23), we use the following change of variables.

$$\widetilde{\varphi}(t,y) \stackrel{\text{def}}{=} \mathrm{Cof}(\nabla X(t,y))^* \varphi(t,X(t,y)). \tag{2.15}$$

Lemma 2.3. *With the notation* (2.15)*, if φ satisfies* (1.20)*–*(1.22) *then*

$$\operatorname{div} \widetilde{\varphi} = 0 \quad in\ \mathcal{F}, \tag{2.16}$$

$$\widetilde{\varphi} = T_{\eta^S}\zeta \quad on\ \partial\mathcal{F}. \tag{2.17}$$

We can then transform the weak formulation (1.23): combining Lemma A.1, Lemma A.2, and Lemma A.3 in the appendix (Section A), we obtain that $\widetilde{v}$ satisfies the weak formulation

$$\begin{aligned}
-\int_{\mathcal{F}} \widetilde{v}^0 \cdot \widetilde{\varphi}(0,\cdot)\, dy - \int_{\mathbb{R}_+}\int_{\mathcal{F}} \widetilde{v}\cdot \partial_t \widetilde{\varphi}\, dy + \int_{\mathcal{F}} \mathcal{M}^{(1)}_{\xi}(\widetilde{v},\nabla\widetilde{v}) : \nabla\widetilde{\varphi}\, dy + \int_{\mathcal{F}} \mathcal{M}^{(2)}_{\xi}(\widetilde{v},\nabla\widetilde{v})\cdot\widetilde{\varphi}\, dy \\
- \int_{\mathcal{F}} \mathcal{B}^{(1)}_{\xi}(\widetilde{v},\widetilde{v}) : \nabla\widetilde{\varphi}\, dy - \int_{\mathcal{F}} \mathcal{B}^{(2)}_{\xi}(\widetilde{v},\widetilde{v})\cdot\widetilde{\varphi}\, dy \\
- \left(\eta^0_2, \partial_t\zeta(0,\cdot)\right)_{\mathcal{H}_{\mathcal{S}}} - \int_{\mathbb{R}_+} \left(\partial_t\eta, \partial_t\zeta\right)_{\mathcal{H}_{\mathcal{S}}} dt + \int_{\mathbb{R}_+} \left(A_1^{1/2}\eta, A_1^{1/2}\zeta\right)_{\mathcal{H}_{\mathcal{S}}} dt \\
-\left(A_2^{1/2}\eta^0_1, A_2^{1/2}\zeta(0)\right)_{\mathcal{H}_{\mathcal{S}}} - \int_{\mathbb{R}_+} \left(A_2^{1/2}\eta, A_2^{1/2}\partial_t\zeta\right)_{\mathcal{H}_{\mathcal{S}}} dt = \int_{\mathbb{R}_+}\int_{\mathcal{F}} (\nabla X)^*(f^S\circ X)\cdot\widetilde{\varphi}\, dy\, dt + \int_{\mathbb{R}_+} \left(g^S,\zeta\right)_{\mathcal{H}_{\mathcal{S}}} dt,
\end{aligned} \tag{2.18}$$

where

$$\xi \stackrel{\text{def}}{=} \eta - \eta^S,$$

where $\widetilde{v}^0 \stackrel{\text{def}}{=} \nabla X(0,y)^* v^0(X(0,y))$, where $\mathcal{M}^{(1)}_{\xi} = \mathcal{M}^{(3)}_{\xi} + \mathcal{M}^{(5)}_{\xi}$, $\mathcal{M}^{(2)}_{\xi} = \mathcal{M}^{(4)}_{\xi} + \mathcal{M}^{(6)}_{\xi}$ are linear mappings depending on ξ given in Lemma A.1 and Lemma A.2, and where $\mathcal{B}^{(1)}_{\xi}$, $\mathcal{B}^{(2)}_{\xi}$ are bilinear mappings depending on ξ given in Lemma A.3. Note that (2.10) is used to transform (1.23) into (2.18).

Since (v^S,η^S) is independent in time, we have

$$\int_{\mathcal{F}} v^S\cdot\widetilde{\varphi}(0,\cdot)\, dy + \int_{\mathbb{R}_+}\int_{\mathcal{F}} v^S\cdot\partial_t\widetilde{\varphi}\, dy\, dt + \left(A_2^{1/2}\eta^S, A_2^{1/2}\zeta(0)\right)_{\mathcal{H}_{\mathcal{S}}} + \int_{\mathbb{R}_+} \left(A_2^{1/2}\eta^S, A_2^{1/2}\partial_t\zeta\right)_{\mathcal{H}_{\mathcal{S}}} dt = 0,$$

for $(\widetilde{\varphi},\zeta) \in C^1_0([0,\infty); C^1(\overline{\mathcal{F}}) \times (C^2([0,1]) \cap L^2_0(0,1)))$.

Using the above relation, (1.27) and (2.18), we deduce that

$$\widetilde{w} \stackrel{\text{def}}{=} \widetilde{v} - v^S \tag{2.19}$$

satisfies

$$\begin{aligned}
-\int_{\mathcal{F}} \widetilde{w}^0 \cdot \widetilde{\varphi}(0,\cdot)\, dy - \int_{\mathbb{R}_+} \int_{\mathcal{F}} \widetilde{w} \cdot \partial_t \widetilde{\varphi}\, dy + \int_{\mathcal{F}} \mathcal{M}_\xi^{(1)}(\widetilde{w} + v^S, \nabla \widetilde{w} + \nabla v^S) : \nabla \widetilde{\varphi}\, dy \\
+ \int_{\mathcal{F}} \mathcal{M}_\xi^{(2)}(\widetilde{w} + v^S, \nabla \widetilde{w} + \nabla v^S) \cdot \widetilde{\varphi}\, dy - 2\nu \int_{\mathcal{F}} D(v^S) : D(\widetilde{\varphi})\, dy \\
- \int_{\mathcal{F}} \mathcal{B}_\xi^{(1)}(\widetilde{w} + v^S, \widetilde{w} + v^S) : \nabla \widetilde{\varphi}\, dy - \int_{\mathcal{F}} \mathcal{B}_\xi^{(2)}(\widetilde{w} + v^S, \widetilde{w} + v^S) \cdot \widetilde{\varphi}\, dy + \int_{\mathcal{F}} (v^S \otimes v^S) : \nabla \widetilde{\varphi}\, dy \\
- \left(\xi_2^0, \partial_t \zeta(0,\cdot)\right)_{\mathcal{H}_{\mathcal{S}}} - \int_{\mathbb{R}_+} \left(\partial_t \xi, \partial_t \zeta\right)_{\mathcal{H}_{\mathcal{S}}} dt + \int_{\mathbb{R}_+} \left(A_1^{1/2} \xi, A_1^{1/2} \zeta\right)_{\mathcal{H}_{\mathcal{S}}} dt \\
- \left(A_2^{1/2} \xi_1^0, A_2^{1/2} \zeta(0)\right)_{\mathcal{H}_{\mathcal{S}}} - \int_{\mathbb{R}_+} \left(A_2^{1/2} \xi, A_2^{1/2} \partial_t \zeta\right)_{\mathcal{H}_{\mathcal{S}}} dt = \int_{\mathbb{R}_+} \int_{\mathcal{F}} \left[(\nabla X)^* (f^S \circ X) - f^S\right] \cdot \widetilde{\varphi}\, dy\, dt,
\end{aligned} \tag{2.20}$$

where

$$\widetilde{w}^0 \stackrel{\text{def}}{=} \widetilde{v}^0 - v^S, \quad \xi_1^0 \stackrel{\text{def}}{=} \eta_1^0 - \eta^S, \quad \xi_2^0 \stackrel{\text{def}}{=} \eta_2^0. \tag{2.21}$$

Now, we can decompose the above operators in a linear part and surlinear part. First, we define the sets of type $\mathcal{Q}_i(\alpha_1, \dots, \alpha_k)$ where $i, k \in \mathbb{N}$. They are the sets of polynomials in the variables $\alpha_1, \dots, \alpha_k$ and with coefficients that are Lipschitz continuous functions of $y \in \mathbb{R}^2$ and of ξ and that vanish in $\overline{\mathcal{F}} \backslash \overline{\mathcal{V}_\alpha}$ (see (2.4)), and such that the degree of its nonzero monomial of lowest degree is greater or equal to i. For instance, we can write

$$\frac{1}{1 + (\partial_{y_2} \theta)\xi} = 1 - (\partial_{y_2} \theta)\xi + \frac{(\partial_{y_2} \theta)^2 (\xi)^2}{1 + (\partial_{y_2} \theta)\xi}.$$

Using (2.7), we deduce from the above relation that

$$\frac{1}{1 + (\partial_{y_2} \theta)\xi} - 1 \in \mathcal{Q}_1(\xi), \quad \frac{1}{1 + (\partial_{y_2} \theta)\xi} - 1 + (\partial_{y_2} \theta)\xi \in \mathcal{Q}_2(\xi).$$

Similarly,

$$\xi (\partial_s \xi)^3 + \frac{1}{1 + (\partial_{y_2} \theta)\xi} (\partial_t \xi)(\partial_s \xi) \in \mathcal{Q}_2(\xi, \partial_s \xi, \partial_t \xi).$$

We also need to consider the partial degree of such terms: for instance if we denote by $r = r(\xi, \partial_s \xi, \partial_t \xi)$ the above polynomial,

$$\deg_2 r = 3, \quad \deg_3 r = 1, \quad \deg_{1,2} r = 4, \quad \deg_{1,3} r = 1.$$

The last expression means the total degree with respect to the first and the third variables.

For the linear part, we also introduce a notation: we write

$$\gamma^{(i)}(\alpha_1, \dots, \alpha_k)$$

the linear mappings that depend on y in a Lipschitz continuous way and that vanish in $\overline{\mathcal{F}} \backslash \overline{\mathcal{V}_\alpha}$ (see (2.4)).

From Lemma 2.2 and Lemma B.1, we obtain

$$\operatorname{div} \widetilde{w} = -\operatorname{div}(\gamma^{(1)}(\xi, \partial_s \xi) v^S) - \operatorname{div}(r^{(1)}(\xi, \partial_s \xi, \widetilde{w})), \tag{2.22}$$

where

$$r^{(1)} \in \mathcal{Q}_2(\alpha_1, \alpha_2, \alpha_3, \alpha_4), \quad \deg_{3,4} r^{(1)} \leqslant 1.$$

To avoid a linear operator in the divergence condition, we consider another change of variable:

$$w \stackrel{\text{def}}{=} \widetilde{w} + \gamma^{(1)}(\xi, \partial_s \xi) v^S. \tag{2.23}$$

Then relation (2.22) transforms into

$$\operatorname{div} w = \operatorname{div}(r^{(2)}(\xi, \partial_s \xi, w)),$$

and relation (2.14) transforms into (see Lemma B.1)

$$w = (\partial_t \xi) e_2 + r^{(2)}(\xi, \partial_s \xi, w) \quad \text{on} \quad \Gamma_{\text{str}},$$

where

$$r^{(2)} \in \mathcal{Q}_2(\alpha_1, \alpha_2, \alpha_3, \alpha_4), \quad \deg_{3,4} r^{(2)} \leqslant 1. \tag{2.24}$$

In what follows, we introduce the new state variable

$$\mathbf{Z} = [w,\, \xi,\, \partial_t \xi]$$

and we write

$$F_{\text{div}}(\mathbf{Z}) = \operatorname{div}(r^{(2)}(\xi, \partial_s \xi, w)). \tag{2.25}$$

Then the divergence condition for w can be rewritten as:

$$\operatorname{div} w = F_{\text{div}}(\mathbf{Z}) \quad \text{in } \mathcal{F}. \tag{2.26}$$

In order to rewrite the boundary condition for w we write

$$F_b(\mathbf{Z}) = r^{(2)}(\xi, \partial_s \xi, w), \tag{2.27}$$

and we define

$$T \stackrel{\text{def}}{=} T_{\eta^S} M \in \mathcal{L}(L^2(0,1), \mathbf{V}^0(\partial \mathcal{F})), \tag{2.28}$$

where M and T_{η^S} are defined by (1.12) and (1.13). Note that the fact that the range of T belongs to $\mathbf{V}^0(\partial \mathcal{F})$ follows from the following calculation:

$$\int_{\partial \mathcal{F}} (T\xi) \cdot n d\gamma = \int_{\Gamma_{\text{str}}} (T_{\eta^S} M \xi) \cdot n d\gamma = \int_0^1 M\xi(s) e_2 \cdot (-\partial_s \eta^S(s) e_1 + e_2) ds = \int_0^1 M\xi(s) ds = 0.$$

Moreover, since the localization operator Ξ is defined by (1.9) from a smooth cut off function ρ supported in Γ_0 we can abusively consider Ξ as an element of $\mathcal{L}(\mathbf{L}^2(\mathcal{F}))$ and (1.9) becomes:

$$\Xi(u) \stackrel{\text{def}}{=} \rho u - \left(\int_{\partial \mathcal{F}} \rho u \cdot n d\gamma \right) \rho n. \tag{2.29}$$

Then the boundary condition for w can be rewritten as

$$w = T(\partial_t \xi) + F_b(\mathbf{Z}) + \Xi(u) \quad \text{on } (0, +\infty) \times \partial \mathcal{F}. \tag{2.30}$$

Next, (2.20) is transformed into

$$\begin{aligned}
- \int_{\mathcal{F}} w^0 \cdot \widetilde{\varphi}(0,\cdot)\, dy - \int_{\mathbb{R}_+} \int_{\mathcal{F}} w \cdot \partial_t \widetilde{\varphi}\, dy - \int_{\mathbb{R}_+} \int_{\mathcal{F}} \gamma^{(1)}(\partial_t \xi, \partial_{ts} \xi) v^S \cdot \widetilde{\varphi}\, dy \\
+ \int_{\mathcal{F}} \mathcal{M}_\xi^{(1)}(w + (1-\gamma^{(1)}) v^S, \nabla w + \nabla((1-\gamma^{(1)}) v^S)) : \nabla \widetilde{\varphi}\, dy \\
+ \int_{\mathcal{F}} \mathcal{M}_\xi^{(2)}(w + (1-\gamma^{(1)}) v^S, \nabla w + \nabla((1-\gamma^{(1)}) v^S)) \cdot \widetilde{\varphi}\, dy - 2\nu \int_{\mathcal{F}} D(v^S) : D(\widetilde{\varphi})\, dy \\
- \int_{\mathcal{F}} \mathcal{B}_\xi^{(1)}(w + (1-\gamma^{(1)}) v^S, w + (1-\gamma^{(1)}) v^S) \cdot \widetilde{\varphi}\, dy \\
- \int_{\mathcal{F}} \mathcal{B}_\xi^{(2)}(w + (1-\gamma^{(1)}) v^S, w + (1-\gamma^{(1)}) v^S) : \nabla \widetilde{\varphi}\, dy + \int_{\mathcal{F}} (v^S \otimes v^S) : \nabla \widetilde{\varphi}\, dy \\
- \Big(\xi_2^0, \partial_t \zeta(0,\cdot)\Big)_{\mathcal{H}_{\mathcal{S}}} - \int_{\mathbb{R}_+} \Big(\partial_t \xi, \partial_t \zeta\Big)_{\mathcal{H}_{\mathcal{S}}} dt + \int_{\mathbb{R}_+} \Big(A_1^{1/2} \xi, A_1^{1/2} \zeta\Big)_{\mathcal{H}_{\mathcal{S}}} dt \\
- \Big(A_2^{1/2} \xi_1^0, A_2^{1/2} \zeta(0)\Big)_{\mathcal{H}_{\mathcal{S}}} - \int_{\mathbb{R}_+} \Big(A_2^{1/2} \xi, A_2^{1/2} \partial_t \zeta\Big)_{\mathcal{H}_{\mathcal{S}}} dt = \int_{\mathbb{R}_+} \int_{\mathcal{F}} \Big[(\nabla X)^*(f^S \circ X) - f^S\Big] \cdot \widetilde{\varphi}\, dy\, dt,
\end{aligned} \tag{2.31}$$

where $w^0 \stackrel{\text{def}}{=} \widetilde{w}^0 + \gamma^{(1)}(\xi_1^0, \partial_s \xi_1^0) v^S$. In the above expression, we have written $\gamma^{(1)}$ instead of $\gamma^{(1)}(\xi, \partial_s \xi)$ to shorten the formula.

From Lemmas B.2, B.3, B.4, B.5, we can write the above relation as

$$
\begin{aligned}
-\int_{\mathcal{F}} w^0 \cdot \widetilde{\varphi}(0,\cdot)\, dy - \int_{\mathbb{R}_+}\int_{\mathcal{F}} w \cdot \partial_t \widetilde{\varphi}\, dy + 2\nu \int_{\mathbb{R}_+}\int_{\mathcal{F}} D(w) : D(\widetilde{\varphi})\, dy\, dt \\
+ \int_{\mathbb{R}_+}\int_{\mathcal{F}} \left[\gamma^{(3)}(\xi, \partial_s\xi, \partial_{ss}\xi, \partial_t\xi) + r^{(3)}(\xi, \partial_s\xi, \partial_{ss}\xi, \partial_t\xi, w, \nabla w)\right] : \nabla\widetilde{\varphi}\, dy\, dt \\
+ \int_{\mathbb{R}_+}\int_{\mathcal{F}} \left[\gamma^{(4)}(\xi, \partial_s\xi, \partial_{ss}\xi, \partial_t\xi, \partial_{ts}\xi) + r^{(4)}(\xi, \partial_s\xi, \partial_{ss}\xi, \partial_t\xi, \partial_{ts}\xi, w, \nabla w)\right] \cdot \widetilde{\varphi}\, dy\, dt \\
- \int_{\mathbb{R}_+}\int_{\mathcal{F}} (w \otimes v^S + v^S \otimes w) : \nabla\widetilde{\varphi}\, dy\, dt \\
- \left(\xi_2^0, \partial_t\zeta(0,\cdot)\right)_{\mathcal{H}_{\mathcal{S}}} - \int_{\mathbb{R}_+} \left(\partial_t\xi, \partial_t\zeta\right)_{\mathcal{H}_{\mathcal{S}}} dt + \int_{\mathbb{R}_+} \left(A_1^{1/2}\xi, A_1^{1/2}\zeta\right)_{\mathcal{H}_{\mathcal{S}}} dt \\
- \left(A_2^{1/2}\xi_1^0, A_2^{1/2}\zeta(0)\right)_{\mathcal{H}_{\mathcal{S}}} - \int_{\mathbb{R}_+} \left(A_2^{1/2}\xi, A_2^{1/2}\partial_t\zeta\right)_{\mathcal{H}_{\mathcal{S}}} dt = 0,
\end{aligned}
\tag{2.32}
$$

where

$$
r^{(3)}(\xi, \partial_s\xi, \partial_{ss}\xi, \partial_t\xi, w, \nabla w) = r^{(5)}(\xi, \partial_s\xi, \partial_{ss}\xi, \partial_t\xi, w) + r^{(6)}(\xi, \partial_s\xi, \nabla w) + r^{(7)}(\xi, \partial_s\xi, w), \tag{2.33}
$$

$$
r^{(5)} \in \mathcal{Q}_2(\alpha_1, \dots, \alpha_6), \quad \deg_{3,4} r^{(5)} \leqslant 1, \quad \deg_{5,6} r^{(5)} \leqslant 1, \tag{2.34}
$$

$$
r^{(6)} \in \mathcal{Q}_2(\alpha_1, \dots, \alpha_6), \quad \deg_{3,\dots,6} r^{(6)} \leqslant 1, \tag{2.35}
$$

$$
r^{(7)} \in \mathcal{Q}_2(\alpha_1, \dots, \alpha_4), \quad \deg_{3,4} r^{(7)} \leqslant 2, \tag{2.36}
$$

and

$$
\begin{aligned}
r^{(4)}(\xi, \partial_s\xi, \partial_{ss}\xi, \partial_t\xi, \partial_{ts}\xi, w, \nabla w) = r^{(8)}(\xi, \partial_s\xi, \partial_{ss}\xi, w) + r^{(9)}(\xi, \partial_s\xi, \partial_{ss}\xi, \nabla w) \\
+ r^{(10)}(\xi, \partial_s\xi, \partial_{ss}\xi, \partial_t\xi, w) + r^{(11)}(\xi, \partial_s\xi, \partial_t\xi, \partial_{ts}\xi, w) + r^{(12)}(\xi, \partial_s\xi, \partial_{ss}\xi, w),
\end{aligned}
\tag{2.37}
$$

$$
r^{(8)} \in \mathcal{Q}_2(\alpha_1, \dots, \alpha_5), \quad \deg_3 r^{(8)} \leqslant 2, \quad \deg_{4,5} r^{(8)} \leqslant 1, \tag{2.38}
$$

$$
r^{(9)} \in \mathcal{Q}_2(\alpha_1, \dots, \alpha_7), \quad \deg_3 r^{(9)} \leqslant 1, \quad \deg_{4,\dots,7} r^{(9)} \leqslant 1, \tag{2.39}
$$

$$
r^{(10)} \in \mathcal{Q}_2(\alpha_1, \dots, \alpha_6), \quad \deg_3 r^{(10)} \leqslant 1, \quad \deg_4 r^{(10)} \leqslant 1, \quad \deg_{5,6} r^{(10)} \leqslant 1, \tag{2.40}
$$

$$
r^{(11)} \in \mathcal{Q}_2(\alpha_1, \dots, \alpha_6), \quad \deg_{3,4} r^{(11)} \leqslant 1, \quad \deg_{5,6} r^{(11)} \leqslant 1, \tag{2.41}
$$

$$
r^{(12)} \in \mathcal{Q}_2(\alpha_1, \dots, \alpha_5), \quad \deg_3 r^{(12)} \leqslant 1, \quad \deg_{4,5} r^{(12)} \leqslant 2. \tag{2.42}
$$

In what follows we write

$$
F(\mathbf{Z}) \stackrel{\text{def}}{=} -r^{(4)}(\xi, \partial_s\xi, \partial_{ss}\xi, \partial_t\xi, \partial_{ts}\xi, w, \nabla w), \quad G(\mathbf{Z}) \stackrel{\text{def}}{=} -r^{(3)}(\xi, \partial_s\xi, \partial_{ss}\xi, \partial_t\xi, w, \nabla w), \tag{2.43}
$$

and we write $\gamma^{(3)}$ and $\gamma^{(4)}$ as operators acting on $\xi_1 = \xi$ and $\xi_2 = \partial_t\xi$, namely

$$
\Lambda^{(1)}(\xi_1, \xi_2) = \gamma^{(3)}(\xi_1, \partial_s\xi_1, \partial_{ss}\xi_1, \xi_2) \quad \text{and} \quad \Lambda^{(2)}(\xi_1, \xi_2) = \gamma^{(4)}(\xi_1, \partial_s\xi_1, \partial_{ss}\xi_1, \xi_2, \partial_s\xi_2). \tag{2.44}
$$

Then (2.32) becomes

$$
\begin{aligned}
-\int_{\mathcal{F}} w^0 \cdot \widetilde{\varphi}(0,\cdot)\, dy - \int_{\mathbb{R}_+}\int_{\mathcal{F}} w \cdot \partial_t \widetilde{\varphi}\, dy + 2\nu \int_{\mathbb{R}_+}\int_{\mathcal{F}} D(w) : D(\widetilde{\varphi})\, dy\, dt \\
+ \int_{\mathbb{R}_+}\int_{\mathcal{F}} \Lambda^{(1)}(\xi, \partial_t\xi) : \nabla\widetilde{\varphi}\, dy\, dt + \int_{\mathbb{R}_+}\int_{\mathcal{F}} \Lambda^{(2)}(\xi, \partial_t\xi) \cdot \widetilde{\varphi}\, dy\, dt - \int_{\mathbb{R}_+}\int_{\mathcal{F}} (w \otimes v^S + v^S \otimes w) : \nabla\widetilde{\varphi}\, dy\, dt \\
- \left(\xi_2^0, \partial_t\zeta(0,\cdot)\right)_{\mathcal{H}_{\mathcal{S}}} - \int_{\mathbb{R}_+} \left(\partial_t\xi, \partial_t\zeta\right)_{\mathcal{H}_{\mathcal{S}}} dt + \int_{\mathbb{R}_+} \left(A_1^{1/2}\xi, A_1^{1/2}\zeta\right)_{\mathcal{H}_{\mathcal{S}}} dt - \left(A_2^{1/2}\xi_1^0, A_2^{1/2}\zeta(0)\right)_{\mathcal{H}_{\mathcal{S}}} \\
- \int_{\mathbb{R}_+} \left(A_2^{1/2}\xi, A_2^{1/2}\partial_t\zeta\right)_{\mathcal{H}_{\mathcal{S}}} dt = \int_{\mathbb{R}_+}\int_{\mathcal{F}} F(\mathbf{Z}) \cdot \widetilde{\varphi} + G(\mathbf{Z}) : \nabla\widetilde{\varphi}\, dx\, dt.
\end{aligned}
\tag{2.45}
$$

Hence, using standard arguments, (2.45) can be rewritten as the following dynamical system: for any $(\varphi, \zeta) \in \mathbf{C}^1(\mathcal{F}) \times C^1([0,1]) \cap L^2_0(0,1)$ satisfying

$$\left\{\begin{array}{l} \operatorname{div} \varphi = 0 \quad \text{in } \mathcal{F} \\ \varphi = T\zeta \quad \text{on } \partial\mathcal{F} \\ \zeta = \partial_s \zeta = 0 \quad \text{on } \{0,1\}, \end{array}\right. \tag{2.46}$$

we have

$$\begin{gathered}
\frac{d}{dt} \int_{\mathcal{F}} w \cdot \varphi \, dy + 2\nu \int_{\mathcal{F}} D(w) : D(\varphi) \, dy - \int_{\mathcal{F}} (w \otimes v^S + v^S \otimes w) : \nabla \varphi \, dy \\
+ \int_{\mathcal{F}} \Lambda^{(1)}(\xi, \partial_t \xi) : \nabla \varphi \, dy + \int_{\mathcal{F}} \Lambda^{(2)}(\xi, \partial_t \xi) \cdot \varphi \, dy \, dt \\
+ \frac{d}{dt} \Big(\partial_t \xi, \zeta \Big)_{\mathcal{H}_S} + \Big(A_2^{1/2} \partial_t \xi, A_2^{1/2} \zeta \Big)_{\mathcal{H}_S} + \Big(A_1^{1/2} \xi, A_1^{1/2} \zeta \Big)_{\mathcal{H}_S} = \int_{\mathcal{F}} F(\mathbf{Z}) \cdot \varphi + G(\mathbf{Z}) : \nabla \varphi \, dy, \\
\xi_1(0) = \xi_1^0 \quad \text{and} \quad \xi_2(0) = \xi_2^0, \quad w(0) = w^0 \quad \text{in } \mathcal{F},
\end{gathered} \tag{2.47}$$

satisfied for all $(\varphi, \zeta) \in \mathbf{H}^1(\Omega) \times H^2(0,1)$ verifying (2.46). The main goal of Sections 3, 4 and 5 will be the construction of a solution of the dynamical system (2.47), (2.26), (2.30).

Finally, the strong formulation of the system (2.47), (2.26), (2.30) is

$$\begin{aligned}
&\partial_t w - \operatorname{div} \mathbb{T}(w,q) - \operatorname{div} \Lambda^{(1)}(\xi, \partial_t \xi) + \Lambda^{(2)}(\xi, \partial_t \xi) \\
&\qquad + (v^S \cdot \nabla) w + (w \cdot \nabla) v^S = F(\mathbf{Z}) - \operatorname{div} G(\mathbf{Z}) \quad \text{in } (0,+\infty) \times \mathcal{F}, && (2.48)\\
&\operatorname{div} w = F_{\mathrm{div}}(\mathbf{Z}) \quad \text{in } (0,+\infty) \times \mathcal{F}, && (2.49)\\
&w = T(\partial_t \xi) + F_b(\mathbf{Z}) + \Xi(u) \quad \text{on } (0,+\infty) \times \partial\mathcal{F}, && (2.50)\\
&\partial_{tt} \xi + A_2 \partial_t \xi + A_1 \xi = -T^* \Big(\mathbb{T}(w,q) n + \Lambda^{(1)}(\xi, \partial_t \xi) n \Big) + T^* G(\mathbf{Z}), \quad t \in (0,+\infty), && (2.51)\\
&\xi_1(0) = \xi_1^0 \quad \text{and} \quad \xi_2(0) = \xi_2^0, \quad w(0) = w^0 \quad \text{in } \mathcal{F}. && (2.52)
\end{aligned}$$

2.4 Properties of the linear operators Ξ, T, $\Lambda^{(1)}$, $\Lambda^{(1)}$

In order to study system (2.48)-(2.52) we need to give some regularity properties of Ξ, T, $\Lambda^{(1)}$, $\Lambda^{(2)}$.

First, observe that $\Xi \in \mathcal{L}(\mathbf{L}^2(\partial\mathcal{F}))$ defined by (2.29) is self-adjoint and since $\partial\mathcal{F}$ is of class $C^{1,1}$,

$$\Xi \in \mathcal{L}(\mathbf{V}^s(\partial\mathcal{F})), \quad s \in [0,1]. \tag{2.53}$$

From a classical interpolation argument and (1.16) we have that for any $s \in [0,2]$, $\mathcal{D}(A_1^{s/2}) \hookrightarrow H^{2s}(0,1) \cap L^2_0(0,1)$ with moreover

$$\mathcal{D}(A_1^{s/2}) = \left\{\begin{array}{ll} H^{2s}(0,1) \cap H^2_0(0,1) \cap L^2_0(0,1) & \text{for } s \in (5/4, 2], \\ H^{2s}_0(0,1) \cap L^2_0(0,1) & \text{for } s \in (1/4, 3/4) \cup (3/4, 5/4), \\ H^{2s}(0,1) \cap L^2_0(0,1) & \text{for } s \in [0, 1/4). \end{array}\right. \tag{2.54}$$

Next, we recall that $T \in \mathcal{L}(L^2(0,1), \mathbf{V}^0(\partial\mathcal{F}))$ is defined by (2.28) (and (1.12), (1.13)). Moreover, it satisfies $(T\xi)_{\Gamma_0} \equiv 0$ for all $\xi \in \mathcal{H}_S$ and

$$\forall \xi \in \mathcal{H}_S, \quad \|T\xi\|_{\mathbf{V}^0(\partial\mathcal{F})} = \|T\xi\|_{\mathbf{L}^2(\Gamma_{\mathrm{str}})} \geqslant C \|\xi\|_{\mathcal{H}_S}. \tag{2.55}$$

As a consequence, using (1.13), the regularity on η^S and an interpolation argument, we can check that

$$\forall s \in [0, 1/2], \quad T \in \mathcal{L}(\mathcal{D}(A_1^s), \mathbf{V}^{4s}(\partial\mathcal{F})). \tag{2.56}$$

Finally, from (1.14) and the regularity on η^S we deduce that $T^* = M T^*_{\eta^S}$ satisfies

$$\forall s \in [0,1], \quad T^* \in \mathcal{L}(\mathbf{H}^s(\partial\mathcal{F}), H^s(0,1) \cap L^2_0(0,1)). \tag{2.57}$$

In particular,

$$\forall s \in [0, 1/8), \quad T^* \in \mathcal{L}(\mathbf{H}^{4s}(\partial\mathcal{F}), \mathcal{D}(A_1^s)). \tag{2.58}$$

We also underline that since $T(L^2(0,1)) \subset \mathbf{V}^0(\mathcal{F})$ we have $T^*(n) = 0$.

