



HAL
open science

Almost mixing of all orders and CLT for some Z^d -actions on subgroups of $F_p^{Z^d}$

Guy Cohen, Jean-Pierre Conze

► **To cite this version:**

Guy Cohen, Jean-Pierre Conze. Almost mixing of all orders and CLT for some Z^d -actions on subgroups of $F_p^{Z^d}$. 2016. hal-01369614

HAL Id: hal-01369614

<https://hal.science/hal-01369614>

Preprint submitted on 21 Sep 2016

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

ALMOST MIXING OF ALL ORDERS AND CLT FOR SOME \mathbb{Z}^d -ACTIONS ON SUBGROUPS OF $\mathbb{F}_p^{\mathbb{Z}^d}$

GUY COHEN AND JEAN-PIERRE CONZE

ABSTRACT. For \mathbb{N}^d -actions by algebraic endomorphisms on compact abelian groups, the existence of non-mixing configurations is related to " S -unit type" equations and plays a role in limit theorems for such actions.

We consider a family of endomorphisms on shift-invariant subgroups of $\mathbb{F}_p^{\mathbb{Z}^d}$ and show that there are few solutions of the corresponding equations. This implies the validity of the Central Limit Theorem for different methods of summation.

CONTENTS

Introduction	2
1. Shift-invariant subgroups of $\mathbb{F}_p^{\mathbb{Z}^d}$ and a class of endomorphism	2
1.1. Shift-invariant subgroups of $\mathbb{F}_p^{\mathbb{Z}^d}$	2
1.2. Endomorphisms of $\mathbb{F}_p^{\mathbb{Z}}$ and their invertible extension	3
1.3. Non r -mixing tuples	6
2. Basic special \mathcal{D}-polynomials	7
2.1. Decomposition of special \mathcal{D} -polynomials	7
2.2. Counting special \mathcal{R} -polynomials	14
3. Application to limit theorems	17
3.1. Preliminaries: variance, cumulants	17
3.2. Counting non zero cumulants	19
3.3. Examples of limit theorems for some shift-invariant groups	21
4. Appendix: endomorphisms of compact abelian groups	24
References	27

Date: 18 September 2016.

2010 Mathematics Subject Classification. Primary: 60F05, 28D05, 22D40, 60G50; Secondary: 47B15, 37A25, 37A30.

Key words and phrases. \mathbb{Z}^d -action, totally disconnected groups, algebraic endomorphisms, r -mixing, limit theorems, S -unit equations, cumulant.

Introduction

Let G be a compact abelian group endowed with its Haar measure μ . If $T_1^{\ell_1}, \dots, T_d^{\ell_d}$, $d \geq 1$, are commuting algebraic automorphisms or surjective endomorphisms of G , they generate a \mathbb{Z}^d or \mathbb{N}^d -action on G : $\underline{\ell} \rightarrow T^{\underline{\ell}} = T_1^{\ell_1} \dots T_d^{\ell_d}$.

Given an "observable" $f : G \rightarrow \mathbb{R}$ with some regularity, one can investigate the statistical behavior of the random field $(T^{\underline{\ell}}f)_{\underline{\ell} \in \mathbb{N}^d}$, in particular the following limits (in distribution with respect to μ):

- ergodic sums for a sequence (D_n) of sets in \mathbb{N}^d : $\lim_n |D_n|^{-\frac{1}{2}} \sum_{\underline{\ell} \in D_n} T^{\underline{\ell}}f$,
- ergodic sums along a random walk $Z_n = Y_0 + \dots + Y_{n-1}$ on \mathbb{Z}^d or \mathbb{N}^d :

$$\lim_n a_n^{-1} \sum_{0 \leq k < n} T^{Z_k(\omega)} f, \text{ for a.e. fixed } \omega, \text{ where } (a_n) \text{ is a normalizing sequence.}$$

The connected case was considered in [3]. Here we are interested in non connected groups G . More precisely we consider in Section 1 some commutative actions by endomorphisms or automorphisms on shift-invariant subgroups of $\mathbb{F}_p^{\mathbb{Z}^d}$ (characteristic p , where $p \geq 2$ is a prime integer).

For these actions, mixing of all orders is not satisfied. Nevertheless, it is possible to show that non-mixing configurations are sparse (Section 2). This was shown for a particular case of our model (Ledrappier's system) by D. Arenas-Carmona, D. Berend and V. Bergelson in [1]. We borrow from their paper, a source of inspiration for us, the term "almost mixing of all orders" used in the title.

The scarcity of non-mixing configurations allows to apply the cumulant method as in [3] to prove the Central Limit Theorem for different methods of summation (Section 3).

The last section (appendix) is devoted to reminders on algebraic endomorphisms of compact abelian groups.

1. Shift-invariant subgroups of $\mathbb{F}_p^{\mathbb{Z}^d}$ and a class of endomorphism

In this section, we recall some facts about shift-invariant subgroups of $\mathbb{F}_p^{\mathbb{Z}^d}$ (cf. [12]) and we define a class of endomorphisms of these groups.

1.1. Shift-invariant subgroups of $\mathbb{F}_p^{\mathbb{Z}^d}$.

Notations

Let $p > 1$ be a prime integer fixed once for all and let \mathbb{F}_p denote the finite field $\mathbb{Z}/p\mathbb{Z}$. For all integers a, b , we have $a^p = a \pmod{p}$, $(a + b)^p = a^p + b^p \pmod{p}$. Underlined symbols will represent vectors or tuples. The element $(0, 0, \dots, 0)$ is represented by $\underline{0}$. For $d \geq 1$, if J is a set of indices, a $|J|$ -tuple of elements of \mathbb{Z}^d or \mathbb{N}^d (\mathbb{N} includes 0) is written $\underline{a}_J = (\underline{a}_j, j \in J)$. The coordinates are denoted by $a_{j,k}$, $j \in J$, $k = 1, \dots, d$. The notation \underline{x}_J represents the variable $(x_j, j \in J)$ or the formal product $\prod_{j \in J} x_j$.

We denote by $G_0^{(d)}$, or simply G_0 , the compact abelian group $\mathbb{F}_p^{\mathbb{Z}^d}$ (with coordinate-wise addition and endowed with the product topology) identified with the ring $\mathcal{S}_d = \mathbb{F}_p[[x_1^{\pm}, \dots, x_d^{\pm}]]$ of formal power series in d variables with coefficients in \mathbb{F}_p .

An element $\zeta = (\zeta_{\underline{k}}, \underline{k} \in \mathbb{Z}^d)$ in G_0 is represented by the formal power series¹ with coefficients in \mathbb{F}_p : $\zeta(\underline{x}) = \sum_{\underline{k} \in \mathbb{Z}^d} \zeta_{\underline{k}} \underline{x}^{-\underline{k}}$. For $j = 1, \dots, d$, the shift σ_j on $\mathbb{F}_p^{\mathbb{Z}^d}$ corresponds to the multiplication by x_j : $\zeta(\underline{x}) \rightarrow x_j \zeta(\underline{x})$.

Polynomials with coefficients in \mathbb{F}_p and characters

The ring $\mathbb{F}_p[x_1^{\pm}, \dots, x_d^{\pm}]$ of Laurent polynomials in d variables with coefficients in \mathbb{F}_p is denoted by \mathcal{P}_d . For $d = 1$, we write simply \mathcal{P} . A Laurent polynomial $P \in \mathcal{P}_d$ reads

$$P(x_1, \dots, x_d) = \sum_{\underline{k} \in S(P)} c(P, \underline{k}) \underline{x}^{\underline{k}},$$

where $S(P)$, called the support of P , is the finite set $\{\underline{k} : c(P, \underline{k}) \neq 0\}$.

For $P \in \mathcal{P}_d$ and $\zeta \in \mathcal{S}_d$, the product $P\zeta$ is well defined:

$$(P\zeta)(\underline{x}) = \left(\sum_{\underline{\ell} \in \mathbb{Z}^d} c(P, \underline{\ell}) \underline{x}^{\underline{\ell}} \right) \left(\sum_{\underline{\ell}' \in \mathbb{Z}^d} \zeta_{\underline{\ell}'} \underline{x}^{-\underline{\ell}'} \right) = \sum_{\underline{k} \in \mathbb{Z}^d} \left(\sum_{\underline{j} \in S(P)} c(P, \underline{j}) \zeta_{\underline{k}+\underline{j}} \right) \underline{x}^{-\underline{k}}.$$

The dual $\widehat{G_0^{(d)}}$ of $G_0^{(d)}$ can be identified with \mathcal{P}_d : for any character χ on $\mathbb{F}_p^{\mathbb{Z}^d}$ there is a polynomial $P \in \mathcal{P}_d$ such that

$$(1) \quad \chi(\zeta) = \chi_P(\zeta) := e^{\frac{2\pi}{p}i \sum_{\underline{\ell} \in S(P)} c(P, \underline{\ell}) \zeta_{\underline{\ell}}} = e^{\frac{2\pi}{p}i c(P\zeta, \underline{0})}.$$

Shift-invariant subgroups of $\mathbb{F}_p^{\mathbb{Z}^d}$

Let $G \subset G_0$ be a shift-invariant closed subgroup of $\mathbb{F}_p^{\mathbb{Z}^d}$. The annihilator G^\perp of G in $\widehat{G_0}$ is $\{P : \chi_P(\zeta) = 1, \forall \zeta \in G\} = \{P : c(P\zeta, \underline{0}) = 0, \forall \zeta \in G\}$. Since G is shift-invariant, if $c(P\zeta, \underline{0}) = 0, \forall \zeta \in G$, the same relation is satisfied for $\underline{x}^{\underline{k}} \zeta(\underline{x}), \forall \underline{k} \in \mathbb{Z}^d$, which implies $P\zeta = 0, \forall \zeta \in G$.

Therefore G^\perp can be identified with the ideal $\mathcal{J} = \{P \in \mathcal{P}_d : P\zeta = 0, \forall \zeta \in G\}$. Since, by duality in $\mathbb{F}_p^{\mathbb{Z}^d}$, we have $G = (G^\perp)^\perp$ (see Appendix), this shows that $G = G_{\mathcal{J}}$ where

$$(2) \quad G_{\mathcal{J}} = \{\zeta : P\zeta = 0, \forall P \in \mathcal{J}\}.$$

Conversely, for every ideal $\mathcal{J} \subset \mathcal{P}_d$, (2) defines a shift-invariant subgroup $G_{\mathcal{J}}$ of G_0 .

The dual of $G_{\mathcal{J}}$ is isomorphic to the quotient $\widehat{G_0}/G_{\mathcal{J}}^\perp$, i.e., $\widehat{G_{\mathcal{J}}} = \mathcal{P}_d/\mathcal{J}$.

1.2. Endomorphisms of $\mathbb{F}_p^{\mathbb{Z}}$ and their invertible extension.

For $R \in \mathcal{P}$, let γ_R be the endomorphism of $K := \mathbb{F}_p^{\mathbb{Z}}$ defined by R . The action of γ_R on characters is the multiplication $P \rightarrow RP$. If $R \neq 0$, the surjectivity of γ_R on K , or equivalently the injectivity of the action of γ_R on the dual group \hat{K} , is clear, since $RP \equiv 0$ if and only if $P \equiv 0$.

The invertible extension of γ_R can be constructed by duality from the action $F \rightarrow RF$ on the ring $\mathcal{F}_R[x^{\pm}]$ of fractions of the form $F(x) = \frac{P(x)}{R(x)^\ell}$, $P \in \mathcal{P}$, $\ell \in \mathbb{N}$. The invertible extension of γ_R is the dual action on the compact group dual of the discrete additive group \mathcal{F}_R .

¹ We write \underline{x} for (x_1, \dots, x_d) as well as for $x_1 \cdots x_d$, $\underline{x}^{-\underline{k}}$ for $x_1^{-k_1} \cdots x_d^{-k_d}$, $\zeta_{\underline{k}}$ or $c(\zeta, \underline{k})$ for the coefficients of the series $\zeta(\underline{x})$.

An isomorphic version of the invertible extension is obtained in the following way. Let us consider the subgroup $G_{\mathcal{J}}$ of $G^{(2)}$, where \mathcal{J} is the ideal in \mathcal{P}_2 generated by the polynomial $x_2 - R(x_1)$. Observe that the homomorphism h_R from \mathcal{P}_2 to \mathcal{F}_R defined by $P(x_1, x_2) \rightarrow P(x_1, R(x_1))$ is surjective and has for kernel the ideal \mathcal{J} (Lemma 1.2 below). Therefore we get an isomorphism between $\mathcal{P}_2 \bmod \mathcal{J}$ and \mathcal{F}_R . The shift σ_2 on the second coordinate is the invertible extension of γ_R .

Multidimensional action

This construction can be extended to a multidimensional action. We start with the group $K = \mathbb{F}_p^{\mathbb{Z}}$ and with d polynomials R_1, \dots, R_d in $\mathbb{F}_p[x^{\pm}]$. We can add the polynomial $R_0(x) = x$ to the list.

The family $\gamma_{R_1}, \dots, \gamma_{R_d}$ generate an \mathbb{N}^d -action on K by algebraic endomorphisms. The dual action on $\hat{K} = \mathcal{P}$ is $\underline{\ell} \rightarrow (P \rightarrow \underline{R}^{\underline{\ell}}P)$, where $\underline{R}^{\underline{\ell}}(x) := R_1^{\ell_1}(x) \dots R_d^{\ell_d}(x)$.

The natural extension is constructed as follows. Let $\mathcal{F}_{\mathcal{R}}[x^{\pm}]$ be the ring of rational fractions in one variable with denominators $R_j(x)$, $j = 1, \dots, d$, i.e., the discrete group of rational fractions with coefficients in \mathbb{F}_p of the form $\frac{V}{R_1^{\ell_1} \dots R_d^{\ell_d}}$, $V \in \mathcal{P}[x^{\pm}]$, $\ell_1, \dots, \ell_d \in \mathbb{N}$.

Using duality, the natural extension \tilde{K} of K (with respect to the endomorphisms $\gamma_{R_1}, \dots, \gamma_{R_d}$) can be built as the dual of $\mathcal{F}_{\mathcal{R}}[x^{\pm}]$ view as an additive group.

As above, we can get an isomorphic version of the invertible extension in a shift-invariant subgroup of $G_0^{(d+1)}$, namely the shift-invariant subgroups $G_{\mathcal{J}}$ of $G_0^{(d+1)}$, where \mathcal{J} is the ideal generated in \mathcal{P}_{d+1} by $x_{j+1} - R_j(x_1)$, $j = 1, \dots, d$.

Lemma 1.1. *The shifts $\sigma_2, \dots, \sigma_{d+1}$ are the invertible extensions of the endomorphisms $\gamma_{R_1}, \dots, \gamma_{R_d}$ acting on $\mathbb{F}_p^{\mathbb{Z}}$ and generate a \mathbb{Z}^d -action on $G_{\mathcal{J}}^{(d+1)}$.*

Proof. If Γ is in $\mathcal{P}_{d+1} = \mathbb{F}_p[x_1^{\pm}, x_2^{\pm}, \dots, x_{d+1}^{\pm}]$, let $h_{\mathcal{R}}(\Gamma)$ be the rational fraction

$$(3) \quad h_{\mathcal{R}}(\Gamma)(x) = \Gamma(x, R_1(x), \dots, R_d(x)).$$

The map $h_{\mathcal{R}}$ is a surjective homomorphism from \mathcal{P}_{d+1} to $\mathcal{F}_{\mathcal{R}}[x^{\pm}]$. The homomorphism \tilde{h} defined by $\tilde{h}(\Gamma \bmod \mathcal{J}) := h(\Gamma)$ is well defined, since $\Gamma \in \mathcal{J}$ implies $h(\Gamma) = 0$. By Lemma 1.2 below, it is an isomorphism between $\mathcal{P}_{d+1} \bmod \mathcal{J}$ and \mathcal{P} .