Finally, since $\Lambda^{(1)}$, $\Lambda^{(2)}$ are defined from $\gamma^{(3)}$ and $\gamma^{(4)}$ in (2.44) and since $\gamma^{(3)}$ and $\gamma^{(4)}$ are linear mappings that depend on y in a Lipschitz continuous way we deduce:

$$\Lambda^{(1)} \in \mathcal{L}(\mathcal{D}(A_1^{1/2}) \times \mathcal{H}_S, (L^2(\mathcal{F}))^{2\times 2}), \quad \Lambda^{(2)} \in \mathcal{L}(\mathcal{D}(A_1^{1/2}) \times \mathcal{D}(A_1^{1/4}), \mathbf{L}^2(\mathcal{F})), \tag{2.59}$$

$$\Lambda^{(1)} \in \mathcal{L}(\mathcal{D}(A_1^{3/4}) \times \mathcal{D}(A_1^{1/4}), (H^1(\mathcal{F}))^{2\times 2}). \tag{2.60}$$

In what follows we introduce the decompositions

$$\Lambda^{(1)}(\xi_1, \xi_2) = \Lambda^{(1,1)}(\xi_1) + \Lambda^{(1,2)}(\xi_2), \quad \Lambda^{(2)}(\xi_1, \xi_2) = \Lambda^{(2,1)}(\xi_1) + \Lambda^{(2,2)}(\xi_2). \tag{2.61}$$

The following lemma is dedicated to some regularity results for their adjoint operators.

Lemma 2.4. *The following regularity properties hold:*

$$\left(\Lambda^{(1,1)}\right)^* \in \mathcal{L}((H^1(\mathcal{F}))^{2\times 2}, \mathcal{D}(A_1^{1/4})')), \quad \left(\Lambda^{(2,1)}\right)^* \in \mathcal{L}(\mathbf{H}^2(\mathcal{F}), \mathcal{H}_S), \tag{2.62}$$

$$\left(\Lambda^{(1,2)}\right)^* \in \mathcal{L}(H^1(\mathcal{F}))^{2\times 2}, \mathcal{D}(A_1^{1/4})), \quad \left(\Lambda^{(2,2)}\right)^* \in \mathcal{L}(\mathbf{H}^2(\mathcal{F}), \mathcal{D}(A_1^{1/4})). \tag{2.63}$$

Proof. First, since $\gamma^{(3)}$ and $\gamma^{(4)}$ are linear mappings that depend on y in a Lipschitz continuous way and that vanish in $\overline{\mathcal{F}}\backslash\overline{\mathcal{V}_\alpha}$ (see (2.4)), there exist $\Gamma_i \in (W^{1,\infty}(\mathcal{F}))^{2\times 2}$, $i = 1, 2, 3$, supported in $\overline{\mathcal{V}_\alpha}$ such that

$$\Lambda^{(1,1)}\xi = \Gamma_1\xi + \Gamma_2\partial_s\xi + \Gamma_2\partial_{ss}\xi.$$

Its adjoint $(\Lambda^{(1,1)})^*$ can be defined as an element of $\mathcal{L}((L^2(\mathcal{F}))^{2\times 2}, \mathcal{D}(A_1^{1/2})')$ through the relation

$$\langle (\Lambda^{(1,1)})^*\Psi, \xi\rangle_{\mathcal{D}(A_1^{1/2})', \mathcal{D}(A_1^{1/2})} = \int_{\mathcal{F}} \xi(\Gamma_1 : \Psi)\, dy + \int_{\mathcal{F}} \partial_s\xi(\Gamma_2 : \Psi)\, dy + \int_{\mathcal{F}} \partial_{ss}\xi(\Gamma_3 : \Psi)\, dy$$
$$(\Psi \in (L^2(\mathcal{F}))^{2\times 2},\ \xi \in \mathcal{D}(A_1^{1/2})). \tag{2.64}$$

Assume now that $\Psi \in (H^1(\mathcal{F}))^{2\times 2})$ and $\xi \in \mathcal{D}(A_1^{1/2})$. Then by integrating by parts

$$\begin{aligned}
\int_{\mathcal{F}} \partial_{ss}\xi(\Gamma_3 : \Psi)\, dy &= \int_{\mathcal{V}_\alpha} \partial_{ss}\xi(\Gamma_3 : \Psi)\, dy = \int_0^1 \partial_{ss}\xi(s) \left(\int_{\eta^S(s)-\alpha}^{\eta^S(s)} \Gamma_3(s, y_2) : \Psi(s, y_2)\, dy_2\right) ds \\
&= \int_0^1 \partial_{ss}\xi(s) \left(\int_{-\alpha}^{0} \Gamma_3(s, y_2 + \eta^S(s)) : \Psi(s, y_2 + \eta^S(s))\, dy_2\right) ds \\
&= -\int_{\mathcal{F}} \partial_s\xi \left(\left(\frac{\partial\Gamma_3}{\partial y_1} + (\partial_s\eta^S)\frac{\partial\Gamma_3}{\partial y_2}\right) : \Psi + \Gamma_3 : \left(\frac{\partial\Psi}{\partial y_1} + (\partial_s\eta^S)\frac{\partial\Psi}{\partial y_2}\right)\right) dy.
\end{aligned}$$

In the above calculations we have used the fact that Γ_3 is supported in $\overline{\mathcal{V}_\alpha}$ and that $\partial_s\xi = 0$ on $\{0, 1\}$. The above relation and (2.64) yield that $(\Lambda^{(1,1)})^*\Psi \in \mathcal{D}(A_1^{1/4})'$ and moreover that

$$(\Lambda^{(1,1)})^* \in \mathcal{L}((H^1(\mathcal{F}))^{2\times 2}), \mathcal{D}(A_1^{1/4})').$$

This gives the first relation of (2.62).

The three other relations can be obtained in a similar way. $\square$

3 Operators for the linear system

3.1 General functional settings

This section is devoted to the study of the linear system

$$\begin{aligned}
\partial_t w - \operatorname{div} \mathbb{T}(w, q) - \operatorname{div} \Lambda^{(1)}(\xi, \partial_t\xi) + \Lambda^{(2)}(\xi, \partial_t\xi)& \\
+(v^S \cdot \nabla)w + (w \cdot \nabla)v^S = F - \operatorname{div} G \quad &\text{in } (0, +\infty) \times \mathcal{F}, \qquad (3.1)\\
\operatorname{div} w = F_{\text{div}} \quad &\text{in } (0, +\infty) \times \mathcal{F}, \qquad (3.2)\\
w = T\xi_2 + F_b + \Xi(u) \quad &\text{on } (0, +\infty) \times \partial\mathcal{F}, \qquad (3.3)
\end{aligned}$$

$$\partial_{tt}\xi + A_2\partial_t\xi + A_1\xi = -T^*\left(\mathbb{T}(w, q)n + \Lambda^{(1)}(\xi, \partial_t\xi)n\right) + T^*G, \quad t \in (0, +\infty), \tag{3.4}$$

where here F, G, $F_{\rm div}$ and F_b are given.

Let us remark that the results given in this section can be obtained for general operators A_1, A_2, T, $\Lambda^{(1)}$ and $\Lambda^{(2)}$. More precisely, we only need to assume that $A_1 : \mathcal{D}(A_1) \subset \mathcal{H}_{\mathcal{S}} \to \mathcal{H}_{\mathcal{S}}$ and $A_2 : \mathcal{D}(A_2) \subset \mathcal{H}_{\mathcal{S}} \to \mathcal{H}_{\mathcal{S}}$ are positive, densely defined, self-adjoint and with compact resolvents and that $\mathcal{D}(A_1^{1/2}) \hookrightarrow \mathcal{D}(A_2)$,

$$D(A_1^{1/4}) = D(A_2^{1/2}). \tag{3.5}$$

Assumption (3.5) is crucial in our analysis since it allows to invoke [17] and to obtain the analyticity of the semigroup generated by the underlying linear operator of system (3.1)-(3.4) (see Proposition 3.11 below). We suppose that $T \in \mathcal{L}(\mathcal{H}_{\mathcal{S}}, \mathbf{V}^0(\partial\mathcal{F}))$ satisfies (2.55), (2.57), (2.56), (2.58), that $(T\xi)_{\Gamma_0} \equiv 0$ for any $\xi \in \mathcal{H}_{\mathcal{S}}$. Finally, the operators $\Lambda^{(1)}$, $\Lambda^{(2)}$ are assumed to satisfy (2.59), (2.60), (2.61), (2.62) and (2.63). Note that the operators A_1, A_2, T, $\Lambda^{(1)}$ and $\Lambda^{(2)}$ defined by (1.16), (1.17), (2.28) and (2.44) satisfy the above conditions.

We still assume that $\Xi \in \mathcal{L}(\mathbf{L}^2(\partial\mathcal{F}))$ is the self-adjoint operator defined by (2.29). We need its precise definition to obtain the adjoint of the control operator (see (3.67) below). We recall that Ξ satisfies (2.53).

We first consider system (3.1)–(3.4) in the case $F = 0$, $G = 0$, $F_{\rm div} = 0$ and $F_b = 0$:

$$\partial_t w - \operatorname{div} \mathbb{T}(w,p) - \operatorname{div} \Lambda^{(1)}(\xi_1,\xi_2) + \Lambda^{(2)}(\xi_1,\xi_2) + (v^S \cdot \nabla) w + (w\cdot\nabla) v^S = 0 \quad \text{in } (0,+\infty)\times\mathcal{F}, \tag{3.6}$$

$$\operatorname{div} w = 0 \quad \text{in } (0,+\infty)\times\mathcal{F}, \tag{3.7}$$

$$w = T\xi_2 + \Xi(u) \quad \text{on } (0,+\infty)\times\partial\mathcal{F}, \tag{3.8}$$

$$\partial_t \xi_1 = \xi_2, \quad t\in(0,+\infty), \tag{3.9}$$

$$\partial_t\xi_2 + A_2\xi_2 + A_1\xi_1 = -T^*\left(\mathbb{T}(w,p)n + \Lambda^{(1)}(\xi_1,\xi_2)n\right), \quad t\in(0,+\infty). \tag{3.10}$$

The above system is completed with the initial conditions

$$\xi_1(0) = \xi_1^0 \quad \text{and} \quad \xi_2(0) = \xi_2^0, \quad w(0) = w^0 \quad \text{in } \mathcal{F}. \tag{3.11}$$

We show that the system (3.6)–(3.11) can be rewritten in the form

$$\mathbb{P}\mathbf{Z}' = A\mathbb{P}\mathbf{Z} + Bu \text{ in } \mathcal{D}(A^*)', \quad \mathbb{P}\mathbf{Z}(0) = \mathbb{P}\mathbf{Z}^0 \tag{3.12}$$

$$(I-\mathbb{P})\mathbf{Z} = (I-\mathbb{P})D_{\mathcal{F}}u, \tag{3.13}$$

where A is the infinitesimal generator of an analytic semigroup. This abstract form is quite standard in the study of the stabilizability for the Navier–Stokes system, see [35].

We consider the space $\mathbf{L}^2(\mathcal{F}) \times \mathcal{D}(A_1^{1/2}) \times \mathcal{H}_{\mathcal{S}}$ equipped with the scalar product:

$$\left\langle \left[w^{(1)},\xi_1^{(1)},\xi_2^{(1)}\right], \left[w^{(2)},\xi_1^{(2)},\xi_2^{(2)}\right]\right\rangle = \int_{\mathcal{F}} w^{(1)}\cdot w^{(2)}\ dy + \left(A_1^{1/2}\xi_1^{(1)}, A_1^{1/2}\xi_1^{(2)}\right)_{\mathcal{H}_{\mathcal{S}}} + \left(\xi_2^{(1)},\xi_2^{(2)}\right)_{\mathcal{H}_{\mathcal{S}}},$$

and we introduce the following spaces:

$$\mathcal{H} \stackrel{\rm def}{=} \left\{[w,\xi_1,\xi_2]\in \mathbf{L}^2(\mathcal{F})\times\mathcal{D}(A_1^{1/2})\times\mathcal{H}_{\mathcal{S}}\ ;\ w\cdot n = (T\xi_2)\cdot n \text{ on } \partial\mathcal{F},\ \operatorname{div}\ w = 0 \text{ in } \mathcal{F}\right\},$$

$$\mathcal{V} \stackrel{\rm def}{=} \left\{[w,\xi_1,\xi_2]\in \mathbf{H}^1(\mathcal{F})\times\mathcal{D}(A_1^{3/4})\times\mathcal{D}(A_1^{1/4})\ ;\ w = T\xi_2 \text{ on } \partial\mathcal{F},\ \operatorname{div}\ w = 0 \text{ in } \mathcal{F}\right\}.$$

Let us define $\mathbb{P}$ the orthogonal projection of $\mathbf{L}^2(\mathcal{F})\times\mathcal{D}(A_1^{1/2})\times\mathcal{H}_{\mathcal{S}}$ onto $\mathcal{H}$.

We have the following characterization of the orthogonal of $\mathcal{H}$ in $\mathbf{L}^2(\mathcal{F})\times\mathcal{D}(A_1^{1/2})\times\mathcal{H}_{\mathcal{S}}$.

Proposition 3.1. *The orthogonal of $\mathcal{H}$ in $\mathbf{L}^2(\mathcal{F})\times\mathcal{D}(A_1^{1/2})\times\mathcal{H}_{\mathcal{S}}$ is given by*

$$\mathcal{H}^\perp = \left\{[\nabla p, 0, -T^*(pn)]\ ;\ p\in H^1(\mathcal{F}),\ \int_{\mathcal{F}} p\ dy = 0\right\}. \tag{3.14}$$

Proof. Assume $[w^{(1)},\xi_1^{(1)},\xi_2^{(1)}]\in \mathbf{L}^2(\mathcal{F})\times\mathcal{D}(A_1^{1/2})\times\mathcal{H}_{\mathcal{S}}$ satisfies for all $[w^{(2)},\xi_1^{(2)},\xi_2^{(2)}]\in\mathcal{H}$:

$$\int_{\mathcal{F}} w^{(1)}\cdot w^{(2)}\ dy + \left(A_1^{1/2}\xi_1^{(1)}, A_1^{1/2}\xi_1^{(2)}\right)_{\mathcal{H}_{\mathcal{S}}} + \left(\xi_2^{(1)},\xi_2^{(2)}\right)_{\mathcal{H}_{\mathcal{S}}} = 0. \tag{3.15}$$

Then we have in particular that $\int_{\mathcal{F}} w^{(1)} \cdot w^{(2)} \ dy = 0$ for all $w^{(2)} \in \mathbf{V}^1_0(\mathcal{F})$ and the De Rham Lemma guarantees that $w^{(1)} = \nabla p$ for some $p \in H^1(\mathcal{F})$ such that $\int_{\mathcal{F}} p \ dy = 0$, see [42, Chap. I, Prop. 1.1 and Rem 1.4]. Thus, by plugging $w^{(1)} = \nabla p$ in (3.15) and integrating by parts, we obtain that

$$\int_{\partial\mathcal{F}} pn \cdot (T\xi_2^{(2)}) \ d\gamma + \left(\xi_2^{(1)}, \xi_2^{(2)}\right)_{\mathcal{H}_{\mathcal{S}}} = 0 \quad \text{and} \quad \left(A_1^{1/2}\xi_1^{(1)}, A_1^{1/2}\xi_1^{(2)}\right)_{\mathcal{H}_{\mathcal{S}}} = 0$$

is satisfied for all $[\xi_1^{(2)}, \xi_2^{(2)}] \in \mathcal{D}(A_1^{1/2}) \times \mathcal{H}_{\mathcal{S}}$, which gives the result. $\square$

Proposition 3.2. *The orthogonal projection operator* $\mathbb{P} : \mathbf{L}^2(\mathcal{F}) \times \mathcal{D}(A_1^{1/2}) \times \mathcal{H}_{\mathcal{S}} \to \mathcal{H}$ *satisfies for* $s \in [0, 1]$*:*

$$\mathbb{P} \in \mathcal{L}(\mathbf{H}^s(\mathcal{F}) \times \mathcal{D}(A_1^{1/2+s/4}) \times \mathcal{D}(A_1^{s/4}), \mathbf{H}^s(\mathcal{F}) \times \mathcal{D}(A_1^{1/2+s/4}) \times \mathcal{D}(A_1^{s/4})). \tag{3.16}$$

Proof. First, by using (3.14) we verify that for any $[w, \xi_1, \xi_2] \in \mathbf{L}^2(\mathcal{F}) \times \mathcal{D}(A_1^{1/2}) \times \mathcal{H}_{\mathcal{S}}$,

$$\mathbb{P} \begin{bmatrix} w \\ \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} w - \nabla p \\ \xi_1 \\ \xi_2 + T^*(pn) \end{bmatrix}$$

where the pressure function $p \in H^1(\mathcal{F})$ obeys $\int_{\mathcal{F}} p \ dy = 0$ and is solution to the Neumann problem:

$$\begin{cases} \Delta p = \operatorname{div} w \text{ in } \mathcal{F}, \\ \dfrac{\partial p}{\partial n} + T(T^*(pn)) \cdot n = w \cdot n - (T\xi_2) \cdot n \text{ on } \partial\mathcal{F}, \end{cases}$$

that is for all $q \in H^1(\mathcal{F})$ such that $\int_{\mathcal{F}} q \ dy = 0$,

$$\int_{\mathcal{F}} \nabla p \cdot \nabla q \ dy + \left(T^*(pn), T^*(qn)\right)_{\mathcal{H}_{\mathcal{S}}} = \int_{\mathcal{F}} w \cdot \nabla q dy - \left(\xi_2, T^*(qn)\right)_{\mathcal{H}_{\mathcal{S}}}. \tag{3.17}$$

From (3.17), we deduce that

$$\|\nabla p\|_{\mathbf{L}^2(\mathcal{F})} + \|T^*(pn)\|_{\mathcal{H}_{\mathcal{S}}} \leqslant C(\|w\|_{\mathbf{L}^2(\mathcal{F})} + \|\xi_2\|_{\mathcal{H}_{\mathcal{S}}}),$$

from which, we obtain (3.16) for $s = 0$.

For $s = 1$, we take $[w, \xi_1, \xi_2] \in \mathbf{H}^1(\mathcal{F}) \times \mathcal{D}(A_1^{3/4}) \times \mathcal{D}(A_1^{1/4})$. Then, we deduce from (2.56) and from the $C^{1,1}$ regularity of $\partial\mathcal{F}$ that $(T\xi_2) \cdot n \in H^{1/2}(\partial\mathcal{F})$. Similarly, from (2.56) and (2.57) we get

$$T(T^*(pn)) \cdot n - w \cdot n \in H^{1/2}(\partial\mathcal{F})$$

and from the regularity of $\partial\mathcal{F}$ and standard elliptic properties of the Neumann problem we deduce that $\mathbb{P}[w, \xi_1, \xi_2] \in \mathbf{H}^1(\mathcal{F}) \times \mathcal{D}(A_1^{3/4}) \times \mathcal{D}(A_1^{1/4})$.

Then the conclusion follows by an interpolation argument. $\square$

As a consequence, since $\mathbb{P}$ is self-adjoint on $\mathcal{H}$, a duality argument yields the following result.

Corollary 3.3. *The orthogonal projection operator* $\mathbb{P} : \mathbf{L}^2(\mathcal{F}) \times \mathcal{D}(A_1^{1/2}) \times \mathcal{H}_{\mathcal{S}} \to \mathcal{H}$ *can be extended as an operator satisfying for* $s \in (0, 1]$*:*

$$\mathbb{P} \in \mathcal{L}(\mathbf{H}^s(\mathcal{F})' \times \mathcal{D}(A_1^{1/2-s/4}) \times \mathcal{D}(A_1^{s/4})', \mathbf{H}^s(\mathcal{F})' \times \mathcal{D}(A_1^{1/2-s/4}) \times \mathcal{D}(A_1^{s/4})'). \tag{3.18}$$

3.2 The operator A_0

First, we define the linear operator $A_0 : \mathcal{D}(A_0) \subset \mathcal{H} \to \mathcal{H}$ as follows: we set

$$\mathcal{D}(A_0) \stackrel{\text{def}}{=} \mathcal{V} \cap \left[\mathbf{H}^2(\mathcal{F}) \times \mathcal{D}(A_1) \times \mathcal{D}(A_1^{1/2})\right], \tag{3.19}$$

and for $\left[w, \xi_1, \xi_2\right] \in \mathcal{D}(A_0)$, we set

$$\widetilde{A}_0 \begin{bmatrix} w \\ \xi_1 \\ \xi_2 \end{bmatrix} \stackrel{\text{def}}{=} \begin{bmatrix} \nu\Delta w \\ \\ \xi_2 \\ \\ -A_1\xi_1 - A_2\xi_2 - T^*(2\nu D(w)n) \end{bmatrix} \tag{3.20}$$

and

$$A_0 \overset{\text{def}}{=} \mathbb{P}\widetilde{A}_0. \tag{3.21}$$

Proposition 3.4. *The operator A_0 defined by* (3.19)–(3.21) *is densely defined with compact resolvent and it is the infinitesimal generator of a strongly continuous semigroup of contractions on $\mathcal{H}$.*

Proof. Standard calculation gives, for all $\mathbf{Z} \in \mathcal{D}(A_0)$, $\langle A_0\mathbf{Z}, \mathbf{Z}\rangle \leqslant 0$ which implies that A_0 is dissipative. Then, we show that $(\lambda - A_0)$ is onto for some $\lambda > 0$: assume $\mathbf{F} = [f, g, h] \in \mathcal{H}$, we have to prove the existence and the uniqueness of $\mathbf{Z} = [w, \xi_1, \xi_2] \in \mathcal{D}(A_0)$ such that

$$\begin{aligned} \lambda w - \nu\Delta w + \nabla q &= f \quad \text{ in } \mathcal{F}, \\ \operatorname{div} w &= 0 \quad \text{ in } \mathcal{F}, \\ w &= T\xi_2 \quad \text{ on } \partial\mathcal{F}, \\ \lambda\xi_1 - \xi_2 &= g, \\ \lambda\xi_2 + A_2\xi_2 + A_1\xi_1 &= -T^*\left(\mathbb{T}(w,q)n\right) + h. \end{aligned} \tag{3.22}$$

Let us consider a variational formulation associated to (3.22): find

$$[w, \xi_2] \in \mathbb{V} \overset{\text{def}}{=} \left\{[w, \xi_2] \in \mathbf{H}^1(\mathcal{F}) \times \mathcal{D}(A_1^{1/2}) \ ; \ \operatorname{div} w = 0, \ w = T\xi_2 \text{ on } \partial\mathcal{F}\right\}, \tag{3.23}$$

such that for any $[\varphi, \zeta_2] \in \mathbb{V}$,

$$\begin{aligned} \lambda\left(\int_{\mathcal{F}} w\cdot\varphi dy + \Big(\xi_2, \zeta_2\Big)_{\mathcal{H}_\mathcal{S}}\right) + 2\nu\int_{\mathcal{F}} Dw : D\varphi \ dy + \Big(A_2^{1/2}\xi_2, A_2^{1/2}\zeta_2\Big)_{\mathcal{H}_\mathcal{S}} \\ + \frac{1}{\lambda}\Big(A_1^{1/2}\xi_2, A_1^{1/2}\zeta_2\Big)_{\mathcal{H}_\mathcal{S}} = \int_{\mathcal{F}} f\cdot\varphi \ dy + \Big(h, \zeta_2\Big)_{\mathcal{H}_\mathcal{S}} - \frac{1}{\lambda}\Big(A_1^{1/2}g, A_1^{1/2}\zeta_2\Big)_{\mathcal{H}_\mathcal{S}}. \end{aligned} \tag{3.24}$$

The Riesz theorem gives the existence and uniqueness of $[w, \xi_2] \in \mathbb{V}$ satisfying (3.24). Taking $\zeta_2 = 0$ in (3.24) and using the De Rham theorem, we obtain the existence of q such that (w, q) is the weak solution of the Stokes system (the three first equations of (3.22)). From (2.56), we deduce $T\xi_2 \in \mathbf{V}^{3/2}(\partial\mathcal{F})$ and thus, since $f \in \mathbf{L}^2(\mathcal{F})$, standard elliptic results on the Stokes system give $w \in \mathbf{H}^2(\mathcal{F})$ and $q \in H^1(\mathcal{F})$. In particular, $\mathbb{T}(w,q)n \in \mathbf{H}^{1/2}(\partial\mathcal{F})$ and thus $T^*\left(\mathbb{T}(w,q)n\right) \in \mathcal{H}_\mathcal{S}$.

We write $\xi_1 = \lambda^{-1}(\xi_2 + g)$ and we use that (w, q) satisfies the Stokes system to transform (3.24) into

$$\Big(A_1^{1/2}\xi_1, A_1^{1/2}\zeta_2\Big)_{\mathcal{H}_\mathcal{S}} = -\lambda\Big(\xi_2, \zeta_2\Big)_{\mathcal{H}_\mathcal{S}} - \Big(A_2\xi_2, \zeta_2\Big)_{\mathcal{H}_\mathcal{S}} - \Big(T^*\left(\mathbb{T}(w,q)n\right), \zeta_2\Big)_{\mathcal{H}_\mathcal{S}} + \Big(h, \zeta_2\Big)_{\mathcal{H}_\mathcal{S}},$$

for all $\zeta_2 \in \mathcal{D}(A_1^{1/2})$. Note that we have used the continuous embedding $D(A_1^{1/2}) \hookrightarrow D(A_2)$. The above system implies that $A_1\xi_1 \in \mathcal{H}_\mathcal{S}$ and thus that $\xi_1 \in \mathcal{D}(A_1)$. Finally, the fact that A_0 is densely defined with compact resolvent is straightforward. ☐

Proposition 3.5. *The semigroup generated by A_0 is exponentially stable on $\mathcal{H}$.*

Proof. To show the exponential stability of the semigroup, we use the classical result of Gearhart (see, for instance, [31, Theorem 1.3.2, p.4]): since $(e^{tA})_{t>0}$ is a C_0-semigroup of contractions on the Hilbert space $\mathcal{H}$ (see Proposition 3.4), then it is exponentially stable if and only if

$$i\mathbb{R} \subset \rho(A_0), \tag{3.25}$$

and

$$\sup_{\tau\in\mathbb{R}} \|(i\tau - A_0)^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty. \tag{3.26}$$

Using that A generates a semigroup of contractions, we have (see, for instance, [32, Corollary 3.6, p.11])

$$\forall\lambda \in \mathbb{C}, \ \Re\lambda > 0 \quad \Re\lambda\|(\lambda - A_0)^{-1}\|_{\mathcal{L}(\mathcal{H})} \leqslant 1. \tag{3.27}$$

In order to prove the exponential stability of $(e^{tA})_{t>0}$, we show the existence of $C > 0$ such that:

$$\forall\lambda \in \mathbb{C}, \ \Re\lambda \in (0,1) \quad \|(\lambda - A_0)^{-1}\|_{\mathcal{L}(\mathcal{H})} \leqslant C. \tag{3.28}$$

The above estimate yields the result: indeed, for $\tau \in \mathbb{R}$ and $\delta \in (0,1)$, one can write the formula

$$(i\tau - A_0) = (i\tau + \delta - A_0)(I - \delta(i\tau + \delta - A_0)^{-1}).$$

Taking $\delta < 1/C$ where C is the constant in (3.28), it yields (3.25) and (3.26).

Now let us prove (3.28): assume $\lambda \in \mathbb{C}$ with $\Re\lambda \in (0,1)$ and assume $(\lambda - A_0)[w, \xi_1, \xi_2] = [f, g, h] \in \mathcal{H}$. This relation can be written as (3.22). Multiplying by $[w, \xi_1, \xi_2]$, we first obtain

$$\Re\lambda \left(\|w\|_{\mathbf{L}^2(\mathcal{F})}^2 + \|\xi_2\|_{\mathcal{H_S}}^2 + \|A_1^{1/2}\xi_1\|_{\mathcal{H_S}}^2 \right) + 2\nu \int_{\mathcal{F}} |Dw|^2 \ dy + \|A_2^{1/2}\xi_2\|_{\mathcal{H_S}}^2 \\ \leqslant C\|[f,g,h]\|_{\mathcal{H}}\|[w,\xi_1,\xi_2]\|_{\mathcal{H}}. \quad (3.29)$$

Moreover, since Γ_0 is a nonempty open subset such that $w = 0$ on Γ_0, we have the Poincaré inequality $\|w\|_{\mathbf{H}^1(\mathcal{F})} \leqslant C\|\nabla w\|_{\mathbf{L}^2(\mathcal{F})}$. Combining this relation with the trace inequality, the Korn inequality and (2.55), we deduce that:

$$\|\xi_2\|_{\mathcal{H_S}}^2 \leqslant C\|T\xi_2\|_{\mathbf{V}^{1/2}(\partial\mathcal{F})}^2 \leqslant C\|w\|_{\mathbf{H}^1(\mathcal{F})}^2 \leqslant C \int_{\mathcal{F}} |Dw|^2 \ dy.$$

Then using the above inequality with (3.29) we obtain

$$\|w\|_{\mathbf{H}^1(\mathcal{F})}^2 + \|T\xi_2\|_{\mathbf{V}^{1/2}(\partial\mathcal{F})}^2 + \|\xi_2\|_{\mathcal{H_S}}^2 \leqslant C\|[f,g,h]\|_{\mathcal{H}}\|[w,\xi_1,\xi_2]\|_{\mathcal{H}}. \quad (3.30)$$

Combining the above inequality with $\lambda\xi_1 = \xi_2 + g$ and (2.56) yields

$$|\lambda|^2\|\xi_1\|_{\mathcal{H_S}}^2 + |\lambda|^2\|T\xi_1\|_{\mathbf{V}^{1/2}(\partial\mathcal{F})}^2 \leqslant C\|[f,g,h]\|_{\mathcal{H}}^2 + C\|[f,g,h]\|_{\mathcal{H}}\|[w,\xi_1,\xi_2]\|_{\mathcal{H}}. \quad (3.31)$$

Next, from the two last equalities in (3.22) we obtain

$$A_1\xi_1 = -T^*\mathbb{T}(w,p)n + h + A_2 g - \lambda A_2\xi_1 + \lambda g - \lambda^2\xi_1.$$

Then by multiplying the above equation by ξ_1 and using $D(A_1^{1/2}) \hookrightarrow D(A_2)$ we deduce

$$\|A_1^{1/2}\xi_1\|_{\mathcal{H_S}}^2 \leqslant C \Big(\|\mathbb{T}(w,p)n\|_{\mathbf{V}^{-1/2}(\partial\mathcal{F})}\|T\xi_1\|_{\mathbf{V}^{1/2}(\partial\mathcal{F})} + \|h\|_{\mathcal{H_S}}\|\xi_1\|_{\mathcal{H_S}} + \|A_1^{1/2}g\|_{\mathcal{H_S}}\|A_1^{1/2}\xi_1\|_{\mathcal{H_S}} \\ + \|A_1^{1/2}\xi_1\|_{\mathcal{H_S}}|\lambda|\|\xi_1\|_{\mathcal{H_S}} + \|g\|_{\mathcal{H_S}}|\lambda|\|\xi_1\|_{\mathcal{H_S}} + |\lambda|^2\|\xi_1\|_{\mathcal{H_S}}^2 \Big). \quad (3.32)$$

The above inequality, (2.56) and (3.31) yield

$$\|A_1^{1/2}\xi_1\|_{\mathcal{H_S}}^2 \leqslant C \left(\|\mathbb{T}(w,p)n\|_{\mathbf{V}^{-1/2}(\partial\mathcal{F})}^2 \frac{1}{1+|\lambda|^2} + \|[f,g,h]\|_{\mathcal{H}}^2 + \|[f,g,h]\|_{\mathcal{H}}\|[w,\xi_1,\xi_2]\|_{\mathcal{H}} \right). \quad (3.33)$$

Moreover, from the following Green formula

$$\forall \varphi \in \mathbf{V}^1(\mathcal{F}) \quad \int_{\partial\mathcal{F}} \mathbb{T}(w,p)n \cdot \varphi d\Gamma = \int_{\mathcal{F}} \operatorname{div} \mathbb{T}(w,p) \cdot \varphi dy + \int_{\mathcal{F}} 2\nu Dw : D\varphi dy \quad (3.34)$$

we deduce

$$\|\mathbb{T}(w,p)n\|_{\mathbf{V}^{-1/2}(\partial\mathcal{F})}^2 \leqslant C \left(\|w\|_{\mathbf{H}^1(\mathcal{F})}^2 + \|\operatorname{div} \mathbb{T}(w,p)\|_{\mathbf{L}^2(\mathcal{F})}^2 \right)$$

and with (3.30) and the first equation in (3.22) we obtain

$$\|\mathbb{T}(w,p)n\|_{\mathbf{V}^{-1/2}(\partial\mathcal{F})}^2 \leqslant C \left(\|[f,g,h]\|_{\mathcal{H}}\|[w,\xi_1,\xi_2]\|_{\mathcal{H}} + \|[f,g,h]\|_{\mathcal{H}}^2 + |\lambda|^2\|w\|_{\mathbf{L}^2(\mathcal{F})}^2 \right).$$

Then combining this last estimate with (3.33) yields

$$\|A_1^{1/2}\xi_1\|_{\mathcal{H_S}}^2 \leqslant C \left(\|w\|_{\mathbf{L}^2(\mathcal{F})}^2 + \|[f,g,h]\|_{\mathcal{H}}^2 + \|[f,g,h]\|_{\mathcal{H}}\|[w,\xi_1,\xi_2]\|_{\mathcal{H}} \right).$$

Moreover, this last estimates with (3.30) yields

$$\|w\|_{\mathbf{L}^2(\mathcal{F})}^2 + \|\xi_2\|_{\mathcal{H_S}}^2 + \|A_1^{1/2}\xi_1\|_{\mathcal{H_S}}^2 \leqslant C \left(\|[f,g,h]\|_{\mathcal{H}}^2 + \|[f,g,h]\|_{\mathcal{H}}\|[w,\xi_1,\xi_2]\|_{\mathcal{H}} \right),$$

and it proves (3.28). □

Remark 3.6. *Assumption* (3.5) *is not used in the proof of Proposition 3.5. It remains true even if* $A_2 = 0$.

We have the following characterization of the adjoint of A_0.

Proposition 3.7. *The adjoint of the operator* A_0 *is given by*

$$\mathcal{D}(A_0^*) = \mathcal{D}(A_0) \tag{3.35}$$

and

$$A_0^* \begin{bmatrix} \varphi \\ \zeta_1 \\ \zeta_2 \end{bmatrix} = \mathbb{P} \begin{bmatrix} \nu\Delta\varphi \\ -\zeta_2 \\ A_1\zeta_1 - A_2\zeta_2 - T^*\left(2\nu D(\varphi)n\right) \end{bmatrix}. \tag{3.36}$$

Proof. Equality (3.36) follows from an integration by parts and (3.35) is obtained from regularity results for the Stokes system as in Proposition 3.4. □

Proposition 3.8. *For* $\alpha \in [0,1]$, *the following equalities hold*

$$\mathcal{D}((-A_0)^\alpha) = [\mathcal{D}(A_0), \mathcal{H}]_{1-\alpha} = [\mathcal{D}(A_0^*), \mathcal{H}]_{1-\alpha} = \mathcal{D}((-A_0^*)^\alpha), \tag{3.37}$$

where $[\cdot,\cdot]_\cdot$ *denotes the complex interpolation method. Moreover, we have*

$$\mathcal{D}((-A_0)^\alpha) = \left[\mathbf{H}^{2\alpha}(\mathcal{F}) \times \mathcal{D}(A_1^{1/2+\alpha/2}) \times \mathcal{D}(A_1^{\alpha/2})\right] \cap \mathcal{H} \quad \textit{if } \alpha \in (0,1/4), \tag{3.38}$$

$$\begin{aligned}&\mathcal{D}((-A_0)^\alpha)\\ &\quad = \left\{[w,\xi_1,\xi_2] \in \left[\mathbf{H}^{2\alpha}(\mathcal{F}) \times \mathcal{D}(A_1^{1/2+\alpha/2}) \times \mathcal{D}(A_1^{\alpha/2})\right] \cap \mathcal{H}\ ;\ w = T\xi_2 \textit{ on } \partial\mathcal{F}\right\} \textit{ if } \alpha \in (1/4,1).\end{aligned} \tag{3.39}$$

Proof. Relations (3.37) are consequences of $\mathcal{D}(A_0^*) = \mathcal{D}(A_0)$ and of the maximal accretivity of $-A_0$, see [11, Prop. 6.1, p170].

To prove the last two relations, we introduce the Dirichlet map defined by $D_0(\xi_2) = z$ where z is the solution of

$$\begin{cases} -\Delta z + \nabla\pi &= 0 \text{ in } \mathcal{F}, \\ \operatorname{div} z &= 0 \text{ in } \mathcal{F}, \\ z &= T\xi_2 \text{ on } \partial\mathcal{F}. \end{cases}$$

Using (2.56) and standard result on the Stokes system, we deduce that for any $\alpha \in [0,1]$,

$$D_0 \in \mathcal{L}\left(\mathcal{D}(A_1^{\alpha/2}), \mathbf{H}^{2\alpha}(\mathcal{F})\right).$$

It is clear that

$$\mathcal{D}(A_0) = \left\{[w,\xi_1,\xi_2] \in \mathbf{H}^2(\mathcal{F}) \times \mathcal{D}(A_1) \times \mathcal{D}(A_1^{1/2})\ ;\ w - D_0\xi_2 \in \mathbf{V}_0^2(\Omega)\right\}$$

and that

$$\mathcal{H} = \left\{[w,\xi_1,\xi_2] \in \mathbf{L}^2(\mathcal{F}) \times \mathcal{D}(A_1^{1/2}) \times \mathcal{H}_{\mathcal{S}}\ ;\ w - D_0\xi_2 \in \mathbf{V}_n^0(\mathcal{F})\right\}.$$

More precisely, $[w,\xi_1,\xi_2] \mapsto [w - D_0\xi_2, \xi_1, \xi_2]$ is an isomorphism from $\mathcal{D}(A_0)$ onto $\mathbf{V}_0^2(\Omega) \times \mathcal{D}(A_1) \times \mathcal{D}(A_1^{1/2})$ as well as from $\mathcal{H}$ onto $\mathbf{V}_n^0(\Omega) \times \mathcal{D}(A_1^{1/2}) \times \mathcal{H}_{\mathcal{S}}$. We deduce by interpolation that for all $\alpha \in [0,1]$:

$$[\mathcal{D}(A_0),\mathcal{H}]_{1-\alpha} = \left\{[w,\xi_1,\xi_2] \in \mathcal{H}\ ;\ (\xi_1,\xi_2) \in \mathcal{D}(A_1^{1/2+\alpha/2}) \times \mathcal{D}(A_1^{\alpha/2}),\ w - D_0\xi_2 \in [\mathbf{V}_0^2(\mathcal{F}), \mathbf{V}_n^0(\mathcal{F})]_{1-\alpha}\right\}.$$

Then the conclusion follows from (3.37), from

$$[\mathbf{V}_0^2(\mathcal{F}), \mathbf{V}_n^0(\mathcal{F})]_{1-\alpha} = [\mathbf{H}^2(\mathcal{F}) \cap \mathbf{H}_0^1(\mathcal{F}), \mathbf{L}^2(\mathcal{F})]_{1-\alpha} \cap \mathbf{V}_n^0(\mathcal{F})$$

(see [22]) and from the characterization of this last interpolation space (see [25]). □

Corollary 3.9. *The following continuous embedding holds:*

$$\mathcal{D}((-A_0)^\alpha)' \hookrightarrow \mathbf{H}^{2\alpha}(\mathcal{F})' \times \mathcal{D}(A_1^{1/2-\alpha/2}) \times \mathcal{D}(A_1^{\alpha/2})' \quad \textit{if } \alpha \in [0,1/4). \tag{3.40}$$

Proof. First, from (3.38) we deduce that for $\alpha \in [0,1/4)$,

$$\mathbb{P} \in \mathcal{L}(\mathbf{H}^{2\alpha}(\mathcal{F}) \times \mathcal{D}(A_1^{1/2+\alpha/2}) \times \mathcal{D}(A_1^{\alpha/2}), \mathcal{D}((-A_0)^\alpha)).$$

Then for $\mathbf{X} \in \mathcal{H}$ and $\mathbf{Y} \in \mathbf{L}^2(\mathcal{F}) \times \mathcal{D}(A_1^{1/2}) \times \mathcal{H}_S$ we have

$$\langle \mathbf{X}, \mathbf{Y} \rangle = \langle \mathbf{X}, \mathbb{P}\mathbf{Y} \rangle \leqslant \|\mathbf{X}\|_{\mathcal{D}((-A_0)^\alpha)'} \|\mathbb{P}\mathbf{Y}\|_{\mathcal{D}((-A_0)^\alpha)} \leqslant C\|\mathbf{X}\|_{\mathcal{D}((-A_0)^\alpha)'} \|\mathbf{Y}\|_{\mathbf{H}^{2\alpha}(\mathcal{F}) \times \mathcal{D}(A_1^{1/2+\alpha/2}) \times \mathcal{D}(A_1^{\alpha/2})}.$$

Then it follows,

$$\|\mathbf{X}\|_{\mathbf{H}^{2\alpha}(\mathcal{F})' \times \mathcal{D}(A_1^{1/2-\alpha/2}) \times \mathcal{D}(A_1^{\alpha/2})'} \leqslant C\|\mathbf{X}\|_{\mathcal{D}((-A_0)^\alpha)'},$$

and we conclude with a density argument. □

We recall a classical result for analytic semigroups (see [32, Thm 5.2, p.61])

Lemma 3.10. *Assume A is the infinitesimal generator of a strongly continuous semigroup on $\mathcal{H}$ with an exponential growth lower or equal to zero i.e. $\sup_{t\geqslant 0} \|e^{tA}\| < +\infty$. If $i\mathbb{R} \subset \rho(A)$ and if there exists $C_0 > 0$ such that,*

$$\left\|(i\tau - A)^{-1}\right\|_{\mathcal{L}(\mathcal{H})} \leqslant \frac{C_0}{|\tau|} \quad (\tau \in \mathbb{R}^*), \tag{3.41}$$

then (e^{tA}) is an analytic semigroup on $\mathcal{H}$.

We recall the proof of this lemma for sake of completeness.

Proof. First we have $0 \in \rho(A)$. Assume $\lambda \in \mathbb{C}$. We write

$$(\lambda - A) = (i\Im\lambda - A)\left(\operatorname{Id} + \Re\lambda(i\Im\lambda - A)^{-1}\right).$$

Assume $\Im\lambda \neq 0$. Then from (3.41)

$$\|(i\Im\lambda - A)^{-1}\|_{\mathcal{L}(\mathcal{H})} \leqslant \frac{C_0}{|\Im\lambda|}.$$

Therefore, if

$$|\Re\lambda| \leqslant \alpha\frac{|\Im\lambda|}{C_0} \tag{3.42}$$

for some $\alpha \in (0,1)$, then $\left(\operatorname{Id} + \Re\lambda(i\Im\lambda - A)^{-1}\right)$ is invertible and

$$\left\|\left(\operatorname{Id} + \Re\lambda(i\Im\lambda - A)^{-1}\right)^{-1}\right\|_{\mathcal{L}(\mathcal{H})} \leqslant \frac{1}{1-\alpha}.$$

Thus any λ satisfying (3.42) belongs to $\rho(A)$ and satisfies

$$\|(\lambda - A)^{-1}\|_{\mathcal{L}(\mathcal{H})} \leqslant \frac{C}{|\lambda|}.$$

Since (e^{tA}) is a strongly continuous semigroup on $\mathcal{H}$ with an exponential growth lower or equal to zero, for λ such that $\Re\lambda > 0$, we have $\lambda \in \rho(A)$ and

$$\|(\lambda - A)^{-1}\|_{\mathcal{L}(\mathcal{H})} \leqslant \frac{C}{\Re\lambda},$$

see [11, Thm 2.5 p 101]. If moreover,

$$\Re\lambda \geqslant \alpha\frac{|\Im\lambda|}{C_0},$$

then we deduce

$$\|(\lambda - A)^{-1}\|_{\mathcal{L}(\mathcal{H})} \leqslant \frac{C}{|\lambda|}.$$

Thus there exists $\delta \in (0, \pi/2)$ such that

$$\rho(A) \supset \Sigma \stackrel{\text{def}}{=} \left\{ \lambda \in \mathbb{C} \ ; \ |\arg \lambda| < \frac{\pi}{2} + \delta \right\} \cup \{0\}$$

and

$$\|(\lambda - A)^{-1}\|_{\mathcal{L}(\mathcal{H})} \leqslant \frac{C}{|\lambda|} \quad (\lambda \in \Sigma \setminus \{0\}).$$

Applying [32, Thm 5.2, p.61], we deduce that (e^{tA}) is an analytic semigroup. □

Proposition 3.11. *The operator A_0 defined by* (3.19), (3.20), (3.21) *is the infinitesimal generator of an analytic semigroup on $\mathcal{H}$.*

Proof. We apply Lemma 3.10. We already know from Proposition 3.5 that $i\mathbb{R} \subset \rho(A_0)$. For $\tau \in \mathbb{R}^*$, we consider the equation $(i\tau - A_0)\mathbf{Z} = \mathbf{F} \in \mathcal{H}$. Setting $\mathbf{Z} = [w, \xi_1, \xi_2]$ and $\mathbf{F} = [f, g, h]$ we can write

$$\begin{aligned} i\tau w - \nu \Delta w + \nabla q &= f \quad \text{ in } \mathcal{F}, \\ \operatorname{div} w &= 0 \quad \text{ in } \mathcal{F}, \\ w &= T\xi_2 \quad \text{ on } \partial\mathcal{F}, \\ i\tau \xi_1 - \xi_2 &= g, \\ i\tau\xi_2 + A_2\xi_2 + A_1\xi_1 &= -T^*(\mathbb{T}(w,q)n) + h. \end{aligned} \tag{3.43}$$

Multiplying the first equation by $\overline{w}$ and performing an integration by parts we obtain:

$$i\tau \left(\int_{\mathcal{F}} |w|^2 dy + \|\xi_2\|^2_{\mathcal{H}_{\mathcal{S}}} \right) + 2\nu \int_{\mathcal{F}} |Dw|^2 \ dy + \|A_2^{1/2}\xi_2\|^2_{\mathcal{H}_{\mathcal{S}}} + \Big(A_1\xi_1, \xi_2\Big)_{\mathcal{H}_{\mathcal{S}}} = \int_{\mathcal{F}} f \cdot \overline{w} \ dy + \Big(h, \xi_2\Big)_{\mathcal{H}_{\mathcal{S}}}. \tag{3.44}$$

Next, using $\xi_2 = i\tau\xi_1 - g$ we deduce

$$\begin{aligned} i\tau \left(\int_{\mathcal{F}} |w|^2 dy + \|\xi_2\|^2_{\mathcal{H}_{\mathcal{S}}} - \|A_1^{1/2}\xi_1\|^2_{\mathcal{H}_{\mathcal{S}}} \right) + 2\nu \int_{\mathcal{F}} |Dw|^2 \ dy + \|A_2^{1/2}\xi_2\|^2_{\mathcal{H}_{\mathcal{S}}} \\ = \int_{\mathcal{F}} f \cdot \overline{w} \ dy + \Big(h, \xi_2\Big)_{\mathcal{H}_{\mathcal{S}}} + \Big(A_1^{1/2}\xi_1, A_1^{1/2} g\Big)_{\mathcal{H}_{\mathcal{S}}}. \end{aligned} \tag{3.45}$$

Then multiplying by τ and taking the imaginary part of the above equation first gives:

$$|\tau|^2 \|[w, \xi_1, \xi_2]\|^2_{\mathcal{H}} = 2|\tau|^2 \|A_1^{1/2}\xi_1\|^2_{\mathcal{H}_{\mathcal{S}}} + \Im\langle [f,g,h], \tau[w,\xi_1,\xi_2]\rangle_{\mathcal{H}},$$

and with the Cauchy-Schwarz inequality, we obtain:

$$|\tau|^2 \|[w, \xi_1, \xi_2]\|^2_{\mathcal{H}} \leqslant 4|\tau|^2 \|A_1^{1/2}\xi_1\|^2_{\mathcal{H}_{\mathcal{S}}} + \|[f,g,h]\|^2_{\mathcal{H}}. \tag{3.46}$$

We now consider the equation of the structure (the last two equations of (3.43)): since the dissipation (the term $A_2\xi_2$) is sufficient, the corresponding system is parabolic. More precisely, since A_2 is a positive, densely defined, self-adjoint operator on $\mathcal{H}_{\mathcal{S}}$ with (3.5), Theorem 1.1 in [17] guarantees that

$$|\tau|(\|A_1^{1/2}\xi_1\|_{\mathcal{H}_{\mathcal{S}}} + \|\xi_2\|_{\mathcal{H}_{\mathcal{S}}}) \leqslant C(\|T^*(\mathbb{T}(w,q)n)\|_{\mathcal{H}_{\mathcal{S}}} + \|A_1^{1/2} g\|_{\mathcal{H}_{\mathcal{S}}} + \|h\|_{\mathcal{H}_{\mathcal{S}}}). \tag{3.47}$$

Then by combining (3.46), (3.47) and the boundedness of $T^* : \mathbf{L}^2(\partial\mathcal{F}) \to \mathcal{H}_{\mathcal{S}}$ we deduce that:

$$|\tau| \|[w, \xi_1, \xi_2]\|_{\mathcal{H}} \leqslant C \left(\|\mathbb{T}(w,p)n\|_{\mathbf{L}^2(\partial\mathcal{F})} + \|[f,g,h]\|_{\mathcal{H}} \right). \tag{3.48}$$

In order to remove the term $\|\mathbb{T}(w,p)n\|_{\mathbf{L}^2(\partial\mathcal{F})}$ in the above estimate, we first use the trace theorem and regularity results for the Stokes system, for $\epsilon \in (0, 1/4)$:

$$\|\mathbb{T}(w,p)n\|_{\mathbf{L}^2(\partial\mathcal{F})} \leqslant C(\|\operatorname{div} \mathbb{T}(w,p)\|_{(\mathbf{H}^{2\epsilon}(\mathcal{F}))'} + \|T\xi_2\|_{\mathbf{V}^{2-2\epsilon}(\partial\mathcal{F})})$$

and then with the first equality in (3.43) and the boundedness of $T : \mathcal{D}(A_1^{1/2-\epsilon/2}) \to \mathbf{V}^{2-2\epsilon}(\partial\mathcal{F})$ we get

$$\|\mathbb{T}(w,p)n\|_{\mathbf{L}^2(\partial\mathcal{F})} \leqslant C \left(\|f\|_{\mathbf{L}^2(\mathcal{F})} + |\tau| \|w\|_{(\mathbf{H}^{2\epsilon}(\mathcal{F}))'} + \|\xi_2\|_{\mathcal{D}(A_1^{1/2-\epsilon/2})} \right). \tag{3.49}$$

Combining the above relation with $\xi_2 = i\tau\xi_1 - g$ we deduce

$$\|\mathbb{T}(w,p)n\|_{\mathbf{L}^2(\partial\mathcal{F})} \leqslant C\left(\|f\|_{\mathbf{L}^2(\mathcal{F})} + \|g\|_{\mathcal{D}(A_1^{1/2})} + |\tau|\|w\|_{(\mathbf{H}^{2\epsilon}(\mathcal{F}))'} + |\tau|\|\xi_1\|_{\mathcal{D}(A_1^{1/2-\epsilon/2})}\right). \tag{3.50}$$

Now let us prove that for $\epsilon \in (0,1/4)$,

$$\|w\|_{(\mathbf{H}^{2\epsilon}(\mathcal{F}))'} + \|\xi_1\|_{\mathcal{D}\left(A_1^{1/2-\epsilon/2}\right)} \leqslant C\|[w,\xi_1,\xi_2]\|_{(\mathcal{D}((-A_0)^{\epsilon}))'}. \tag{3.51}$$

Assume $(\varphi,\zeta_1) \in \mathbf{H}^{2\epsilon}(\mathcal{F}) \times \mathcal{D}(A_1^{1/2+\epsilon/2})$ with $\epsilon \in (0,1/4)$. Using Proposition 3.2 and Proposition 3.8, we have $\mathbb{P}[\varphi,\zeta_1,0] \in \mathcal{D}((-A_0)^{\epsilon})$. Then we can write

$$\begin{aligned}
\left|\int_{\mathcal{F}} w\cdot\varphi\, dy + \left(A_1^{1/2}\xi_1, A_1^{1/2}\zeta_1\right)_{\mathcal{H}_{\mathcal{S}}}\right| &= \left|\langle [w,\xi_1,\xi_2],[\varphi,\zeta_1,0]\rangle_{\mathbf{L}^2(\mathcal{F})\times\mathcal{D}(A_1^{1/2})\times\mathcal{H}_{\mathcal{S}}}\right| \\
&= |\langle [w,\xi_1,\xi_2],\mathbb{P}[\varphi,\zeta_1,0]\rangle_{\mathcal{H}}| \\
&\leqslant \|[w,\xi_1,\xi_2]\|_{(\mathcal{D}((-A_0)^{\epsilon}))'}\|\mathbb{P}[\varphi,\zeta_1,0]\|_{\mathcal{D}((-A_0)^{\epsilon}))} \\
&\leqslant C\|[w,\xi_1,\xi_2]\|_{(\mathcal{D}((-A_0)^{\epsilon}))'}\left(\|\varphi\|_{\mathbf{H}^{2\epsilon}(\mathcal{F})} + \|\zeta_1\|_{\mathcal{D}(A_1^{1/2+\epsilon/2})}\right).
\end{aligned}$$

Consequently, we deduce (3.51) and combining it with (3.50) yields

$$\|\mathbb{T}(w,p)n\|_{\mathbf{L}^2(\partial\mathcal{F})} \leqslant C(\|[f,g,h]\|_{\mathcal{H}} + |\tau|\|(-A_0)^{-\epsilon}[w,\xi_1,\xi_2]\|_{\mathcal{H}}).$$

The above relation and (3.48) imply

$$|\tau|\|[w,\xi_1,\xi_2]\|_{\mathcal{H}} \leqslant C(|\tau|\|(-A_0)^{-\epsilon}[w,\xi_1,\xi_2]\|_{\mathcal{H}} + \|[f,g,h]\|_{\mathcal{H}}).$$

Recalling $(i\tau - A_0)\mathbf{Z} = \mathbf{F}$, this can be written

$$\|i\tau(i\tau - A_0)^{-1}\mathbf{F}\|_{\mathcal{H}} \leqslant C(\|i\tau(-A_0)^{-\epsilon}(i\tau - A_0)^{-1}\mathbf{F}\|_{\mathcal{H}} + \|\mathbf{F}\|_{\mathcal{H}}). \tag{3.52}$$

Thus, remarking that $(-A_0)^{-\epsilon}(i\tau - A_0)^{-1}\mathbf{F} = (i\tau - A_0)^{-1}(-A_0)^{-\epsilon}\mathbf{F}$, by using (3.52) with $(-A_0)^{-\epsilon}\mathbf{F}$ instead of $\mathbf{F}$, we deduce that

$$\|i\tau(i\tau - A_0)^{-1}\mathbf{F}\|_{\mathcal{H}} \leqslant C(\|i\tau(-A_0)^{-2\epsilon}(i\tau - A_0)^{-1}\mathbf{F}\|_{\mathcal{H}} + \|\mathbf{F}\|_{\mathcal{H}}).$$

Then by iterating the argument we finally prove that for all $n \in \mathbb{N}^*$ there exists $C_n > 0$ such that

$$\|i\tau(i\tau - A_0)^{-1}\mathbf{F}\|_{\mathcal{H}} \leqslant C_n(\|i\tau(-A_0)^{-n\epsilon}(i\tau - A_0)^{-1}\mathbf{F}\|_{\mathcal{H}} + \|\mathbf{F}\|_{\mathcal{H}}),$$

and for $n \geqslant 1/\epsilon$, the above relation with $i\tau(i\tau - A_0)^{-1}\mathbf{F} = A_0(i\tau - A_0)^{-1}\mathbf{F} + \mathbf{F}$ and (3.26) finally yields $\|i\tau(i\tau - A_0)^{-1}\mathbf{F}\|_{\mathcal{H}} \leqslant C\|\mathbf{F}\|_{\mathcal{H}}$ which gives the result. ☐

3.3 The operator A

Now we define the operator A of our system:

$$\mathcal{D}(A) = \mathcal{V} \cap \left[\mathbf{H}^2(\mathcal{F}) \times \mathcal{D}(A_1) \times \mathcal{D}(A_1^{1/2})\right], \tag{3.53}$$

and for $[w,\xi_1,\xi_2] \in \mathcal{D}(A)$, we set

$$\widetilde{A}\begin{bmatrix} w \\ \xi_1 \\ \xi_2 \end{bmatrix} \stackrel{\text{def}}{=} \begin{bmatrix} \nu\Delta w + \operatorname{div}\Lambda^{(1)}(\xi_1,\xi_2) - \Lambda^{(2)}(\xi_1,\xi_2) - (v^S\cdot\nabla)w - (w\cdot\nabla)v^S \\ \xi_2 \\ -A_1\xi_1 - A_2\xi_2 - T^*(2\nu D(w)n + \Lambda^{(1)}(\xi_1,\xi_2)n) \end{bmatrix} \tag{3.54}$$

and

$$A \stackrel{\text{def}}{=} \mathbb{P}\widetilde{A}, \tag{3.55}$$

where $\mathbb{P} : \mathbf{L}^2(\mathcal{F}) \times \mathcal{D}(A_1^{1/2}) \times \mathcal{H}_{\mathcal{S}} \to \mathcal{H}$ is the orthogonal projection operator.

Proposition 3.12. *The operator A defined by* (3.53), (3.54), *and* (3.55) *is densely defined with compact resolvent, and it is the infinitesimal generator of an analytic semigroup on $\mathcal{H}$. Moreover, the adjoint of the operator A is given by $\mathcal{D}(A^*) = \mathcal{D}(A)$ and*

$$A^* \begin{bmatrix} \varphi \\ \zeta_1 \\ \zeta_2 \end{bmatrix} = \mathbb{P} \begin{bmatrix} \nu\Delta\varphi + (v^S \cdot \nabla)\varphi - (\nabla v^S)^*\varphi \\ -\zeta_2 - A_1^{-1}\left(\Lambda^{(1,1)}\right)^*(\nabla\varphi) - A_1^{-1}\left(\Lambda^{(2,1)}\right)^*(\varphi) \\ A_1\zeta_1 - A_2\zeta_2 - T^*\left(2\nu D(\varphi)n\right) - \left(\Lambda^{(1,2)}\right)^*(\nabla\varphi) - \left(\Lambda^{(2,2)}\right)^*(\varphi) \end{bmatrix}. \tag{3.56}$$

Proof. We write

$$A = A_0 + A_\flat$$

where A_0 is defined by (3.19), (3.20), (3.21) and is the infinitesimal generator of an analytic semigroup on $\mathcal{H}$ (Proposition 3.11). Relations (2.59), (2.60) and (1.34) combined with Proposition 3.8 yield that $\mathcal{D}(A_\flat) \subset \mathcal{D}((-A_0)^{1/2})$. Using a perturbation argument (see, for instance, [32, Corollary 2.4, p. 81]), we deduce the first part of the Proposition.

To characterize the adjoint of A, assume $\left[w, \xi_1, \xi_2\right], \left[\varphi, \zeta_1, \zeta_2\right] \in \mathcal{D}(A)$ and we observe that:

$$\begin{aligned} \langle A\left[w, \xi_1, \xi_2\right], [\varphi, \zeta_1, \zeta_2]\rangle = \langle [w, \xi_1, \xi_2], A_0^*\left[\varphi, \zeta_1, \zeta_2\right]\rangle - \left(T^*(\Lambda^{(1)}(\xi_1, \xi_2)n), \zeta_2\right)_{\mathcal{H}_S} \\ + \int_{\mathcal{F}} \left(\operatorname{div}\Lambda^{(1)}(\xi_1, \xi_2) - \Lambda^{(2)}(\xi_1, \xi_2) - (v^S \cdot \nabla)w - (w \cdot \nabla)v^S\right) \cdot \varphi \, dy. \end{aligned} \tag{3.57}$$

Thus

$$\begin{aligned} \langle A\left[w, \xi_1, \xi_2\right], [\varphi, \zeta_1, \zeta_2]\rangle = \langle [w, \xi_1, \xi_2], A_0^*\left[\varphi, \zeta_1, \zeta_2\right]\rangle - \int_{\mathcal{F}} \Lambda^{(1)}(\xi_1, \xi_2) : \nabla\varphi \, dy - \int_{\mathcal{F}} \Lambda^{(2)}(\xi_1, \xi_2) \cdot \varphi \, dy \\ + \int_{\mathcal{F}} \left((v^S \cdot \nabla)\varphi - (\nabla v^S)^*\varphi\right) \cdot w \, dy. \end{aligned} \tag{3.58}$$

Here we have used that $v^S = 0$ on Γ_{str} and $\varphi = 0$ on Γ_0. Using (2.61) we deduce the result. ☐

Let us fix $\lambda_0 > 0$ large enough so that $\lambda_0 - A$ is positive and $(\lambda_0 - A)^\alpha$ is well defined for $\alpha \in (0,1)$. We deduce from Proposition 3.12 and similarly as for Proposition 3.8 the following result.

Proposition 3.13. *For $\alpha \in [0,1]$, the following equalities hold*

$$\mathcal{D}((\lambda_0 - A)^\alpha) = [\mathcal{D}(A), \mathcal{H}]_{1-\alpha} = [\mathcal{D}(A^*), \mathcal{H}]_{1-\alpha} = \mathcal{D}((\lambda_0 - A^*)^\alpha), \tag{3.59}$$

In particular, $\mathcal{D}((\lambda_0 - A)^\alpha) = \mathcal{D}((\lambda_0 - A^*)^\alpha) = \mathcal{D}((-A_0)^\alpha)$ is characterized by (3.38) *and* (3.39).