The multiplication by x_{j+1} on $\mathcal{P}_{d+1} \bmod \mathcal{J}$ corresponds by \tilde{h} to the multiplication by R_j on \mathcal{P} and the \mathbb{Z}^d -action generated by the shifts $\sigma_2, \dots, \sigma_{d+1}$ on $G_{\mathcal{J}}$ has the \mathbb{N}^d -action generated by $\gamma_{R_1}, \dots, \gamma_{R_d}$ on K as a factor through the map \tilde{h} . \square

Lemma 1.2. *The polynomials $L_j(\underline{x}) := x_{j+1} - R_j(x_1)$, $j = 1, \dots, d$, form a basis of $\text{Ker } h_{\mathcal{R}}$.*

Proof. Let us take for simplicity $d = 3$. If P is in $\text{Ker } h_{\mathcal{R}}$, then $x_2 = R_2(x)$ is a root of the polynomial $Q_x(x_2)$ in x_2 defined by $Q_x(x_2) = P(x, x_2, R_3(x)) - P(x, R_2(x), R_3(x))$. Therefore, there is V_x such that $P(x, x_2, R_3(x)) = P(x, R_2(x), R_3(x)) + V_x(x_2)(x_2 - R_2(x)) = V_x(x_2)(x_2 - R_2(x))$. The last equality follows from $P \in \text{Ker } h_{\mathcal{R}}$.

Now, $R_3(x)$ is a root of the polynomial $Q_{x, x_2}(x_3)$ in x_3 defined by $P(x, x_2, x_3) - P(x, x_2, R_3(x))$. There is W_{x, x_2} such that $P(x, x_2, x_3) = P(x, x_2, R_3(x)) + W_{x, x_2}(x_3)(x_3 - R_3(x))$. Put together, it gives: $P(x, x_2, x_3) = V_x(x_2)(x_2 - R_2(x)) + W_{x, x_2}(x_3)(x_3 - R_3(x))$.

V_x and W_{x, x_2} can be written as polynomials, respectively $V(x, x_2)$ and $W(x, x_2, x_3)$. We have $P = V L_2 + W L_3$. \square

Ledrappier's example ([9]) corresponds to $p = 2$, $d = 1$, $R_1(x) = 1 + x$. In this case, the invertible extension of γ_R is given by the shift action on the second coordinate for the shift-invariant group $G_{\mathcal{J}}$ associated to the ideal \mathcal{J} generated by the polynomial $1 + x_1 + x_2$. The group $G_{\mathcal{J}}$ is the set of configurations ζ in $\mathbb{F}_2^{\mathbb{Z}^2}$ such that $\zeta_{n,m} + \zeta_{n+1,m} + \zeta_{n,m+1} = 0 \pmod{2}$, $\forall (n, m) \in \mathbb{Z}^2$. The \mathbb{Z}^2 -shift-action on $G_{\mathcal{J}}$ endowed with its Haar measure is not r -mixing for $r \geq 3$, a fact which is general for the model described above.

Generalization: endomorphisms of $G_{\mathcal{J}}$ and their invertible extension

For an integer $d_1 \geq 1$, let $G_{\mathcal{J}}$ be a shift-invariant subgroup of $G_0^{(d_1)}$.

Every polynomial R in \mathcal{P}_d defines an endomorphism of $G_{\mathcal{J}}$, $\gamma_R : \zeta(\underline{x}) \rightarrow R(\underline{x})\zeta(\underline{x})$. Indeed, if ζ is such that $P\zeta = 0$, then $PR\zeta = RP\zeta = 0$. The dual action of γ_R on $\hat{G}_{\mathcal{J}}$ is the map $P \pmod{\mathcal{J}} \rightarrow RP \pmod{\mathcal{J}}$.

Any family R_1, \dots, R_{d_2} , $d_2 \geq 1$, of Laurent polynomials in $\underline{x} = (x_1, \dots, x_{d_1})$ defines an \mathbb{N}^{d_2} -action by commuting endomorphisms γ_{R_j} of $G_{\mathcal{J}}$.

The natural invertible extension of this action to a \mathbb{Z}^{d_2} -action by algebraic automorphisms of an extension of $G_{\mathcal{J}}$ can be obtained as above in the following way.

Let $x_{d_1+1}, \dots, x_{d_1+d_2}$ be additional coordinates and consider $G_0^{(d_1+d_2)} = \mathbb{F}_p^{\mathbb{Z}^{d_1+d_2}}$. The ideal \mathcal{J}' in $\mathcal{P}_{d_1+d_2}$ generated by \mathcal{J} (embedded in $\mathcal{P}_{d_1+d_2}$) and by the polynomials $x_{d_1+1} - R_1(\underline{x}), \dots, x_{d_1+d_2} - R_{d_2}(\underline{x})$ defines a shift-invariant subgroup $G_{\mathcal{J}'}$ of $G_0^{(d_1+d_2)}$.

Let us consider the surjective homomorphism h from the ring $\mathcal{P}_{d_1+d_2}$ of polynomials in $d_1 + d_2$ variables to the ring \mathcal{P}_{d_1} of polynomials in d_1 variables defined by $h(Q)(\underline{x}) = Q(\underline{x}, R_1(\underline{x}), \dots, R_{d_2}(\underline{x}))$.

The homomorphism \tilde{h} defined by $\tilde{h}(Q \pmod{\mathcal{J}'}) := h(Q) \pmod{\mathcal{J}}$ is well defined, since $Q \in \mathcal{J}'$ implies $h(Q) \in \mathcal{J}$. Using an extension of Lemma 1.2 below, it can be shown that it is an isomorphism between $\mathcal{P}_{d_1+d_2} \pmod{\mathcal{J}'}$ and $\mathcal{P}_{d_1} \pmod{\mathcal{J}}$.

The multiplication by x_{d+j} on $\mathcal{P}_{d_1+d_2} \pmod{\mathcal{J}'}$ corresponds by \tilde{h} to the multiplication by R_j on $\mathcal{P}_{d_1} \pmod{\mathcal{J}}$. In other words, the \mathbb{Z}^{d_2} -action generated by the shifts $\sigma_{d_1+1}, \dots, \sigma_{d_1+d_2}$ on $G_{\mathcal{J}'}$ has the \mathbb{N}^{d_2} -action generated by the endomorphisms $R_{d_1+1}, \dots, R_{d_1+d_2}$ on $G_{\mathcal{J}}$ as a factor through \tilde{h} .

The action of the shifts on $G_{\mathcal{J}'}$ generate a $\mathbb{Z}^{d_1+d_2}$ -action, invertible extension of the action generated on $G_{\mathcal{J}}$ by multiplication by $x_1, \dots, x_{d_1}, R_1(\underline{x}), \dots, R_{d_2}(\underline{x})$.

In the sequel we restrict the previous model to the case $\mathcal{J} = \{0\}$. Moreover, although we think that the methods used below can be extended to $d_1 > 1$, we take $d_1 = 1$.

Total ergodicity

Suppose that the polynomials $R_j, j = 1, \dots, d$, are pairwise relatively prime of degree ≥ 1 . Then the family $(R_j, j = 1, \dots, d)$ generates an \mathbb{Z}^d -action on $K = \mathbb{F}_p^{\mathbb{Z}^d}$ by endomorphisms, which extends to a \mathbb{Z}^d -action $(A^{\underline{\ell}}, \underline{\ell} \in \mathbb{Z}^d)$ on the natural extension \tilde{K} of K which is *totally ergodic* (i.e. such that $A^{\underline{\ell}}$ on $(\tilde{K}, \tilde{\mu})$ is ergodic for every $\underline{\ell} \in \mathbb{Z}^d \setminus \{0\}$).

Example: (with $d = 3$) We take $p = 2$, $R_0(x) = x$, $R_1(x) = 1 + x$, $R_2(x) = 1 + x + x^2$. The orbits on the set of non trivial characters of the generated \mathbb{Z}^3 -action are infinite by

primality of the polynomials $x, 1 + x, 1 + x + x^2$. Therefore, we get a 2-mixing \mathbb{Z}^3 -action, hence a \mathbb{Z}^3 -action with Lebesgue spectrum on $L_0^2(\tilde{\mu})$, where $\tilde{\mu}$ is the Haar measure on \tilde{K} .

1.3. Non r -mixing tuples.

Let us briefly recall the relation between r -mixing for an action by algebraic endomorphisms and S -unit equations. A general measure preserving \mathbb{N}^d -action $(T^{\underline{\ell}})_{\underline{\ell} \in \mathbb{N}^d}$ on a probability measure space (X, μ) is mixing of order $r \geq 2$ if, for any r -tuple of bounded measurable functions f_1, \dots, f_r on X with 0 integral and for every $\varepsilon > 0$, there is $M \geq 1$ such that

$$(4) \quad \|\underline{\ell}_j - \underline{\ell}_{j'}\| \geq M, \forall j \neq j' \Rightarrow \left| \int T^{\underline{\ell}_1} f_1 \dots T^{\underline{\ell}_r} f_r d\mu \right| < \varepsilon.$$

When (X, μ) is a compact abelian group G with its Haar measure, one easily checks by approximation that mixing of order r for an \mathbb{N}^d -action generated by algebraic endomorphisms T_1, \dots, T_d is equivalent to: *for every set $\mathcal{K} = \{\chi_1, \dots, \chi_r\}$ of r characters different from the trivial character χ_0 , there is $M \geq 1$ such that $\|\underline{\ell}_j - \underline{\ell}_{j'}\| \geq M$ for $j \neq j'$ implies $T^{\underline{\ell}_1} \chi_1 \dots T^{\underline{\ell}_r} \chi_r \neq \chi_0$.*

The "non-mixing" r -tuples in \mathbb{N}^d for \mathcal{K} are the r -tuples in the set

$$(5) \quad \Phi(\mathcal{K}, r) := \{(\underline{\ell}_1, \dots, \underline{\ell}_r) : T^{\underline{\ell}_1} \chi_1 \dots T^{\underline{\ell}_r} \chi_r = \chi_0\}.$$

Example: action by $\times 2, \times 3$ on \mathbb{T}^1

Let us illustrate the question of mixing on an example in the connected case: the action $\times 2, \times 3$ on \mathbb{T}^1 . A set \mathcal{K} of non zero characters on the torus is given by an r -tuple $\{k_1, \dots, k_r\}$ of non zero integers. By putting $\underline{\ell}_j = (a_j, b_j)$, Equation (5) for the action by 2 and 3 reads $k_1 2^{a_1} 3^{b_1} + \dots + k_r 2^{a_r} 3^{b_r} = 0$, which leads to consider equations of the form:

$$(6) \quad k_1 2^{a_1} 3^{b_1} + \dots + k_r 2^{a_r} 3^{b_r} = 1, ((a_1, b_1), \dots, (a_r, b_r)) \in (\mathbb{Z}^2)^r.$$

It is known that, for a given set $\{k_1, \dots, k_r\}$, the number of r -tuples $((a_1, b_1), \dots, (a_r, b_r))$ solutions of (6), such that no proper subsum vanishes, is finite (cf. Theorem 1.4). It implies that the \mathbb{Z}^2 -action generated by the invertible extension $\times 2, \times 3$ is mixing of all orders. This mixing result is a special case of a general theorem of K. Schmidt and T. Ward (1992):

Theorem 1.3. ([13]) *Every 2-mixing \mathbb{Z}^d -action by automorphisms on a compact connected abelian group G is mixing of all orders.*

The proof of Theorem 1.3 relies on a result on S -unit equations (Schlickewei (1990)). Let us mention the following version of results on S -unit equations in characteristic 0:

Let \mathbb{F} be an algebraically closed field of characteristic 0, \mathbb{F}^* its multiplicative group of nonzero elements. Let Γ be a subgroup of $(\mathbb{F}^*)^r$.

Theorem 1.4. (*J.-H. Evertse, J.-H. Schlickewei, W. M. Schmidt [5]*) *If the rank of Γ is finite, for $(k_1, \dots, k_r) \in (\mathbb{F}^*)^r$, the number of solutions $(\gamma_1, \dots, \gamma_r) \in \Gamma$ of equation*

$$(7) \quad k_1 \gamma_1 + \dots + k_r \gamma_r = 1,$$

such that $\sum_{i \in I} k_i \gamma_i \neq 0$ for every nonempty subset I of $\{1, \dots, r\}$, is finite.

"Non-mixing" r -tuples for the action of R_1, \dots, R_d

The situation is different in characteristic $\neq 0$, where there can exist infinitely many solutions for equations of the type (7). In the non connected case (for example for endomorphisms of shift-invariant subgroups of $\mathbb{F}_p^{\mathbb{Z}^d}$), this implies the existence of infinitely many non-mixing r -tuples, for $r \geq 3$.

Our goal is to show that, however, these non-mixing r -tuples for the \mathbb{N}^d -actions described in Subsection 1.2 above are rare in a sense (hence these actions are "almost mixing of all order") (cf. [1] and D. Masser's works about non-mixing r -tuples).

Our framework is the setting introduced previously. We consider the \mathbb{N}^d -action on $\mathbb{F}_p^{\mathbb{Z}}$ defined by $\mathcal{R} = (R_1, \dots, R_d)$. A finite set of characters is given by a finite family of polynomials P_1, \dots, P_r . For such a set, a non-mixing r -tuple of the action is an r -tuple $(\underline{a}_1, \dots, \underline{a}_r) \in (\mathbb{N}^d)^r$ such that in $\mathbb{F}_p[x^\pm]$

$$(8) \quad P_1(x) \prod_{i=1}^d R_i(x)^{a_{1,i}} + \dots + P_r(x) \prod_{i=1}^d R_i(x)^{a_{r,i}} = 0.$$

Equation (8) is analogous to the previous S -unit equation (7), but in characteristic $p \neq 0$. Observe that, for a given family (P_j) , the equation can be reduced to the case where the P_j 's are scalars: it suffices to enlarge the family \mathcal{R} by adding the P_j 's to \mathcal{R} .

Replacing R_j in $\mathbb{F}_p[x^\pm]$ by \tilde{R}_j , the polynomial in $\mathbb{F}_p[x]$ such that $\tilde{R}_j(x) = x^\ell R_j(x)$ where $\ell \geq 0$ is minimal, we can also suppose that the polynomials R_j are in $\mathbb{F}_p[x]$.

To count non-mixing r -tuples (for the action of \mathcal{R} on K or of the shifts on the natural invertible extension), in the next section we will study polynomials Γ which belong to $\text{Ker}(h_{\mathcal{R}})$ where $h_{\mathcal{R}}$ is the homomorphism defined by (3).

2. Basic special \mathcal{D} -polynomials

2.1. Decomposition of special \mathcal{D} -polynomials.

2.1.1. Preliminary notations and results.

In this section, we extend results shown for Ledrappier's example of [1] to the general model introduced in the first section. We start the proof of the main theorem (Theorem 2.4) with some notations and preliminary results.

Notations: We denote by Υ the set of all monic (i.e., with leading coefficient equal to 1) prime polynomials in one variable over \mathbb{F}_p .

For $U \in \mathcal{P}$, $\Upsilon(U)$ denotes the set of its prime monic factors. If U is a constant $\neq 0$, we set $\Upsilon(U) = \{1\}$. If \mathcal{S} is a family of polynomials in one variable, $\Upsilon(\mathcal{S}) := \bigcup_{U \in \mathcal{S}} \Upsilon(U)$ is the set of their prime monic factors.

We denote by \mathcal{Q}_0 the ring of (Laurent) polynomials, with coefficients in \mathbb{F}_p , in the variables x_ρ indexed by $\rho \in \Upsilon$. By definition, for every Γ in \mathcal{Q}_0 , there is a finite subset $J(\Gamma)$ of Υ such that Γ is a polynomial in the variables x_ρ , $\rho \in J(\Gamma)$, and reads (in reduced form):

$$(9) \quad \Gamma(\underline{x}) = \sum_{\underline{a} \in \mathbb{Z}^{J(\Gamma)}} d(\underline{a}) \prod_{\rho \in J(\Gamma)} x_\rho^{a_\rho}, \text{ with } d(\underline{a}) \in \mathbb{F}_p.$$

The term “reduced” means that a product $\prod_{\rho \in J(\Gamma)} x_\rho^{a_\rho}$ in the above formula appears only once for a given $\underline{a} \in \mathbb{N}^{J(\Gamma)}$ with a coefficient $d(\underline{a}) \neq 0$, except for the 0 polynomial. Most of the time it will be enough to consider polynomials with non negative exponents.