Proof. Equalities (3.59) are consequences of $\mathcal{D}(A) = \mathcal{D}(A^*)$ and of the maximal accretivity of $\lambda - A$ for $\lambda > 0$ large enough, see [11, Prop. 6.1, p170]. Note that to obtain $\langle(\lambda - A)\mathbf{Z}, \mathbf{Z}\rangle \geqslant 0$ we have to control the terms coming from $\Lambda^{(1)}$, $\Lambda^{(2)}$ and this can be done by using (2.59), (2.60) and (2.63). In particular we use the fact that $\Lambda^{(2,2)} \in \mathcal{L}(\mathcal{H}_S, (\mathbf{H}^1(\mathcal{F}))')$ which follows from (2.59) and (2.63) with an interpolation argument. ☐

3.4 The operator B

Next, we introduce the Dirichlet operator $D_{\mathcal{F}} : \mathbf{V}^0(\partial\mathcal{F}) \to \mathbf{L}^2(\mathcal{F}) \times \mathcal{D}(A_1^{1/2}) \times \mathcal{H}_S$ defined as follows: for $u \in \mathbf{V}^0(\partial\mathcal{F})$ we denote by $D_{\mathcal{F}}u \overset{\text{def}}{=} [w_u \ \xi_{1,u} \ \xi_{2,u}]$ the unique solution of

$$\left\{\begin{array}{rcll} \lambda_0 w - \nu\Delta w - \operatorname{div}\Lambda^{(1)}(\xi_1, \xi_2) + \Lambda^{(2)}(\xi_1, \xi_2) & & & \\ +(v^S \cdot \nabla)w + (w \cdot \nabla)v^S + \nabla q & = & 0 & \text{in } \mathcal{F}, \\ \operatorname{div} w & = & 0 & \text{in } \mathcal{F}, \\ w & = & T\xi_2 + \Xi(u) & \text{on } \partial\mathcal{F}, \\ \lambda_0\xi_1 - \xi_2 & = & 0, & \\ \lambda_0\xi_2 + A_1\xi_1 + A_2\xi_2 + T^*(\mathbb{T}(w, q)n + \Lambda^{(1)}(\xi_1, \xi_2)n) & = & 0. & \end{array}\right. \tag{3.60}$$

Proposition 3.14. *The mapping $D_{\mathcal{F}}$ defined above satisfies the following boundedness property:*

$$D_{\mathcal{F}} \in \mathcal{L}(\mathbf{V}^s(\partial\mathcal{F}), \mathbf{H}^{s+\frac{1}{2}}(\mathcal{F}) \times \mathcal{D}(A_1^{s/4+5/8}) \times \mathcal{D}(A_1^{s/4+1/8})) \quad s \in \left[-\frac{1}{2}, \frac{1}{2}\right]. \tag{3.61}$$

Proof. To obtain (3.61) it suffices to prove it for $s = 1/2$ and $s = -1/2$ and then use an interpolation argument. We first consider the case $s = 1/2$ and use a lifting argument: according to [1, Cor. 3.8] there exists $z \in \mathbf{H}^1(\mathcal{F})$ such that $\operatorname{div} z = 0$ in $\mathcal{F}$, $z = \Xi u$ on $\partial\mathcal{F}$ and satisfying $\|z\|_{\mathbf{H}^1(\mathcal{F})} \leqslant C\|u\|_{\mathbf{V}^{\frac{1}{2}}(\partial\mathcal{F})}$. Then setting $w = \widetilde{w} + z$, we see that (3.60) writes

$$(\lambda_0 - A) \begin{bmatrix} \widetilde{w} \\ \xi_1 \\ \xi_1 \end{bmatrix} = \mathbb{P} \begin{bmatrix} -\lambda_0 z + \nu\Delta z - (v^S \cdot \nabla)z - (z \cdot \nabla)v^S \\ 0 \\ -T^*(2\nu D(z)n) \end{bmatrix} \in \mathcal{V}'. \tag{3.62}$$

By definition of λ_0, there exists a unique $[\widetilde{w}, \xi_1, \xi_2] \in \mathcal{V} \subset \mathbf{H}^1(\mathcal{F}) \times D(A_1^{3/4}) \times D(A_1^{1/4})$ solution of (3.62).

To prove the case $s = -1/2$, we recall that in that case $D_{\mathcal{F}}u$ is defined by duality as follows: for any $[f, a_f, b_f] \in \mathbf{L}^2(\mathcal{F}) \times \mathcal{D}(A_1^{1/2}) \times \mathcal{H}_{\mathcal{S}}$,

$$\langle D_{\mathcal{F}}u, [f, a_f, b_f]\rangle \overset{\text{def}}{=} -\langle u, \Xi(\mathbb{T}(\varphi, \pi)n)\rangle_{\mathbf{V}^{-1/2}(\partial\mathcal{F}), \mathbf{V}^{1/2}(\partial\mathcal{F})}, \tag{3.63}$$

where $[\varphi, \xi, \zeta] \in \mathcal{D}(A^*)$ and $\pi \in H^1(\mathcal{F})$ such that $\int_{\mathcal{F}} \pi dy = 0$ satisfy

$$\begin{cases} \lambda_0\varphi - \nu\Delta\varphi + (\nabla v^S)^*\varphi - (v^S \cdot \nabla)\varphi + \nabla\pi &= f \quad \text{in } \mathcal{F}, \\ \operatorname{div}\varphi &= 0 \quad \text{in } \mathcal{F} \\ \varphi &= T\zeta_2 \quad \text{on } \partial\mathcal{F}, \\ \lambda_0\zeta_1 + \zeta_2 + A_1^{-1}\left(\Lambda^{(1,1)}\right)^*(\nabla\varphi) + A_1^{-1}\left(\Lambda^{(2,1)}\right)^*(\varphi) &= a_f, \\ \lambda_0\zeta_2 - A_1\zeta_1 + A_2\zeta_2 + T^*(\mathbb{T}(\varphi, \pi)n) + \left(\Lambda^{(1,2)}\right)^*(\nabla\varphi) + \left(\Lambda^{(2,2)}\right)^*(\varphi) &= b_f. \end{cases} \tag{3.64}$$

Using (3.56), we see that system (3.64) can be written as

$$(\lambda_0 - A^*)[\varphi, \zeta_1, \zeta_2] = \mathbb{P}[f, a_f, b_f].$$

Then for any $[f, a_f, b_f] \in \mathbf{L}^2(\mathcal{F}) \times \mathcal{D}(A_1^{1/2}) \times \mathcal{H}_{\mathcal{S}}$, there exists a unique $[\varphi, \zeta_1, \zeta_2] \in \mathcal{D}(A^*)$, as well as a corresponding pressure $\pi \in H^1(\mathcal{F})$ such that $\int_{\mathcal{F}} \pi dy = 0$, solution of the above equation. Moreover, we have from standard trace inequalities and Stokes regularity,

$$\|\mathbb{T}(\varphi, \pi)n\|_{\mathbf{V}^{\frac{1}{2}}(\partial\mathcal{F})} \leqslant C\|[f, a_f, b_f]\|_{\mathbf{L}^2(\mathcal{F}) \times \mathcal{D}(A_1^{1/2}) \times \mathcal{H}_{\mathcal{S}}}$$

which yields the result for $s = -\frac{1}{2}$.

□

Next, we define the input operator

$$B : \mathbf{V}^0(\partial\mathcal{F}) \to [\mathcal{D}(A^*)]', \quad Bu = (\lambda_0 - A)\mathbb{P}D_{\mathcal{F}}u. \tag{3.65}$$

Proposition 3.15. *The operator B defined by* (3.65) *satisfies:*

$$(\lambda_0 - A)^{-1+\epsilon}B \in \mathcal{L}(\mathbf{V}^0(\partial\mathcal{F}), \mathcal{H}), \quad \epsilon \in \left(0, \frac{1}{4}\right). \tag{3.66}$$

Moreover, the adjoint of B is defined by

$$B^*\left[\varphi, \zeta_1, \zeta_2\right] = -\Xi(\mathbb{T}(\varphi, \pi)n), \tag{3.67}$$

where $\pi \in H^1(\mathcal{F})$ satisfies $\int_{\mathcal{F}} \pi \ dy = 0$ and

$$\begin{bmatrix} \nabla\pi \\ 0 \\ -T^*(\pi n) \end{bmatrix} = -(I - \mathbb{P}) \begin{bmatrix} -\nu\Delta\varphi + (\nabla v^S)^*\varphi - (v^S \cdot \nabla)\varphi \\ \zeta_2 + A_1^{-1}\left(\Lambda^{(1,1)}\right)^*(\nabla\varphi) + A_1^{-1}\left(\Lambda^{(2,1)}\right)^*(\varphi) \\ -A_1\zeta_1 + A_2\zeta_2 + T^*(2\nu D(\varphi)n) + \left(\Lambda^{(1,2)}\right)^*(\nabla\varphi) + \left(\Lambda^{(2,2)}\right)^*(\varphi) \end{bmatrix}. \tag{3.68}$$

Proof. The regularity property (3.66) is a direct consequence of Proposition 3.2, Proposition 3.13 and of (3.61). The characterization (3.67) follows directly from (3.63). $\square$

We are now in position to deduce formulation (3.12)-(3.13) from (3.6)-(3.10) and (3.11). First, since we have $\mathbf{Z} - D_{\mathcal{F}} u \in \mathcal{H}$ we have $(I - \mathbb{P})(\mathbf{Z} - D_{\mathcal{F}} u) = 0$ which implies that $\mathbf{Z}$ satisfies (3.13). Moreover, assume that $\mathbf{Z} = [w, \xi_1, \xi_2]$ is a regular solution of (3.6)-(3.10) and assume that $\mathbf{\Phi} = [\varphi, \zeta_1, \zeta_2] \in \mathcal{D}(A^*)$. We multiply (3.6) with φ, (3.9) by ζ_1 and (3.10) by ζ_2. Then, after some integrating by parts and using (3.13), we obtain

$$\frac{d}{dt} \langle \mathbb{P}\mathbf{Z}(t), \mathbf{\Phi} \rangle - \langle \mathbb{P}\mathbf{Z}(t), A^* \mathbf{\Phi} \rangle = \langle u(t), B^* \mathbf{\Phi} \rangle. \tag{3.69}$$

Since the above equality is satisfied for any $\mathbf{\Phi} \in \mathcal{D}(A^*)$ we obtain the first equality of (3.12).

4 Feedback Stabilizability of the linear system

4.1 Stabilizability of the homogeneous linear system

The goal of this subsection is to prove, for a fixed rate of decrease $\sigma > 0$, the existence of a feedback control

$$u(t) = \sum_{j=1}^{N_\sigma} \left(\int_{\mathcal{F}} w(t) \cdot \varphi_j dy + \left(A_1^{1/2} \xi_1(t), A_1^{1/2} \zeta_{1,j} \right)_{\mathcal{H}_{\mathcal{S}}} + \left(\xi_2(t), \zeta_{2,j} \right)_{\mathcal{H}_{\mathcal{S}}} \right) v_j \tag{4.1}$$

such that solutions of (3.6)-(3.10) tends to zero as $t \to +\infty$ with an exponential rate of decrease $\sigma > 0$. For that, we are going to show the existence of families $(\varphi_j, \zeta_{1,j}, \zeta_{2,j})$ and v_j, $j = 1, \ldots, N_\sigma$ such that the underlying closed-loop linear operator of (3.6)-(3.10) with (4.1) generates and analytic and exponentially stable semigroup of type lower than $-\sigma$ (see [11, II-1, (2.8) and Cor. 2.1]). It then permits to deduce results for nonhomogeneous system (3.1)-(3.4) that are used in the next subsection to construct solutions of the nonlinear system (2.48)-(2.51) with a fixed-point argument.

Proposition 4.1. *For $\sigma > 0$, there exist $N_\sigma \in \mathbb{N}^*$ and families $[\varphi_j, \zeta_{1,j}, \zeta_{2,j}] \in \mathcal{D}(A^*)$ and $v_j \in \mathbf{V}^2(\partial\mathcal{F})$, $j = 1, \ldots, N_\sigma$, and a corresponding feedback operator $F_\sigma : \mathcal{H} \to \mathbf{V}^2(\partial\mathcal{F})$ defined by*

$$F_\sigma[w, \xi_1, \xi_2] = \sum_{j=1}^{N_\sigma} \left(\int_{\mathcal{F}} w \cdot \varphi_j dy + \left(A_1^{1/2} \xi_1, A_1^{1/2} \zeta_{1,j} \right)_{\mathcal{H}_{\mathcal{S}}} + \left(\xi_2, \zeta_{2,j} \right)_{\mathcal{H}_{\mathcal{S}}} \right) v_j \tag{4.2}$$

such that the linear operator $A_\sigma \overset{def}{=} A + BF_\sigma$ with domain $\mathcal{D}(A_\sigma) \overset{def}{=} \{\mathbf{Z} \in \mathcal{H} \mid A\mathbf{Z} + BF_\sigma \mathbf{Z} \in \mathcal{H}\}$ is the infinitesimal generator of an analytic and exponentially stable semigroup on $\mathcal{H}$ of type lower than $-\sigma$. Moreover, for $\alpha \in [0, 1]$ we have $\mathcal{D}((-A_\sigma)^\alpha) \hookrightarrow [\mathbf{H}^{2\alpha}(\mathcal{F}) \times \mathcal{D}(A_1^{1/2+\alpha/2}) \times \mathcal{D}(A_1^{\alpha/2})] \cap \mathcal{H}$ and $\mathcal{D}((-A_\sigma^*)^\alpha) = \mathcal{D}((\lambda_0 - A^*)^\alpha)$, and for $\alpha \in [0, 1/4)$ we have $\mathcal{D}((-A_\sigma^*)^\alpha)') \hookrightarrow \mathbf{H}^{2\alpha}(\mathcal{F})' \times \mathcal{D}(A_1^{1/2-\alpha/2}) \times \mathcal{D}(A_1^{\alpha/2})'$.

Proof. The proof of the above proposition relies on the Hautus-Fattorini stabilizability criterion, see [5, Theorem 1] or [8]. Since A has compact resolvent and generates an analytic semigroup on $\mathcal{H}$, and since B is relatively bounded with respect to A, then the homogeneous linear system is stabilizable by finite dimensional feedback control for any rate of decrease if and only if the following criterion is satisfied for all $\lambda \in \mathbb{C}$:

$$\lambda \mathbf{\Phi} - A^* \mathbf{\Phi} = 0 \quad \text{and} \quad B^* \mathbf{\Phi} = 0 \quad \Longrightarrow \quad \mathbf{\Phi} = 0. \tag{4.3}$$

Assume $\mathbf{\Phi} = [\varphi, \zeta_1, \zeta_2] \in \mathcal{D}(A^*)$ satisfies the two first relations (4.3). From (3.56) and (3.67), it implies that

$$\left\{ \begin{aligned} \lambda\varphi - \nu\Delta\varphi + (\nabla v^S)^* \varphi - (v^S \cdot \nabla)\varphi + \nabla\pi &= 0 \quad \text{in } \mathcal{F}, \\ \operatorname{div} \varphi &= 0 \quad \text{in } \mathcal{F}, \\ \varphi &= T\zeta_2 \quad \text{on } \partial\mathcal{F}, \\ \lambda\zeta_1 + \zeta_2 + A_1^{-1} \left(\Lambda^{(1,1)} \right)^* (\nabla\varphi) + A_1^{-1} \left(\Lambda^{(2,1)} \right)^* (\varphi) &= 0, \\ \lambda\zeta_2 - A_1\zeta_1 + A_2\zeta_2 + T^*(\mathbb{T}(\varphi, \pi)n) + \left(\Lambda^{(1,2)} \right)^* (\nabla\varphi) + \left(\Lambda^{(2,2)} \right)^* (\varphi) &= 0, \end{aligned} \right. \tag{4.4}$$

and

$$\rho\mathbb{T}(\varphi,\pi)n = \left(\int_{\partial\mathcal{F}} \rho\mathbb{T}(\varphi,\pi)n\cdot n d\gamma\right)\rho n \quad \text{on } \partial\mathcal{F}. \tag{4.5}$$

In what follows we denote $c(\varphi,\pi) \stackrel{\text{def}}{=} \int_{\partial\mathcal{F}} \rho\mathbb{T}(\varphi,\pi)n\cdot n d\gamma$. Combining (4.5) and the classical uniqueness result of [21] for Stokes type systems (see also [8, Appendix A]) we deduce that $\varphi = 0$ and $\pi - c(\varphi,\pi) = 0$ in $\mathcal{F}$. It implies in particular that $\mathbb{T}(\varphi,\pi - c(\varphi,\pi))n = 0$ on $\partial\mathcal{F}$ and $T\zeta_2 = 0$ on $\partial\mathcal{F}$. Using (2.55), we deduce that $\zeta_2 = 0$. Moreover, since we have $T^*(n) = 0$, from $\mathbb{T}(\varphi,\pi - c(\varphi,\pi))n = 0$ we deduce $T^*(\mathbb{T}(\varphi,\pi)n) = 0$ on $\partial\mathcal{F}$. Then using the last equation of (4.4) we obtain $A_1\zeta_1 = 0$ and then $\zeta_1 = 0$. We have obtained $\mathbf{\Phi} = 0$.

Then the general framework of [5, 8] can be applied and for a given $\sigma > 0$, there exist families

$$[\varphi_j, \zeta_{1,j}, \zeta_{2,j}] \in \mathcal{D}(A^*)$$

and $v_j \in \mathbf{V}^0(\partial\mathcal{F})$, $j = 1,\dots,N_\sigma$, and a feedback law of the form (4.2) such that the conclusions of the proposition hold. Moreover, each v_j can be chosen in $\mathbf{V}^2(\partial\mathcal{F})$. This comes from the fact that the set of admissible families (v_j) is a nonempty open set of $(\mathbf{V}^0(\partial\mathcal{F}))^{N_\sigma}$ (see [5, Theorem 5] or [8, Theorem 6]). Indeed, if a family $(\tilde{v}_j)$ is admissible then all families in a neighborhood of $(\tilde{v}_j)$ in $(\mathbf{V}^0(\partial\mathcal{F}))^{N_\sigma}$ are admissible. Then the conclusion follows from the density of $\mathbf{V}^2(\partial\mathcal{F})$ in $\mathbf{V}^0(\partial\mathcal{F})$.

Finally, the statements concerning $\mathcal{D}((-A_\sigma)^\alpha)$ and $\mathcal{D}((-A_\sigma^*)^\alpha)$ are obtained as in [6] and the statement concerning $\mathcal{D}((-A_\sigma^*)^\alpha)'$ then follows from (3.40). □

Remark 4.2. *From the definition* (4.2) *we can extend F_σ to an operator from $\mathbf{L}^2(\mathcal{F}) \times \mathcal{D}(A_1^{1/2}) \times \mathcal{H}_\mathcal{S}$ to $\mathbf{V}^2(\partial\mathcal{F})$ by using the same formula* (4.2). *Moreover, since $[\varphi_j, \zeta_{1,j}, \zeta_{2,j}] \in \mathcal{H}$, it yields that $F_\sigma = F_\sigma\mathbb{P}$ and that $F_\sigma = 0$ on $\mathcal{H}^\perp$. We can also extend F_σ as an operator from $\mathcal{D}(A_\sigma^*)'$ to $\mathbf{V}^2(\partial\mathcal{F})$ by setting*

$$F_\sigma[w,\xi_1,\xi_2] = \sum_{j=1}^{N_\sigma} \langle [w,\xi_1,\xi_2], [\varphi_j,\zeta_{1,j},\zeta_{2,j}]\rangle_{\mathcal{D}(A_\sigma^*)',\mathcal{D}(A_\sigma^*)}\, v_j.$$

4.2 Stabilizability of the non homogeneous linear system

The goal of this section is to obtain regularity results for the following nonhomogeneous linear system:

$$\begin{aligned}
&\partial_t w - \operatorname{div}\mathbb{T}(w,p) - \operatorname{div}\Lambda^{(1)}(\xi_1,\xi_2) + \Lambda^{(2)}(\xi_1,\xi_2) \\
&\qquad + (v^S\cdot\nabla)w + (w\cdot\nabla)v^S = F - \operatorname{div} G \quad \text{in } (0,+\infty)\times\mathcal{F}, &(4.6)\\
&\operatorname{div} w = F_{\text{div}} \quad \text{in } (0,+\infty)\times\mathcal{F}, &(4.7)\\
&w = T\xi_2 + \Xi(F_\sigma([w,\xi_1,\xi_2])) + F_b \quad \text{on } (0,+\infty)\times\partial\mathcal{F}, &(4.8)\\
&\partial_t\xi_1 = \xi_2, \quad t\in(0,+\infty), &(4.9)\\
&\partial_t\xi_2 + A_2\xi_2 + A_1\xi_1 = -T^*\left(\mathbb{T}(w,p)n + \Lambda^{(1)}(\xi_1,\xi_2)n\right) + T^*Gn, \quad t\in(0,+\infty), &(4.10)
\end{aligned}$$

with the initial conditions

$$\xi_1(0) = \xi_1^0 \quad \text{and} \quad \xi_2(0) = \xi_2^0, \quad w(0) = w^0 \quad \text{in } \mathcal{F}. \tag{4.11}$$

In above settings we have extended the feedback operator F_σ to $\mathbf{L}^2(\mathcal{F}) \times \mathcal{D}(A_1^{1/2}) \times \mathcal{H}_\mathcal{S}$ (see Remark 4.2) and F, G, F_{div}, F_b are nonhomogenous right-hand terms which play the role of the nonlinearities $F(\mathbf{Z})$, $G(\mathbf{Z})$, $F_{\text{div}}(\mathbf{Z})$, $F_b(\mathbf{Z})$ in (2.48)-(2.51).

Suppose for the moment that $(F_{\text{div}}, F_b) = (0,0)$. By taking into account (4.1) in formulation (3.12)-(3.13) complemented with the nonhomogeneous right-hand terms F, G, we deduce that the above system with $F_{\text{div}} = 0$ and $F_b = 0$ can be rewritten as

$$\begin{aligned}
\mathbb{P}\mathbf{Z}' &= A_\sigma\mathbb{P}\mathbf{Z} + \mathbb{P}(F - \operatorname{div} G) \ \text{ in } \mathcal{D}(A^*)',\ \mathbb{P}\mathbf{Z}(0) = \mathbb{P}\mathbf{Z}^0 &(4.12)\\
(I-\mathbb{P})\mathbf{Z} &= (I-\mathbb{P})D_\mathcal{F}F_\sigma\mathbb{P}\mathbf{Z}. &(4.13)
\end{aligned}$$

Here we have used that F_σ vanishes on $\mathcal{H}^\perp$. The notation $F - \operatorname{div} G$ means here the operator

$$\langle F - \operatorname{div} G, [\varphi,\zeta_1,\zeta_2]\rangle_{\mathcal{D}(A^*)',\mathcal{D}(A^*)} \stackrel{\text{def}}{=} \langle F,\varphi\rangle_{\mathbf{H}^1(\mathcal{F})',\mathbf{H}^1(\mathcal{F})} + \int_\mathcal{F} G : \nabla\varphi\, dy \quad ([\varphi,\zeta_1,\zeta_2] \in \mathcal{D}(A^*)). \tag{4.14}$$

In what follows, we recall that we use the notation (2.1). We have the following result.

Proposition 4.3. *Assume* $[w^0, \xi_1^0, \xi_2^0] \in \mathbf{L}^2(\mathcal{F}) \times \mathcal{D}(A_1^{1/2}) \times \mathcal{H}_S$*,* $F \in L^2_\sigma(\mathbf{H}^1(\mathcal{F})')$*,* $G \in L^2_\sigma(\mathbf{L}^2(\mathcal{F}))$ *and* $(F_{\mathrm{div}}, F_b) = (0,0)$*. Then system* (4.6)-(4.11) *admits a unique solution*

$$[w, \xi_1, \xi_2] \in W_\sigma(\mathcal{D}((-A_\sigma)^{1/2}), \mathcal{D}((-A_\sigma^*)^{1/2})') + H^1_\sigma(\mathbf{H}^1(\mathcal{F}) \times \mathcal{D}(A_1^{3/4}) \times \mathcal{D}(A_1^{1/4}))$$

and we have

$$\|\mathbb{P}[w, \xi_1, \xi_2]\|_{W_\sigma(\mathcal{D}((-A_\sigma)^{1/2}), \mathcal{D}((-A_\sigma^*)^{1/2})')} + \|(I - \mathbb{P})[w, \xi_1, \xi_2]\|_{H^1_\sigma(\mathbf{H}^1(\mathcal{F}) \times \mathcal{D}(A_1^{3/4}) \times \mathcal{D}(A_1^{1/4}))}$$
$$\leqslant C \left(\|[F, G]\|_{L^2_\sigma(\mathbf{H}^1(\mathcal{F})' \times \mathbf{L}^2(\mathcal{F}))} + \|\mathbb{P}[w^0, \xi_1^0, \xi_2^0]\|_{\mathcal{H}} \right).$$

Proof. We write system (4.6)-(4.11) as (4.12)-(4.13). By using (4.14) we have

$$\mathbb{P}(F - \operatorname{div} G) \in L^2_\sigma(\mathcal{D}((-A_\sigma^*)^{1/2})'). \tag{4.15}$$

Since A_σ generates an analytic semigroup on $\mathcal{H}$, from maximal regularity results applied to equation (4.12), we deduce from (4.15) and $\mathbb{P}\mathbf{Z}^0 \in \mathcal{H}$ that $\mathbb{P}\mathbf{Z} \in W_\sigma(\mathcal{D}((-A_\sigma)^{1/2}), \mathcal{D}((-A_\sigma^*)^{1/2})')$. Finally, from the definition (4.2) of the operator F_σ, from Proposition 3.14 and from Proposition 3.2, equality (4.13) yields $(I - \mathbb{P})\mathbf{Z} \in H^1_\sigma(\mathbf{H}^1(\mathcal{F}) \times \mathcal{D}(A_1^{3/4}) \times \mathcal{D}(A_1^{1/4}))$. □

Let us now consider the case of non zero nonhomegeneous terms F_{div} and F_b. For that we need to introduce a lifting operator for the divergence condition which is compatible with the feedback condition, namely we set $L_{\mathrm{div}}[g, h] = [w, \xi_1, 0]$ with (w, ξ_1) satisfying

$$-\operatorname{div} \mathbb{T}(w, p) - \operatorname{div} \Lambda^{(1)}(\xi_1, 0) + \Lambda^{(2)}(\xi_1, 0) + (v^S \cdot \nabla)w + (w \cdot \nabla)v^S = 0 \quad \text{in } \mathcal{F}, \tag{4.16}$$
$$\operatorname{div} w = h \quad \text{in } \mathcal{F}, \tag{4.17}$$
$$w = \Xi(F_\sigma([w, \xi_1, 0])) + g \quad \text{on } \partial\mathcal{F}, \tag{4.18}$$
$$A_1 \xi_1 = -T^* \left(\mathbb{T}(w, p)n + \Lambda^{(1)}(\xi_1, 0)n \right). \tag{4.19}$$

To state regularity properties for L_{div} we need the functional framework introduced in [39]. For $s \in [-1/2, 2]$ we define

$$\mathbf{H}^{s,\sigma}_{\partial\mathcal{F},\mathcal{F}} \overset{\text{def}}{=} \left\{ (g, h) \in \mathbf{H}^s(\partial\mathcal{F}) \times H^\sigma(\mathcal{F}) \ ; \ \langle g \cdot n, 1 \rangle_{H^s(\partial\mathcal{F}), H^{-s}(\partial\mathcal{F})} = \int_{\mathcal{F}} h \, dy \right\} \quad \text{if } \sigma \geqslant 0,$$
$$\mathbf{H}^{s,\sigma}_{\partial\mathcal{F},\mathcal{F}} \overset{\text{def}}{=} \left\{ (g, h) \in \mathbf{H}^s(\partial\mathcal{F}) \times H^{-\sigma}(\mathcal{F})' \ ; \ \langle g \cdot n, 1 \rangle_{H^s(\partial\mathcal{F}), H^{-s}(\partial\mathcal{F})} = \langle h, 1 \rangle_{H^{-\sigma}(\mathcal{F})', H^{-\sigma}(\mathcal{F})} \right\} \quad \text{if } \sigma < 0.$$

In what follows, we need another assumption than the ones introduced in Section 3.1: for some $\varepsilon \in (0, 1/8)$:

$$\forall \xi \in \mathcal{D}(A_1^{1/2+\varepsilon/2}), \quad \|A_1^{\varepsilon/2} A_2 \xi\|_{\mathcal{H}_S} \leqslant \|A_1^{1/2+\varepsilon/2} \xi\|_{\mathcal{H}_S}. \tag{4.20}$$

Inequality (4.20) is only needed to prove (4.21) for $s \in [-2\varepsilon, 0)$ in Proposition 4.4 below. Note that the operators A_1, A_2 defined by (1.16), (1.17) satisfy the above condition. It is an easy consequence of (2.54).

Proposition 4.4. *Let* $\varepsilon \in (0, 1/8)$ *be given in* (4.20)*. The mapping* L_{div} *defined above satisfies:*

$$L_{\mathrm{div}} \in \mathcal{L}(\mathbf{H}^{s-1/2, s-1}_{\partial\mathcal{F},\mathcal{F}}, \mathbf{H}^s(\mathcal{F}) \times \mathcal{D}(A_1^{1/2+s/4}) \times \{0\}) \quad s \in [-2\varepsilon, 2]. \tag{4.21}$$

Proof. This existence and properties of L_{div} are obtained by a duality argument. First we consider $[f, \zeta_1^f, \zeta_2^f] \in \mathbf{L}^2(\mathcal{F}) \times \mathcal{D}(A_1^{1/2}) \times \mathcal{H}_S$. From Proposition 4.1, there exists a unique solution $[\varphi, \zeta_1, \zeta_2] \in \mathcal{D}(A^*)$ of

$$-A_\sigma^*[\varphi, \zeta_1, \zeta_2] = [f, \zeta_1^f, \zeta_2^f].$$

Moreover we have the estimate

$$\|[\varphi, \zeta_1, \zeta_2]\|_{\mathbf{H}^2(\mathcal{F}) \times \mathcal{D}(A_1) \times \mathcal{D}(A_1^{1/2})} \leqslant C \|\mathbb{P}[f, \zeta_1^f, \zeta_2^f]\|_{\mathcal{H}}. \tag{4.22}$$

From (3.56) and (4.2), it means that we have the existence and uniqueness of the solution of

$$\left\{\begin{array}{rcl}
-\nu\Delta\varphi+(\nabla v^S)^*\varphi-(v^S\cdot\nabla)\varphi+\nabla\chi & = & \displaystyle\sum_{j=1}^{N_\sigma}\varphi_j\int_{\partial\mathcal{F}} v_j\cdot B^*[\varphi,\zeta_1,\zeta_2]d\gamma+f \quad \text{in } \mathcal{F},\\
\operatorname{div}\varphi & = & 0 \quad \text{in } \mathcal{F},\\
\varphi & = & T\zeta_2 \quad \text{on } \partial\mathcal{F},\\
\zeta_2+A_1^{-1}\left(\Lambda^{(1,1)}\right)^*(\nabla\varphi)+A_1^{-1}\left(\Lambda^{(2,1)}\right)^*(\varphi) & = & \displaystyle\sum_{j=1}^{N_\sigma}\zeta_{1,j}\int_{\partial\mathcal{F}} v_j\cdot B^*[\varphi,\zeta_1,\zeta_2]d\gamma+\zeta_1^f,\\
-A_1\zeta_1+A_2\zeta_2+\left(\Lambda^{(1,2)}\right)^*(\nabla\varphi)+\left(\Lambda^{(2,2)}\right)^*(\varphi) & & \\
=-T^*(\mathbb{T}(\varphi,\chi)n) & + & \displaystyle\sum_{j=1}^{N_\sigma}\zeta_{2,j}\int_{\partial\mathcal{F}} v_j\cdot B^*[\varphi,\zeta_1,\zeta_2]d\gamma+\zeta_2^f.
\end{array}\right. \tag{4.23}$$

Using that $[\varphi_j,\zeta_{1,j},\zeta_{2,j}]\in\mathcal{D}(A^*)$ and Proposition 3.1, we deduce that the pressure χ satisfies

$$\begin{bmatrix}\nabla\chi\\ 0\\ -T^*(\chi n)\end{bmatrix}=(I-\mathbb{P})\begin{bmatrix}\nu\Delta\varphi-(\nabla v^S)^*\varphi+(v^S\cdot\nabla)\varphi+f\\ -\zeta_2-A_1^{-1}\left(\Lambda^{(1,1)}\right)^*(\nabla\varphi)-A_1^{-1}\left(\Lambda^{(2,1)}\right)^*(\varphi)+\zeta_1^f\\ A_1\zeta_1-A_2\zeta_2-T^*(2\nu(D\varphi)n)-\left(\Lambda^{(1,2)}\right)^*(\nabla\varphi)-\left(\Lambda^{(2,2)}\right)^*(\varphi)+\zeta_2^f\end{bmatrix}. \tag{4.24}$$

We can assume $\chi\in L^2_0(\mathcal{F})$ and in that case, using the Poincaré-Wirtinger inequality, Lemma 2.4 and (4.22), we obtain

$$\|\chi\|_{H^1(\mathcal{F})}\leqslant C\|[f,\zeta_1^f,\zeta_2^f]\|_{\mathbf{L}^2(\mathcal{F})\times\mathcal{D}(A_1^{1/2})\times\mathcal{H}_S}.$$