If a polynomial Γ is expressed in a non reduced form, its expression in reduced form (possibly the 0 polynomial) is

$$\text{red}(\Gamma)(\underline{x}) = \sum_{\underline{b} \in \mathbb{Z}^{J(\Gamma)}} \left(\sum_{\underline{a}: \underline{a}=\underline{b}} d(\underline{a}) \right) \prod_{\rho \in J(\Gamma)} x_\rho^{b_\rho}.$$

An element $(\alpha_1, \dots, \alpha_d)$ of $\{0, \dots, p-1\}^d$ (identified to \mathbb{F}_p^d) is denoted by $\underline{\alpha}$. For Γ given by (9), we call $\underline{\alpha}$ -homogeneous component of Γ , for $\underline{\alpha} \in \mathbb{F}_p^{J(\Gamma)}$ the sum:

$$(10) \quad \Gamma_{\underline{\alpha}}(\underline{x}) = \sum_{\underline{b} \in \mathbb{Z}^{J(\Gamma)}} d(p\underline{b} + \underline{\alpha}) \prod_{\rho \in J(\Gamma)} x_\rho^{p b_\rho + \alpha_\rho}.$$

There is a homomorphism $h : \Gamma \rightarrow h(\Gamma)$, denoted also $\Gamma \rightarrow \widehat{\Gamma}$, from \mathcal{Q}_0 to \mathcal{P} , defined by

$$(11) \quad \Gamma(\underline{x}) = \sum_{\underline{a} \in \mathbb{Z}^{J(\Gamma)}} d(\underline{a}) \prod_{\rho \in J(\Gamma)} x_\rho^{a_\rho} \rightarrow h(\Gamma)(x) := \sum_{\underline{a} \in \mathbb{Z}^{J(\Gamma)}} d(\underline{a}) \prod_{\rho \in J(\Gamma)} \rho(x)^{a_\rho}.$$

We consider also the ring \mathcal{Q}_1 of polynomials Γ in the variables $x_\rho, \rho \in \Upsilon$, with coefficients in $\mathbb{F}_p[x]$:

$$(12) \quad \Gamma(x, \underline{x}) = \sum_{\underline{a} \in \mathbb{Z}^{J(\Gamma)}} d(\underline{a}) U_{\underline{a}}(\underline{x}) \prod_{\rho \in J(\Gamma)} x_\rho^{a_\rho}.$$

Definitions: If \mathcal{D} is a finite subset of Υ (i.e., a finite set of prime polynomials), a polynomial in \mathcal{Q}_1 of the form

$$(13) \quad \Gamma(x, \underline{x}) = \sum_{\underline{a} \in \mathbb{Z}^{\mathcal{D}}} d(\underline{a}) U_{\underline{a}}(\underline{x}) \prod_{\rho \in \mathcal{D}} x_\rho^{a_\rho}, \text{ with } d(\underline{a}) \in \mathbb{F}_p, U_{\underline{a}}(\underline{x}) \text{ monic,}$$

is called a \mathcal{D} -polynomial. It is called a *special \mathcal{D} -polynomial* if it satisfies

$$(14) \quad \sum_{\underline{a} \in \mathbb{Z}^{\mathcal{D}}} d(\underline{a}) U_{\underline{a}}(\underline{x}) \prod_{\rho \in \mathcal{D}} \rho(x)^{a_\rho} = 0.$$

For $U \in \mathcal{P}$, if $U(x) = c(U) \prod_{\rho \in \Upsilon(U)} \rho(x)^{\theta_\rho(U)}$, $c(U) \in \mathbb{F}_p$, is the factorization of U into prime monic factors, we put

$$\Psi(U)(\underline{x}) = c(U) \prod_{\rho \in \Upsilon(U)} x_\rho^{\theta_\rho(U)}.$$

For example, for $p = 2$ and $U(x) = x^3 + x^5$, denoting by ρ_1, ρ_2 the polynomials x and $1 + x$, we get $\Psi(U)(\underline{x}) = x_{\rho_1}^3 x_{\rho_2}^2$.

Observe that $\Psi(U)(\underline{x}) - U(x)$ is a special $\Upsilon(U)$ -polynomial.

We define now a map $\Gamma \rightarrow \Psi(\Gamma)$, also denoted $\Gamma \rightarrow \widetilde{\Gamma}$, from \mathcal{Q}_1 to \mathcal{Q}_0 , which maps Γ given by (12) to the (not necessarily reduced) polynomial $\widetilde{\Gamma}$:

$$(15) \quad \Psi(\Gamma)(\underline{x}) = \widetilde{\Gamma}(\underline{x}) = \sum_{\underline{a} \in \mathbb{Z}^{J(\Gamma)}} d(\underline{a}) \Psi(U_{\underline{a}})(\underline{x}) \prod_{\rho \in J(\Gamma)} x_\rho^{a_\rho}.$$

If Γ is a special \mathcal{D} -polynomial, then $\tilde{\Gamma}$ is a special $\mathcal{D} \cup_{\underline{a}} \Upsilon(U_{\underline{a}})$ -polynomial. Denoting by $r(\Gamma)$ the number of terms of Γ and $S(\Gamma)$ its support, observe that $\tilde{\Gamma} - \Gamma$ is a sum of $r(\Gamma)$ special $\bigcup_{\underline{a}} \Upsilon(U_{\underline{a}})$ -polynomials:

$$(16) \quad \tilde{\Gamma}(\underline{x}) - \Gamma(\underline{x}) = \sum_{\underline{a} \in S(\Gamma)} c(\underline{a}) \left[\prod_{\rho \in \Upsilon(U_{\underline{a}})} x_{\rho}^{\theta_{\rho}(U_{\underline{a}})} - U_{\underline{a}}(x) \right] \prod_{\rho \in J(\Gamma)} x_{\rho}^{\alpha_{\rho}}.$$

Basic special \mathcal{D} -polynomials

Let \mathcal{D} be any family of prime polynomials containing the polynomial $x \rightarrow x$. The polynomials $x_{\rho} - \rho(x)$, $\rho \in \mathcal{D}$, are called *basic special \mathcal{D} -polynomials* (abbreviated in "bs \mathcal{D} -polynomial"). We say that a polynomial Γ is *shifted* from Γ_0 if $\Gamma(\underline{x}) = \underline{x}^{\underline{a}} \Gamma_0(\underline{x})$ for some monomial $\underline{x}^{\underline{a}}$. We will use the following elementary lemma:

Lemma 2.1. *For any monic polynomial U in one variable, $\Psi(U)(\underline{x}) - U(x)$ is a sum of polynomials shifted from basic special $\Upsilon(U)$ -polynomials.*

Proof. If U is a power of a prime polynomial, $U(x) = \rho(x)^b$, $b \geq 1$, then we use:

$$x_{\rho}^b - \rho(x)^b = \left(\sum_{k=0}^{b-1} x_{\rho}^{b-k-1} \rho(x)^k \right) (x_{\rho} - \rho(x)).$$

The general case follows from the formula $Y^b Z^c - y^b z^c = (Y^b - y^b) Z^c + y^b (Z^c - z^c)$ by induction. \square

A polynomial Λ is called *generalized basic special \mathcal{D} -polynomial* (abbreviated in "gbs \mathcal{D} -polynomial"), if it is obtained from a basic special \mathcal{D} -polynomial Δ by shift and dilation (exponentiation with a power of p as exponent).

Therefore Λ is a gbs \mathcal{D} -polynomial if there are $\underline{a} \in \mathbb{Z}^d$, $t \geq 0$ and a bs \mathcal{D} -polynomial $\Delta = x_{\rho} - \rho(x)$ such that:

$$\Lambda(\underline{x}) = \underline{x}^{\underline{a}} (\Delta(\underline{x}))^{p^t} = \underline{x}^{\underline{a}} (x_{\rho}^{p^t} + (-\rho(x))^{p^t}).$$

In the sequel, $\mathcal{R} = (R_j, j = 1, \dots, d)$ will be a *fixed finite family* of $d \geq 2$ distinct prime polynomials in one variable over \mathbb{F}_p . If the polynomial $x \rightarrow x$ is not included in the family \mathcal{R} , we add it to the list.

For this fixed family, it is convenient to introduce another notation for polynomials in \mathcal{Q}_0 depending on the variables $x_{\rho} \in \mathcal{R}$. We write them as polynomials in d variables x_i :

$$(17) \quad \Gamma(\underline{x}) = \sum_{\underline{a} \in \mathbb{N}^d} d(\underline{a}) \prod_{i=1}^d x_i^{\alpha_i}.$$

The variable x_i corresponds to the polynomial R_i . We will use the equivalent notations $\underline{x}^{\underline{a}}$, $\prod_{i=1}^d x_i^{\alpha_i}$ or $\prod_{\rho \in \mathcal{R}} x_{\rho}^{\alpha_{\rho}}$ (here $\Upsilon(\mathcal{R}) = \mathcal{R}$, since the R_j 's are prime polynomials).

Γ in (17) is written in its reduced form (a product $\prod_{i=1}^d x_i^{\alpha_i}$ appears only once for a given $\underline{a} \in \mathbb{N}^d$). As above, Γ reads as a sum of $\underline{\alpha}$ -homogeneous components:

$$(18) \quad \Gamma(\underline{x}) = \sum_{\underline{\alpha} \in \mathbb{F}_p^d} \Gamma_{\underline{\alpha}}(\underline{x}) = \sum_{\underline{\alpha} \in \mathbb{F}_p^d} \left[\sum_{\underline{b} \in \mathbb{N}^d} c_{\underline{b}, \underline{\alpha}} \prod_{i=1}^d x_i^{p b_i + \alpha_i} \right] = \sum_{\underline{\alpha} \in \mathbb{F}_p^d} \left(\prod_{i=1}^d x_i^{\alpha_i} \right) \bar{\Gamma}_{\underline{\alpha}}(\underline{x}),$$

$$(19) \quad \text{with } \bar{\Gamma}_{\underline{\alpha}}(\underline{x}) := \left(\prod_{i=1}^d x_i^{-\alpha_i} \right) \Gamma_{\underline{\alpha}}(\underline{x}) = \sum_{\underline{b} \in \mathbb{N}^d} c_{\underline{b}, \underline{\alpha}} \prod_{i=1}^d x_i^{p b_i}.$$

We denote by $r(\Gamma_{\underline{\alpha}})$ the number of monomials in the sum $\Gamma_{\underline{\alpha}}$. The length of Γ is the number $r(\Gamma)$ of its monomials. It is the cardinal of the support of Γ .

The map $\Gamma \rightarrow \check{\Gamma}$

In case the R_i 's are monic polynomials non necessarily prime, we use the reduction to the prime case given by the following map. Let $R_i = \prod_{\rho \in \Upsilon(\mathcal{R})} \rho^{b_{i,\rho}}$. The map $\Gamma \rightarrow \check{\Gamma}$ is defined by

$$(20) \quad \Gamma(\underline{x}) = \sum_{\underline{a} \in S(\Gamma)} c(\underline{a}) \prod_{i=1}^d x_i^{a_i} \rightarrow \check{\Gamma}(\underline{x}_{\Upsilon(\mathcal{R})}) = \text{red} \left(\sum_{\underline{a} \in S(\Gamma)} c(\underline{a}) \prod_{\rho \in \Upsilon(\mathcal{R})} x_{\rho}^{\sum_{i=1}^d a_i b_{i,\rho}} \right).$$

If Γ is such that $h_{\mathcal{R}}(\Gamma) = 0$, i.e., $\sum_{\underline{a} \in S(\Gamma)} c(\underline{a}) \prod R_i(x)^{a_i} = 0$, then $\check{\Gamma}$ is a special $\Upsilon(\mathcal{R})$ -polynomial.

The goal of this section is the study of the set of special \mathcal{R} -polynomials. Theorem 2.4 will show that, for every family \mathcal{R} of polynomials and every r , there is a finite constant $t(r, \mathcal{R})$ and a finite family \mathcal{E} of polynomials in one variable containing \mathcal{R} such that every special \mathcal{R} -polynomial Γ of length r is a sum of at most $t(r, \mathcal{R})$ gbs \mathcal{E} -polynomials. The constant $t(r, \mathcal{R})$ does not depend on the degree of the polynomial Γ .

Let us now recall or mention some facts about polynomials over \mathbb{F}_p .

Lemma 2.2. a) For any polynomials A, B , we have $(AB^p)' = A' B^p$.

b) A product of pairwise relatively prime polynomials is a p -th power if and only if each factor is a p -th power.

c) If P is a (reduced) polynomial in one variable, then $P' = 0$ if and only if $P = U^p$ for some polynomial U .

d) If V_1, \dots, V_n are pairwise relatively prime polynomials which are not p -th powers, then $(\prod_{i=1}^n V_i)' \neq 0$.

Proof. a), b) are clear. For c), suppose that $P' = 0$, with $P(x) = \sum_k \sum_{\ell=0}^{p-1} c(k, \ell) x^{pk+\ell}$, then $0 = P'(x) = \sum_k \sum_{\ell=1}^{p-1} \ell c(k, \ell) x^{pk+\ell}$ hence $P(x) = [\sum_k c(k, 0) x^k]^p$.

For d), observe that $(\prod_{i=1}^n V_i)' = 0$ implies that $\prod_{i=1}^n V_i$ is equal to U^p for some polynomial U by c), which is impossible by the hypotheses on the V_j 's and b). \square

2.1.2. Decomposition of special \mathcal{R} -polynomials.

Let Γ be a polynomial as in (18). With the notation (19), for $\underline{\beta} \in \mathbb{F}_p^d$ we put

$$(21) \quad A_{\underline{\beta}}(\Gamma)(\underline{x}) := \sum_{\underline{\alpha}} \left(\prod_{i=1}^d R_i^{\alpha_i + \beta_i}(x) \right)' \bar{\Gamma}_{\underline{\alpha}}(\underline{x}),$$

$$(22) \quad B_{\underline{\beta}}(\Gamma)(\underline{x}) := - \left(\prod_i R_i^{\beta_i}(x) \right) A_{\underline{0}}(\Gamma)(\underline{x}) = - \left(\prod_i R_i^{\beta_i}(x) \right) \sum_{\underline{\alpha}} \left(\prod_{i=1}^d R_i^{\alpha_i}(x) \right)' \bar{\Gamma}_{\underline{\alpha}}(\underline{x}),$$

$$(23) \quad \Pi_{\underline{\beta}}(\Gamma)(\underline{x}) := \left(\prod_i R_i^{\beta_i}(x) \right)' \sum_{\underline{\alpha}} \left(\prod_{i=1}^d R_i^{\alpha_i}(x) \right) \bar{\Gamma}_{\underline{\alpha}}(\underline{x}).$$

We assume that Γ is a special \mathcal{R} -polynomial, i.e., $\widehat{\Gamma} = 0$.

It follows that $A_{\underline{\beta}}(\Gamma)$ (hence also $B_{\underline{\beta}}(\Gamma)$) is a special \mathcal{R} -polynomial. Indeed we have by Lemma 2.2 a):

$$\begin{aligned} \widehat{A}_{\underline{\beta}}(\Gamma) &= \sum_{\underline{\alpha}} \sum_{\underline{a}} c(\underline{b}, \underline{\alpha}) \left(\prod_i R_i^{\alpha_i + \beta_i} \right)' \prod_i R_i^{pb_i} \\ &= \left[\sum_{\underline{\alpha}} \sum_{\underline{a}} c(\underline{b}, \underline{\alpha}) \left(\prod_i R_i^{\alpha_i + \beta_i} \right) \prod_i R_i^{pb_i} \right]' = \left[\left(\prod_i R_i^{\beta_i} \right) \widehat{\Gamma} \right]' = 0. \end{aligned}$$

From the identity $\left(\prod_i R_i^{\beta_i + \alpha_i} \right)' = \left(\prod_i R_i^{\beta_i} \right)' \left(\prod_i R_i^{\alpha_i} \right) + \left(\prod_i R_i^{\beta_i} \right) \left(\prod_i R_i^{\alpha_i} \right)'$, we get

$$\Pi_{\underline{\beta}}(\Gamma) = A_{\underline{\beta}}(\Gamma) + B_{\underline{\beta}}(\Gamma).$$

Notation: For a finite family of prime polynomials $\mathcal{D} = \{S_i, i \in I(\mathcal{D})\}$ and $\underline{\beta} = (\beta_1, \dots, \beta_{I(\mathcal{D})})$, we put

$$(24) \quad D_{\mathcal{D}, \underline{\beta}, 0} := \prod_{i \in I(\mathcal{D})} S_i^{\beta_i}, \quad D_{\mathcal{D}, \underline{\beta}, 1} := \left(\prod_{i \in I(\mathcal{D})} S_i^{\beta_i} \right)',$$

$$(25) \quad \zeta(\mathcal{D}) := \mathcal{D} \cup \bigcup_{\underline{\beta} \in \mathbb{F}_p^{I(\mathcal{D})}} \Upsilon(D_{\mathcal{D}, \underline{\beta}, 1}).$$

If we iterate k -times the map $\zeta : \mathcal{D} \rightarrow \zeta(\mathcal{D})$ starting from a finite family of prime polynomials \mathcal{R} , we get a finite family of prime polynomials denoted by $\zeta^k(\mathcal{R})$.

Remark that, if the derivatives of order 1 of products of polynomials in a family of prime polynomials \mathcal{R} do not contain prime factors $\notin \mathcal{R}$, then $\zeta(\mathcal{R}) = \mathcal{R}$. This the case in few examples like for $p = 2$: $\mathcal{R} = \{x, 1+x\}$ (Ledrappier's example), $\mathcal{R} = \{x, 1+x, 1+x+x^2\}$.

The map Ψ (also denoted by $\tilde{}$) defined in (15) gives for $\Pi_{\underline{\beta}}(\Gamma), A_{\underline{\beta}}(\Gamma), B_{\underline{\beta}}(\Gamma)$:

$$(26) \quad \begin{aligned} \Pi_{\underline{\beta}}(\Gamma)(\underline{x}) &= D_{\mathcal{R},\underline{\beta},1}(x) \sum_{\underline{\alpha}} \left(\prod_i R_i^{\alpha_i}(x) \right) \bar{\Gamma}_{\underline{\alpha}}(\underline{x}) \rightarrow \\ \tilde{\Pi}_{\underline{\beta}}(\Gamma)(\underline{x}) &= \left(\prod_{\rho \in \Upsilon(D_{\mathcal{R},\underline{\beta},1})} x_{\rho}^{\theta_{\rho}(D_{\mathcal{R},\underline{\beta},1})} \right) \sum_{\underline{\alpha}} \left(\prod_i x_i^{\alpha_i} \right) \bar{\Gamma}_{\underline{\alpha}}(\underline{x}) = \Psi(D_{\mathcal{R},\underline{\beta},1})(\underline{x}) \sum_{\underline{\alpha}} \Gamma_{\underline{\alpha}}(\underline{x}), \end{aligned}$$

$$(27) \quad \begin{aligned} A_{\underline{\beta}}(\Gamma)(\underline{x}) &= \sum_{\underline{\alpha}} D_{\mathcal{R},\underline{\beta}+\underline{\alpha},1}(x) \bar{\Gamma}_{\underline{\alpha}}(\underline{x}) \rightarrow \\ \tilde{A}_{\underline{\beta}}(\Gamma)(\underline{x}) &= \sum_{\underline{\alpha}} \left(\prod_{\rho \in \Upsilon(\mathcal{R}, D_{\underline{\beta}+\underline{\alpha},1})} x_{\rho}^{\theta_{\rho}(D_{\mathcal{R},\underline{\beta}+\underline{\alpha},1})} \right) \bar{\Gamma}_{\underline{\alpha}}(\underline{x}) = \sum_{\underline{\alpha}} \Psi(D_{\mathcal{R},\underline{\beta}+\underline{\alpha},1})(\underline{x}) \bar{\Gamma}_{\underline{\alpha}}(\underline{x}), \end{aligned}$$

$$(28) \quad \begin{aligned} B_{\underline{\beta}}(\Gamma)(\underline{x}) &= D_{\mathcal{R},\underline{\beta},0}(x) A_{\underline{0}}(\underline{x}) = D_{\mathcal{R},\underline{\beta},0}(x) \sum_{\underline{\alpha}} D_{\mathcal{R},\underline{\alpha},1}(x) \bar{\Gamma}_{\underline{\alpha}}(\underline{x}) \rightarrow \\ \tilde{B}_{\underline{\beta}}(\Gamma)(\underline{x}) &= \Psi(D_{\mathcal{R},\underline{\beta},0})(\underline{x}) \sum_{\underline{\alpha}} \Psi(\mathcal{R}, D_{\mathcal{R},\underline{\alpha},1})(\underline{x}) \bar{\Gamma}_{\underline{\alpha}}(\underline{x}). \end{aligned}$$

The polynomials $\tilde{A}_{\underline{\beta}}(\Gamma), \tilde{B}_{\underline{\beta}}(\Gamma)$ are special $\zeta(\mathcal{R})$ -polynomials (with more variables than Γ in general). This follows from (16) and from the fact that $A_{\underline{\beta}}(\Gamma), B_{\underline{\beta}}(\Gamma)$ are special \mathcal{R} -polynomials, as was shown above.

Reduction of the number of terms

For $\gamma \in \mathbb{F}_p^d$, we define $u(\gamma)$ by $u(\gamma)_i = 0$ if $\gamma_i = 0$, $u(\gamma)_i = p - \gamma_i$ if $\gamma_i = 1, \dots, p-1$, $i = 1, \dots, d$. We have: $D_{\mathcal{R},\underline{0},1}(x) = 0$, $D_{\mathcal{R},\underline{\beta}_1+\underline{\alpha},1}(x) = 0$, for $\underline{\alpha} = u(\underline{\beta}_1)$.

If Γ is not reduced to 0, by shifting Γ by a monomial, we can assume that $\Gamma_{\underline{0}} \neq 0$. If Γ does not reduce to the single component $\Gamma_{\underline{0}}$, there is $\underline{\beta}_1 \neq \underline{0}$ such that $\Gamma_{\underline{\beta}_1} \neq 0$.

If Γ does not reduce to a single homogeneous component, we can optimize the choices of components in the decomposition (see the proof of Theorem 2.4). There are at most p^d non zero homogeneous components. We get

$$(29) \quad r(\tilde{A}_{\underline{0}}(\Gamma)) \leq (1 - \lambda_r) r, \quad r(\tilde{A}_{\underline{\beta}'_1}) \leq (1 - \mu_r) r,$$

$$(30) \quad \text{with } \lambda_r = \max(p^{-d}, r^{-1}), \quad \mu_r = \max((1 - \lambda_r)p^{-d}, r^{-1}).$$

Suppose that $\Gamma_{\underline{0}}, \Gamma_{\underline{\beta}_1} \neq 0$. Let $\underline{\beta} = u(\underline{\beta}_1)$. The polynomials $\tilde{A}_{\underline{\beta}}, \tilde{B}_{\underline{\beta}}(\Gamma)$, are special $\zeta(\mathcal{R})$ -polynomials with strictly less terms than Γ .

For a family \mathcal{R} , we get from the differences $\tilde{\Pi}_{\underline{\beta}}(\Gamma) - \Pi_{\underline{\beta}}(\Gamma)$, $\tilde{A}_{\underline{\beta}}(\Gamma) - A_{\underline{\beta}}(\Gamma)$, $\tilde{B}_{\underline{\beta}}(\Gamma) - B_{\underline{\beta}}(\Gamma)$ respectively the following special polynomial:

$$(31) \quad \Delta_{\mathcal{R},\underline{\alpha},0}^{\underline{\beta}}(\underline{x}) := \Psi(D_{\mathcal{R},\underline{\beta},1})(\underline{x}) \prod_{i=1}^d x_i^{\alpha_i} - D_{\mathcal{R},\underline{\beta},1}(x) \prod_{i=1}^d R_i^{\alpha_i}(x), \quad \underline{\alpha} \in \mathbb{F}^d,$$

$$(32) \quad \Delta_{\mathcal{R},\underline{\beta}+\underline{\alpha},1}(\underline{x}) := \Psi(D_{\mathcal{R},\underline{\beta}+\underline{\alpha},1})(\underline{x}) - D_{\mathcal{R},\underline{\beta}+\underline{\alpha},1}(x), \quad \underline{\alpha} \in \mathbb{F}^d, \underline{\alpha} \neq \underline{\beta}',$$

$$(33) \quad \Delta_{\mathcal{R},\underline{\alpha},k}^{\underline{\beta}}(\underline{x}) := \Psi(D_{\mathcal{R},\underline{\beta},0})(\underline{x}) \Psi(D_{\mathcal{R},\underline{\alpha},1})(\underline{x}) - D_{\mathcal{R},\underline{\beta},0}(x) D_{\mathcal{R},\underline{\alpha},1}(x), \quad \underline{\alpha} \in \mathbb{F}^d \setminus \{\underline{0}\}.$$

Some polynomials in the list can be 0 and there can be redundancy. With the notation used in (30), the number of these polynomials is

$$\leq \sum_{\underline{\alpha}} r(\Gamma_{\underline{\alpha}}) + \sum_{\underline{\alpha} \neq \underline{\beta}'} r(\Gamma_{\underline{\alpha}}) + \sum_{\underline{\alpha} \neq \underline{0}} r(\Gamma_{\underline{\alpha}}) = r + r(1 - \mu_r) + r(1 - \lambda_r) \leq 3r(\Gamma).$$

By Lemma 2.1 each of them can be expressed as a sum of shifted basic polynomials, with a number of terms bounded by a constant C . They are then shifted by the corresponding $\bar{\Gamma}_{\underline{\alpha}}$ associated to the $\underline{\alpha}$ -homogeneous component of Γ .

The results of these preliminaries are summarized in the following lemma:

Lemma 2.3. *Let Γ be a special \mathcal{R} -polynomial of length r .*

Let $A_{\underline{\beta}}(\Gamma), \Pi_{\underline{\beta}}(\Gamma), B_{\underline{\beta}}(\Gamma), \tilde{\Pi}_{\underline{\beta}}(\Gamma), \tilde{A}_{\underline{\beta}}(\Gamma), \tilde{B}_{\underline{\beta}}(\Gamma)$ be defined respectively by (21), (23), (22), (26), (27), (28). Then we have

$$(34) \quad \Psi(D_{\mathcal{R}, \underline{\beta}, 1}) \Gamma = \tilde{\Pi}_{\underline{\beta}}(\Gamma)$$

$$(35) \quad = \tilde{A}_{\underline{\beta}}(\Gamma) + \tilde{B}_{\underline{\beta}}(\Gamma) + \tilde{\Pi}_{\underline{\beta}}(\Gamma) - \Pi_{\underline{\beta}}(\Gamma) + A_{\underline{\beta}}(\Gamma) - \tilde{A}_{\underline{\beta}}(\Gamma) + B_{\underline{\beta}}(\Gamma) - \tilde{B}_{\underline{\beta}}(\Gamma).$$

$\tilde{A}_{\underline{\beta}}(\Gamma)$ and $\tilde{B}_{\underline{\beta}}(\Gamma)$ are special $\zeta(\mathcal{R})$ -polynomials with a number of terms strictly less than the number of terms of Γ .

The differences $\tilde{\Pi}_{\underline{\beta}}(\Gamma) - \Pi_{\underline{\beta}}(\Gamma)$, $A_{\underline{\beta}}(\Gamma) - \tilde{A}_{\underline{\beta}}(\Gamma)$, $B_{\underline{\beta}}(\Gamma) - \tilde{B}_{\underline{\beta}}(\Gamma)$ are sums of at most Cr gbs $\zeta(\mathcal{R})$ -polynomials.

The polynomial $\tilde{\Pi}_{\underline{\beta}}(\Gamma)$ differs from $\tilde{A}_{\underline{\beta}}(\Gamma) + \tilde{B}_{\underline{\beta}}(\Gamma)$ by at most $3Cr(\Gamma)$ gbs $\zeta(\mathcal{R})$ -polynomials given by the decomposition of (31), (32), (33).

Now we prove the main result of this section, which will be used to show that the non-mixing configurations are sparse for the actions that we consider.

Theorem 2.4. *Let r be an integer ≥ 2 . For every family $\mathcal{R} = (R_j, j = 1, \dots, d)$ of $d \geq 1$ polynomials, there is a finite constant $t(r, \mathcal{R})$ and a finite family \mathcal{E} of polynomials in one variable containing \mathcal{R} such that every special \mathcal{R} -polynomial of length $\leq r$ is a sum of at most $t(r, \mathcal{R})$ gbs \mathcal{E} -polynomials.*

Moreover, $\mathcal{E} = \zeta^{r_1}(\mathcal{R})$ for some $r_1 \leq r$ and there are two constants $K > 0$, $\theta \geq 2$, such that $t(r, \mathcal{R}) \leq Kr^\theta$.

Proof. Let $H(r_0)$ be the property that, for every non empty family \mathcal{D} of polynomials, every special \mathcal{D} -polynomial Γ of length $r \leq r_0$ is a sum of at most $Cr_0 2^{r_0}$ gbs $\zeta^r(\mathcal{D})$ -polynomials, where C is the constant introduced before Lemma 2.3.

Let $\mathcal{R} = (R_j, j = 1, \dots, d)$ be a family of d polynomials. Let Γ be an \mathcal{R} -polynomial Γ of length $r(\Gamma) = r_0 + 1$. By applying the map $\Gamma \rightarrow \check{\Gamma}$ to Γ which preserves the number of terms, we can assume that the R_i 's are prime and distinct.

The property $H(2)$ is satisfied (the null polynomial is the only reduced special \mathcal{R} -polynomial of length ≤ 2), if the R_i 's are pairwise relatively prime. Let us show that $H(r_0)$ implies $H(r_0 + 1)$.

We use the fact that, if Γ is a p th power of a special \mathcal{R} -polynomial which is a sum of at most $t(r, \zeta^r(\mathcal{R}))$ gbs \mathcal{S} -polynomials, then Γ has the same property (with the same $t(r, \zeta^r(\mathcal{R}))$) since the p th power of a sum is the sum of the p th power of its terms.

Therefore, we can write $\Gamma = \underline{x}^{\underline{s}} \Lambda^{p^t}$, for some $\underline{s} \in \mathbb{Z}^d$ and some special \mathcal{R} -polynomial Λ (with the same number of terms: $r(\Lambda) = r(\Gamma)$) containing at least two non zero homogeneous components, $\Lambda_{\underline{\beta}_0}, \Lambda_{\underline{\beta}_1}$. Multiplying by a monomial, one can assume $\underline{\beta}_0 = \underline{0}$. Let $\underline{\beta} = u(\underline{\beta}_1)$.

We apply Lemma 2.3 to Λ . With the previous notations, $\tilde{\Pi}_{\underline{\beta}}(\Lambda)$ differs from $\tilde{A}_{\underline{\beta}}(\Lambda) + \tilde{B}_{\underline{\beta}}(\Lambda)$ by at most $3Cr(\Gamma)$ gbs $\zeta(\mathcal{R})$ -polynomials.

$\tilde{A}_{\underline{\beta}}(\Lambda)$ and $\tilde{B}_{\underline{\beta}}(\Lambda)$ are special $\zeta(\mathcal{R})$ -polynomials with a number of terms $\leq r_0$. Therefore, by the induction hypothesis (applied with $\mathcal{D} = \zeta(\mathcal{R})$), they are sum of at most $r_0 2^{r_0}$ gbs $\zeta^{r_0}(\zeta(\mathcal{R}))$ -polynomials.

For $r_0 \geq 3$, since $\zeta^{r_0}(\zeta(\mathcal{R})) = \zeta^{r_0+1}(\mathcal{R})$, $\tilde{\Pi}$ is a sum of at most $2Cr_0 2^{r_0} + 3C(r_0 + 1) \leq C(r_0 + 1) 2^{r_0+1}$ gbs $\zeta^{r_0+1}(\mathcal{R})$ -polynomials.

Using (34), after multiplication of $\tilde{\Pi}_{\underline{\beta}}(\Lambda)$ by $(\prod_{\rho \in \Upsilon(D_{\zeta^{r_0}(\mathcal{R}), \beta, 1})} x_{\rho}^{\theta_{\rho}(D_{\zeta^{r_0}(\mathcal{R}), \beta, 1})})^{-1}$, the inverse of $\Psi(D_{r, \beta, 1})$, to obtain Λ , this shows that $H(r_0 + 1)$ is true (a product of distinct prime polynomials is not a p -th power, hence its derivative is not zero, cf. Lemma 2.2).