Note that χ can be decomposed as $\chi=\pi+p^f$ with $\pi,p^f\in H^1(\mathcal{F})\cap L^2_0(\mathcal{F})$ defined by

$$\begin{bmatrix}\nabla\pi\\ 0\\ -T^*(\pi n)\end{bmatrix}=(I-\mathbb{P})\begin{bmatrix}\nu\Delta\varphi-(\nabla v^S)^*\varphi+(v^S\cdot\nabla)\varphi\\ -\zeta_2-A_1^{-1}\left(\Lambda^{(1,1)}\right)^*(\nabla\varphi)-A_1^{-1}\left(\Lambda^{(2,1)}\right)^*(\varphi)\\ A_1\zeta_1-A_2\zeta_2-T^*(2\nu(D\varphi)n)-\left(\Lambda^{(1,2)}\right)^*(\nabla\varphi)-\left(\Lambda^{(2,2)}\right)^*(\varphi)\end{bmatrix}, \tag{4.25}$$

and

$$\begin{bmatrix}\nabla p^f\\ 0\\ -T^*(p^f n)\end{bmatrix}=(I-\mathbb{P})\begin{bmatrix}f\\ \zeta_1^f\\ \zeta_2^f\end{bmatrix}. \tag{4.26}$$

Now, let us assume that

$$[f,\zeta_1^f,\zeta_2^f]\in\left[\mathbf{H}^{2\varepsilon}(\mathcal{F})\times\mathcal{D}(A_1^{1/2+\varepsilon/2})\times\mathcal{D}(A_1^{\varepsilon/2})\right]. \tag{4.27}$$

From (2.56) and (4.22), combined with the above assumption and standard elliptic regularity for the Stokes system, we deduce

$$\begin{aligned}\|\varphi\|_{\mathbf{H}^{2+2\varepsilon}(\mathcal{F})}+\|\chi\|_{H^{1+2\varepsilon}(\mathcal{F})}&\leqslant C(\|f\|_{\mathbf{H}^{2\varepsilon}(\mathcal{F})}+\|B^*[\varphi,\zeta_1,\zeta_2]\|_{\mathbf{V}^0(\mathcal{F})}+\|T\zeta_2\|_{\mathbf{H}^{3/2+2\varepsilon}(\partial\mathcal{F})})\\ &\leqslant C(\|f\|_{\mathbf{H}^{2\varepsilon}(\mathcal{F})}+\|[f,\zeta_1^f,\zeta_2^f]\|_{\mathbf{L}^2(\mathcal{F})\times\mathcal{D}(A_1^{1/2})\times\mathcal{H}_S}).\end{aligned}$$

Using the forth equation of (4.23) and (2.62), we deduce that $\zeta_2\in\mathcal{D}(A_1^{1/2+\varepsilon/2})$. Then combining (2.63), (4.20) and (2.58) and the above regularity for (φ,χ,ζ_2), we deduce that $\zeta_1\in\mathcal{D}(A_1^{1+\varepsilon/2})$ with the estimate

$$\|\chi\|_{H^{1+2\varepsilon}(\mathcal{F})}+\|[\varphi,\zeta_1,\zeta_2]\|_{\mathbf{H}^{2+2\varepsilon}(\mathcal{F})\times\mathcal{D}(A_1^{1+\varepsilon/2})\times\mathcal{D}(A_1^{1/2+\varepsilon/2})}\leqslant C\|[f,\zeta_1^f,\zeta_2^f]\|_{\mathbf{H}^{2\varepsilon}(\mathcal{F})\times\mathcal{D}(A_1^{1/2+\varepsilon/2})\times\mathcal{D}(A_1^{\varepsilon/2})}. \tag{4.28}$$

We can now prove the well-posedness of (4.16)-(4.19) by a duality argument. First we rewrite this system as

$$-\operatorname{div}\mathbb{T}(w,p)-\operatorname{div}\Lambda^{(1)}(\xi_1,\xi_2)+\Lambda^{(2)}(\xi_1,\xi_2)+(v^S\cdot\nabla)w+(w\cdot\nabla)v^S=0\quad\text{in }\mathcal{F},\tag{4.29}$$

$$\operatorname{div}w=h\quad\text{in }\mathcal{F},\tag{4.30}$$

$$w=T\xi_2+\Xi(F_\sigma([w,\xi_1,\xi_2]))+g\quad\text{on }\partial\mathcal{F},\tag{4.31}$$

$$-\xi_2=0,\tag{4.32}$$

$$A_2\xi_2+A_1\xi_1=-T^*\left(\mathbb{T}(w,p)n+\Lambda^{(1)}(\xi_1,\xi_2)n\right).\tag{4.33}$$

Assume now that $[w,\xi_1,\xi_2]$ is a regular solution of the above system and $[\varphi,\zeta_1,\zeta_2]\in\mathcal{D}(A^*)$ is the solution (4.23). We multiply the first equation of (4.23) by w and (4.29) by φ. After some calculation, we obtain

$$\langle[w,\xi_1,\xi_2],[f,\zeta_1^f,\zeta_2^f]\rangle=-\int_{\mathcal{F}}\chi h dy-\int_{\partial\mathcal{F}}(\mathbb{T}(\varphi,\chi)n\cdot g\,d\gamma-\int_{\partial\mathcal{F}}(\Xi(\mathbb{T}(\varphi,\chi)n)+B^*[\varphi,\zeta_1,\zeta_2])\cdot F_\sigma[w,\xi_1,\xi_2]\,d\gamma.\tag{4.34}$$

From (3.67), (3.68) and (4.25), we have

$$B^*[\varphi,\zeta_1,\zeta_2]=-\Xi(\mathbb{T}(\varphi,\pi)n).$$

Combining the two above relations with the fact that $\int_{\mathcal{F}}h dy=\int_{\partial\mathcal{F}}g\cdot n d\gamma$ leads to

$$\begin{aligned}\langle[w,\xi_1,\xi_2],[f,\zeta_1^f,\zeta_2^f]\rangle=-\int_{\mathcal{F}}(\chi-k(\varphi,\chi))h dy-\int_{\partial\mathcal{F}}(\mathbb{T}(\varphi,\chi)n+k(\varphi,\chi)n)\cdot g\,d\gamma\\ -\int_{\partial\mathcal{F}}p^f n\cdot\Xi(F_\sigma\mathbb{P}[w,\xi_1,\xi_2])\,d\gamma,\end{aligned}\tag{4.35}$$

for any $[f,\zeta_1^f,\zeta_2^f]$ satisfying (4.27) and where

$$k(\varphi,\chi)\stackrel{\text{def}}{=}\frac{1}{|\partial\mathcal{F}|+|\mathcal{F}|}\left(\int_{\mathcal{F}}\chi\,dy-\int_{\partial\mathcal{F}}\mathbb{T}(\varphi,\chi)n\cdot n\,d\gamma\right).$$

In order to prove the existence and uniqueness of $[w,\xi_1,\xi_2]\in(\mathbf{H}^{2\varepsilon}(\mathcal{F}))'\times\mathcal{D}(A_1^{1/2-\varepsilon/2})\times\mathcal{D}(A_1^{\varepsilon/2})'$ satisfying (4.35), we proceed as follows: first there exists a unique $[\widehat{w},\widehat{\xi}_1,\widehat{\xi}_2]\in\mathcal{D}(A_0^{\varepsilon/2})'$ such that for any $[f,\zeta_1^f,\zeta_2^f]\in\mathcal{D}(A_0^{\varepsilon/2})$,

$$\begin{aligned}&\langle[\widehat{w},\widehat{\xi}_1,\widehat{\xi}_2],[f,\zeta_1^f,\zeta_2^f]\rangle_{\mathcal{D}(A_0^{\varepsilon/2})',\mathcal{D}(A_0^{\varepsilon/2})}\\&\quad=-\left\langle(\mathbb{T}(\varphi,\chi)n+k(\varphi,\chi)n,\chi-k(\varphi,\chi)),(g,h)\right\rangle_{\mathbf{H}^{1/2+2\varepsilon,1+2\varepsilon}_{\partial\mathcal{F},\mathcal{F}},\mathbf{H}^{-1/2-2\varepsilon,-1-2\varepsilon}_{\partial\mathcal{F},\mathcal{F}}},\end{aligned}\tag{4.36}$$

where $(\varphi,\chi,\zeta_1,\zeta_2)$ is the solution of (4.23) associated with $[f,\zeta_1^f,\zeta_2^f]$. The existence and uniqueness for this problem is a consequence of (4.28).

Second, there exists a unique $[w,\xi_1,\xi_2]\in(\mathbf{H}^{2\varepsilon}(\mathcal{F}))'\times\mathcal{D}(A_1^{1/2-\varepsilon/2})\times\mathcal{D}(A_1^{\varepsilon/2})'$ such that for any

$$[f,\zeta_1^f,\zeta_2^f]\in\mathbf{H}^{2\varepsilon}(\mathcal{F})\times\mathcal{D}(A_1^{1/2+\varepsilon/2})\times\mathcal{D}(A_1^{\varepsilon/2}),$$

$$\begin{aligned}&\left\langle[w,\xi_1,\xi_2],[f,\zeta_1^f,\zeta_2^f]\right\rangle_{(\mathbf{H}^{2\varepsilon}(\mathcal{F}))'\times\mathcal{D}(A_1^{1/2-\varepsilon/2})\times(\mathcal{D}(A_1^{\varepsilon/2}))',\mathbf{H}^{2\varepsilon}(\mathcal{F})\times\mathcal{D}(A_1^{1/2+\varepsilon/2})\times\mathcal{D}(A_1^{\varepsilon/2})}\\&\quad=-\left\langle(\mathbb{T}(\varphi,\chi)n+k(\varphi,\chi)n,\chi-k(\varphi,\chi)),(g,h)\right\rangle_{\mathbf{H}^{1/2+2\varepsilon,1+2\varepsilon}_{\partial\mathcal{F},\mathcal{F}},\mathbf{H}^{-1/2-2\varepsilon,-1-2\varepsilon}_{\partial\mathcal{F},\mathcal{F}}}\\&\qquad-\int_{\partial\mathcal{F}}p^f n\cdot\Xi(F_\sigma[\widehat{w},\widehat{\xi}_1,\widehat{\xi}_2])\,d\gamma\end{aligned}\tag{4.37}$$

where $(\varphi,\chi,\zeta_1,\zeta_2)$ is the solution of (4.23) associated with $[f,\zeta_1^f,\zeta_2^f]$ and p_f is defined by (4.26). The existence and uniqueness for this problem is a consequence of (4.28). Taking $[f,\zeta_1^f,\zeta_2^f]\in\mathcal{D}(A_0^{\varepsilon/2})$ in (4.37) yields that $p^f=0$ (see (4.26)) and thus that

$$\mathbb{P}[w,\xi_1,\xi_2]=[\widehat{w},\widehat{\xi}_1,\widehat{\xi}_2].$$

Consequently, $[w, \xi_1, \xi_2]$ is the solution of (4.35) and we can define

$$L_{\mathrm{div}} : \mathbf{H}^{-1/2-2\varepsilon,-1-2\varepsilon}_{\partial\mathcal{F},\mathcal{F}} \to (\mathbf{H}^{2\varepsilon}(\mathcal{F}))' \times \mathcal{D}(A_1^{1/2-\varepsilon/2}) \times \mathcal{H}_S$$

as $L_{\mathrm{div}}(g,h) \stackrel{\text{def}}{=} [w, \xi_1, \xi_2]$ for any $(g,h) \in \mathbf{H}^{-1/2-2\varepsilon,-1-2\varepsilon}_{\partial\mathcal{F},\mathcal{F}}$, where $[w, \xi_1, \xi_2]$ is the solution of (4.35). Using (4.28), from (4.37)-(4.36) we deduce that,

$$\|L_{\mathrm{div}}(g,h)\|_{(\mathbf{H}^{2\varepsilon}(\mathcal{F}))' \times \mathcal{D}(A_1^{1/2-\varepsilon/2})} \leqslant C \|(g,h)\|_{\mathbf{H}^{-1/2-2\varepsilon,-1-2\varepsilon}_{\partial\mathcal{F},\mathcal{F}}}. \tag{4.38}$$

Finally, assume $\zeta_1 \in \mathcal{D}(A_1^{1+\varepsilon/2})$ and set

$$[f, \zeta_1^f, \zeta_2^f] = -F_\sigma^* B^*[0, \zeta_1, 0] + [0, 0, -A_1\zeta_1].$$

Then, the corresponding solution of (4.23) is $\varphi = 0$, $\zeta_2 = 0$ and $\chi = 0$. In that case (see (4.25)),

$$B^*[0, \zeta_1, 0] = \Xi(\pi n), \quad \text{where} \quad \begin{bmatrix} \nabla\pi \\ 0 \\ -T^*(\pi n) \end{bmatrix} = (I - \mathbb{P}) \begin{bmatrix} 0 \\ 0 \\ A_1\zeta_1 \end{bmatrix},$$

and since $[\varphi_j, \zeta_{1,j}, \zeta_{2,j}] \in \mathcal{D}(A^*)$, $p^f = -\pi$ (see (4.26)). Consequently, (4.37) reduces to

$$\langle \xi_2, A_1\zeta_1 \rangle = 0.$$

Since the above relation holds for any $\zeta_1 \in \mathcal{D}(A_1^{1+\varepsilon/2})$, a density argument implies $\xi_2 = 0$.

Now let us prove (4.21) in the case $s = 2$. For that, we assume $(g,h) \in \mathbf{H}^{3/2,1}_{\partial\mathcal{F},\mathcal{F}}$ and we use the elliptic regularity for the Stokes system with nonhomogeneous divergence and boundary conditions. More precisely, by performing the above calculations in the case $\varepsilon = 0$ we first obtain $(w, \xi_1) \in \mathbf{L}^2(\mathcal{F}) \times \mathcal{D}(A_1^{1/2})$, and from (2.59) we have in particular $\Lambda^{(1)}(\xi_1, 0) \in (L^2(\mathcal{F}))^{2\times 2}$. Then Stokes regularity result applied to system (4.16)-(4.19) yields $w \in \mathbf{H}^1(\mathcal{F})$. We also have $\operatorname{div}(\mathbb{T}(w,p) + \Lambda^{(1)}(\xi_1, 0)) \in \mathbf{L}^2(\mathcal{F})$ which guarantees that $(\mathbb{T}(w,p) + \Lambda^{(1)}(\xi_1, 0))n \in \mathbf{V}^{-1/2}(\mathcal{F})$. Then (2.56) for $s = 1/8$ yields $T^*\left[(\mathbb{T}(w,p) + \Lambda^{(1)}(\xi_1, 0))n\right] \in \mathcal{D}(A_1^{1/8})'$ and from equation (4.19) we deduce $\xi_1 \in \mathcal{D}(A_1^{7/8})$. Finally, using again Stokes regularity results with the fact that $(w, \xi_1) \in \mathbf{H}^1(\mathcal{F}) \times \mathcal{D}(A_1^{3/4})$ yields (4.21) for $s = 2$.

The case $s \in (-2\varepsilon, 2)$ then follows by interpolation. □

Using the lifting operator L_{div}, system (4.6)–(4.10), (4.11) can be written as

$$\begin{aligned}
\mathbf{Z} &= \widehat{\mathbf{Z}} + L_{\mathrm{div}}(F_b, F_{\mathrm{div}}), && (4.39)\\
\mathbb{P}\widehat{\mathbf{Z}}' &= A_\sigma \mathbb{P}\widehat{\mathbf{Z}} + \mathbb{P}(F - \operatorname{div} G) - \mathbb{P}L_{\mathrm{div}}(F_b, F_{\mathrm{div}})' \ \text{ in } \mathcal{D}(A^*)', && (4.40)\\
\mathbb{P}\widehat{\mathbf{Z}}(0) &= \mathbb{P}(\mathbf{Z}^0 - L_{\mathrm{div}}(F_b(0), F_{\mathrm{div}}(0))), && (4.41)\\
(I - \mathbb{P})\widehat{\mathbf{Z}} &= (I - \mathbb{P}) D_{\mathcal{F}} F_\sigma \mathbb{P}\widehat{\mathbf{Z}}. && (4.42)
\end{aligned}$$

In order to analyze (4.40), we need the following proposition.

Proposition 4.5. *Assume $\mathbf{F} \in H_\sigma^{1/2+\delta}(\mathcal{D}((-A_\sigma^*)^\delta)')$ for some $\delta \in (0, 1/2)$. Then, the solution of*

$$\mathbf{W}' = A_\sigma \mathbf{W} + \mathbf{F}', \quad \mathbf{W}(0) = 0 \tag{4.43}$$

belongs to $H_\sigma^{1/2+\delta}(\mathcal{D}((-A_\sigma)^\delta)') \cap L^2_\sigma(\mathcal{D}((-A_\sigma)^{1/2}))$ and satisfies

$$\|\mathbf{W}\|_{H_\sigma^{1/2+\delta}(\mathcal{D}((-A_\sigma^*)^\delta)')} + \|\mathbf{W}\|_{L^2_\sigma(\mathcal{D}((-A_\sigma)^{1/2}))} \leqslant C \|\mathbf{F}\|_{H_\sigma^{1/2+\delta}(\mathcal{D}((-A_\sigma^*)^\delta)')}. \tag{4.44}$$

Proof. First, since $A_\sigma - \sigma I$ is of negative type and is the infinitesimal generator of an analytic semigroup, for any $\mathbf{F} \in H^1_\sigma(\mathcal{D}((-A_\sigma^*)^\delta)')$ the solution of (4.43) satisfies

$$\|\mathbf{W}\|_{L^2_\sigma(\mathcal{D}((-A_\sigma)^{1-\delta}))} + \|\mathbf{W}\|_{H^1_\sigma(\mathcal{D}((-A_\sigma^*)^\delta)')} \leqslant C \|\mathbf{F}\|_{H^1_\sigma(\mathcal{D}((-A_\sigma^*)^\delta)')}. \tag{4.45}$$

Second, assume $\mathbf{F} \in L^2_\sigma(\mathcal{D}((-A_\sigma^*)^\delta)')$. There exists a sequence $(\mathbf{F}_n)$ in $C^1_c(\mathcal{H})$ that converges to $\mathbf{F}$ in $L^2_\sigma(\mathcal{D}((-A_\sigma^*)^\delta)')$. We remark that the solution $\mathbf{W}_n$ of (4.43) corresponding to $\mathbf{F}_n$ satisfies

$$(\mathbf{W}_n - \mathbf{F}_n)' = A_\sigma(\mathbf{W}_n - \mathbf{F}_n) + A_\sigma \mathbf{F}_n, \quad (\mathbf{W}_n - \mathbf{F}_n)(0) = 0,$$

and using again the maximal regularity results, we find

$$\|\mathbf{W}_n - \mathbf{F}_n\|_{L^2_\sigma(\mathcal{D}((-A_\sigma)^\delta)')} \leqslant C\|A_\sigma \mathbf{F}_n\|_{L^2_\sigma(\mathcal{D}((-A^*_\sigma)^{1+\delta})')} \leqslant C\|\mathbf{F}_n\|_{L^2_\sigma(\mathcal{D}((-A_\sigma)^\delta)')}.$$

Thus, passing to the limit as $n \to +\infty$ we obtain $\|\mathbf{W}\|_{L^2_\sigma(\mathcal{D}((-A_\sigma)^\delta)')} \leqslant C\|\mathbf{F}\|_{L^2_\sigma(\mathcal{D}((-A_\sigma)^\delta)')}$. Then we conclude with (4.45) and with an interpolation argument.

□

Next, for $\varepsilon \in (0, 1/8)$ given in (4.20) let us define the following functional spaces:

$$\mathcal{E} \overset{\text{def}}{=} L^2_\sigma(\mathbf{H}^1(\mathcal{F})') \times L^2_\sigma(\mathbf{L}^2(\mathcal{F})) \times \left[H^{1/2+\varepsilon}_\sigma(\mathbf{H}^{-1/2-2\varepsilon,-1-2\varepsilon}_{\partial\mathcal{F},\mathcal{F}}) \cap L^2_\sigma(\mathbf{H}^{1/2,0}_{\partial\mathcal{F},\mathcal{F}})\right], \tag{4.46}$$

$$\mathcal{G} \overset{\text{def}}{=} \left[H^{1/2+\varepsilon}_\sigma(\mathbf{H}^{2\varepsilon}(\mathcal{F})' \times \mathcal{D}(A_1^{1/2-\varepsilon/2}) \times \mathcal{D}(A_1^{\varepsilon/2})')\right] \cap \left[L^2_\sigma(\mathbf{H}^1(\mathcal{F}) \times \mathcal{D}(A_1^{3/4}) \times \mathcal{D}(A_1^{1/4}))\right]. \tag{4.47}$$

Notice that (2.2) yields the following continuous embedding:

$$\mathcal{G} \hookrightarrow \left[H^{1/2}_\sigma(\mathbf{L}^2(\mathcal{F}) \times \mathcal{D}(A_1^{1/2}) \times \mathcal{H}_S)\right] \cap \left[C_{b,\sigma}(\mathbf{L}^2(\mathcal{F}) \times \mathcal{D}(A_1^{1/2}) \times \mathcal{H}_S)\right]. \tag{4.48}$$

We are now in position to state the main result of this section.

Corollary 4.6. *Assume* $[w^0, \xi_1^0, \xi_2^0] \in \mathbf{L}^2(\mathcal{F}) \times \mathcal{D}(A_1^{1/2}) \times \mathcal{H}_S$ *and* $[F, G, F_b, F_{\mathrm{div}}] \in \mathcal{E}$. *Then system* (4.6)-(4.11) *admits a unique solution* $[w, \xi_1, \xi_2] \in \mathcal{G}$ *and we have*

$$\|[w, \xi_1, \xi_2]\|_{\mathcal{G}} \leqslant C\left(\|[F, G, F_b, F_{\mathrm{div}}]\|_{\mathcal{E}} + \|[w^0, \xi_1^0, \xi_2^0]\|_{\mathbf{L}^2(\mathcal{F}) \times \mathcal{D}(A_0^{1/2}) \times \mathcal{H}_S}\right). \tag{4.49}$$

Proof. We write system (4.6)-(4.11) as (4.39)-(4.42) with $\mathbf{Z} = [w, \xi_1, \xi_2]$. Since

$$[F_b, F_{\mathrm{div}}] \in H^{1/2+\varepsilon}_\sigma(\mathbf{H}^{-1/2-2\varepsilon,-1-2\varepsilon}_{\partial\mathcal{F},\mathcal{F}}) \cap L^2_\sigma(\mathbf{H}^{1/2,0}_{\partial\mathcal{F},\mathcal{F}})$$

we deduce from Proposition 4.4, that

$$L_{\mathrm{div}}(F_b, F_{\mathrm{div}}) \in \mathcal{G}.$$

Using Corollary 3.3 and (4.48) we deduce from the above relation

$$\mathbb{P}L_{\mathrm{div}}(F_b, F_{\mathrm{div}}) \in H^{1/2+\varepsilon}_\sigma(\mathbf{H}^{2\varepsilon}(\mathcal{F})' \times \mathcal{D}(A_1^{1/2-\varepsilon/2}) \times \mathcal{D}(A_1^{\varepsilon/2})') \cap C_{b,\sigma}(\mathcal{H}).$$

Using Proposition 4.1, Proposition 3.13 and (3.38) we deduce from the above relation,

$$\mathbb{P}L_{\mathrm{div}}(F_b, F_{\mathrm{div}}) \in H^{1/2+\varepsilon}_\sigma(\mathcal{D}((-A^*_\sigma)^\varepsilon)') \cap C_{b,\sigma}(\mathcal{H}). \tag{4.50}$$

From the hypotheses on the initial conditions, and from the above relation, we obtain

$$\mathbb{P}(\mathbf{Z}^0 - L_{\mathrm{div}}(F_b(0), F_{\mathrm{div}}(0)) \in \mathcal{H}, \tag{4.51}$$

where $\mathbf{Z}^0 \overset{\text{def}}{=} [w^0, \xi_1^0, \xi_2^0]$.

Gathering (4.50), (4.51) and applying Proposition 4.5 and Proposition 4.3 with the fact that

$$W_\sigma(\mathcal{D}((-A_\sigma)^{1/2}), \mathcal{D}((-A^*_\sigma)^{1/2})') \hookrightarrow H^{1/2+\varepsilon}_\sigma(\mathcal{D}((-A^*_\sigma)^\varepsilon)') \cap L^2_\sigma(\mathcal{D}((-A_\sigma)^{1/2}))$$

we deduce that

$$\mathbb{P}\widehat{\mathbf{Z}} \in H^{1/2+\varepsilon}_\sigma(\mathcal{D}((-A^*_\sigma)^\varepsilon)') \cap L^2_\sigma(\mathcal{D}((-A_\sigma)^{1/2})) \hookrightarrow \mathcal{G}.$$

We underline that the last above embedding is well justified by Proposition 4.1. In particular the embedding $\mathcal{D}((-A^*_\sigma)^\varepsilon)' \hookrightarrow \mathbf{H}^{2\varepsilon}(\mathcal{F})' \times \mathcal{D}(A_1^{1/2-\varepsilon/2}) \times \mathcal{D}(A_1^{\varepsilon/2})'$ is true since $\varepsilon < 1/4$.

We deduce from the above relation, from the definition (4.2) of the operator F_σ, from Proposition 3.14 and from Proposition 3.2 that

$$(I - \mathbb{P})D_{\mathcal{F}}F_\sigma\mathbb{P}\widehat{\mathbf{Z}} \in H^{1/2+\varepsilon}_\sigma(\mathbf{H}^1(\mathcal{F}) \times D(A_1^{1/2}) \times \mathcal{H}_S).$$

Combining the above relations, we deduce that

$$[w, \xi_1, \xi_2] = \mathbf{Z} = \mathbb{P}\widehat{\mathbf{Z}} + (I - \mathbb{P})\widehat{\mathbf{Z}} + L_{\mathrm{div}}(F_b, F_{\mathrm{div}}) \in \mathcal{G}$$

with the estimate (4.49).

□

5 Fixed point

5.1 Proof of Theorem 1.2

In order to prove Theorem 1.2, we consider the Banach spaces $\mathcal{E}$ and $\mathcal{G}$ defined by (4.46) and (4.47) and the following mapping defined on a closed ball of $\mathcal{E}$ of radius $R > 0$,

$$\Psi : B_{\mathcal{E}}(0,R) \to \mathcal{E}, \quad \left[F, G, F_b, F_{\mathrm{div}}\right] \mapsto \left[F(\mathbf{Z}), G(\mathbf{Z}), F_b(\mathbf{Z}), F_{\mathrm{div}}(\mathbf{Z})\right]$$

where $\mathbf{Z} = \left[w, \xi_1, \xi_2\right] \in \mathcal{G}$ is the solution of (4.6)-(4.11) given by Corollary 4.6 and where $F(\mathbf{Z})$, $G(\mathbf{Z})$, $F_b(\mathbf{Z})$, $F_{\mathrm{div}}(\mathbf{Z})$ are defined by (2.43), (2.25) and (2.27).

We remark that if $\left[F, G, F_b, F_{\mathrm{div}}\right]$ is a fixed point of the mapping Ψ, then the corresponding solution $\left[w, \xi_1, \xi_2\right]$ of (4.6)-(4.11) is a solution of (2.48)-(2.52). Consequently, we are reduced to show that Ψ admits a fixed point. We prove that for R small enough, Ψ is well-defined from $B_{\mathcal{E}}(0,R)$ onto itself and that the restriction of Ψ on this closed ball is a contraction mapping.

First, we notice that (4.49) implies (2.8) provided that $\left[F, G, F_b, F_{\mathrm{div}}\right] \in B_{\mathcal{E}}(0,R)$ with R small enough and that $[w^0, \xi_1^0, \xi_2^0]$ has a norm small enough in $\mathbf{L}^2(\mathcal{F}) \times \mathcal{D}(A_1^{1/2}) \times \mathcal{H}_S$. In particular, the changes of variables X and Y are well-defined as well as $F(\mathbf{Z})$, $G(\mathbf{Z})$, $F_b(\mathbf{Z})$, $F_{\mathrm{div}}(\mathbf{Z})$.

Second, we use several technical results whose proofs are given in the next subsections. To simplify the notation, in what follows, we assume

$$R + \|[w^0, \xi_1^0, \xi_2^0]\|_{\mathbf{L}^2(\mathcal{F}) \times \mathcal{D}(A_1^{1/2}) \times \mathcal{H}_S} \leqslant 1. \tag{5.1}$$

Proposition 5.1. *There exists $C_\# > 0$ such that for all $R > 0$ and $[w^0, \xi_1^0, \xi_2^0]$ satisfying* (5.1)*, and all $\left[F, G, F_{\mathrm{div}}, F_b\right] \in B_{\mathcal{E}}(0,R)$,*

$$\|\Psi(\left[F, G, F_b, F_{\mathrm{div}}\right])\|_{\mathcal{E}} \leqslant C_\# \left(R + \|[w^0, \xi_1^0, \xi_2^0]\|_{\mathbf{L}^2(\mathcal{F}) \times \mathcal{D}(A_1^{1/2}) \times \mathcal{H}_S}\right)^2.$$

From the above proposition, we remark that if

$$\|[w^0, \xi_1^0, \xi_2^0]\|_{\mathbf{L}^2(\mathcal{F}) \times \mathcal{D}(A_1^{1/2}) \times \mathcal{H}_S} \leqslant R, \tag{5.2}$$

and R is small enough so that

$$4 C_\# R \leqslant 1, \tag{5.3}$$

then Ψ is well-defined from $B_{\mathcal{E}}(0,R)$ into itself.

The second important technical result we need is the following:

Proposition 5.2. *There exists $C_\# > 0$ such that for all $R > 0$ and $[w^0, \xi_1^0, \xi_2^0]$ satisfying* (5.1)*, and all $\left[F^{(1)}, G^{(1)}, F_b^{(1)}, F_{\mathrm{div}}^{(1)}\right]$ and $\left[F^{(2)}, G^{(2)}, F_b^{(2)}, F_{\mathrm{div}}^{(2)}\right]$ in $B_{\mathcal{E}}(0,R)$,*

$$\begin{aligned} &\left\| \Psi\left(\left[F^{(1)}, G^{(1)}, F_b^{(1)}, F_{\mathrm{div}}^{(1)}\right]\right) - \Psi\left(\left[F^{(2)}, G^{(2)}, F_b^{(2)}, F_{\mathrm{div}}^{(2)}\right]\right)\right\|_{\mathcal{E}} \\ &\qquad \leqslant C_\# \left(R + \|[w^0, \xi_1^0, \xi_2^0]\|_{\mathbf{L}^2(\mathcal{F}) \times \mathcal{D}(A_0^{1/2}) \times \mathcal{H}_S}\right) \left\| \left[F^{(1)}, G^{(1)}, F_b^{(1)}, F_{\mathrm{div}}^{(1)}\right] - \left[F^{(2)}, G^{(2)}, F_b^{(2)}, F_{\mathrm{div}}^{(2)}\right] \right\|_{\mathcal{E}}. \end{aligned}$$

With the same conditions (5.2) and (5.3), we deduce that the restriction of Ψ on $B_{\mathcal{E}}(0,R)$ is a contraction mapping. The classical Banach fixed point theorem allows us to deduce the existence of a solution.