The previous computation suffices to give an effective bound for the number of generalized basic special \mathcal{E} -polynomials in the decomposition of a polynomial Γ of a given length $r(\Gamma)$. The following more precise estimation gives a polynomial bound.

First we take the $\underline{\beta}_0$ -homogeneous component of Λ which contains the biggest number of terms. Let $r\lambda_r$ be this number. Altogether, the other components contain $r(1 - \lambda_r)$ terms. Then we take the $\underline{\beta}_1$ -homogeneous component which contains the second biggest number of terms (denoted by $r\mu_r$).

Let $c := p^{-d}$. As there are at most p^d nonempty homogeneous components, we have $r\lambda_r \leq r - 1$ (hence $1 - \lambda_r \geq r^{-1}$) and $\lambda_r \geq c$, $\mu_r \geq c(1 - \lambda_r)$.

If $\theta > 2$ is such that $(1 - c)^{\theta-1} \leq \frac{c}{2}$, then $(1 - \lambda_r)^{\theta} + (1 - \mu_r)^{\theta} + \frac{c}{2} r^{1-\theta} \leq 1$, since

$$\begin{aligned} & (1 - \lambda_r)^{\theta} + (1 - \mu_r)^{\theta} + \frac{c}{2} r^{1-\theta} \leq \frac{c}{2} (1 - \lambda_r) + (1 - \mu_r)^{\theta} + \frac{c}{2} r^{1-\theta} \\ & \leq \frac{c}{2} (1 - \lambda_r) + (1 - \mu_r) + \frac{c}{2} r^{1-\theta} \leq 1 + (1 - \lambda_r) \left(-\frac{c}{2}\right) + \frac{c}{2} r^{1-\theta} \leq 1 - \left[\frac{c}{2} r^{-1} - \frac{c}{2} r^{1-\theta}\right] \leq 1. \end{aligned}$$

For this choice of θ and $K = 6/c$, we have $K(r(1 - \lambda_r))^{\theta} + K(r(1 - \mu_r))^{\theta} + 3r \leq Kr^{\theta}$. Therefore this shows, by induction, that the number of needed gbs $\zeta^r(\mathcal{R})$ -polynomials for the decomposition of Γ is $\leq Kr^{\theta}$. \square

2.2. Counting special \mathcal{R} -polynomials.

We need an auxiliary lemma.

Lemma 2.5. *Let h be an integer, F a finite set of non zero integers and p an integer > 1 . For $h \geq 1$, let $W_h \subset \mathbb{Z}$ be the set of integers which can be written as a sum $L = \sum_{i=1}^h v_i p^{t_i}$, $t_i \in \mathbb{N}$, $v_i \in F$. There is a constant K depending on F, h such that, for all $N \geq 1$, the cardinal of the set $D_h \cap [-N, N]$ is less than $K(\log N)^h$.*

Proof. Taking an element $L \neq 0$ in $W_h \cap [-N, N]$, we can write $L = \sum_{i=1}^{h_1} v_i p^{t_i}$, $t_i \in \mathbb{Z}^+$, where we can assume that the set $\{t_j, j = 1, \dots, h_1\}$ is written in increasing order and $h_1 \leq h$ is such that $\sum_{i=k}^{h_1} v_i p^{t_i} \neq 0$, for all $1 \leq k \leq h_1$.

We have $|L| = p^{t_1} |v_1 + \sum_{j=2}^{h_1} v_j p^{t_j - t_1}|$; hence $p^{t_1} \leq |L| \leq N$; therefore: $t_1 \leq \log N / \log p$.

Since $\sum_{j=2}^{h_1} v_j p^{t_j - t_1} \neq 0$, we have $1 \leq |\sum_{j=2}^{h_1} v_j p^{t_j - t_1}| \leq |L| + M$, where M denotes the maximum of $|u|$, for $u \in F$; hence: $p^{t_2 - t_1} |v_2 + \sum_{j=3}^{h_1} v_j p^{t_j - t_2}| \leq |L| + M$, which implies $t_2 = t_2 - t_1 + t_1 \leq \log(|L| + M)/\log p + \log |L|/\log p \leq \log(N + M)/\log p + \log N/\log p$. By iteration, we obtain $h_1 \leq h$ and a constant C_h depending only on h such that

$$L = \sum_{i=1}^{h_1} v_i p^{t_i} \text{ and } t_1 \leq t_2 \leq \dots \leq t_{h_1} \leq C_h \log N / \log p.$$

Therefore L can take at most $|2F|^h (C_h \log N)^h = K (\log N)^h$ different values. \square

In the statement of the next theorem, \mathcal{R} is a family of polynomials (R_1, \dots, R_d) and $t(r)$ is the constant $t(r) = t(r, \mathcal{R})$ introduced in Theorem 2.4,

Theorem 2.6. *The number $\theta(D, r)$ of reduced special \mathcal{R} -polynomials Γ with r terms, supported in a domain D , satisfies for a constant $\gamma(r)$*

$$(36) \quad \theta(D, r) = O(|D|^{r/3} (\log \text{diam } D)^{\gamma(r)}).$$

Proof. Let $\Gamma(\underline{x}) = \sum_{\underline{a} \in S(\Gamma)} c(\underline{a}) \underline{x}^{\underline{a}}$ be a reduced special \mathcal{R} -polynomials with r terms such that $S(\Gamma) \subset D$.

By Theorem 2.4, there are a finite family of polynomials \mathcal{E} and $t = t(r, \mathcal{R})$ such that $\Gamma = \sum_{j=1}^t \Delta_j$, where each Δ_j is a gbs \mathcal{E} -polynomials,

$$\Delta_j(\underline{x}) = \sum_{\underline{b} \in S(\Delta_j)} d(j, \underline{b}) \underline{x}^{\underline{b}}.$$

In the above formula, we have $\underline{b} \in \mathbb{Z}^{d'}$ for some $d' \geq d$. If $d' > d$ we embed \mathbb{Z}^d into $\mathbb{Z}^{d'}$ by completing by 0 the missing coordinated. We can view the elements \underline{a} of the support $S(\Gamma)$ of Γ as points in $\mathbb{Z}^{d'}$, with the last $d' - d$ coordinates equal to 0. The decomposition of Γ reads more explicitly:

$$\Gamma(\underline{x}) = \sum_{\underline{a} \in S(\Gamma)} c(\underline{a}) \underline{x}^{\underline{a}} = \sum_j \left(\sum_{\underline{b} \in S(\Delta_j)} d(j, \underline{b}) \underline{x}^{\underline{b}} \right) = \sum_{\underline{g} \in \mathbb{Z}^{d'}} \left[\sum_j \sum_{\underline{b} \in S(\Delta_j): \underline{b}=\underline{g}} d(j, \underline{b}) \right] \underline{x}^{\underline{g}}.$$

Putting $u(\underline{g}) = \sum_j \sum_{\underline{b} \in S(\Delta_j): \underline{b}=\underline{g}} d(j, \underline{b})$, for $\underline{g} \in \mathbb{Z}^{d'}$, the formula reads in reduced form:

$$\Gamma(\underline{x}) = \sum_{\underline{g} \in \mathbb{Z}^{d'}: u(\underline{g}) \neq 0} u(\underline{g}) \underline{x}^{\underline{g}}.$$

With the above embedding of \mathbb{Z}^d into $\mathbb{Z}^{d'}$, we get

$$u(\underline{g}) = 0 \text{ for } \underline{g} \notin S(\Gamma), \quad u(\underline{g}) = c(\underline{g}) \text{ for } \underline{g} \in S(\Gamma).$$

Let us denote by Φ the family of the Δ_j 's and write $d(\Delta, \underline{b})$ instead of $d(j, \underline{b})$ for the coefficients of $\Delta = \Delta_j \in \Phi$.

Using an idea of [1], we put a graph structure on Φ by saying that there is an arrow between Δ and Δ' if $S(\Delta) \cap S(\Delta') \neq \emptyset$. For this graph structure, Φ decomposes in connected components denoted by Φ_k .

Let $S_k := \bigcup_{\Delta \in \Phi_k} S(\Delta)$. Since $S(\Delta)$ and $S(\Delta')$ are disjoint for Δ, Δ' in different components, the sets S_k are disjoint.

It follows that the above definition of u can be written

$$u(\underline{g}) = \sum_{\Delta \in \Phi_k} \sum_{\underline{b} \in S(\Delta): \underline{b}=\underline{g}} d(\Delta, \underline{b}), \text{ for } \underline{g} \in S_k.$$

We have $\Gamma(\underline{x}) = \sum_k \Gamma_k(\underline{x})$ with

$$\begin{aligned} \Gamma_k(\underline{x}) &= \sum_{\underline{g} \in S_k} u(\underline{g}) \underline{x}^{\underline{g}} = \sum_{\underline{g} \in S_k} \left[\sum_{\Delta \in \Phi_k} \sum_{\underline{b} \in S(\Delta): \underline{b}=\underline{g}} d(\Delta, \underline{b}) \right] \underline{x}^{\underline{g}} \\ &= \sum_{\Delta \in \Phi_k} \left[\sum_{\underline{b} \in S(\Delta)} d(\Delta, \underline{b}) \right] \underline{x}^{\underline{b}} = \sum_{\Delta \in \Phi_k} \Delta(\underline{x}). \end{aligned}$$

Therefore the supports $S(\Gamma_k)$ are pairwise disjoint and each Γ_k is a special \mathcal{E} -polynomial (actually, once reduced, a special \mathcal{R} -polynomial).

We say that a reduced special \mathcal{R} -polynomial $\Gamma = \sum_{\underline{a} \in S(\Gamma)} c(\underline{a}) \underline{x}^{\underline{a}}$ is \mathcal{R} -minimal if, for every set S_1 strictly contained in $S(\Gamma)$, the polynomial $\sum_{\underline{a} \in S_1} c(\underline{a}) \underline{x}^{\underline{a}}$ is not a special \mathcal{R} -polynomial.

Let us assume first that the polynomial Γ is \mathcal{R} -minimal. The disjointness of the supports $S(\Gamma_k)$ implies that Φ is a connected graph.

The support of the gbs polynomials are sites of the form $\underline{b} + p^k \underline{v}_t$, $t \in J$, where J is a finite set of indices corresponding to the collection of all basic special polynomials. Suppose that Δ, Δ' are two gbs polynomials with a common site in their support. This site reads $\underline{b} + p^k \underline{v}_t = \underline{b}' + p^{k'} \underline{v}_{t'}$, since it belongs to Δ and Δ' . Therefore, $\underline{b}' - \underline{b} = p^k \underline{v}_t - p^{k'} \underline{v}_{t'}$.

If \underline{c}_1 and \underline{c}_2 belong respectively to Δ and Δ' , then we have: $\underline{c}_1 = \underline{b} + p^{k_1} \underline{v}_{t_1}$, $\underline{c}_2 = \underline{b}' + p^{k_2} \underline{v}_{t_2}$; hence: $\underline{c}_2 - \underline{c}_1 = \underline{b}' + p^{k_2} \underline{v}_{t_2} - (\underline{b} + p^{k_1} \underline{v}_{t_1}) = p^k \underline{v}_t - p^{k'} \underline{v}_{t'} + p^{k_2} \underline{v}_{t_2} - p^{k_1} \underline{v}_{t_1}$.

It follows that, if \underline{c}' belongs to a connected chain (starting at \underline{c}) of gbs polynomials Δ_j (i.e., two consecutive Δ, Δ' in the chain have a common site in their support), the difference $\underline{c}' - \underline{c}$ has the form:

$$(37) \quad \underline{c}' - \underline{c} = \sum_i (p^{k_i} \underline{v}_{t_i} - p^{k'_i} \underline{v}_{t'_i}).$$

There are $|D|$ choices for \underline{c} multiplied by $p - 1$ (the cardinal of $\mathbb{F}_p \setminus \{0\}$). We obtain all minimal special \mathcal{R} -polynomials starting from \underline{c} by constructing all possible connected chains of gbs \mathcal{E} -polynomials.

Since, in view of (37), $S(\Gamma) \subset \prod_{i=1}^d (W_{2t_i(r)} + c_i)$, using Lemma 2.5 for each coordinate, we obtain that the number of choices is at most, for a given starting point \underline{c} , $[K (\log \text{diam}(D))^{t_1(r)}]^d$, where $t_1(r)$ is a constant.

This implies that the number $\theta(D, r)$ of minimal special \mathcal{R} -polynomials Γ with s terms, $s \leq r$, supported in a domain D , satisfies the bound

$$(38) \quad \theta(D, r) = O(|D| (\log \text{diam } D)^{d t_1(r)}).$$

If Γ is not \mathcal{R} -minimal, then there is S_1 strictly contained in $S(\Gamma)$ such that $\sum_{\underline{a} \in S_1} c(\underline{a}) \underline{x}^{\underline{a}}$ is a special \mathcal{R} -polynomial. Since $\sum_{\underline{a} \in S(\Gamma) \setminus S_1} c(\underline{a}) \underline{x}^{\underline{a}}$ is also a special \mathcal{R} -polynomial, by iteration of this decomposition, any special \mathcal{D} -polynomial decomposes as a sum of minimal ones with disjoint supports. As the length of a minimal polynomial is at least 3, (36) follows from (38). \square

3. Application to limit theorems

3.1. Preliminaries: variance, cumulants.

We need some general facts about variance, summation sequences, cumulants. (See [3] for more details.) Recall that, if $\mathcal{S} = (T^{\underline{\ell}}, \underline{\ell} \in \mathbb{Z}^d)$ is an abelian group isomorphic to \mathbb{Z}^d of unitary operators on a Hilbert space \mathcal{H} , for every $f \in \mathcal{H}$ there is a positive finite measure ν_f on \mathbb{T}^d , the spectral measure of f , with Fourier coefficients $\widehat{\nu}_f(\underline{\ell}) = \langle T^{\underline{\ell}}f, f \rangle$, $\underline{\ell} \in \mathbb{Z}^d$. When ν_f is absolutely continuous, its density is denoted by φ_f .

We assume that \mathcal{S} has the Lebesgue spectrum property for its action on \mathcal{H} , i.e., there exists a closed subspace \mathcal{K}_0 such that $\{T^{\underline{\ell}}\mathcal{K}_0, \underline{\ell} \in \mathbb{Z}^d\}$ is a family of pairwise orthogonal subspaces spanning a dense subspace in \mathcal{H} . If $(\psi_j)_{j \in \mathcal{J}}$ is an orthonormal basis of \mathcal{K}_0 , $\{T^{\underline{\ell}}\psi_j, j \in \mathcal{J}, \underline{\ell} \in \mathbb{Z}^d\}$ is an orthonormal basis of \mathcal{H} . For every $f \in \mathcal{H}$, ν_f has a density φ_f in $L^1(d\underline{t})$.

Summation sequence

Definitions: We call *summation sequence* any sequence $(w_n)_{n \geq 1}$ of functions from \mathbb{Z}^d to \mathbb{R}^+ with $0 < \sum_{\underline{\ell} \in \mathbb{Z}^d} w_n(\underline{\ell}) < +\infty$, $\forall n \geq 1$. Given $\mathcal{S} = \{T^{\underline{\ell}}, \underline{\ell} \in \mathbb{Z}^d\}$ and $f \in \mathcal{H}$, the associated sums are $\sum_{\underline{\ell} \in \mathbb{Z}^d} w_n(\underline{\ell}) T^{\underline{\ell}}f$.

We say that (w_n) is ζ -regular, if ζ is a probability measure on \mathbb{T}^d and the sequence of nonnegative kernel \tilde{w}_n defined by

$$(39) \quad \tilde{w}_n(\underline{t}) = \frac{|\sum_{\underline{\ell} \in \mathbb{Z}^d} w_n(\underline{\ell}) e^{2\pi i \langle \underline{\ell}, \underline{t} \rangle}|^2}{\sum_{\underline{\ell} \in \mathbb{Z}^d} |w_n(\underline{\ell})|^2}, \quad \underline{t} \in \mathbb{T}^d,$$

weakly converges to ζ when n tends to infinity. This is equivalent to

$$\widehat{\zeta}(\underline{p}) = \lim_{n \rightarrow \infty} \int \tilde{w}_n(\underline{t}) e^{-2\pi i \langle \underline{p}, \underline{t} \rangle} d\underline{t}, \quad \forall \underline{p} \in \mathbb{Z}^d.$$

When the spectral density is continuous, $\varphi_f \rightarrow (\zeta(\varphi_f))^{\frac{1}{2}}$ satisfies the triangular inequality.

Variance for summation sequences

If (w_n) is a ζ -regular summation sequence and f in \mathcal{H} with a continuous spectral density φ_f . By the spectral theorem, we have for $\underline{\theta} = (\theta_1, \dots, \theta_d) \in \mathbb{T}^d$:

$$(40) \quad \left(\sum_{\underline{\ell}} w_n^2(\underline{\ell}) \right)^{-1} \left\| \sum_{\underline{\ell} \in \mathbb{Z}^d} w_n(\underline{\ell}) e^{2\pi i \langle \underline{\ell}, \underline{\theta} \rangle} T^{\underline{\ell}}f \right\|_2^2 = (\tilde{w}_n * \varphi_f)(\underline{\theta}) \xrightarrow{n \rightarrow \infty} (\zeta * \varphi_f)(\underline{\theta}).$$

For example, if (D_n) is a Følner sequence of sets in \mathbb{Z}^d , then $w_n(\underline{\ell}) = 1_{D_n}(\underline{\ell})$, $\zeta = \delta_{\underline{0}}$ and the usual asymptotic variance $\sigma^2(f)$ is $\varphi_f(\underline{0})$.

Moments, cumulants and the CLT

Let us recall now some general results on mixing of order r , moments and cumulants (see [10]). In what follows, we assume the random variables to be uniformly bounded.

Let (X_1, \dots, X_r) be a random vector. For any subset $I = \{i_1, \dots, i_p\} \subset J_r := \{1, \dots, r\}$, we put $m(I) = m(i_1, \dots, i_p) := \mathbb{E}(X_{i_1} \cdots X_{i_p})$. Cumulants are computed from moments by

$$(41) \quad C(X_1, \dots, X_r) = \sum_{\pi \in \mathcal{P}} (-1)^{p(\pi)-1} (p(\pi) - 1)! m(I_1) \cdots m(I_{p(\pi)}),$$

where $\pi = \{I_1, I_2, \dots, I_{p(\pi)}\}$ runs through the set \mathcal{P} of partitions of $J_r = \{1, \dots, r\}$ into nonempty subsets and $p(\pi)$ is the number of elements of π .

Putting $s(I) := C(X_{i_1}, \dots, X_{i_p})$ for $I = \{i_1, \dots, i_p\}$, we have

$$(42) \quad \mathbb{E}(X_1 \cdots X_r) = \sum_{\pi \in \mathcal{P}} s(I_1) \cdots s(I_{p(\pi)}).$$

For a single random variable Y , we define $C^{(r)}(Y) := C(Y, \dots, Y)$, where (Y, \dots, Y) is the vector with r components equal to Y . If Y is centered, $C^{(2)}(Y)$ coincides with $\|Y\|_2^2$.

Let be given a random field of real random variables $(X_{\underline{k}})_{\underline{k} \in \mathbb{Z}^d}$ and a summable weight w from \mathbb{Z}^d to \mathbb{R} . For $Y := \sum_{\underline{\ell} \in \mathbb{Z}^d} w(\underline{\ell}) X_{\underline{\ell}}$, using the multilinearity of the cumulants, we obtain:

$$(43) \quad C^{(r)}(Y) = \sum_{(\underline{\ell}_1, \dots, \underline{\ell}_r) \in (\mathbb{Z}^d)^r} w(\underline{\ell}_1) \cdots w(\underline{\ell}_r) C(X_{\underline{\ell}_1}, \dots, X_{\underline{\ell}_r}).$$

Lemma 3.1. *The number $\gamma(p, r)$ of partitions of J_r into $p \leq r$ nonempty subsets satisfies*

$$(44) \quad \sum_{p=1}^r (-1)^{p-1} (p-1)! \gamma(p, r) = 0.$$

Proof. (44) follows by induction from the following formula: $\gamma(p, r) = \gamma(p-1, r-1) + p\gamma(p, r-1)$, $p = 1, \dots, r$, $r \geq 1$. \square

Theorem 3.2. (cf. [11], Theorem 7) *Let $(X_{\underline{k}})_{\underline{k} \in \mathbb{Z}^d}$ be a random process and $(w_n)_{n \geq 1}$ a summation sequence on \mathbb{Z}^d . Let $Y^{(n)} = \sum_{\underline{\ell}} w_n(\underline{\ell}) X_{\underline{\ell}}$, $n \geq 1$. If $\|Y^{(n)}\|_2 \neq 0$ and*

$$(45) \quad \sum_{(\underline{\ell}_1, \dots, \underline{\ell}_r) \in (\mathbb{Z}^d)^r} w_n(\underline{\ell}_1) \cdots w_n(\underline{\ell}_r) C(X_{\underline{\ell}_1}, \dots, X_{\underline{\ell}_r}) = o(\|Y^{(n)}\|_2^r), \forall r \geq 3,$$

then $\frac{Y^{(n)}}{\|Y^{(n)}\|_2}$ tends in distribution to $\mathcal{N}(0, 1)$ when n tends to ∞ .

Proof. Let $\beta_n := \|Y^{(n)}\|_2 = \|\sum_{\underline{\ell}} w_n(\underline{\ell}) X_{\underline{\ell}}\|_2$ and $Z^{(n)} = \beta_n^{-1} Y^{(n)}$. In view of (43), we have $C^{(r)}(Z^{(n)}) = \beta_n^{-r} \sum_{(\underline{\ell}_1, \dots, \underline{\ell}_r) \in (\mathbb{Z}^d)^r} w_n(\underline{\ell}_1) \cdots w_n(\underline{\ell}_r) C(X_{\underline{\ell}_1}, \dots, X_{\underline{\ell}_r})$, hence by (45):

$$(46) \quad \lim_n C^{(2)}(Z^{(n)}) = 1, \quad \lim_n C^{(r)}(Z^{(n)}) = 0, \forall r \geq 3.$$

Using the formula linking moments and cumulants, the theorem follows from the result of [6] applied to $(Z^{(n)})_{n \geq 1}$. \square

Algebraic framework

Coming back to the framework of a compact abelian group G , we consider a totally ergodic \mathbb{N}^d -action $\underline{\ell} \rightarrow T^{\underline{\ell}}$ by algebraic commuting endomorphisms on G , or its invertible \mathbb{Z}^d -extension, with the Lebesgue spectrum property.

Below a function f on G will be called a "regular function" if f belongs to the space $AC_0(G)$, i.e., has an absolutely convergent Fourier series. Recall that, if f is regular, its spectral density φ_f is continuous on \mathbb{T}^d and for every $\varepsilon > 0$ there is a trigonometric polynomial P defined on G such that $\|\varphi_f - P\|_{\infty} \leq \varepsilon$.

The proof of the CLT given in [11] for a single ergodic endomorphism of a compact abelian group G is based on the computation of the moments of the ergodic sums of trigonometric polynomials and uses mixing of all orders. As mentioned in Section 1.2, for \mathbb{Z}^d -actions by

automorphisms on G , mixing of all orders is satisfied when G is connected, but may fail for non connected groups like shift-invariant subgroups of $\mathbb{F}_p^{\mathbb{Z}^d}$. Nevertheless, when the non-mixing configurations are sparse enough, the moment method can be applied.

Non-mixing r -tuples

Let $f = \sum_{j \in J} c_j \chi_j$ be a trigonometric polynomial and $\Phi = (\chi_j, j \in J)$. We defined the set of “non-mixing” r -tuples for $\Phi = (\chi_j, j \in J)$ by

$$(47) \quad \mathcal{N}(\Phi, r) := \{(\underline{a}_1, \dots, \underline{a}_r) : \exists \chi_{j_1}, \dots, \chi_{j_r} \in \Phi : C(T^{\underline{a}_1} \chi_{j_1}, \dots, T^{\underline{a}_r} \chi_{j_r}) \neq 0\}.$$

In view of (54) (appendix) and (41), if $(\underline{a}_1, \dots, \underline{a}_r) \in \mathcal{N}(\Phi, r)$, we have $T^{\underline{a}_1} \chi_{j_1} \dots T^{\underline{a}_r} \chi_{j_r} = \chi_0$, for some $(\chi_{j_1}, \dots, \chi_{j_r}) \in \Phi$. We will use the results of the subsection 2.2 to show that the sets $\mathcal{N}(\Phi, r)$ are small in some sense.

3.2. Counting non zero cumulants.

Now we consider the action by endomorphisms discussed in the first section. For $d \geq 2$, R_1, R_2, \dots, R_d are d polynomials of degree ≥ 1 in x over \mathbb{F}_p , fixed once for all. Recall that for $\underline{a}_j \in \mathbb{Z}^d$, the action of $T^{\underline{a}_j}$ on a character χ_{Q_j} associated to a polynomial Q_j is the multiplication of Q_j by $\underline{R}^{\underline{a}_j} = \prod_{i=1}^d R_i^{a_{j,i}}$.

For $\tilde{Q} = (Q_1, \dots, Q_r)$, the corresponding cumulant is $C_{\tilde{Q}}(A) = C(T^{\underline{a}_1} \chi_{Q_1}, \dots, T^{\underline{a}_r} \chi_{Q_r})$. Let χ_1, \dots, χ_r be characters on $\mathbb{F}_p^{\mathbb{Z}}$. They correspond to a set of polynomials in one variable $\tilde{Q} = \{Q_1, \dots, Q_r\}$. For an r -tuple $A = (\underline{a}_1, \dots, \underline{a}_r) \in (\mathbb{Z}^d)^r$ the relation $T^{\underline{a}_1} \chi_1 \dots T^{\underline{a}_r} \chi_r = \chi_0$ is equivalent to the relation

$$(48) \quad \sum_{j=1}^r Q_j \prod_{i=1}^d R_i^{a_{j,i}} = 0.$$

In the present framework, the formula for cumulants is used for the random variables $X_j = T^{\underline{a}_j} \chi_j$, where the characters $\chi_j = \chi_{Q_j}$ are associated by (1) to *non zero given fixed polynomials (over \mathbb{F}_p)* Q_j , $i = 1, \dots, r$.

For a domain $D \subset \mathbb{Z}^d$, D^r denotes the set of r -tuples \underline{A} of elements of D .

Let $Q := \sum_i Q_i \underline{R}^{\underline{a}_i}$. The moments read as the integral (actually a finite discrete sum)

$$\int e^{\frac{2\pi}{p} i \sum_{k \in S(Q)} c(Q,k) \zeta_k} d\zeta = \prod_{k \in S(Q)} \frac{1}{p} \sum_{j=0}^{p-1} e^{\frac{2\pi}{p} i c(Q,k) j}.$$

They are equal to 1 if $\sum_i Q_i \underline{R}^{\underline{a}_i} = 0$ and to 0 else (mod p).

Proposition 3.3. *For each $r \geq 3$, there are constants γ, K (dependent on \tilde{Q}) such that*

$$(49) \quad \#\{\underline{A} \in D^r : C_{\tilde{Q}}(\underline{A}) \neq 0\} \leq K |D|^{\frac{r}{2} - \frac{1}{2}} (\log \text{diam } D)^\gamma.$$

Proof. If $C(T^{\underline{a}_1} \chi_1, \dots, T^{\underline{a}_r} \chi_r) \neq 0$, by (41) there exists a partition $\pi = \{I_1, \dots, I_p\}$ of $J = \{1, \dots, r\}$ such that $\sum_{j \in I_k} Q_j \underline{R}^{\underline{a}_j} = 0$, $k = 1, \dots, p$. This implies $\sum_{j \in J} Q_j \underline{R}^{\underline{a}_j} = 0$. The polynomial $\Lambda(x, \underline{x}) = \sum_{j=1}^r Q_j(x) \prod_{i=1}^d x_i^{a_{j,i}}$ satisfies (48) when R_i is substituted to x_i .

Let $\Upsilon(\tilde{Q})$ be the set of prime factors of the polynomials Q_j in \tilde{Q} . In $\Upsilon(\tilde{Q})$ it may exist prime factors belonging to \mathcal{R} and possibly new prime factors denoted by R_i , $i = d+1, \dots, \delta$.

We enlarge the set \mathcal{R} to $\tilde{\mathcal{R}} = \mathcal{R} \cup \Upsilon(\tilde{Q})$ by adding to \mathcal{R} the prime factors of the Q_j 's, i.e., we consider the set of prime polynomials $\tilde{\mathcal{R}} = \{R_1, \dots, R_d, R_{d+1}, \dots, R_{d+\delta}\}$.

The factorization of Q_j in prime monic polynomials (with $d(Q_j) \in \mathbb{F}_p$) is

$$Q_j(x) = d(Q_j) \prod_{\rho \in \Upsilon(\tilde{Q})} \rho^{g_{j,\rho}} = d(Q_j) \prod_{i=1}^d R_i^{g_{j,i}} \prod_{i=d+1}^{d+\delta} R_i^{g_{j,i}}.$$

Some of the $g_{j,i}$ may be zero. Equation (48) reads

$$\sum_{j=1}^r d(Q_j) \prod_{i=1}^d R_i^{g_{j,i}} \prod_{i=d+1}^{d+\delta} R_i^{g_{j,i}} \prod_{i=1}^d R_i^{a_{j,i}} = 0.$$

Putting $a_{j,i} = 0$ for $i = d+1, \dots, d+\delta$, the new r -tuple $B = (b_1, \dots, b_r)$ in $(\mathbb{Z}^{d'})^r$ is given by $b_{j,i} = a_{j,i} + g_{j,i}$, $i = 1, \dots, d+\delta$. We get a polynomial with $d' \geq d$ variables, $\sum_{j=1}^r c(\underline{b}_j) \prod_i^{d+\delta} x_i^{b_{j,i}}$ which is a (not necessarily reduced) special $\tilde{\mathcal{R}}$ -polynomial. We have

$$\sum_{j \in J} Q_j \underline{R}^{a_j} = \sum_{\underline{b}} c(\underline{b}) \tilde{\underline{R}}^{\underline{b}}, \text{ with } c(\underline{b}) = \sum_{j: \underline{a}_j + \underline{g}_j = \underline{b}} d(Q_j).$$

The r -tuple \underline{A} can be viewed as a collection of r vectors in \mathbb{Z}^d which is divided into the two following subsets: $\underline{A}_0 := \{\underline{a}_j : c(\underline{a}_j + \underline{g}_j) = 0\}$, $\underline{A}_1 := \{\underline{a}_j : c(\underline{a}_j + \underline{g}_j) \neq 0\}$. The terms corresponding to $\underline{a}_j \in \underline{A}_0$ disappear. The sum $\Gamma(\underline{x}) = \sum_{\underline{b}: c(\underline{b}) \neq 0} c(\underline{b}) \underline{x}^{\underline{b}}$ is reduced.

Once the sets $\underline{A}_0, \underline{A}_1$ are chosen, \underline{A} is determined up to a permutation which introduces a bounded factor in the counting of the configurations \underline{A}_0 .

In what follows, K will a generic constant which may change from an inequality to another. \tilde{D} is the domain obtained from D when the a 's are replaced by the b 's. Its cardinal and its diameter are less than a constant times the cardinal and the diameter of D .

Let us say that \underline{a}_j is equivalent to $\underline{a}_{j'}$ if $\underline{a}_j + \underline{g}_j = \underline{a}_{j'} + \underline{g}_{j'}$. All elements in the same equivalence class are at bounded distance from each other (their mutual distance is bounded by $\max_{j,j'} \|g_j - g_{j'}\|$). Once an element is chosen in a class, there is an uniformly bounded number of choices for the other elements. The classes of \underline{a}_j 's such that $c(\underline{a}_j + \underline{g}_j) = 0$ have at least two elements.

Let $t \in [0, r]$ be the number of elements in \underline{A}_0 . The number of choices for the elements of \underline{A} belonging to \underline{A}_0 is at most $K |D|^{t/2}$. The polynomial Γ is reduced and has less than $r - t$ terms. By Theorem 2.6 the number of choices of such polynomials is less than $K |\tilde{D}|^{(r-t)/3} (\log \text{diam } \tilde{D})^{\gamma(r)}$.