5.2 Proof of Proposition 5.1

Since $[F, G, F_b, F_{\mathrm{div}}]$ satisfies,

$$\|\,[F, G, F_b, F_{\mathrm{div}}]\,\|_{\mathcal{E}} \leqslant R, \tag{5.4}$$

then from (4.49) we deduce that the corresponding solution of (4.6)-(4.11) obeys

$$\|[w, \xi_1, \xi_2]\|_{\mathcal{G}} \leqslant C \left(R + \|[w^0, \xi_1^0, \xi_2^0]\|_{\mathbf{L}^2(\mathcal{F}) \times H^2(0,1) \times L^2(0,1)}\right). \tag{5.5}$$

By definition of $\mathcal{G}$ (see (4.47)) and from (4.48), we deduce for $\varepsilon \in (0, 1/8)$,

$$\xi_1 \in H^{1/2+\varepsilon}_\sigma(H^{2-2\varepsilon}_0(0,1)), \tag{5.6}$$

$$\xi_2 \in H^{1/2+\varepsilon}_\sigma(H^{-2\varepsilon}(0,1)), \tag{5.7}$$

$$w \in H^{1/2}_\sigma(\mathbf{L}^2(\mathcal{F})) \cap C_{b,\sigma}(\mathbf{L}^2(\mathcal{F})) \cap L^2_\sigma(\mathbf{H}^1(\mathcal{F})), \tag{5.8}$$

$$\xi_1 \in H^{1/2}_\sigma(H^2_0(0,1)) \cap C_{b,\sigma}(H^2_0(0,1)) \cap L^2_\sigma(H^3(0,1)), \tag{5.9}$$

$$\xi_2 \in H^{1/2}_\sigma(L^2(0,1)) \cap C_{b,\sigma}(L^2(0,1)) \cap L^2_\sigma(H^1_0(0,1)), \tag{5.10}$$

and since $\xi_2 = \partial_t \xi_1$

$$\xi_1 \in H^{3/2}_\sigma(L^2(0,1)) \cap C^1_{b,\sigma}(L^2(0,1)) \cap H^1_\sigma(H^1_0(0,1)). \tag{5.11}$$

Using interpolation arguments and Sobolev embeddings we deduce the following embeddings,

$$H^{1/2+\varepsilon}_\sigma(H^{1-2\varepsilon}(0,1)) \hookrightarrow C_{b,\sigma}(L^\infty(0,1)),$$

$$H^{1/2}_\sigma(L^2(0,1)) \cap L^2_\sigma(H^1(0,1)) \hookrightarrow H^{1/4}_\sigma(H^{1/2}(0,1)) \hookrightarrow L^4_\sigma(L^4(0,1)),$$
$$H^{1/2}_\sigma(L^2(0,1)) \cap L^2_\sigma(H^1(0,1)) \hookrightarrow H^{3/8}_\sigma(H^{1/4}(0,1)) \hookrightarrow L^8_\sigma(L^4(0,1)),$$
$$H^{1/2}_\sigma(\mathbf{L}^2_\sigma(\mathcal{F})) \cap L^2_\sigma(\mathbf{H}^1_\sigma(\mathcal{F})) \hookrightarrow H^{1/4}_\sigma(\mathbf{H}^{1/2}(\mathcal{F})) \hookrightarrow L^4_\sigma(\mathbf{L}^4(\mathcal{F})),$$

from which we deduce

$$\xi_1, \partial_s \xi_1 \in C_{b,\sigma}(L^\infty(0,1)), \tag{5.12}$$

$$\partial_{ss}\xi_1, \xi_2 \in L^4_\sigma(L^4(0,1)), \tag{5.13}$$

$$\partial_{ss}\xi_1, \xi_2 \in L^8_\sigma(L^4(0,1)), \tag{5.14}$$

$$w \in L^4_\sigma(\mathbf{L}^4(\mathcal{F})), \tag{5.15}$$

with the corresponding estimate:

$$\begin{aligned}\|\xi_1\|_{H^{1/2+\varepsilon}_\sigma(H^{2-2\varepsilon}_0(0,1))} + \|\xi_2\|_{H^{1/2+\varepsilon}_\sigma(H^{-2\varepsilon}(0,1))} + \|\xi_1\|_{C_{b,\sigma}(L^\infty(0,1))} + \|\partial_s\xi_1\|_{C_{b,\sigma}(L^\infty(0,1))} \\ + \|\partial_{ss}\xi_1\|_{L^4_\sigma(L^4(0,1))} + \|\xi_2\|_{L^4_\sigma(L^4(0,1))} + \|\partial_{ss}\xi_1\|_{L^8_\sigma(L^4(0,1))} + \|\xi_2\|_{L^8_\sigma(L^4(0,1))} \\ + \|w\|_{L^4_\sigma(\mathbf{L}^4(\mathcal{F}))} \leqslant C\left(R + \|[w^0,\xi^0_1,\xi^0_2]\|_{\mathbf{L}^2(\mathcal{F})\times H^2(0,1)\times L^2(0,1)}\right).\end{aligned} \tag{5.16}$$

Note that ξ_1 and ξ_2 can be considered as functions defined on $\mathcal{F}$ but only depending on y_1 and equal to zero outside $(0,1)$, and, in particular by using the fact that $\partial_s\xi_1 = \xi_1 = 0$ on $\{0,1\}$, the above spacial norms in $(0,1)$ can be replaced by spacial norms in $\mathcal{F}$. Moreover, by combining (4.8), (5.7), (4.2) and $T \in \mathcal{L}(H^{-2\varepsilon}(0,1), \mathbf{H}^{-1/2-2\varepsilon}(\partial\mathcal{F}))$, (obtained from (2.57) with duality argument) we deduce

$$w \in H^{1/2+\varepsilon}_\sigma(\mathbf{H}^{-1/2-2\varepsilon}(\partial\mathcal{F})),$$

and with (5.4), (5.16) and (5.5) we obtain

$$\|w\|_{H^{1/2+\varepsilon}_\sigma(\mathbf{H}^{-1/2-2\varepsilon}(\partial\mathcal{F}))} \leqslant C\left(R + \|[w^0,\xi^0_1,\xi^0_2]\|_{\mathbf{L}^2(\mathcal{F})\times H^2(0,1)\times L^2(0,1)}\right). \tag{5.17}$$

Lemma 5.3. *The maps F and G defined by* (2.43), (2.33) *and* (2.37) *satisfy*

$$\|G(\mathbf{Z})\|_{L^2_\sigma(\mathbf{L}^2(\mathcal{F}))} + \|F(\mathbf{Z})\|_{L^2_\sigma((\mathbf{H}^1(\mathcal{F})')} \leqslant C\left(R + \|[w^0,\xi^0_1,\xi^0_2]\|_{\mathbf{L}^2(\mathcal{F})\times H^2(0,1)\times L^2(0,1)}\right)^2.$$

Proof. We first estimate $r^{(5)}(\xi_1, \partial_s\xi_1, \partial_{ss}\xi_1, \xi_2, w)$. Using (2.34), we see in particular that we have to estimate terms of the form

$$a(x,\xi_1)\xi_1^{n_1}(\partial_s\xi_1)^{n_2}\partial_{ss}\xi_1 w_i \quad (i=1,2),$$

or

$$a(x,\xi_1)\xi_1^{n_1}(\partial_s\xi_1)^{n_2}\xi_2 w_i \quad (i=1,2),$$

where $n_1, n_2 \in \mathbb{N}$ and where a is a Lipschitz continuous function. For that we use (5.12), (5.13) and (5.15). Thus, with (5.1) and (5.16) we finally deduce

$$\|r^{(5)}(\xi_1, \partial_s\xi_1, \partial_{ss}\xi_1, \xi_2, w)\|_{L^2_\sigma(\mathbf{L}^2(\mathcal{F}))} \leqslant C\left(R + \|[w^0,\xi^0_1,\xi^0_2]\|_{\mathbf{L}^2(\mathcal{F})\times H^2(0,1)\times L^2(0,1)}\right)^2. \tag{5.18}$$

Similarly, using (2.35), (2.36) and (5.8), (5.12), (5.15), with (5.1) and (5.16) we obtain

$$\|r^{(6)}(\xi_1,\partial_s\xi_1,\nabla w)\|_{L^2_\sigma(\mathbf{L}^2(\mathcal{F}))}+\|r^{(7)}(\xi_1,\partial_s\xi_1,w)\|_{L^2_\sigma(\mathbf{L}^2(\mathcal{F}))} \leqslant C\left(R+\|[w^0,\xi_1^0,\xi_2^0]\|_{\mathbf{L}^2(\mathcal{F})\times H^2(0,1)\times L^2(0,1)}\right)^2. \quad (5.19)$$

Combining, (2.33), (2.43) with (5.18) and (5.19) we deduce the result for $G(\mathbf{Z})$.

Then, we estimate $F(\mathbf{Z})$. First, using (5.14) and (5.15), we deduce that

$$|\partial_{ss}\xi_1|^2|w|\in L^2_\sigma(\mathbf{L}^{4/3}(\mathcal{F})).$$

Thus, using (2.38), (5.12) and $L^2_\sigma(\mathbf{L}^{4/3}(\mathcal{F}))\hookrightarrow L^2_\sigma(\mathbf{H}^1(\mathcal{F})')$, with (5.1) and (5.16) we obtain

$$\|r^{(8)}(\xi_1,\partial_s\xi_1,\partial_{ss}\xi_1,w)\|_{L^2_\sigma(\mathbf{H}^1(\mathcal{F})')}\leqslant C\left(R+\|[w^0,\xi_1^0,\xi_2^0]\|_{\mathbf{L}^2(\mathcal{F})\times H^2(0,1)\times L^2(0,1)}\right)^2. \quad (5.20)$$

Moreover, with the same proof,

$$\|r^{(10)}(\xi_1,\partial_s\xi_1,\partial_{ss}\xi_1,\xi_2,w)\|_{L^2_\sigma(\mathbf{H}^1(\mathcal{F})')}\leqslant C\left(R+\|[w^0,\xi_1^0,\xi_2^0]\|_{\mathbf{L}^2(\mathcal{F})\times H^2(0,1)\times L^2(0,1)}\right)^2. \quad (5.21)$$

Next, using Lemma C.2, we have for any function $\tau\in W^{1,\infty}(\mathbb{R}^3)$,

$$\tau(\xi_1,\partial_s\xi_1)(\partial_{ss}\xi_1)(\nabla w)\in L^2_\sigma(H^1(\mathcal{F})')^{2\times 2}),$$

using Lemma C.3, we have for any functions $\tau_1,\tau_2\in W^{1,\infty}(\mathbb{R}^3)$,

$$\tau_1(\xi_1,\partial_s\xi_1)\xi_2 w+\tau_2(\xi_1,\partial_s\xi_1)(\partial_s\xi_2)w\in L^2_\sigma(\mathbf{H}^1(\mathcal{F})'),$$

and using Lemma C.4, we have for any function $\tau\in W^{1,\infty}(\mathbb{R}^3)$, and for any i,j,

$$\tau(\xi_1,\partial_s\xi_1)(\partial_{ss}\xi_1)(w_iw_j)\in L^2_\sigma(H^1(\mathcal{F})').$$

Then gathering the corresponding of above results with (2.39), (2.41), (2.42), (5.1) and (5.16), we deduce

$$\begin{aligned}\|r^{(9)}(\xi_1,\partial_s\xi_1,\partial_{ss}\xi_1,\nabla w)\|_{L^2_\sigma(\mathbf{H}^1(\mathcal{F})')} &+\|r^{(11)}(\xi_1,\partial_s\xi_1,\xi_2,\partial_s\xi_2,w)\|_{L^2_\sigma(\mathbf{H}^1(\mathcal{F})')}+\|r^{(12)}(\xi_1,\partial_s\xi_1,\partial_{ss}\xi_1,w)\|_{L^2_\sigma(\mathbf{H}^1(\mathcal{F})')}\\ &\leqslant C\left(R+\|[w^0,\xi_1^0,\xi_2^0]\|_{\mathbf{L}^2(\mathcal{F})\times H^2(0,1)\times L^2(0,1)}\right)^2.\end{aligned}$$

These estimates with (2.43) and (2.37) give the result for $F(\mathbf{Z})$. □

Lemma 5.4. *The maps $F_b(\cdot)$ and $F_{\mathrm{div}}(\cdot)$ defined by* (2.25)*,* (2.27) *and* (2.24) *satisfy*

$$\|(F_b(\mathbf{Z}),F_{\mathrm{div}}(\mathbf{Z}))\|_{L^2_\sigma(\mathbf{H}^{1/2,0}_{\partial\mathcal{F},\mathcal{F}})}+\|(F_b(\mathbf{Z}),F_{\mathrm{div}}(\mathbf{Z}))\|_{H^{1/2+\varepsilon}_\sigma(\mathbf{H}^{-1/2-2\varepsilon,-1-2\varepsilon}_{\partial\mathcal{F},\mathcal{F}})} \leqslant C\left(R+\|[w^0,\xi_1^0,\xi_2^0]\|_{\mathbf{L}^2(\mathcal{F})\times H^2(0,1)\times L^2(0,1)}\right)^2$$

Proof. Using (5.8), (5.12) and (5.13), (5.15) we deduce that

$$(\partial_s\xi_1)\nabla w\in L^2_\sigma(L^2(\mathcal{F})^{2\times 2})\quad\text{and}\quad(\partial_{ss}\xi_1)w\in L^2_\sigma(\mathbf{L}^2(\mathcal{F})),$$

so that, with (2.24), (5.8), (5.12), (5.15) and (5.1), (5.16) we deduce $r^{(2)}(\xi_1,\partial_s\xi_1,w)\in L^2_\sigma(\mathbf{H}^1(\mathcal{F}))$ and

$$\|r^{(2)}(\xi_1,\partial_s\xi_1,w)\|_{L^2_\sigma(\mathbf{H}^1(\mathcal{F}))}\leqslant C\left(R+\|[w^0,\xi_1^0,\xi_2^0]\|_{\mathbf{L}^2(\mathcal{F})\times H^2(0,1)\times L^2(0,1)}\right)^2.$$

In particular,

$$(F_b(\mathbf{Z}),F_{\mathrm{div}}(\mathbf{Z}))\in L^2_\sigma(\mathbf{H}^{1/2,0}_{\partial\mathcal{F},\mathcal{F}}),$$

with

$$\|(F_b(\mathbf{Z}),F_{\mathrm{div}}(\mathbf{Z}))\|_{L^2_\sigma(\mathbf{H}^{1/2,0}_{\partial\mathcal{F},\mathcal{F}})}\leqslant C\left(R+\|[w^0,\xi_1^0,\xi_2^0]\|_{\mathbf{L}^2(\mathcal{F})\times H^2(0,1)\times L^2(0,1)}\right)^2. \quad (5.22)$$

Next, assume $(g,h) \in \mathbf{H}^{1/2+2\varepsilon,1+2\varepsilon}_{\partial\mathcal{F},\mathcal{F}}$. For a.e. in $(0,\infty)$, we have

$$\langle (F_b(\mathbf{Z}), F_{\mathrm{div}}(\mathbf{Z})), (g,h)\rangle_{\mathbf{H}^{-1/2-2\varepsilon,-1-2\varepsilon}_{\partial\mathcal{F},\mathcal{F}}, \mathbf{H}^{1/2+2\varepsilon,1+2\varepsilon}_{\partial\mathcal{F},\mathcal{F}}} = \int_{\partial\mathcal{F}} r^{(2)} \cdot g \, d\gamma + \int_{\mathcal{F}} (\operatorname{div} r^{(2)}) h \, dy$$
$$= \int_{\partial\mathcal{F}} r^{(2)} \cdot (g + hn) \, d\gamma - \int_{\mathcal{F}} r^{(2)} \cdot \nabla h \, dy. \quad (5.23)$$

On the other hand, since we have (5.17), we can apply Lemma C.5 to obtain,

$$\|r^{(2)}(\xi_1, \partial_s\xi_1, w)\|_{H^{1/2+\varepsilon}_\sigma(\mathbf{H}^{-1/2-2\varepsilon}(\partial\mathcal{F}))} \leqslant C \left(R + \|[w^0, \xi_1^0, \xi_2^0]\|_{\mathbf{L}^2(\mathcal{F})\times H^2(0,1)\times L^2(0,1)}\right)^2. \quad (5.24)$$

Note that assumption $\varepsilon \in (0, 1/8)$ is needed in Lemma C.5 to obtain $\partial_s\xi_1 \in H^{1/2+\varepsilon}_\sigma(H^{1/2+2\varepsilon}(0,1))$ from (5.6). Moreover, from $\xi_1, \partial_s\xi_1 \in H^{1/2+\varepsilon}_\sigma(H^{1/2+2\varepsilon}(\mathcal{F}))$ we also deduce from Lemma C.5

$$\|r^{(2)}(\xi_1, \partial_s\xi_1, w)\|_{H^{1/2+\varepsilon}_\sigma(\mathbf{H}^{2\varepsilon}(\mathcal{F})')} \leqslant C \left(R + \|[w^0, \xi_1^0, \xi_2^0]\|_{\mathbf{L}^2(\mathcal{F})\times H^2(0,1)\times L^2(0,1)}\right)^2.$$

Combining the above inequality with (5.24) and (5.23) yields

$$\|(F_b(\mathbf{Z}), F_{\mathrm{div}}(\mathbf{Z}))\|_{H^{1/2+\varepsilon}_\sigma(\mathbf{H}^{-1/2-2\varepsilon,-1-2\varepsilon}_{\partial\mathcal{F},\mathcal{F}})} \leqslant C \left(R + \|[w^0, \xi_1^0, \xi_2^0]\|_{\mathbf{L}^2(\mathcal{F})\times H^2(0,1)\times L^2(0,1)}\right)^2.$$

The above estimate and (5.22) give the result. □

5.3 Proof of Proposition 5.2

First, $\left[F^{(i)}, G^{(i)}, F_b^{(i)}, F_{\mathrm{div}}^{(i)}\right]$, $i = 1, 2$ and the corresponding solutions of (4.6)-(4.11) satisfy (5.4)-(5.17).

Moreover, if we write

$$[w, \xi_1, \xi_2] = [w^{(1)}, \xi_1^{(1)}, \xi_2^{(1)}] - [w^{(2)}, \xi_1^{(2)}, \xi_2^{(2)}],$$
$$\left[F, G, F_b, F_{\mathrm{div}}\right] = \left[F^{(1)}, G^{(1)}, F_b^{(1)}, F_{\mathrm{div}}^{(1)}\right] - \left[F^{(2)}, G^{(2)}, F_b^{(2)}, F_{\mathrm{div}}^{(2)}\right],$$

then we have from Corollary 4.6,

$$\|[w, \xi_1, \xi_2]\|_{\mathcal{G}} \leqslant C \| \left[F, G, F_b, F_{\mathrm{div}}\right] \|_{\mathcal{E}}, \quad (5.25)$$

and as in the previous section we thus deduce

$$\|\xi_1\|_{H^{1/2+\varepsilon}_\sigma(H^{2-2\varepsilon}_0(0,1))} + \|\xi_2\|_{H^{1/2+\varepsilon}_\sigma(H^{-2\varepsilon}(0,1))} + \|\xi_1\|_{C_{b,\sigma}(L^\infty(0,1))} + \|\partial_s\xi_1\|_{C_{b,\sigma}(L^\infty(0,1))}$$
$$+ \|\partial_{ss}\xi_1\|_{L^4_\sigma(L^4(0,1))} + \|\xi_2\|_{L^4_\sigma(L^4(0,1))} + \|\partial_{ss}\xi_1\|_{L^8_\sigma(L^4(0,1))} + \|\xi_2\|_{L^8_\sigma(L^4(0,1))}$$
$$+ \|w\|_{L^4_\sigma(\mathbf{L}^4(\mathcal{F}))} + \|w\|_{H^{1/2+\varepsilon}_\sigma(\mathbf{H}^{-1/2-2\varepsilon}(\partial\mathcal{F}))} \leqslant C \| \left[F, G, F_b, F_{\mathrm{div}}\right] \|_{\mathcal{E}}.$$

In the same way as for Lemma 5.3 and Lemma 5.4 we can prove the following Lemma.

Lemma 5.5. *The maps F and G defined by* (2.43), (2.33) *and* (2.37) *satisfy*

$$\|G(\mathbf{Z}^{(1)}) - G(\mathbf{Z}^{(2)})\|_{L^2_\sigma(\mathbf{L}^2(\mathcal{F})} + \|F(\mathbf{Z}^{(1)}) - F(\mathbf{Z}^{(2)})\|_{L^2_\sigma((\mathbf{H}^1(\mathcal{F})')}$$
$$\leqslant C \left(R + \|[w^0, \xi_1^0, \xi_2^0]\|_{\mathbf{L}^2(\mathcal{F})\times H^2(0,1)\times L^2(0,1)}\right) \left\| \left[F, G, F_b, F_{\mathrm{div}}\right] \right\|_{\mathcal{E}}.$$

The maps $F_b(\cdot)$ and $F_{\mathrm{div}}(\cdot)$ defined by (2.25), (2.27) *and* (2.24) *satisfy*

$$\|(F_b(\mathbf{Z}^{(1)}), F_{\mathrm{div}}(\mathbf{Z}^{(1)})) - (F_b(\mathbf{Z}^{(2)}), F_{\mathrm{div}}(\mathbf{Z}^{(2)}))\|_{L^2_\sigma(\mathbf{H}^{1/2,0}_{\partial\mathcal{F},\mathcal{F}})}$$
$$+ \|(F_b(\mathbf{Z}^{(1)}), F_{\mathrm{div}}(\mathbf{Z}^{(1)})) - (F_b(\mathbf{Z}^{(2)}), F_{\mathrm{div}}(\mathbf{Z}^{(2)}))\|_{H^{1/2+\varepsilon}_\sigma(\mathbf{H}^{-1/2-2\varepsilon,-1-2\varepsilon}_{\partial\mathcal{F},\mathcal{F}})}$$
$$\leqslant C \left(R + \|[w^0, \xi_1^0, \xi_2^0]\|_{\mathbf{L}^2(\mathcal{F})\times H^2_0(0,1)\cap L^2_0(0,1)\times L^2_0(0,1)}\right) \left\| \left[F, G, F_b, F_{\mathrm{div}}\right] \right\|_{\mathcal{E}}.$$

A Calculation for the change of variables

In this section, we gather several lemmas and several proofs for the change of variables (Section 2).

A.1 Proof of Lemma 2.2 and of Lemma 2.3

Proof of Lemma 2.2. First we write (2.11) as

$$v_i = \sum_j \frac{\partial Y_j}{\partial x_i} \widetilde{v}_j(Y). \tag{A.1}$$

Thus

$$\frac{\partial v_i}{\partial x_k} = \sum_j \frac{\partial^2 Y_j}{\partial x_i \partial x_k} \widetilde{v}_j(Y) + \sum_{j,\ell} \frac{\partial Y_j}{\partial x_i} \frac{\partial Y_\ell}{\partial x_k} \frac{\partial \widetilde{v}_j}{\partial y_\ell}(Y). \tag{A.2}$$

and in particular

$$\sum_i \frac{\partial v_i}{\partial x_i} = \sum_{i,j} \frac{\partial^2 Y_j}{\partial x_i^2} \widetilde{v}_j(Y) + \sum_{i,j,\ell} \frac{\partial Y_j}{\partial x_i} \frac{\partial Y_\ell}{\partial x_i} \frac{\partial \widetilde{v}_j}{\partial y_\ell}(Y). \tag{A.3}$$

We thus deduce that

$$\det(\nabla X)(\operatorname{div} v) \circ X = \det(\nabla X)(\Delta Y)(X) \cdot \widetilde{v} + \det(\nabla X)[\nabla Y](X)[\nabla Y]^*(X) : \nabla \widetilde{v}. \tag{A.4}$$

We set

$$K \stackrel{\text{def}}{=} \det(\nabla X)[\nabla Y](X)[\nabla Y]^*(X). \tag{A.5}$$

We recall that for a matrix-valued function M,

$$(\operatorname{div} M)_i \stackrel{\text{def}}{=} \sum_j \frac{\partial M_{ij}}{\partial y_j}$$

and that for a vector-valued function b,

$$\operatorname{div}(Mb) = \operatorname{div}(M^*) \cdot b + M^* : \nabla b.$$

After some calculation, we have

$$\operatorname{div}(K) = \det(\nabla X) \Delta Y(X). \tag{A.6}$$

Thus,

$$\det(\nabla X)(\operatorname{div} v) \circ X = \operatorname{div}(K \widetilde{v}). \tag{A.7}$$

Finally,

$$\widetilde{v} = (\nabla X)^*(\partial_t \eta) e_2 \quad \text{on } \Gamma_{\text{str}},$$

and on $\mathcal{V}_{\alpha/2}$ (c.f. (2.4)),

$$X(y_1, y_2) = \Big(y_1, y_2 + (\eta(y_1) - \eta^S(y_1))\Big).$$

Thus on Γ_{str},

$$\operatorname{Cof}(\nabla X)^*(y_1, y_2) = \begin{bmatrix} 1 & 0 \\ -(\partial_s \eta(y_1) - \partial_s \eta^S(y_1)) & 1 \end{bmatrix}.$$

and (2.14) follows from

$$[\operatorname{Cof}(\nabla X)]^{-*} e_2 = e_2.$$

□

Proof of Lemma 2.3. Relation (2.16) is proved in [14, Lemma 3.1] and (2.17) follows from the equality $[\operatorname{Cof}(\nabla X)]^* e_2 = e_2$. □

A.2 Calculation of $\mathcal{M}_\xi^{(i)}$ and $\mathcal{B}_\xi^{(i)}$

Lemma A.1. *Assume* (2.11) *and* (2.15). *Then*

$$2\nu \int_{\mathcal{F}(\eta)} D(v) : D(\varphi)\ dx = \int_{\mathcal{F}} \mathcal{M}_\xi^{(3)}(\widetilde{v}, \nabla\widetilde{v}) : \nabla\widetilde{\varphi}\ dx + \int_{\mathcal{F}} \mathcal{M}_\xi^{(4)}(\widetilde{v}, \nabla\widetilde{v}) \cdot \widetilde{\varphi}\ dx, \tag{A.8}$$

where

$$\left[\mathcal{M}_\xi^{(3)}(\widetilde{v}, \nabla\widetilde{v})\right]_{\alpha,\beta} \overset{def}{=} 2\nu \left[(D\widetilde{v})(\nabla Y)(X)(\nabla Y)(X)^*\right]_{\alpha,\beta} + 2\nu \sum_{i,j,k} \frac{\partial X_i}{\partial y_\alpha} \frac{\partial^2 Y_j}{\partial x_i \partial x_k}(X) \frac{\partial Y_\beta}{\partial x_k}(X)\widetilde{v}_j, \tag{A.9}$$

and

$$\begin{aligned}\left[\mathcal{M}_\xi^{(4)}(\widetilde{v}, \nabla\widetilde{v})\right]_k \overset{def}{=} 2\nu \sum_{i,j,\alpha,\ell,k} \frac{\partial^2 Y_\alpha}{\partial x_i \partial x_j}(X) \left(\sum_m \frac{\partial X_\ell}{\partial y_m} \frac{\partial^2 Y_m}{\partial x_\ell \partial x_j}(X) \frac{\partial X_i}{\partial y_k} + \frac{\partial Y_\ell}{\partial x_j}(X) \frac{\partial^2 X_i}{\partial y_k \partial y_\ell}\right) \widetilde{v}_\alpha \\ + 2\nu \sum_{i,j,\alpha,\beta,\ell} \frac{\partial Y_\alpha}{\partial x_i}(X)(D\widetilde{v})_{\alpha,\beta} \frac{\partial Y_\beta}{\partial x_j}(X) \left(\sum_m \frac{\partial X_\ell}{\partial y_m} \frac{\partial^2 Y_m}{\partial x_\ell \partial x_j}(X) \frac{\partial X_i}{\partial y_k} + \frac{\partial Y_\ell}{\partial x_j}(X) \frac{\partial^2 X_i}{\partial y_k \partial y_\ell}\right).\end{aligned} \tag{A.10}$$

Proof. We first write

$$\begin{aligned}2\nu \int_{\mathcal{F}(\eta)} D(v) : D(\varphi)\ dx &= 2\nu \int_{\mathcal{F}} D(v) \circ X : D(\varphi) \circ X\ \det(\nabla X)\ dy \\ &= 2\nu \int_{\mathcal{F}} D(v) \circ X : (\nabla\varphi) \circ X\ \det(\nabla X)\ dy.\end{aligned} \tag{A.11}$$

From (2.15)

$$\varphi(t,x) = \operatorname{Cof}(\nabla Y(t,x))^* \widetilde{\varphi}(t, Y(t,x)). \tag{A.12}$$

Thus

$$\frac{\partial \varphi_i}{\partial x_j} = \sum_k \frac{\partial \operatorname{Cof}(\nabla Y)_{ki}}{\partial x_j} \widetilde{\varphi}_k(Y) + \sum_{k,\ell} \operatorname{Cof}(\nabla Y)_{ki} \frac{\partial Y_\ell}{\partial x_j} \frac{\partial \widetilde{\varphi}_k}{\partial y_\ell}(Y).$$

and

$$\begin{aligned}(\nabla\varphi) \circ X\ \det(\nabla X) &= \left(\det(\nabla X) \sum_k \frac{\partial \operatorname{Cof}(\nabla Y)_{ki}}{\partial x_j}(X)\widetilde{\varphi}_k\right)_{i,j} + \det(\nabla X) \operatorname{Cof}(\nabla Y)(X)^* (\nabla\widetilde{\varphi})(\nabla Y)(X) \\ &= \left(\det(\nabla X) \sum_k \frac{\partial \operatorname{Cof}(\nabla Y)_{ki}}{\partial x_j}(X)\widetilde{\varphi}_k\right)_{i,j} + (\nabla X)(\nabla\widetilde{\varphi})(\nabla Y)(X).\end{aligned} \tag{A.13}$$

From (A.2), we have

$$(\nabla v)(X) = \left(\sum_j \frac{\partial^2 Y_j}{\partial x_i \partial x_k}(X)\widetilde{v}_j\right)_{ik} + (\nabla Y)^*(X)(\nabla\widetilde{v})(\nabla Y)(X). \tag{A.14}$$

We deduce

$$(Dv)(X) = \left(\sum_j \frac{\partial^2 Y_j}{\partial x_i \partial x_k}(X)\widetilde{v}_j\right)_{ik} + (\nabla Y)^*(X)(D\widetilde{v})(\nabla Y)(X). \tag{A.15}$$

Gathering (A.11), (A.13) and (A.15), we obtain

$$
\begin{aligned}
2\nu \int_{\mathcal{F}(\eta)} D(v) : D(\varphi)\, dx
&= 2\nu \int_{\mathcal{F}} \left(\sum_j \frac{\partial^2 Y_j}{\partial x_i \partial x_k}(X) \widetilde{v}_j \right)_{ik} : \left(\det(\nabla X) \sum_k \frac{\partial \operatorname{Cof}(\nabla Y)_{ki}}{\partial x_j}(X) \widetilde{\varphi}_k \right)_{i,j} dy \\
&\quad + 2\nu \int_{\mathcal{F}} \left(\sum_j \frac{\partial^2 Y_j}{\partial x_i \partial x_k}(X) \widetilde{v}_j \right)_{ik} : (\nabla X)(\nabla \widetilde{\varphi})(\nabla Y)(X)\, dy \\
&\quad + 2\nu \int_{\mathcal{F}} (\nabla Y)^*(X)(D\widetilde{v})(\nabla Y)(X) : \left(\det(\nabla X) \sum_k \frac{\partial \operatorname{Cof}(\nabla Y)_{ki}}{\partial x_j}(X) \widetilde{\varphi}_k \right)_{i,j} dy \\
&\quad + 2\nu \int_{\mathcal{F}} (\nabla Y)^*(X)(D\widetilde{v})(\nabla Y)(X) : (\nabla X)(\nabla \widetilde{\varphi})(\nabla Y)(X)\, dy.
\end{aligned} \tag{A.16}
$$

Standard calculation gives

$$
2\nu \int_{\mathcal{F}} (\nabla Y)^*(X)(D\widetilde{v})(\nabla Y)(X) : (\nabla X)(\nabla \widetilde{\varphi})(\nabla Y)(X)\, dy = 2\nu \int_{\mathcal{F}} (D\widetilde{v})(\nabla Y)(X)(\nabla Y)(X)^* : (\nabla \widetilde{\varphi})\, dy, \tag{A.17}
$$

and

$$
\begin{aligned}
2\nu \int_{\mathcal{F}} \left(\sum_j \frac{\partial^2 Y_j}{\partial x_i \partial x_k}(X) \widetilde{v}_j \right)_{ik} &: (\nabla X)(\nabla \widetilde{\varphi})(\nabla Y)(X)\, dy \\
&= 2\nu \int_{\mathcal{F}} \left(\sum_{i,j,k} \frac{\partial X_i}{\partial y_\alpha} \frac{\partial^2 Y_j}{\partial x_i \partial x_k}(X) \widetilde{v}_j \frac{\partial Y_\beta}{\partial x_k}(X) \right)_{\alpha\beta} : (\nabla \widetilde{\varphi})\, dy.
\end{aligned} \tag{A.18}
$$