Therefore the total number of choices for \underline{A} is at most $K |\tilde{D}|^{t/2+(r-t)/3} (\log \text{diam } \tilde{D})^{\gamma(r)} = K |\tilde{D}|^{r/3+t/6} (\log \text{diam } \tilde{D})^{\gamma(r)}$.

If \underline{A}_1 is not empty, then we have $r - t \geq 3$, since a reduced special \mathcal{R} -polynomial has at least 3 terms, and the above upper bound is less than $K |\tilde{D}|^{r/2-1/2} (\log \text{diam } \tilde{D})^{\gamma(r)}$.

If \underline{A}_1 is empty, then $t = r$. If each class is composed only of pairs of 2 elements, then $r = 2r'$ is even and the computation of the cumulant corresponds exactly (for r' instead of r) to the case where all moments are equal to 1. By (44) the cumulant is 0. It shows that this case does not appear in the computation for (49). Therefore there is a class

containing at least 3 elements and we have a bound by $K |\tilde{D}|^{(r-3)/2+1} (\log \text{diam } \tilde{D})^{\theta(r)} \leq K |\tilde{D}|^{r/2-1/2} (\log \text{diam } \tilde{D})^{\gamma(r)}$. which is less than $K |D|^{r/2-1/2} (\log \text{diam } D)^{\gamma(r)}$, for a new constant K . \square

Example: For $r = 4$, the cumulants are given by

$$\begin{aligned} & \int T^{\underline{a}_1} \chi_1 T^{\underline{a}_2} \chi_2 T^{\underline{a}_3} \chi_3 T^{\underline{a}_4} \chi_4 - [\int T^{\underline{a}_1} \chi_1 T^{\underline{a}_2} \chi_2 \int T^{\underline{a}_3} \chi_3 T^{\underline{a}_4} \chi_4 \\ & + \int T^{\underline{a}_1} \chi_1 T^{\underline{a}_3} \chi_3 \int T^{\underline{a}_2} \chi_2 T^{\underline{a}_4} \chi_4 + \int T^{\underline{a}_1} \chi_1 T^{\underline{a}_4} \chi_4 \int T^{\underline{a}_2} \chi_2 T^{\underline{a}_3} \chi_3]. \end{aligned}$$

The characters are given by polynomials Q_i . The integrals and their products take the value 0 or 1. Each time an integral is 1, we have relations of the form $\sum_{i \in I} Q_i R^{\underline{a}_i} = 0$. There are 3 cases:

- a) $Q_1 R^{\underline{a}_1} + Q_2 R^{\underline{a}_2} + Q_3 R^{\underline{a}_3} + Q_4 R^{\underline{a}_4} = 0$ (and no vanishing subsums),
- b) $Q_1 R^{\underline{a}_1} + Q_2 R^{\underline{a}_2} = 0$ and $Q_3 R^{\underline{a}_3} + Q_4 R^{\underline{a}_4} = 0$, or the analogous relations obtained by permutation,
- c) $[Q_1 R^{\underline{a}_1} + Q_2 R^{\underline{a}_2} = 0, Q_3 R^{\underline{a}_3} + Q_4 R^{\underline{a}_4} = 0]$, $[Q_1 R^{\underline{a}_1} + Q_3 R^{\underline{a}_3} = 0$ and $Q_2 R^{\underline{a}_2} + Q_4 R^{\underline{a}_4} = 0]$, or the analogous relations obtained by permutation.

In case c) we see that \underline{a}_1 and \underline{a}_2 are close together as well as \underline{a}_3 and \underline{a}_4 and \underline{a}_2 and \underline{a}_4 . It follows that the four elements $\underline{a}_1, \underline{a}_2, \underline{a}_3, \underline{a}_4$ are close together and there is only one degree of freedom for the choice of A if A belongs to this type of 4-tuple.

If we are in case b), but not in case c), then we have the relations $Q_1 R^{\underline{a}_1} + Q_2 R^{\underline{a}_2} = 0$ and $Q_3 R^{\underline{a}_3} + Q_4 R^{\underline{a}_4} = 0$ (hence $Q_1 R^{\underline{a}_1} + Q_2 R^{\underline{a}_2} + Q_3 R^{\underline{a}_3} + Q_4 R^{\underline{a}_4} = 0$). The cumulant reduces to $\int T^{\underline{a}_1} \chi_1 T^{\underline{a}_2} \chi_2 T^{\underline{a}_3} \chi_3 T^{\underline{a}_4} \chi_4 - \int T^{\underline{a}_1} \chi_1 T^{\underline{a}_2} \chi_2 \int T^{\underline{a}_3} \chi_3 T^{\underline{a}_4} \chi_4 = 1 - 1 = 0$.

If we are in case a), but not b) or c), the cumulant is $\int T^{\underline{a}_1} \chi_1 T^{\underline{a}_2} \chi_2 T^{\underline{a}_3} \chi_3 T^{\underline{a}_4} \chi_4 = 1$. The relation $Q_1 R^{\underline{a}_1} + Q_2 R^{\underline{a}_2} + Q_3 R^{\underline{a}_3} + Q_4 R^{\underline{a}_4} = 0$ may be reducible, but we find at least 3 terms in the irreducible relations of the decomposition. The number of 4-tuples $A = (\underline{a}_1, \underline{a}_2, \underline{a}_3, \underline{a}_4)$ belonging to types corresponding to case a) is less than $O(|D| (\log \text{diam } D)^\theta)$ for some constant θ .

3.3. Examples of limit theorems for some shift-invariant groups.

If $(w_n)_{n \geq 1}$ is a summation sequence on \mathbb{Z}^d , for $f \in L^2(G)$, we put $\sigma_n(f) := \|\sum_{\underline{\ell}} w_n(\underline{\ell}) T^{\underline{\ell}} f\|_2$ and assume $\sigma_n^2(f) \neq 0$, for n big enough. We suppose that (w_n) is ζ -regular.

We can suppose $\zeta(\varphi_f) > 0$, since otherwise the limiting distribution is δ_0 . By ζ -regularity we have $\sigma_n^2(f) \sim (\sum_{\underline{\ell}} w_n^2(\underline{\ell})) \zeta(\varphi_f)$ with $\zeta(\varphi_f) > 0$.

Theorem 3.4. *Let $(w_n)_{n \geq 1}$ be a summation sequence on \mathbb{Z}^d which is ζ -regular (cf. definition in Subsection 3.1). Let f be a regular function with spectral density φ_f such that $\zeta(\varphi_f) > 0$. The condition*

$$(50) \quad \sum_{(\underline{\ell}_1, \dots, \underline{\ell}_r) \in \mathcal{N}(\Phi, r)} \prod_{j=1}^r w_n(\underline{\ell}_j) = o\left(\left(\sum_{\underline{\ell} \in \mathbb{Z}^d} w_n^2(\underline{\ell})\right)^{\frac{r}{2}}\right), \forall \text{ finite family } \Phi \text{ of characters}, \forall r \geq 3,$$

implies

$$(51) \quad \left(\sum_{\underline{\ell} \in \mathbb{Z}^d} w_n^2(\underline{\ell})\right)^{-\frac{1}{2}} \sum_{\underline{\ell} \in \mathbb{Z}^d} w_n(\underline{\ell}) f(T^{\underline{\ell}} \cdot) \xrightarrow[n \rightarrow \infty]{\text{distr}} \mathcal{N}(0, \zeta(\varphi_f)).$$

Proof. a) First let us take for f a trigonometric polynomial. Let us check (45) of Theorem 3.2, i.e., in view of (43),

$$(52) \quad \left| \sum_{(\underline{\ell}_1, \dots, \underline{\ell}_r) \in (\mathbb{Z}^d)^r} C(T^{\underline{\ell}_1} f, \dots, T^{\underline{\ell}_r} f) w_n(\underline{\ell}_1) \dots w_n(\underline{\ell}_r) \right| = o\left(\left(\sum_{\underline{\ell} \in \mathbb{Z}^d} w_n^2(\underline{\ell})\right)^{\frac{r}{2}}\right), \forall r \geq 3.$$

If the cumulant $C(T^{\underline{\ell}_1} f, \dots, T^{\underline{\ell}_r} f)$ is $\neq 0$, then $(\underline{\ell}_1, \dots, \underline{\ell}_r)$ is a non-mixing r -tuple for the set Φ of characters which appear in the expansion of f ; hence (cf. notation (47)):

$$\sum_{(\underline{\ell}_1, \dots, \underline{\ell}_r) \in (\mathbb{Z}^d)^r} C(T^{\underline{\ell}_1} f, \dots, T^{\underline{\ell}_r} f) \prod_{j=1}^r w_n(\underline{\ell}_j) = \sum_{(\underline{\ell}_1, \dots, \underline{\ell}_r) \in \mathcal{N}(\Phi, r)} C(T^{\underline{\ell}_1} f, \dots, T^{\underline{\ell}_r} f) \prod_{j=1}^r w_n(\underline{\ell}_j).$$

Since the cumulants are bounded, the sums in the previous formula are bounded by $C \sum_{(\underline{\ell}_1, \dots, \underline{\ell}_r) \in \mathcal{N}(\Phi, r)} \prod_{j=1}^r w_n(\underline{\ell}_j)$. Therefore, in view of (50), the condition of Theorem 3.2 is satisfied. This implies the CLT when f is a trigonometric polynomial.

b) Now, for a regular function by the ζ -regularity of (w_n) , we have:

$$\left(\sum_{\underline{\ell} \in \mathbb{Z}^d} w_n^2(\underline{\ell})\right)^{-1} \left\| \sum_{\underline{\ell} \in \mathbb{Z}^d} w_n(\underline{\ell}) T^{\underline{\ell}} f \right\|_2^2 = \int_{\mathbb{T}^d} \tilde{w}_n \varphi_f dt \xrightarrow{n \rightarrow \infty} \zeta(\varphi_f).$$

If (ε_k) a sequence of positive numbers tending to 0, there is a sequence of trigonometric polynomials (f_k) such that: $\|\varphi_{f-f_k}\|_\infty \leq \varepsilon_k$. Let us consider the processes defined respectively by

$$U_n^{(k)} := \left(\sum_{\underline{\ell} \in \mathbb{Z}^d} w_n^2(\underline{\ell})\right)^{-\frac{1}{2}} \sum_{\underline{\ell} \in \mathbb{Z}^d} w_n(\underline{\ell}) f_k(T^{\underline{\ell}}), \quad U_n := \left(\sum_{\underline{\ell} \in \mathbb{Z}^d} w_n^2(\underline{\ell})\right)^{-\frac{1}{2}} \sum_{\underline{\ell} \in \mathbb{Z}^d} w_n(\underline{\ell}) f(T^{\underline{\ell}}).$$

We have $\zeta(\varphi_{f-f_k}) \rightarrow 0$. It follows that $\zeta(\varphi_{f_k}) \neq 0$ for k big enough.

Since $\sigma_n^2(f_k) \sim \left(\sum_{\underline{\ell}} w_n^2(\underline{\ell})\right) \zeta(\varphi_{f_k})$ with $\zeta(\varphi_{f_k}) > 0$, it follows from the result in a) for the trigonometric polynomials f_k : $U_n^{(k)} \xrightarrow[n \rightarrow \infty]{distr} \mathcal{N}(0, \zeta(\varphi_{f_k}))$ for every fixed k . Moreover, since

$$\lim_n \int |U_n^{(k)} - U_n|_2^2 d\mu = \lim_n \int_{\mathbb{T}^d} \tilde{w}_n \varphi_{f-f_k} d\underline{t} = \zeta(\varphi_{f-f_k}) \leq \varepsilon_k,$$

we have $\limsup_n \mu[|U_n(k) - U_n| > \varepsilon] \leq \varepsilon^{-2} \limsup_n \int |U_n(k) - U_n|_2^2 d\mu \xrightarrow{k \rightarrow \infty} 0$ for every $\varepsilon > 0$.

Therefore the condition $\lim_k \limsup_n \mu[|U_n(k) - U_n| > \varepsilon] = 0, \forall \varepsilon > 0$, is satisfied and the conclusion $U_n \xrightarrow[n \rightarrow \infty]{distr} \mathcal{N}(0, \zeta(\varphi_f))$ follows from Theorem 3.2 in [2]. \square

Application to shift-invariant subgroups

The limit theorems shown in [3] hold in the present framework of shift-invariant subgroups. We restrict the presentation to two examples.

Let us consider a family $(R_j, j \in J)$ of polynomials of degree ≥ 1 and $\gamma_j = \gamma_{R_j}$ the corresponding endomorphisms of $K = \mathbb{F}_p^{\mathbb{Z}}$. As in Section 1, taking the natural invertible extension, we extend them to automorphisms of the shift-invariant subgroup $G_{\mathcal{J}}$ of $G^{(d+1)}$ defined by the ideal $\mathcal{J} = \text{Ker}(h_{\mathcal{R}})$. The (R_j) 's are chosen to be algebraically independent. Therefore we have a totally ergodic \mathbb{Z}^d -action $(T^{\underline{\ell}}, \underline{\ell} \in \mathbb{Z}^d)$ on $G_{\mathcal{J}}$, with $T^{\underline{\ell}} = T_1^{\underline{\ell}_1} \dots T_d^{\underline{\ell}_d}$ and T_j the composition by the shift σ_{j+1} .

Example 1: Følner sequence in \mathbb{N}^d

Theorem 3.5. *Let $(D_n)_{n \geq 1}$ be a Følner sequence of sets in \mathbb{N}^d . If f is a regular function, we have $\sigma^2(f) = \lim_n \|\sum_{\underline{\ell} \in D_n} T^{\underline{\ell}} f\|_2^2 / |D_n| = \varphi_f(0)$. If moreover $\log \text{diam } D_n = O(|D_n|^\delta), \forall \delta > 0$, then*

$$|D_n|^{-\frac{1}{2}} \sum_{\underline{\ell} \in D_n} T^{\underline{\ell}} f(\cdot) \xrightarrow[n \rightarrow \infty]{\text{distr}} \mathcal{N}(0, \sigma^2(f)).$$

Proof. The sequence $w_n(\underline{\ell}) = 1_{D_n}(\underline{\ell})$ is ζ -regular, with $\zeta = \delta_0$. Suppose that $\varphi_f(0) \neq 0$. We have $\sigma_n^2(f) \sim |D_n| \varphi_f(0)$ and $w_n(\underline{\ell}) = 0$ or 1 . Condition (50) reads here

$$\sum_{(\underline{\ell}_1, \dots, \underline{\ell}_r) \in \mathcal{N}(\Phi, r)} \prod_{j=1}^r 1_{\underline{\ell}_j \in D_n} = o(|D_n|^{\frac{r}{2}}), \text{ for } r \geq 3.$$

For $r \geq 3$, by Proposition 3.3 we have

$$\#\{\underline{A} \in D_n^r : C_{\tilde{P}}(\underline{A}) \neq 0\} = O(|D_n|^{\frac{r}{2} - \frac{1}{2}} (\log \text{diam } D_n)^{\theta(r)}).$$

By the hypothesis on the diameter, this bound implies (50) and the result follows from Theorem 3.4. \square

Remark: For the case of rectangles, see Theorem 4.3.

Example 2: Random walks and quenched CLT

Using the notations and results of [3], now we apply the previous sections to random walks of commuting endomorphisms or automorphisms on a shift-invariant subgroup G .

Let us present the result for $d = 2$. We take two polynomials (R_1, R_2) with $\gamma_{R_1}, \gamma_{R_2}$ the corresponding endomorphisms of $K = \mathbb{F}_p^{\mathbb{Z}}$ generating a 2-dimensional action with Lebesgue spectrum. Taking the natural invertible extension, we extend them to automorphisms (the shifts σ_1, σ_2) of the shift-invariant subgroup $G_{\mathcal{J}}$ of $G_0^{(3)}$ defined by the ideal \mathcal{J} generated in \mathcal{P}_3 by $x_2 - R_1(x_1), x_3 - R_2(x_1)$.

Let $(X_k)_{k \in \mathbb{Z}}$ be a sequence of i.i.d. \mathbb{Z}^2 -valued random variables generating a reduced aperiodic random walk.