On the other hand,

$$
\frac{\partial \operatorname{Cof}(\nabla Y)_{ki}}{\partial x_j} = \frac{\partial}{\partial x_j} \left(\det(\nabla Y)[\nabla X]_{ik}(Y) \right) = \frac{\partial \det(\nabla Y)}{\partial x_j} \frac{\partial X_i}{\partial y_k}(Y) + \det(\nabla Y) \sum_\ell \frac{\partial Y_\ell}{\partial x_j} \frac{\partial^2 X_i}{\partial y_k \partial y_\ell}(Y). \tag{A.19}
$$

We can compute the first part of the above right hand side by using

$$
\frac{\partial \det(\nabla Y)}{\partial x_j} = \sum_{m,\ell} \frac{\partial \det}{\partial M_{m,\ell}}(\nabla Y) \frac{\partial^2 Y_m}{\partial x_\ell \partial x_j} = \sum_{m,\ell} \det(\nabla Y) \frac{\partial X_\ell}{\partial y_m}(Y) \frac{\partial^2 Y_m}{\partial x_\ell \partial x_j}. \tag{A.20}
$$

Combining (A.19) and (A.20), we deduce

$$
\det(\nabla X) \frac{\partial \operatorname{Cof}(\nabla Y)_{ki}}{\partial x_j}(X) = \sum_{m,\ell} \frac{\partial X_\ell}{\partial y_m} \frac{\partial^2 Y_m}{\partial x_\ell \partial x_j}(X) \frac{\partial X_i}{\partial y_k} + \sum_\ell \frac{\partial Y_\ell}{\partial x_j}(X) \frac{\partial^2 X_i}{\partial y_k \partial y_\ell}. \tag{A.21}
$$

Using the above relation, we obtain

$$
\begin{aligned}
2\nu \int_{\mathcal{F}} &\left(\sum_j \frac{\partial^2 Y_j}{\partial x_i \partial x_k}(X) \widetilde{v}_j \right)_{ik} : \left(\det(\nabla X) \sum_k \frac{\partial \operatorname{Cof}(\nabla Y)_{ki}}{\partial x_j}(X) \widetilde{\varphi}_k \right)_{i,j} dy \\
&= 2\nu \int_{\mathcal{F}} \sum_{i,j,\alpha,\ell,k} \frac{\partial^2 Y_\alpha}{\partial x_i \partial x_j}(X) \left(\sum_m \frac{\partial X_\ell}{\partial y_m} \frac{\partial^2 Y_m}{\partial x_\ell \partial x_j}(X) \frac{\partial X_i}{\partial y_k} + \frac{\partial Y_\ell}{\partial x_j}(X) \frac{\partial^2 X_i}{\partial y_k \partial y_\ell} \right) \widetilde{v}_\alpha \widetilde{\varphi}_k\, dy
\end{aligned} \tag{A.22}
$$

The third term in (A.16) is

$$
\begin{aligned}
2\nu \int_{\mathcal{F}} &(\nabla Y)^*(X)(D\widetilde{v})(\nabla Y)(X) : \left(\left(\sum_{m,\ell,k} \frac{\partial X_\ell}{\partial y_m} \frac{\partial^2 Y_m}{\partial x_\ell \partial x_j}(X) \frac{\partial X_i}{\partial y_k} + \sum_{\ell,k} \frac{\partial Y_\ell}{\partial x_j}(X) \frac{\partial^2 X_i}{\partial y_k \partial y_\ell} \right) \widetilde{\varphi}_k \right)_{i,j} dy \\
&= 2\nu \int_{\mathcal{F}} \sum_{i,j,\alpha,\beta,\ell,k} \frac{\partial Y_\alpha}{\partial x_i}(X)(D\widetilde{v})_{\alpha,\beta} \frac{\partial Y_\beta}{\partial x_j}(X) \left(\sum_m \frac{\partial X_\ell}{\partial y_m} \frac{\partial^2 Y_m}{\partial x_\ell \partial x_j}(X) \frac{\partial X_i}{\partial y_k} + \frac{\partial Y_\ell}{\partial x_j}(X) \frac{\partial^2 X_i}{\partial y_k \partial y_\ell} \right) \widetilde{\varphi}_k\, dy.
\end{aligned} \tag{A.23}
$$

Finally, gathering (A.16), (A.17), (A.18), (A.22) and (A.23), we deduce the result. $\square$

Lemma A.2. *Assume* (2.11) *and* (2.15). *Then*

$$\int_{\mathcal{F}(\eta)} v\cdot\partial_t\varphi\,dx=\int_{\mathcal{F}}\widetilde{v}\cdot\partial_t\widetilde{\varphi}\,dy+\int_{\mathcal{F}}\mathcal{M}_\xi^{(5)}(\widetilde{v}):\nabla\widetilde{\varphi}\,dy+\int_{\mathcal{F}}\mathcal{M}_\xi^{(6)}(\widetilde{v})\cdot\widetilde{\varphi}\,dy,\tag{A.24}$$

where

$$\mathcal{M}_\xi^{(5)}(\widetilde{v})\overset{def}{=}-\widetilde{v}\otimes(\nabla Y)(X)\,(\partial_t X),$$

and

$$\begin{aligned}\left[\mathcal{M}_\xi^{(6)}(\widetilde{v})\right]_j\overset{def}{=}&\left[\partial_t\left[\frac{1}{\det(\nabla X)}[\nabla X]^*\right]\operatorname{Cof}(\nabla X)\widetilde{v}\right]_j\\&-\sum_{k,\ell,i,j,\alpha}\frac{\partial Y_\alpha}{\partial x_i}(X)\partial_t X_k\left(\sum_m\frac{\partial X_\ell}{\partial y_m}\frac{\partial^2 Y_m}{\partial x_\ell\partial x_k}(X)\frac{\partial X_i}{\partial y_j}+\frac{\partial Y_\ell}{\partial x_k}(X)\frac{\partial^2 X_i}{\partial y_j\partial y_\ell}\right)\widetilde{v}_\alpha.\end{aligned}$$

Proof. We first write

$$\int_{\mathcal{F}(\eta)} v\cdot\partial_t\varphi\,dx=\int_{\mathcal{F}}(v\circ X)\cdot(\partial_t\varphi)\circ X(\det\nabla X)\,dy.\tag{A.25}$$

Therefore,

$$\int_{\mathcal{F}(\eta)} v\cdot\partial_t\varphi\,dx=\int_{\mathcal{F}}\widetilde{v}\cdot(\nabla Y)(X)(\partial_t\varphi)\circ X(\det\nabla X)\,dy.\tag{A.26}$$

Then, using (2.15), we have

$$\partial_t\varphi=\left[\partial_t\operatorname{Cof}(\nabla Y)^*\right]\widetilde{\varphi}(Y)+\operatorname{Cof}(\nabla Y)^*\left(\partial_t Y\cdot\nabla\right)\widetilde{\varphi}(Y)+\operatorname{Cof}(\nabla Y)^*\partial_t\widetilde{\varphi}(Y).\tag{A.27}$$

and thus

$$\begin{aligned}(\nabla Y)(X)(\partial_t\varphi)\circ X(\det\nabla X)=&\operatorname{Cof}(\nabla X)^*\left[\partial_t\operatorname{Cof}(\nabla Y)^*\right](X)\widetilde{\varphi}\\&+\operatorname{Cof}(\nabla X)^*\operatorname{Cof}(\nabla Y)^*(X)\left(\partial_t Y(X)\cdot\nabla\right)\widetilde{\varphi}+\operatorname{Cof}(\nabla X)^*\operatorname{Cof}(\nabla Y)^*(X)\partial_t\widetilde{\varphi}.\end{aligned}\tag{A.28}$$

The above relation yields

$$(\nabla Y)(X)(\partial_t\varphi)\circ X(\det\nabla X)=\operatorname{Cof}(\nabla X)^*\left[\partial_t\operatorname{Cof}(\nabla Y)^*\right](X)\widetilde{\varphi}+\left(\partial_t Y(X)\cdot\nabla\right)\widetilde{\varphi}+\partial_t\widetilde{\varphi}.\tag{A.29}$$

Then, we use

$$(\partial_t Y)(X)=-(\nabla Y)(X)\,(\partial_t X)\tag{A.30}$$

and

$$\left[\partial_t\operatorname{Cof}(\nabla Y)^*\right]_{i,j}(X)=\partial_t\left[\operatorname{Cof}(\nabla Y)^*(X)\right]_{i,j}-\sum_k\partial_t X_k\frac{\partial\operatorname{Cof}(\nabla Y)_{j,i}}{\partial x_k}(X).\tag{A.31}$$

From (A.21),

$$\sum_k\partial_t X_k\frac{\partial\operatorname{Cof}(\nabla Y)_{j,i}}{\partial x_k}(X)=\frac{1}{\det\nabla X}\sum_{k,\ell}\partial_t X_k\left(\sum_m\frac{\partial X_\ell}{\partial y_m}\frac{\partial^2 Y_m}{\partial x_\ell\partial x_k}(X)\frac{\partial X_i}{\partial y_j}+\frac{\partial Y_\ell}{\partial x_k}(X)\frac{\partial^2 X_i}{\partial y_j\partial y_\ell}\right).\tag{A.32}$$

We also have

$$\partial_t\left[\operatorname{Cof}(\nabla Y)^*(X)\right]=\partial_t\left[\frac{1}{\det(\nabla X)}[\nabla X]\right].\tag{A.33}$$

Finally, we conclude

$$\begin{aligned}\int_{\mathcal{F}(\eta)} v\cdot\partial_t\varphi\,dx=&\int_{\mathcal{F}}\widetilde{v}\cdot\partial_t\widetilde{\varphi}\,dy+\int_{\mathcal{F}}-\widetilde{v}\otimes(\nabla Y)(X)\,(\partial_t X):\nabla\widetilde{\varphi}\,dy\\&+\int_{\mathcal{F}}\partial_t\left[\frac{1}{\det(\nabla X)}[\nabla X]^*\right]\operatorname{Cof}(\nabla X)\,\widetilde{v}\cdot\widetilde{\varphi}\,dy\\&-\int_{\mathcal{F}}\sum_{k,\ell,i,j,\alpha}\frac{\partial Y_\alpha}{\partial x_i}(X)\partial_t X_k\left(\sum_m\frac{\partial X_\ell}{\partial y_m}\frac{\partial^2 Y_m}{\partial x_\ell\partial x_k}(X)\frac{\partial X_i}{\partial y_j}+\frac{\partial Y_\ell}{\partial x_k}(X)\frac{\partial^2 X_i}{\partial y_j\partial y_\ell}\right)\widetilde{\varphi}_j\widetilde{v}_\alpha\,dy.\end{aligned}\tag{A.34}$$

$\square$

Lemma A.3. *Assume* (2.11) *and* (2.15). *Then*

$$\int_{\mathcal{F}(\eta)} v\cdot((v\cdot\nabla)\varphi)\ dx = \int_{\mathcal{F}} \mathcal{B}^{(1)}_\xi(\widetilde{v},\widetilde{v})\cdot\widetilde{\varphi} + \mathcal{B}^{(2)}_\xi(\widetilde{v},\widetilde{v}) : (\nabla\widetilde{\varphi})\ dy, \tag{A.35}$$

where

$$\left[\mathcal{B}^{(1)}_\xi(\widetilde{v},\widetilde{v})\right]_k \overset{def}{=} \sum_{i,j,\alpha,\beta} \frac{\partial Y_\alpha}{\partial x_i}(X)\widetilde{v}_\alpha\widetilde{v}_\beta\frac{\partial Y_\beta}{\partial x_j}(X)\left(\sum_{m,\ell}\frac{\partial X_\ell}{\partial y_m}\frac{\partial^2 Y_m}{\partial x_\ell\partial x_j}(X)\frac{\partial X_i}{\partial y_k} + \sum_\ell \frac{\partial Y_\ell}{\partial x_j}(X)\frac{\partial^2 X_i}{\partial y_k\partial y_\ell}\right) \tag{A.36}$$

and

$$\mathcal{B}^{(2)}_\xi(\widetilde{v},\widetilde{v}) \overset{def}{=} [\widetilde{v}\otimes\widetilde{v}]\,(\nabla Y)(X)(\nabla Y)(X)^*. \tag{A.37}$$

Proof. We first write

$$\int_{\mathcal{F}(\eta)} v\cdot((v\cdot\nabla)\varphi)\ dx = \int_{\mathcal{F}(\eta)} v\otimes v : (\nabla\varphi)\ dx = \int_{\mathcal{F}} (v\circ X)\otimes(v\circ X) : (\nabla\varphi)\circ X(\det(\nabla X))\ dy.$$

We have

$$(v\circ X)\otimes(v\circ X) = (\nabla Y)(X)^*\,[\widetilde{v}\otimes\widetilde{v}]\,(\nabla Y)(X)$$

and from (A.13) and (A.21)

$$(\nabla\varphi)\circ X\ \det(\nabla X) = \left(\sum_k\left(\sum_{m,\ell}\frac{\partial X_\ell}{\partial y_m}\frac{\partial^2 Y_m}{\partial x_\ell\partial x_j}(X)\frac{\partial X_i}{\partial y_k} + \sum_\ell\frac{\partial Y_\ell}{\partial x_j}(X)\frac{\partial^2 X_i}{\partial y_k\partial y_\ell}\right)\widetilde{\varphi}_k\right)_{i,j} + (\nabla X)(\nabla\widetilde{\varphi})(\nabla Y)(X). \tag{A.38}$$

Thus

$$\begin{aligned}&(v\circ X)\otimes(v\circ X) : (\nabla\varphi)\circ X(\det(\nabla X))\\ &\quad= \sum_{i,j,\alpha,\beta}\frac{\partial Y_\alpha}{\partial x_i}(X)\widetilde{v}_\alpha\widetilde{v}_\beta\frac{\partial Y_\beta}{\partial x_j}(X)\sum_k\left(\sum_{m,\ell}\frac{\partial X_\ell}{\partial y_m}\frac{\partial^2 Y_m}{\partial x_\ell\partial x_j}(X)\frac{\partial X_i}{\partial y_k} + \sum_\ell\frac{\partial Y_\ell}{\partial x_j}(X)\frac{\partial^2 X_i}{\partial y_k\partial y_\ell}\right)\widetilde{\varphi}_k\\ &\qquad + [\widetilde{v}\otimes\widetilde{v}]\,(\nabla Y)(X)(\nabla Y)(X)^* : (\nabla\widetilde{\varphi}).\end{aligned} \tag{A.39}$$

This yields the result.

□

B Calculation for the linearization

Here we suppose that (2.7) is satisfied. We recall that

$$X(t,y_1,y_2) = \begin{bmatrix} y_1 \\ y_2 + \theta(y_1,y_2)\xi(t,y_1)\end{bmatrix},$$

where $\theta\in C^3(\mathbb{R}^2)$ is defined in Section 2.2 and where $\xi = \eta - \eta^S$, which originally belongs to $H^2_0(0,1)$, has been extended by zero outside $(0,1)$ to a function of $H^2(\mathbb{R})$ while keeping the same notation.

In what follows, we recall that $\gamma^{(i)}(y,\cdot)$ are linear mappings that depend on y in a Lipschitz continuous way and that vanish in $\overline{\mathcal{F}}\backslash\overline{\mathcal{V}_\alpha}$ (see (2.4)). We also recall that $\mathcal{Q}_2(\alpha_1,\dots,\alpha_k)$ where $k\in\mathbb{N}$ denote the set of polynomials in the variables $\alpha_1,\dots,\alpha_k$ and with coefficients that are Lipschitz continuous functions of $y\in\mathbb{R}^2$ and of ξ and that vanish in $\overline{\mathcal{F}}\backslash\overline{\mathcal{V}_\alpha}$, and such that the degree of its nonzero monomial of lowest degree is greater or equal to 2.

We have

$$(\nabla X) = \begin{bmatrix} 1 & 0 \\ (\partial_{y_1}\theta)\xi + \theta(\partial_s\xi) & 1+(\partial_{y_2}\theta)\xi\end{bmatrix} = I_2 + \gamma^{(13)}(y,\xi,\partial_s\xi). \tag{B.1}$$

We also have
$$\frac{\partial^2 X_i}{\partial y_j \partial y_k} = \gamma^{(14)}_{i,j,k}(y, \xi, \partial_s \xi, \partial_{ss}\xi). \tag{B.2}$$
We deduce from (B.1)
$$(\nabla Y)(X) = \frac{1}{1+(\partial_{y_2}\theta)\xi} \begin{bmatrix} 1+(\partial_{y_2}\theta)\xi & 0 \\ -(\partial_{y_1}\theta)\xi - \theta(\partial_s \xi) & 1 \end{bmatrix}. \tag{B.3}$$
We can write
$$\frac{1}{1+(\partial_{y_2}\theta)\xi} = 1 + \frac{-(\partial_{y_2}\theta)\xi}{1+(\partial_{y_2}\theta)\xi} = 1 - (\partial_{y_2}\theta)\xi + \frac{(\partial_{y_2}\theta)^2(\xi)^2}{1+(\partial_{y_2}\theta)\xi}$$
which implies the following relations
$$\begin{gathered} \frac{1}{1+(\partial_{y_2}\theta)\xi} = 1 + r^{(15)}(\xi) = 1 + \gamma^{(16)}(\xi) + r^{(16)}(\xi), \\ \text{with} \quad r^{(15)} \in \mathcal{Q}_1(\alpha_1), \quad r^{(16)} \in \mathcal{Q}_2(\alpha_1). \end{gathered} \tag{B.4}$$
We deduce from (B.3) and (B.4)
$$(\nabla Y)(X) = I_2 + \gamma^{(17)}(\xi, \partial_s\xi) + r^{(17)}(\xi, \partial_s \xi), \quad r^{(17)} \in \mathcal{Q}_2(\alpha_1, \alpha_2). \tag{B.5}$$
From (B.3), some standard calculation yields
$$\frac{\partial}{\partial y_1} \nabla Y(X) = \frac{\gamma^{(18)}(\xi, \partial_s\xi)}{(1+(\partial_{y_2}\theta)\xi)^2}\left(I_2 + \gamma^{(19)}(\xi,\partial_s\xi)\right) + \frac{1}{1+(\partial_{y_2}\theta)\xi}\gamma^{(20)}(\xi, \partial_s\xi, \partial_{ss}\xi),$$
$$\frac{\partial}{\partial y_2} \nabla Y(X) = \frac{\gamma^{(21)}(\xi)}{(1+(\partial_{y_2}\theta)\xi)^2}\left(I_2 + \gamma^{(19)}(\xi,\partial_s\xi)\right) + \frac{1}{1+(\partial_{y_2}\theta)\xi}\gamma^{(22)}(\xi, \partial_s\xi),$$
which leads to
$$\begin{gathered} \sum_j \frac{\partial X_j}{\partial y_\ell} \frac{\partial^2 Y_i}{\partial x_j \partial x_k}(X) = \gamma^{(23)}_{\ell,i,k}(\xi, \partial_s\xi, \partial_{ss}\xi) + r^{(23)}(\xi, \partial_s\xi, \partial_{ss}\xi), \\ \text{with} \quad r^{(23)} \in \mathcal{Q}_2(\alpha_1, \alpha_2, \alpha_3), \quad \deg_3(r^{(23)}) \leqslant 1. \end{gathered} \tag{B.6}$$
Relations (B.5) and (B.6) imply
$$\begin{gathered} \frac{\partial^2 Y_i}{\partial x_j \partial x_k}(X) = \gamma^{(24)}_{i,j,k}(\xi, \partial_s\xi, \partial_{ss}\xi) + +r^{(24)}(\xi, \partial_s\xi, \partial_{ss}\xi), \\ \text{with} \quad r^{(24)} \in \mathcal{Q}_2(\alpha_1, \alpha_2, \alpha_3), \quad \deg_3(r^{(24)}) \leqslant 1. \end{gathered} \tag{B.7}$$
We also have
$$\partial_t X(t, y_1, y_2) = (\partial_t \xi)\theta e_2. \tag{B.8}$$
We deduce from (B.1) that
$$\begin{gathered} \partial_t \left[\frac{1}{\det(\nabla X)}[\nabla X]^*\right] = \gamma^{(25)}(\partial_t\xi, \partial_{ts}\xi) + r^{(25)}(\xi, \partial_s\xi, \partial_t\xi, \partial_{ts}\xi),, \\ \text{with} \quad r^{(25)} \in \mathcal{Q}_2(\alpha_1, \alpha_2, \alpha_3, \alpha_4), \quad \deg_{3,4}(r^{(25)}) \leqslant 1. \end{gathered} \tag{B.9}$$

We use now the above decomposition in order to linearize the operators appearing with the change of variables. We recall that $\widetilde{v}$ is defined by (2.11) and $\widetilde{w}$ by (2.19). We also recall that (2.7) is satisfied and that we assume $v^S \in W^{2,\infty}(\mathcal{F})$, $\operatorname{div} v^S = 0$ and $f^S \in W^{2,\infty}(\mathcal{F})$.

First we deal with the linearization of the condition on the divergence.

Lemma B.1. *The following equality holds:*

$$\begin{aligned} K(\widetilde{w}+v^S)-v^S &= \widetilde{w}+\gamma^{(1)}(\xi,\partial_s\xi)v^S+r^{(1)}(\xi,\partial_s\xi,\widetilde{w}),\\ \textit{with}\quad r^{(1)} &\in \mathcal{Q}_2(\alpha_1,\alpha_2,\alpha_3,\alpha_4),\quad \deg_{3,4} r^{(1)}\leqslant 1. \end{aligned} \tag{B.10}$$

In particular,

$$\operatorname{div}\Big(K(\widetilde{w}+v^S)\Big)=\operatorname{div}\widetilde{w}+\operatorname{div}(\gamma^{(1)}(\xi,\partial_s\xi)v^S)+\operatorname{div}(r^{(1)}(\xi,\partial_s\xi,\widetilde{w}))\quad \textit{in } \mathcal{F} \tag{B.11}$$

and

$$\widetilde{w}=(\partial_t\xi)e_2-r^{(1)}(\xi,\partial_s\xi,\widetilde{w})\quad \textit{on } \partial\mathcal{F}. \tag{B.12}$$

Proof. We write $K(\widetilde{w}+v^S)-v^S=\widetilde{w}+(K-I_2)(\widetilde{w}+v^S)$ and using (see (B.1) and (B.5))

$$\begin{aligned} K=\det(\nabla X)[\nabla Y](X)[\nabla Y]^*(X)=\operatorname{Cof}(\nabla X)^*[\nabla Y]^*(X)=I_2+\gamma^{(26)}(\xi,\partial_s\xi)+r^{(26)}(\xi,\partial_s\xi)\\ \text{with}\quad r^{(26)}\in\mathcal{Q}_2(\alpha_1,\alpha_2). \end{aligned} \tag{B.13}$$

we deduce (B.10) and (B.11). To obtain (B.12), we start from (2.14):

$$K\widetilde{v}=(\partial_t\eta)e_2=(\partial_t\xi)e_2.$$

Then, since $\widetilde{v}=\widetilde{w}+v^S$ and since $v^S=0$ on $\partial\mathcal{F}$, the result follows from (B.10).

□

We then consider w define by (2.23), i.e. $w=\widetilde{w}+\gamma^{(1)}(\xi,\partial_s\xi)v^S$. Using (B.1)-(B.9) with lemmas A.1, A.2 and A.3 we deduce the following lemmas.

Lemma B.2. *The following equalities hold:*

$$\begin{aligned} \mathcal{M}_\xi^{(3)}(w+(1-\gamma^{(1)})v^S,\nabla w+\nabla((1-\gamma^{(1)})v^S)) &= 2\nu Dv^S+2\nu Dw+\gamma^{(27)}(\xi,\partial_s\xi,\partial_{ss}\xi)\\ &\quad + r^{(27)}(\xi,\partial_s\xi,\partial_{ss}\xi,w)+r^{(28)}(\xi,\partial_s\xi,\nabla w) \end{aligned} \tag{B.14}$$

with

$$r^{(27)}\in\mathcal{Q}_2(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5)\quad \deg_3 r^{(27)}\leqslant 1,\quad \deg_{4,5} r^{(27)}\leqslant 1, \tag{B.15}$$

$$r^{(28)}\in\mathcal{Q}_2(\alpha_1,\alpha_2,\alpha_3,\dots,\alpha_6)\quad \deg_{3,\dots,6} r^{(28)}\leqslant 1, \tag{B.16}$$

and

$$\begin{aligned} \mathcal{M}_\xi^{(4)}(w+(1-\gamma^{(1)})v^S,\nabla w+\nabla((1-\gamma^{(1)})v^S)) &= \gamma^{(29)}(\xi,\partial_s\xi,\partial_{ss}\xi)\\ &\quad + r^{(29)}(\xi,\partial_s\xi,\partial_{ss}\xi,w)+r^{(30)}(\xi,\partial_s\xi,\partial_{ss}\xi,\nabla w) \end{aligned} \tag{B.17}$$

with

$$r^{(29)}\in\mathcal{Q}_2(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5)\quad \deg_3 r^{(29)}\leqslant 2,\quad \deg_{4,5} r^{(29)}\leqslant 1, \tag{B.18}$$

$$r^{(30)}\in\mathcal{Q}_2(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5,\alpha_6,\alpha_7)\quad \deg_3 r^{(30)}\leqslant 1,\quad \deg_{4,\dots,7} r^{(30)}\leqslant 1. \tag{B.19}$$

Lemma B.3. *The following equalities hold:*

$$\mathcal{M}_\xi^{(5)}(w+(1-\gamma^{(1)})v^S)=\gamma^{(31)}(\partial_t\xi)+r^{(31)}(\xi,\partial_s\xi,\partial_t\xi,w), \tag{B.20}$$

with

$$r^{(31)}\in\mathcal{Q}_2(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5)\quad \deg_3 r^{(31)}\leqslant 1,\quad \deg_{4,5} r^{(31)}\leqslant 1, \tag{B.21}$$

and

$$\mathcal{M}_\xi^{(6)}(w+(1-\gamma^{(1)})v^S)=\gamma^{(32)}(\partial_t\xi,\partial_{ts}\xi)+r^{(32)}(\xi,\partial_s\xi,\partial_t\xi,\partial_{ts}\xi,w)+r^{(33)}(\xi,\partial_s\xi,\partial_{ss}\xi,\partial_t\xi,w) \tag{B.22}$$

with

$$r^{(32)}\in\mathcal{Q}_2(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5,\alpha_6)\quad \deg_{3,4} r^{(32)}\leqslant 1,\quad \deg_{5,6} r^{(32)}\leqslant 1, \tag{B.23}$$

$$r^{(33)}\in\mathcal{Q}_2(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5,\alpha_6)\quad \deg_3 r^{(33)}\leqslant 1,\quad \deg_4 r^{(33)}\leqslant 1,\quad \deg_{5,6} r^{(33)}\leqslant 1. \tag{B.24}$$

Lemma B.4. *The following equalities hold:*

$$\mathcal{B}_\xi^{(1)}(w+(1-\gamma^{(1)})v^S, w+(1-\gamma^{(1)})v^S) = \gamma^{(34)}(\xi,\partial_s\xi,\partial_{ss}\xi) + r^{(34)}(\xi,\partial_s\xi,\partial_{ss}\xi,w) \tag{B.25}$$

$$\mathcal{B}_\xi^{(2)}(w+(1-\gamma^{(1)})v^S, w+(1-\gamma^{(1)})v^S) = v^S\otimes v^S + w\otimes v^S + v^S\otimes w + \gamma^{(35)}(\xi,\partial_s\xi) + r^{(35)}(\xi,\partial_s\xi,w) \tag{B.26}$$

with

$$r^{(34)} \in \mathcal{Q}_2(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5) \quad \deg_3 r^{(34)} \leqslant 1, \quad \deg_{4,5} r^{(34)} \leqslant 2, \tag{B.27}$$

$$r^{(35)} \in \mathcal{Q}_2(\alpha_1,\alpha_2,\alpha_3,\alpha_4) \quad \deg_{3,4} r^{(35)} \leqslant 2. \tag{B.28}$$

Lemma B.5. *The following equality holds:*

$$\begin{aligned}(\nabla X)^*(f^S\circ X) - f^S &= \gamma^{(36)}(\xi,\partial_s\xi) + r^{(36)}(\xi,\partial_s\xi),\\ &\textit{with} \quad r^{(36)} \in \mathcal{Q}_2(\alpha_1,\alpha_2).\end{aligned} \tag{B.29}$$

Proof. We write

$$(\nabla X)^*(f^S\circ X) - f^S = [(\nabla X)^* - I_2]f^S + (\nabla X)^*\left[(f^S\circ X) - f^S\right]$$

and the result follows from

$$f^S(X(y)) = f^S(y) + \theta\xi\nabla f^S(y)\cdot e_2 + \int_0^1 (1-s)\nabla^2 f^S(y+s\theta\xi e_2)\theta\xi e_2\cdot\theta\xi e_2\ ds.$$

□

C Anisotropic Estimates

In this section, we suppose that $[w,\xi_1,\xi_2]\in\mathcal{G}$ and $w\in H_\sigma^{1/2+\varepsilon}(\mathbf{H}^{-1/2-2\varepsilon}(\partial\mathcal{F}))$, where $\mathcal{G}$ is defined by (4.47), with the estimates

$$\|[w,\xi_1,\xi_2]\|_{\mathcal{G}} + \|w\|_{H_\sigma^{1/2+\varepsilon}(\mathbf{H}^{-1/2-2\varepsilon}(\partial\mathcal{F}))} \leqslant R_0, \tag{C.1}$$

for $R_0\in(0,1]$. This implies (5.6)-(5.15) with the corresponding bound in term of R_0. From (5.12) and from

$$\xi_1 \in H_\sigma^{3/2}(L^2(0,1))\cap H_\sigma^{1/2}(H_0^2(0,1)) \hookrightarrow H_\sigma^{5/8}(H_0^{7/4}(0,1)),$$

$$\xi_1 \in H_\sigma^{1}(H_0^1(0,1))\cap L_\sigma^{2}(H^3(0,1)\cap H_0^2(0,1)) \hookrightarrow H_\sigma^{3/8}(H_0^{9/4}(0,1)) \hookrightarrow L_\sigma^{8}(H_0^{9/4}(0,1)),$$

$$\xi_2 \in H_\sigma^{1/2}(L^2(0,1))\cap L_\sigma^{2}(H_0^1(0,1)) \hookrightarrow H_\sigma^{1/8}(H_0^{3/4}(0,1)) \hookrightarrow L_\sigma^{8/3}(H_0^{3/4}(0,1)),$$

$$\xi_2 \in H_\sigma^{1/2}(L^2(0,1))\cap L_\sigma^{2}(H_0^1(0,1)) \hookrightarrow H_\sigma^{3/8}(H^{1/4}(0,1)) \hookrightarrow L_\sigma^{8}(H^{1/4}(0,1)),$$

$$w \in H_\sigma^{1/2}(\mathbf{L}^2(\mathcal{F}))\cap L_\sigma^{2}(\mathbf{H}^1(\mathcal{F})) \hookrightarrow H_\sigma^{1/8}(\mathbf{H}^{3/4}(\mathcal{F})) \hookrightarrow L_\sigma^{8/3}(\mathbf{H}^{3/4}(\mathcal{F})),$$

$$w \in H_\sigma^{1/2}(\mathbf{L}^2(\mathcal{F}))\cap L_\sigma^{2}(\mathbf{H}^1(\mathcal{F})) \hookrightarrow H_\sigma^{3/8}(\mathbf{H}^{1/4}(\mathcal{F})) \hookrightarrow L_\sigma^{8}(\mathbf{H}^{1/4}(\mathcal{F})),$$

we obtain the following estimate:

$$\begin{aligned}&\|\xi_1\|_{H_\sigma^{5/8}(H^{7/4}(0,1))} + \|\xi_1\|_{L_\sigma^8(H^{9/4}(0,1))} + \|\xi_2\|_{L_\sigma^{8/3}(H^{3/4}(0,1))} + \|\xi_2\|_{L_\sigma^8(H^{1/4}(0,1))}\\ &\quad + \|w\|_{L_\sigma^{8/3}(\mathbf{H}^{3/4}(\mathcal{F}))} + \|w\|_{L_\sigma^8(\mathbf{H}^{1/4}(\mathcal{F}))} + \|\xi_1\|_{L_\sigma^\infty(L^\infty(0,1))} + \|\partial_s\xi_1\|_{L_\sigma^\infty(L^\infty(0,1))} \leqslant CR_0.\end{aligned} \tag{C.2}$$

In what follows, we suppose $R_0>0$ small enough so that

$$\|\xi_1\|_{L^\infty(L^\infty(0,1))} \leqslant c_0 \quad \text{for } c_0>0 \text{ given in (2.8)}. \tag{C.3}$$

Let $\mathcal{T}=[-L,L]\times[-L,L]$ be a rectangle such that $\overline{\mathcal{F}}\subset\mathcal{T}$. There exists an extension operator E which is continuous from $L^2(\mathcal{F})$ into $L^2(\mathcal{T})$ as well as from $H^1(\mathcal{F})$ into $H_0^1(\mathcal{T})$. Note that an interpolation argument guarantees that we also have $E\in\mathcal{L}(H^s(\mathcal{F}),H_0^s(\mathcal{T}))$ for $s\in(0,1)$. Then using this extension operator, any function in $H^s(\mathcal{F})$, $s\in[0,1]$ can be considered as a function in $H_0^s(\mathcal{T})$: in what follows we will extend w_i,

$i = 1, 2$ or some test functions φ, but for simplicity we will keep the same name w_i and φ instead of $E(w_i)$ and $E(\varphi)$. We will also freely use the boundedness properties of E without recalling it.

Moreover, any function defined on $(0, 1)$ is considered as a function defined on $\mathcal{T}$ but only depending on y_1 and equal to zero outside $(0, 1)$: by this way we can consider ξ_1 and ξ_2 as functions defined on $\mathcal{T}$.

For $p \in [1, +\infty)$, $s \geqslant -1$ and a function $f : \mathcal{T} \to \mathbb{R}$ we write

$$\|f\|_{L^p_{y_2}(H^s_{y_1})} \stackrel{\text{def}}{=} \left(\int_{-L}^{L} \|f(\cdot, y_2)\|^p_{H^s(-L,L)} dy_2 \right)^{1/p}$$

and

$$\|f\|_{L^\infty_{y_2}(H^s_{y_1})} \stackrel{\text{def}}{=} \operatorname{supess}_{y_2 \in (-L,L)} \|f(\cdot, y_2)\|_{H^s(-L,L)}.$$

We also define the $\| \cdot \|_{L^p_{y_1}(H^s_{y_2})}$ analogously, just by reversing the role of y_1, y_2.

Let us stress some basic properties that we use below. First for $s \geqslant 0$ and $f \in H^s(\mathcal{T})$ we have

$$\|f\|_{L^2_{y_i}(H^s_{y_j})} \leqslant \|f\|_{H^s(\mathcal{T})}, \quad (i, j \in \{1, 2\},\ i \neq j), \tag{C.4}$$

and (see [26, Cor 1.4.4.5]),

$$\|\mathbf{1}_{\mathcal{F}} f\|_{H^s(\mathcal{T})} \leqslant C\|f\|_{H^s(\mathcal{T})} \quad \text{if } s \in [0, 1/2). \tag{C.5}$$

Above and in what follows, $\mathbf{1}_{\mathcal{F}}$ denotes the characteristic functions of $\mathcal{F}$. Moreover, for $p \in [1, +\infty]$, $s \in [0, 2] \backslash \{\frac{1}{2}, \frac{3}{2}\}$ and $g \in H^s_0(0, 1)$,

$$\|g\|_{L^p_{y_2}(H^s_{y_1})} \leqslant C\|g\|_{H^s_0(0,1)}. \tag{C.6}$$

We also have the following result.

Lemma C.1. *Assume $p \in [1, +\infty]$, $s \in [0, 1/2)$ and $g \in H^s(0, 1)$. Then we have,*

$$\|\mathbf{1}_{\mathcal{F}} g\|_{L^p_{y_2}(H^s_{y_1})} \leqslant C\|g\|_{H^s(0,1)}.$$

Proof. First, for any nonempty open interval I and any function $f \in H^s(I)$, we denote its extension by 0 outside I by $E_I(f)$. Thus, we define $\tilde{H}^s(I)$ as the space of functions f in $H^s(I)$ such that $E_I(f) \in H^s(\mathbb{R})$ equipped with the norm $\|f\|_{\tilde{H}^s(I)} \stackrel{\text{def}}{=} \|E_I(f)\|_{H^s(\mathbb{R})}$. Since $s < 1/2$, one can prove (see [26, Cor. 1.4.4.5]),

$$\|f\|_{\tilde{H}^s(I)} \leqslant C\|f\|_{H^s(I)}. \tag{C.7}$$

Moreover, we can verify that the above constant C is independent of I. Let us detail the argument. We recall that if X is an open subset of $\mathbb{R}$ the norm of $H^s(X)$ is defined by

$$\|f\|_{H^s(X)} \stackrel{\text{def}}{=} \int_X \int_X \frac{|f(y_1) - f(\tilde{y}_1)|^2}{|y_1 - \tilde{y}_1|^{1+2s}} dy_1 d\tilde{y}_1.$$

Then if we assume $I = (0, A)$ with $A > 0$, from easy calculations we deduce

$$\|f\|^2_{\tilde{H}^s(I)} \stackrel{\text{def}}{=} \|E_I(f)\|^2_{H^s(\mathbb{R})} = \|f\|^2_{H^s(I)} + \int_0^A |f(y_1)|^2 \rho_{s,A}(y_1) dy_1,$$

with

$$\rho_{s,A}(y_1) \stackrel{\text{def}}{=} 2 \int_{(-\infty,0) \cup (A,+\infty)} \frac{du}{|y_1 - u|^{1+2s}} = \frac{1}{s} \left(\frac{1}{y_1^{2s}} + \frac{1}{(A - y_1)^{2s}} \right).$$

Thus, by observing that $\rho_{s,A}(y_1) = A^{-2s}\rho_{s,1}(y_1/A)$ and by using Hardy's inequality we obtain

$$\begin{aligned}
\int_0^A |f(y_1)|^2 \rho_{1,A}(y_1) dy_1 &= A^{-1} \int_0^1 |f(Au)|^2 \rho_{1,1}(u) du \\
&\leqslant C_* A^{-1} \int_0^1 |Af'(Au)|^2 du = C_* \int_0^A |f'(y_1)|^2 dy_1 = C_* \|f\|^2_{H^1_0(I)}
\end{aligned}$$

for a constant $C_* > 0$ which is independent on A. Thus, with an interpolation argument (see [43, Thm. 1.18.5, p. 130]) we deduce that

$$\int_0^A |f(y_1)|^2 (\rho_{1,A}(y_1))^s dy_1 \leqslant C_*^s \|f\|^2_{H^s_0(I)},$$

and (C.7) follows from $(\rho_{1,A}(y_1))^s \geqslant s2^{s-1}\rho_{s,A}(y_1)$ and from $H^s(I) = H^s_0(I)$ (because $s \in [0,1/2)$) and $\|\cdot\|_{H^s(I)} = \|\cdot\|_{H^s_0(I)}$.

Next, the lemma is a consequence of the following inequality: for a.e. $y_2 \in (-L, L)$,

$$\|\mathbf{1}_{\mathcal{F}}(\cdot, y_2)g\|_{H^s(-L,L)} \leqslant C\|g\|_{H^s(-L,L)}, \tag{C.8}$$

for a constant $C > 0$ independent on y_2. In order to prove this inequality, we define for $y_2 \in \mathbb{R}$ the set $\Delta_{y_2} \overset{\text{def}}{=} \{y_1 \in (-L, L) \,;(y_1, y_2) \in \mathcal{F}\}$. Since $\mathcal{F}$ is smooth, the set Δ_{y_2} is a finite union of open segments: $\Delta_{y_2} = \cup_{j=1}^{N_{y_2}} I_j$ with N_{y_2} bounded independently on y_2. Finally, using (C.7), we can prove (C.8):

$$\begin{aligned}\|\mathbf{1}_{\mathcal{F}}(\cdot, y_2)g\|_{H^s(-L,L)} \leqslant \|\mathbf{1}_{\mathcal{F}}(\cdot, y_2)g\|_{H^s(\mathbb{R})} = \left\|\sum_{j=1}^{N_{y_2}} \mathbf{1}_{I_j} g\right\|_{H^s(\mathbb{R})} \leqslant \sum_{j=1}^{N_{y_2}} \left\|\mathbf{1}_{I_j} g\right\|_{H^s(\mathbb{R})} = \sum_{j=1}^{N_{y_2}} \|g\|_{\tilde{H}^s(I_j)} \\ \leqslant C\sum_{j=1}^{N_{y_2}} \|g\|_{H^s(I_j)} \leqslant CN_{y_2}\|g\|_{H^s(\mathbb{R})} = CN_{y_2}\|g\|_{\tilde{H}^s(0,1)} \leqslant C\|g\|_{H^s(0,1)}.\end{aligned}$$

Here, $\mathbf{1}_{I_j}$ denote the characteristic functions of I_j. Above we have use the fact that g is zero outside $(0, 1)$. □

In Lemmas C.2 to C.4 below τ is a bounded Lipschitz continuous function of $\mathbb{R}^2 \times [-c_0, c_0] \times \mathbb{R}$ with values in $\mathbb{R}$ for $c_0 > 0$ given in (2.8). From (C.3) we deduce

$$\|\tau(y, \xi_1, \partial_s\xi_1)\|_{L^\infty_\sigma(L^\infty_{y_2}(H^1_{y_1}))} + \|\partial_{y_2}\tau(y, \xi_1, \partial_s\xi_1)\|_{L^\infty_\sigma(L^\infty(\mathcal{T}))} \leqslant C. \tag{C.9}$$

Lemma C.2. *Assume* (C.9)*. For $i, j = 1, 2$ the following estimate holds,*

$$\left\|\tau(y, \xi_1, \partial_s\xi_1)(\partial_{ss}\xi_1)\partial_{y_j} w_i\right\|_{L^2_\sigma((H^1(\mathcal{F})')} \leqslant CR_0^2.$$

Proof. We start with the case $j = 1$. Assume $\varphi \in H^1(\mathcal{F})$. First, we have for a.e. $t \in (0,\infty)$,

$$\begin{aligned}\left|\int_{\mathcal{F}} \tau(y, \xi_1, \partial_s\xi_1)(\partial_{ss}\xi_1)\partial_{y_1} w_i\varphi dy\right| &= \left|\int_{\mathcal{T}} \mathbf{1}_{\mathcal{F}}\tau(y, \xi_1, \partial_s\xi_1)(\partial_{ss}\xi_1)\partial_{y_1} w_i\varphi dy\right| \\ &\leqslant \|\mathbf{1}_{\mathcal{F}}\tau(y, \xi_1, \partial_s\xi_1)(\partial_{ss}\xi_1)\varphi\|_{L^2_{y_2}(H^{1/4}_{y_1})}\|\partial_{y_1} w_i\|_{L^2_{y_2}(H^{-1/4}_{y_1})}.\end{aligned}$$

Since

$$\|g_1g_2\|_{H^{1/4}(-L,L)} \leqslant C\|g_1\|_{H^1(-L,L)}\|g_2\|_{H^{1/4}(-L,L)}, \tag{C.10}$$

and

$$\|\partial_{y_1} w_i\|_{L^2_{y_2}(H^{-1/4}_{y_1})} \leqslant C\|w_i\|_{L^2_{y_2}(H^{3/4}_{y_1})}$$

we deduce

$$\left|\int_{\mathcal{F}} \tau(y, \xi_1, \partial_s\xi_1)(\partial_{ss}\xi_1)\partial_{y_1} w_i\varphi dy\right| \leqslant C\|\mathbf{1}_{\mathcal{F}}\tau(y, \xi_1, \partial_s\xi_1)(\partial_{ss}\xi_1)\|_{L^\infty_{y_2}(H^{1/4}_{y_1})}\|\varphi\|_{L^2_{y_2}(H^1_{y_1})}\|w_i\|_{L^2_{y_2}(H^{3/4}_{y_1})}.$$

Now using (C.4) and Lemma C.1, we obtain

$$\begin{aligned}\left|\int_{\mathcal{F}} \tau(y, \xi_1, \partial_s\xi_1)(\partial_{ss}\xi_1)\partial_{y_1} w_i\varphi dy\right| &\leqslant C\|\tau(y, \xi_1, \partial_s\xi_1)\|_{L^\infty_{y_2}(H^1_{y_1})}\|(\mathbf{1}_{\mathcal{F}}\partial_{ss}\xi_1)\|_{L^\infty_{y_2}(H^{1/4}_{y_1})}\|\varphi\|_{H^1(\mathcal{T})}\|w_i\|_{H^{3/4}(\mathcal{T})} \\ &\leqslant C\|\tau(y, \xi_1, \partial_s\xi_1)\|_{L^\infty_{y_2}(H^1_{y_1})}\|\partial_{ss}\xi_1\|_{H^{1/4}(0,1)}\|\varphi\|_{H^1(\mathcal{T})}\|w_i\|_{H^{3/4}(\mathcal{T})}\end{aligned}$$

and thus with (C.9),

$$\|\tau(y, \xi_1, \partial_s\xi_1)(\partial_{ss}\xi_1)\partial_{y_1} w_i\|_{L^2_\sigma(H^1(\mathcal{F})')} \leqslant C\|\xi_1\|_{L^8_\sigma(H^{9/4}_0(0,1))}\|w_i\|_{L^{8/3}_\sigma(H^{3/4}(\mathcal{F}))}.$$

We conclude by using (C.2).

Finally, the case $j = 2$ follows more easily from an integration by parts with (5.13)-(5.15), by taking into account the estimate of $\partial_{y_2}\tau$ in (C.9) and the fact that ξ_1 is independent on y_2. The details are left to the reader. □

Lemma C.3. *Assume* (C.9). *For* $i = 1, 2$ *the following estimate holds,*

$$\|\tau(y, \xi_1, \partial_s\xi_1)(\partial_s\xi_2)w_i\|_{L^2_\sigma((H^1(\mathcal{F})')} + \|\tau(y, \xi_1, \partial_s\xi_1)\xi_2 w_i\|_{L^2_\sigma((H^1(\mathcal{F})'))} \leqslant CR_0^2.$$

Proof. We only prove the first estimate, the second can be obtained more easily. Assume $\varphi \in H^1(\mathcal{F})$. First, using (C.10), (C.4), (C.6), (C.9) and (C.5) we have for a.e. $t \in (0, \infty)$,

$$\begin{aligned}
\left|\int_{\mathcal{F}} \tau(y, \xi_1, \partial_s\xi_1)(\partial_s\xi_2)w_i\varphi dy\right| &= \left|\int_{\mathcal{T}} \tau(y, \xi_1, \partial_s\xi_1)(\partial_s\xi_2)(\mathbf{1}_{\mathcal{F}}w_i)\varphi dy\right| \\
&\leqslant C\,\|\tau(y, \xi_1, \partial_s\xi_1)(\mathbf{1}_{\mathcal{F}}w_i)\varphi\|_{L^1_{y_2}(H^{1/4}_{y_1})}\,\|\partial_s\xi_2\|_{L^\infty_{y_2}(H^{-1/4}_{y_1})} \\
&\leqslant C\,\|\tau(y, \xi_1, \partial_s\xi_1)(\mathbf{1}_{\mathcal{F}}w_i)\|_{L^2_{y_2}(H^{1/4}_{y_1})}\,\|\varphi\|_{L^2_{y_2}(H^1_{y_1})}\,\|\xi_2\|_{L^\infty_{y_2}(H^{3/4}_{y_1})} \\
&\leqslant C\,\|\tau(y, \xi_1, \partial_s\xi_1)\|_{L^\infty_{y_2}(H^1_{y_1})}\,\|(\mathbf{1}_{\mathcal{F}}w_i)\|_{L^2_{y_2}(H^{1/4}_{y_1})}\,\|\xi_2\|_{H^{3/4}_0(0,1)}\,\|\varphi\|_{H^1(\mathcal{T})} \\
&\leqslant C\,\|(\mathbf{1}_{\mathcal{F}}w_i)\|_{H^{1/4}(\mathcal{F})}\,\|\xi_2\|_{H^{3/4}_0(0,1)}\,\|\varphi\|_{H^1(\mathcal{F})} \leqslant C\,\|w_i\|_{H^{1/4}(\mathcal{F})}\,\|\xi_2\|_{H^{3/4}_0(0,1)}\,\|\varphi\|_{H^1(\mathcal{F})}.
\end{aligned}$$

The above estimate yields

$$\|\tau(y, \xi_1, \partial_s\xi_1)(\partial_s\xi_2)w_i\|_{L^2_\sigma((H^1(\mathcal{F})')} \leqslant C\,\|w_i\|_{L^8_\sigma(H^{1/4}(\mathcal{F}))}\,\|\xi_2\|_{L^{8/3}_\sigma(H^{3/4}(0,1))}$$

and we conclude by using (C.2). ☐

Lemma C.4. *Assume* (C.9). *For* $i, j = 1, 2$ *the following estimate holds,*

$$\|\tau(y, \xi_1, \partial_s\xi_1)(\partial_{ss}\xi_1)w_iw_j\|_{L^2_\sigma(H^1(\mathcal{F})')} \leqslant CR_0^2.$$

Proof. Assume $\varphi \in H^1(\mathcal{F})$. First, we have for a.e. $t \in (0, \infty)$,

$$\begin{aligned}
\left|\int_{\mathcal{F}} \tau(y, \xi_1, \partial_s\xi_1)(\partial_{ss}\xi_1)w_iw_j\varphi dy\right| &= \left|\int_{\mathcal{T}} \tau(y, \xi_1, \partial_s\xi_1)(\partial_{ss}\xi_1)(\mathbf{1}_{\mathcal{F}}w_i)w_j\varphi dy\right| \\
&\leqslant C\,\|\tau(y, \xi_1, \partial_s\xi_1)(\mathbf{1}_{\mathcal{F}}w_i)w_j\varphi\|_{L^1_{y_2}(L^2_{y_1})}\,\|\partial_{ss}\xi_1\|_{L^\infty_{y_2}(L^2_{y_1})} \\
&\leqslant C\,\|\tau(y, \xi_1, \partial_s\xi_1)(\mathbf{1}_{\mathcal{F}}w_i)w_j\|_{L^2_{y_2}(L^2_{y_1})}\,\|\varphi\|_{L^2_{y_2}(H^1_{y_1})}\,\|\xi_1\|_{H^2_0(0,1)} \\
&\leqslant C\|\tau(y, \xi_1, \partial_s\xi_1)\|_{L^\infty_{y_2}(H^1_{y_1})}\|(\mathbf{1}_{\mathcal{F}}w_i)w_j\|_{L^2(\mathcal{T})}\,\|\xi_1\|_{H^2_0(0,1)}\,\|\varphi\|_{H^1(\mathcal{F})}. \qquad (C.11)
\end{aligned}$$

On the other hand,

$$\|(\mathbf{1}_{\mathcal{F}}w_i)w_j\|_{L^2(\mathcal{T})} \leqslant C\,\|(\mathbf{1}_{\mathcal{F}}w_i)\|_{H^{1/4}(\mathcal{F})}\,\|w_j\|_{H^{3/4}(\mathcal{F})}.$$

Combining the above equation and (C.11), with (C.5) and (C.9) we deduce

$$\|\tau(y, \xi_1, \partial_s\xi_1)(\partial_{ss}\xi_1)w_iw_j\|_{L^2_\sigma(H^1(\mathcal{F})')} \leqslant C\,\|\xi_1\|_{L^\infty_\sigma(H^2_0(0,1))}\,\|w_i\|_{L^8_\sigma(H^{1/4}(\mathcal{F}))}\,\|w_j\|_{L^{8/3}_\sigma(H^{3/4}(\mathcal{F}))},$$

and we conclude by using (C.2). ☐

In Lemma C.5 below, τ is a bounded Lipschitz continuous function of $\mathbb{R}^2 \times [-c_0, c_0]$ with values in $\mathbb{R}$ for $c_0 > 0$ given in (2.8). From (C.2) and (C.3) we deduce that $t \mapsto \tau(y, \xi_1(t, y_1))$ satisfies

$$\|\tau(y, \xi_1)\|_{L^\infty(W^{1,\infty}(\mathcal{T}))} \leqslant C. \qquad (C.12)$$

Moreover, we also assume that $\tau(y, \cdot)$ is zero for $y \in \overline{\mathcal{F}}\backslash\overline{\mathcal{V}_\alpha}$ (see (2.4)). By this way,

$$\forall (y, \zeta) \in (\partial\mathcal{F}\backslash\overline{\Gamma}_{\mathrm{str}}) \times [-c_0, c_0] \quad \tau(y, \zeta) = 0. \qquad (C.13)$$

Lemma C.5. *Let* $n_1, n_2 \in \mathbb{N}$ *such that* $n_1 + n_2 \geqslant 1$ *and assume* (C.12)-(C.13). *Then*

$$\|\tau(y, \xi_1)\xi_1^{n_1}(\partial_s\xi_1)^{n_2}w_i\|_{H^{1/2+\varepsilon}_\sigma(H^{2\varepsilon}(\mathcal{F})')} + \|\tau(y, \xi_1)\xi_1^{n_1}(\partial_s\xi_1)^{n_2}w_i\|_{H^{1/2+\varepsilon}_\sigma(H^{-1/2-2\varepsilon}(\partial\mathcal{F}))} \leqslant CR_0^2. \qquad (C.14)$$

Proof. We start by estimating the first term in the left-hand side of (C.14). First we have the following relations

$$H^{2\varepsilon}(\mathcal{T}) = [H^1(\mathcal{T}), L^2(\mathcal{T})]_{1-2\varepsilon} = [H^1_{y_1}(L^2_{y_2}(-L,L)) \cap L^2_{y_1}(H^1_{y_2}(-L,L)), L^2_{y_1}(L^2_{y_2}(-L,L))]_{1-2\varepsilon}$$
$$= H^{2\varepsilon}_{y_1}(L^2_{y_2}(-L,L)) \cap L^2_{y_1}(H^{2\varepsilon}_{y_2}(-L,L)).$$

Each of the above equalities follows from classical arguments (see for instance[25]). The last relation can be written

$$H^{2\varepsilon}(\mathcal{T}) = L^2_{y_1}(H^{2\varepsilon}_{y_2}(-L,L)) \cap L^2_{y_2}(H^{2\varepsilon}_{y_1}(-L,L)).$$

Assume now that $\varphi \in H^{2\varepsilon}(\mathcal{F})$. Using (C.12) and the above relation, we have for a.e. $t \in (0,\infty)$,

$$\begin{aligned}
\left|\int_{\mathcal{F}} \tau(y,\xi_1)\xi_1^{n_1}(\partial_s\xi_1)^{n_2} w_i\varphi dy\right| &= \left|\langle w_i, \tau(y,\xi_1)\xi_1^{n_1}(\partial_s\xi_1)^{n_2}\varphi\rangle_{H^{2\varepsilon}(\mathcal{F})',H^{2\varepsilon}(\mathcal{F})}\right| \\
&\leqslant C\left\|w_i\right\|_{H^{2\varepsilon}(\mathcal{F})'}\left\|\tau(y,\xi_1)\xi_1^{n_1}(\partial_s\xi_1)^{n_2}\varphi\right\|_{H^{2\varepsilon}(\mathcal{F})} \\
&\leqslant C\left\|w_i\right\|_{H^{2\varepsilon}(\mathcal{F})'}\left\|\xi_1^{n_1}(\partial_s\xi_1)^{n_2}\mathbf{1}_{\mathcal{F}}\varphi\right\|_{H^{2\varepsilon}(\mathcal{T})} \\
&\leqslant C\left\|w_i\right\|_{H^{2\varepsilon}(\mathcal{F})'}\left(\left\|\xi_1^{n_1}(\partial_s\xi_1)^{n_2}\mathbf{1}_{\mathcal{F}}\varphi\right\|_{L^2_{y_2}(H^{2\varepsilon}_{y_1})} + \left\|\xi_1^{n_1}(\partial_s\xi_1)^{n_2}\mathbf{1}_{\mathcal{F}}\varphi\right\|_{L^2_{y_1}(H^{2\varepsilon}_{y_2})}\right). \quad \text{(C.15)}
\end{aligned}$$

Moreover, using (C.4), (C.6), (C.5) we deduce,

$$\begin{aligned}
\left\|\xi_1^{n_1}(\partial_s\xi_1)^{n_2}\mathbf{1}_{\mathcal{F}}\varphi\right\|_{L^2_{y_2}(H^{2\varepsilon}_{y_1})} &\leqslant C\left\|\xi_1^{n_1}(\partial_s\xi_1)^{n_2}\right\|_{L^\infty_{y_2}(H^{1/2+2\varepsilon}_{y_1})}\left\|\mathbf{1}_{\mathcal{F}}\varphi\right\|_{L^2_{y_2}(H^{2\varepsilon}_{y_1})} \\
&\leqslant C\|\xi_1\|^{n_1}_{L^\infty_{y_2}(H^{1/2+2\varepsilon}_{y_1})}\|\partial_s\xi_1\|^{n_2}_{L^\infty_{y_2}(H^{1/2+2\varepsilon}_{y_1}}\left\|\mathbf{1}_{\mathcal{F}}\varphi\right\|_{H^{2\varepsilon}(\mathcal{T})} \\
&\leqslant C\|\xi_1\|^{n_1}_{H^{1/2+2\varepsilon}_0(0,1)}\|\partial_s\xi_1\|^{n_2}_{H^{1/2+2\varepsilon}_0(0,1)}\left\|\varphi\right\|_{H^{2\varepsilon}(\mathcal{T})}
\end{aligned}$$

and using (C.4) and (C.5) with the fact that ξ_1 is independent on y_2 we deduce,

$$\begin{aligned}
\left\|\xi_1^{n_1}(\partial_s\xi_1)^{n_2}\mathbf{1}_{\mathcal{F}}\varphi\right\|_{L^2_{y_1}(H^{2\varepsilon}_{y_2})} &\leqslant C\left\|\xi_1^{n_1}(\partial_s\xi_1)^{n_2}\right\|_{L^\infty_{y_1}(H^{1/2+2\varepsilon}_{y_2})}\left\|\mathbf{1}_{\mathcal{F}}\varphi\right\|_{L^2_{y_1}(H^{2\varepsilon}_{y_2})} \\
&\leqslant C\|\xi_1\|^{n_1}_{L^\infty(0,1)}\|\partial_s\xi_1\|^{n_2}_{L^\infty(0,1)}\left\|\varphi\right\|_{H^{2\varepsilon}(\mathcal{T})}.
\end{aligned}$$

Finally, since $\varepsilon \in (0,1/8)$ we have $H^{1-2\varepsilon}_0(0,1) \hookrightarrow H^{1/2+2\varepsilon}_0(0,1)) \hookrightarrow L^\infty(0,1)$ and combining the above inequalities with (C.15) the result follows from (C.1), (C.2).

For the second relation, assume $\varphi \in H^{1/2+2\varepsilon}(\partial\mathcal{F})$. For a.e. $t \in (0,\infty)$ we have,

$$\begin{aligned}
\left|\int_{\partial\mathcal{F}} \tau(y,\xi_1)\xi_1^{n_1}(\partial_s\xi_1)^{n_2} w_i\varphi d\gamma\right| &= \left|\langle w_i, \tau(y,\xi_1)\xi_1^{n_1}(\partial_s\xi_1)^{n_2}\varphi\rangle_{H^{-1/2-2\varepsilon}(\partial\mathcal{F}),H^{1/2+2\varepsilon}(\partial\mathcal{F})}\right| \\
&\leqslant C\left\|w_i\right\|_{H^{-1/2-2\varepsilon}(\partial\mathcal{F})}\left\|\tau(y,\xi_1)\xi_1^{n_1}(\partial_s\xi_1)^{n_2}\right\|_{H^{1/2+2\varepsilon}(\partial\mathcal{F})}\left\|\varphi\right\|_{H^{1/2+2\varepsilon}(\partial\mathcal{F})}. \quad \text{(C.16)}
\end{aligned}$$

Thus, taking (C.13) into account we deduce that

$$\begin{aligned}
\left\|\tau(y,\xi_1)\xi_1^{n_1}(\partial_s\xi_1)^{n_2}\right\|_{H^{1/2+2\varepsilon}(\partial\mathcal{F})} &= \left\|\tau(y,\xi_1)\xi_1^{n_1}(\partial_s\xi_1)^{n_2}\right\|_{H^{1/2+2\varepsilon}(\Gamma_{\text{str}})} \\
&\leqslant C\left\|\tau(y,\xi_1)\right\|_{W^{1,\infty}(\Gamma_{\text{str}})}\left\|\xi_1\right\|^{n_1}_{H^{1/2+2\varepsilon}(\Gamma_{\text{str}})}\left\|\partial_s\xi_1\right\|^{n_2}_{H^{1/2+2\varepsilon}(\Gamma_{\text{str}})}. \quad \text{(C.17)}
\end{aligned}$$

Moreover, we remark that for a function g defined on $\mathcal{T}$ and only depending on y_1 we have

$$\begin{aligned}
\|g\|^2_{H^{1/2+2\varepsilon}(\Gamma_{\text{str}})} &= \int_{\Gamma_{\text{str}}} |g(y)|d\gamma(y) + \int_{\Gamma_{\text{str}}}\int_{\Gamma_{\text{str}}} \frac{|g(y)-g(\tilde{y})|^2}{|y-\tilde{y}|^{2+4\varepsilon}} d\gamma(y)d\gamma(\tilde{y}) \\
&\leqslant \int_{\Gamma_{\text{str}}} |g(y_1)|d\gamma(y) + \int_{\Gamma_{\text{str}}}\int_{\Gamma_{\text{str}}} \frac{|g(y_1)-g(\tilde{y}_1)|^2}{|y_1-\tilde{y}_1|^{2+4\varepsilon}} d\gamma(y)d\gamma(\tilde{y}) \\
&\leqslant \int_0^1 |g(s)|\ell(s)ds + \int_0^1\int_0^1 \frac{|g(s)-g(\tau)|^2}{|s-\tau|^{2+4\varepsilon}}\ell(s)\ell(\tau)dsd\tau \leqslant \|\ell\|^2_{L^\infty(0,1)}\|g\|^2_{H^{1/2+2\varepsilon}(0,1)},
\end{aligned}$$

where we have used the notation $\ell(s) = \sqrt{1 + (\partial_s \eta^S(s))^2}$. Then combining (C.16) and (C.17) with the above estimate for $g = \xi_1$ and $g = \partial_s \xi_1$, with (C.12) we obtain for a.e. $t \in (0, +\infty)$,

$$\|\tau(y, \xi_1)\xi_1^{n_1}(\partial_s \xi_1)^{n_2} w_i\|_{H^{-1/2-2\varepsilon}(\partial \mathcal{F})} \leqslant C \|\xi_1\|^{n_1}_{H^{1/2+2\varepsilon}(0,1)} \|\partial_s \xi_1\|^{n_2}_{H^{1/2+2\varepsilon}(0,1)} \|w_i\|_{H^{-1/2-2\varepsilon}(\partial \mathcal{F})}.$$

Finally, since $\varepsilon \in (0, 1/8)$ we have $H^{1-2\varepsilon}(0,1) \hookrightarrow H^{1/2+2\varepsilon}(0,1)$ and the conclusion follows from (C.1). □

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