Theorem 3.6. *Suppose that W has a finite moment of order 2 on \mathbb{Z}^2 . Let $\underline{\ell} \rightarrow T^{\underline{\ell}}$ be a \mathbb{Z}^2 -action generated by shifts σ_2, σ_3 on $G_{\mathcal{J}}$. Let f be in $AC_0(G_{\mathcal{J}})$ with spectral density φ_f such that $\varphi_f(0) \neq 0$. Then, there exists a constant C such that, for a.e. ω ,*

$$(Cn \text{Log } n)^{-\frac{1}{2}} \sum_{k=0}^{n-1} T^{Z_k(\omega)} f(\cdot) \xrightarrow[n \rightarrow \infty]{\text{distr}} \mathcal{N}(0, 1).$$

Proof. Theorem 4.16 in [3] gives the $\delta(0)$ -regularity for the r.w. summation $(w_n(\omega, \underline{\ell}))_{n \geq 1} = (\sum_{k=0}^{n-1} 1_{Z_k(\omega) = \underline{\ell}})_{n \geq 1}$.

For a recurrent 2-dimensional r.w., for a.e. ω , $\sum_{\underline{\ell}} w_n^2(\omega, \underline{\ell}) \sim \mathbb{E} \sum_{\underline{\ell}} w_n^2(\cdot, \underline{\ell}) \sim Cn \text{Log } n$. To concluded, we need the bound:

$$(53) \quad \sum_{(\underline{\ell}_1, \dots, \underline{\ell}_r) \in \mathcal{N}(\Phi, r)} \prod_{j=1}^r w_n(\underline{\ell}_j) = o((n \text{Log } n)^{r/2}).$$

For every $\delta > 0$, by the law of iterated logarithm there is a finite constant $C(\omega)$ such that

$$\|\underline{\ell}\| > C(\omega)n^{\frac{1}{2}+\delta} \Rightarrow w_n(\omega, \underline{\ell}) = 0.$$

Therefore, the previous sum can be restricted to $\underline{\ell}_j$ in a ball of radius $C(\omega)n^{\frac{1}{2}+\delta}$. Moreover, we know that $\sup_{\underline{\ell}} w_n(\underline{\ell}) = o(n^\varepsilon)$, $\forall \varepsilon > 0$ (Proposition 4.1, in [3]). It follows that the lhs of (53) is less than $n^{r\varepsilon}$ multiplied by the cardinal of r -tuples in the set $\mathcal{N}(\Phi, r)$ supported in the ball $B(0, C(\omega)n^{\frac{1}{2}+\delta})$, for which a bound is given by Proposition 3.3.

This bound is less than $C n^{r\varepsilon} n^{2(\frac{1}{2}+\delta)(\frac{r}{2}-\frac{1}{2})}$, up to a logarithmic factor. Taking into account only the powers of n , on the left hand, we find for the power of n : $r\varepsilon + (1 + 2\delta)(\frac{r}{2} - \frac{1}{2})$ which is $< r/2$, if $\varepsilon + \delta < \frac{1}{2}r^{-1}$. \square

4. Appendix: endomorphisms of compact abelian groups

We recall here some properties of endomorphisms of compact abelian groups. G denotes a compact abelian group, \widehat{G} the dual group of characters on G , χ_0 the trivial character.

The following fact has been used in the first section: let H be a closed subgroup of G , L a subgroup of \widehat{G} . If $H^\perp = \{\chi \in \widehat{G} : \chi(h) = 1, \forall h \in H\}$ denotes the subgroup of \widehat{G} annihilator of H and if $L^\perp = \{h \in G : \chi(h) = 1, \forall \chi \in L\}$ denotes the closed subgroup of G annihilator of L , then $(H^\perp)^\perp = H$, $(L^\perp)^\perp = L$.

Let $d \geq 1$ be an integer and (T_1, \dots, T_d) commuting endomorphisms of G . If $\underline{\ell} = (\ell_1, \dots, \ell_d)$ is in \mathbb{N}^d , we write $T^\underline{\ell}$ for $T_1^{\ell_1} \dots T_d^{\ell_d}$. If f is function on G , $T^\underline{\ell}f$ stands for $f \circ T^\underline{\ell}$.

If necessary, we lift the action to an invertible \mathbb{Z}^d -action by commuting automorphisms of an extension of G . The \mathbb{Z}^d -action is said to be *totally ergodic* if $T^\underline{\ell}$ is ergodic for every $\underline{\ell} \in \mathbb{Z}^d \setminus \{0\}$. It is equivalent to: $T^\underline{\ell}\chi \neq \chi$ for $\underline{\ell} \neq 0$ and any character $\chi \neq \chi_0$, to the Lebesgue spectrum property, as well as to 2-mixing.

Let (f_1, \dots, f_r) be a finite set of trigonometric polynomials and $\Phi = (\chi_j, j \in J)$ be the finite set of characters $\neq \chi_0$ on G such that $f_i = \sum_{j \in J} c_{i,j}(f_i) \chi_j$, $j = 1, \dots, r$. For $(\underline{a}_1, \dots, \underline{a}_r) \in (\mathbb{Z}^d)^r$, we have

$$(54) \quad \int f_1(T^{\underline{a}_1}x) \dots f_r(T^{\underline{a}_r}x) dx = \sum_{j_1, \dots, j_r \in J} c_{1,j_1} \dots c_{r,j_r} 1_{T^{\underline{a}_1}\chi_{j_1} \dots T^{\underline{a}_r}\chi_{j_r} = \chi_0}.$$

Exactness

If γ is a surjective algebraic endomorphism of G , its action on \widehat{G} , still denoted by γ , is injective. The operator of composition by γ on $L^1(G)$ is denoted by T_γ . In what follows we consider endomorphisms with *finite kernel*.

The adjoint operator Π_γ of T_γ is defined by

$$\int_G T_\gamma f g d\mu = \int_G f \Pi_\gamma g d\mu, \quad f \in L^1, g \in L^\infty.$$

It is a contraction of $L^\infty(G)$ and it extends to a contraction of $L^2(G)$. It can be expressed for $f = \sum_{\chi \in \widehat{G}} c_f(\chi) \chi(x) \in L^2(G)$ as

$$(55) \quad \Pi_\gamma f(x) = \frac{1}{|K_\gamma|} \sum_{y: \gamma y = x} f(y) = \sum_{\chi \in \widehat{G}} c_f(\gamma\chi) \chi(x).$$

It follows from (55) that

$$(56) \quad \Pi_\gamma \chi_1 = 0 \text{ if } \chi_1 \notin \gamma \hat{G}, \quad \Pi_\gamma \chi_1 = \chi_2 \text{ if there is } \chi_2 \in \hat{G} \text{ such that } \gamma \chi_2 = \chi_1.$$

By injectivity, χ_2 is uniquely defined in the second case.

Recall that an endomorphism γ is exact (as a measure preserving map on (G, μ)), if

$$(57) \quad \lim_n \|\Pi_\gamma^n f\|_2 = 0, \quad \forall f \in L_0^2(\mu).$$

Exactness of an (algebraic) endomorphism γ is equivalent to:

$$(58) \quad \forall \chi \neq \chi_0, \quad \exists N(\chi) \text{ such that } \Pi_\gamma^n \chi = 0, \text{ for } n \geq N(\chi).$$

Let R be in $\mathbb{F}_p[x]$. It defines an endomorphisms γ_R of the group $\mathbb{F}_p^{\mathbb{Z}^+}$. By (55), the transfer operator $\Pi = \Pi_{\gamma_R}$ acts on a function formally defined by its Fourier series as follows:

$$f = \sum_{P \in \mathbb{F}_p[x]} c(f, \chi_P) \chi_P \rightarrow \Pi f = \sum_{P \in \mathbb{F}_p[x]} c(f, \chi_{PQ}) \chi_Q.$$

Therefore we have: $\Pi^n f = \sum_Q c(f, \chi_{R^n Q}) \chi_Q$ and $\|\Pi^n f\|^2 = \sum_Q |c(f, \chi_{R^n Q})|^2$.

A character $\chi = \chi_P$ associated to a polynomial $P \in \mathbb{F}_p[x^\pm]$, belongs to $R(x)^n \mathbb{F}_p[x^\pm]$ if P is divisible by $R(x)^n$. Therefore, either $R(x) = cx^\varepsilon$, with $c \neq 0$ in \mathbb{F}_p and $\varepsilon = \pm 1$ or γ_R is *exact*, since then, for every a polynomial P , there is $N(P)$ such that P is not divisible by $R(x)^n$ for $n \geq N(P)$.

Complete commutation

Proposition 4.1. *Let γ_1, γ_2 be commuting surjective endomorphisms of G such that $\text{Ker}(\gamma_1)$ is finite. The following conditions are equivalent ²:*

$$(59) \quad T_{\gamma_2} \Pi_{\gamma_1} = \Pi_{\gamma_1} T_{\gamma_2},$$

$$(60) \quad \gamma_2 \chi \in \gamma_1 \hat{G} \Rightarrow \chi \in \gamma_1 \hat{G},$$

$$(61) \quad \text{Ker}(\gamma_1) \cap \text{Ker}(\gamma_2) = \{0\}.$$

Proof. Condition (59) is equivalent to $T_{\gamma_2} \Pi_{\gamma_1} \chi = \Pi_{\gamma_1} T_{\gamma_2} \chi$, for every $\chi \in \hat{G}$.

Using (55), we have $T_{\gamma_2} \Pi_{\gamma_1} \chi = 0$ if $\chi \notin \gamma_1 \hat{G}$, $= \gamma_2 \zeta$ if $\chi = \gamma_1 \zeta$, with $\zeta \in \hat{G}$. Likewise, we have $\Pi_{\gamma_1} \gamma_2 \chi = 0$ if $\gamma_2 \chi \notin \gamma_1 \hat{G}$, $= \eta$ if $\gamma_2 \chi = \gamma_1 \eta$, with $\eta \in \hat{G}$.

Therefore, (59) is equivalent to: $\gamma_2 \chi \notin \gamma_1 \hat{G} \Leftrightarrow \chi \notin \gamma_1 \hat{G}$, i.e., to (61), since the implication \Leftarrow is always satisfied by commutativity.

The annihilator of $\gamma_1 \hat{G}$ is the kernel of γ_1 . By commutation of γ_1 and γ_2 , the kernel $\text{Ker}(\gamma_1)$ is mapped into itself by γ_2 .

By (60), $\text{Ker}(\gamma_1)$ and $\gamma_2 \text{Ker}(\gamma_1)$ have the same annihilator, hence they coincide. The equality $\gamma_2 \text{Ker}(\gamma_1) = \text{Ker}(\gamma_1)$ implies that γ_2 is surjective on $\text{Ker} \gamma_1$. Since the kernel is finite, injectivity and surjectivity of the restriction of γ_2 to $\text{K}(\gamma_1)$ are equivalent. Therefore, injectivity holds.

Now let $u \in \text{Ker}(\gamma_1) \cap \text{Ker}(\gamma_2)$. It satisfies $u \in \text{Ker}(\gamma_1)$ and $\gamma_2 u = 0$. By injectivity of the restriction of γ_2 to $\text{Ker}(\gamma_1)$, this implies $u = 0$.

² Property (59) is the notion of complete commutation used by M. Gordin [7]. See also [8]. This property can be viewed as a primality condition between γ_1 and γ_2 .

Conversely, the condition $\text{Ker}(\gamma_1) \cap \text{Ker}(\gamma_2) = \{0\}$ implies injectivity, hence surjectivity and (60) follows. \square

The symmetry in Condition (61) implies: $T_{\gamma_1} \Pi_{\gamma_2} = \Pi_{\gamma_2} T_{\gamma_1}$. Observe that if $\gamma_1 = \gamma_2$ the equivalent conditions are satisfied if and only if γ_1 is an automorphisms.

Example 1: (Endomorphisms of \mathbb{T}^ρ , $\rho > 1$) Let A, B be two commuting non singular matrices $d \times d$ with coefficients in \mathbb{Z} . A sufficient condition for (61) for the endomorphisms defined by A and B on \mathbb{T}^ρ is that, in the decomposition of \mathbb{R}^d into irreducible (over \mathbb{Z}) spaces V_j under A (and B), for each V_j the determinants of the restriction of A and B are relatively prime. See also [4].

Example 2: (Endomorphisms of $\mathbb{F}_p^{\mathbb{Z}^d}$) If R_1, R_2 are two relatively prime polynomials in one variable, the endomorphisms γ_{R_1} and γ_{R_2} acting on $\mathbb{F}_p^{\mathbb{Z}^d}$ endowed with its Haar measure are completely commuting. This follows from Bezout relation and (60).

Regular functions on $\mathbb{F}_p^{\mathbb{Z}^d}$

A distance ρ on $\mathbb{F}_p^{\mathbb{Z}^d}$ is defined by $\rho(\zeta, \zeta') = \sum_{k \in \mathbb{Z}^d} 2^{-\|k\|} |\zeta_k - \zeta'_k|$. Let D_n be the square $\{\underline{\ell} : |\ell_1| \leq n, \dots, |\ell_d| \leq n\}$. The regularity of a function f on G_0 or on a subset of G_0 (how it depends on the remote coordinates) is measured by the variations

$$V_n(f) := \sup_{\zeta, \zeta' : \zeta_{\underline{\ell}} = \zeta'_{\underline{\ell}}, \forall \underline{\ell} \in D_n} |f(\zeta) - f(\zeta')|, \quad n \geq 1.$$

If D is a finite set in \mathbb{Z}^d , the space $\mathcal{F}(D)$ of complex valued functions on the finite group \mathbb{F}_p^D can be viewed as the subspace of the space $\mathcal{C}(\mathbb{F}_p^{\mathbb{Z}^d})$ of complex continuous functions on $\mathbb{F}_p^{\mathbb{Z}^d}$ depending only on the coordinates $\zeta_{\underline{\ell}}$ for $\underline{\ell} \in D$.

To a point ζ in $\mathbb{F}_p^{\mathbb{Z}^d}$, let us associate the point $\pi_D(\zeta)$ whose coordinates coincide with the coordinates of ζ on D and are equal to 0 outside D . If f is a function on $\mathbb{F}_p^{\mathbb{Z}^d}$, we denote by $\Pi_D f$ the function in $\mathcal{F}(D)$ defined by $\Pi_D f(\zeta) = f(\pi_D(\zeta))$.

If $(D_n)_{n \geq 0}$ is an increasing sequence of domains in \mathbb{Z}^d such that $\bigcup_n D_n = \mathbb{Z}^d$, then for every continuous function f on $\mathbb{F}_p^{\mathbb{Z}^d}$ we have: $\lim_n \|f - \Pi_{D_n} f\|_\infty = 0$. It follows that $f = \sum_{n=0}^\infty \varphi_n$, where $\varphi_n = \Pi_{D_n} f - \Pi_{D_{n-1}} f \in \mathcal{F}(D_n)$ and the series is converging in the uniform norm.

An approximation of f depending only on coordinates in $\{0, \dots, n-1\}$ is yield by replacing f by the function φ_n such that $\varphi_n(\zeta) = f(\pi_n(\zeta))$. Clearly we have $\|f - \varphi_n\|_\infty \leq V_n(f)$.

Suppose that φ_n is a function on $\mathbb{F}_p^{\mathbb{Z}^d}$ depending only on coordinates in $\{0, \dots, n-1\}$. Then φ_n is a finite sum of characters supported on subsets of $\{0, \dots, n-1\}$. If the polynomial R has degree at least 1, $R^n Q$ has degree $\geq n + \deg(Q)$. Polynomials of degree respectively $\geq n + \deg(Q)$ and $< n$ define orthogonal characters. It follows that φ_n is orthogonal to characters of the form $\chi_{R^n Q}$.

Lemma 4.2. *Let f satisfy $V_n(f) = O(\lambda^n)$, for $\lambda < p^{-1}$. Then f belongs to $AC_0(\mathbb{F}_p^{\mathbb{Z}^d})$ and for every $R \in \mathcal{P}[x]$ of degree ≥ 1 , $\|\Pi^n f\|_\infty \leq C' \lambda^n$, for a constant C .*

Proof. We have: $\sum_R |c(\chi_Q, f)| \leq \sum_n \sum_{\deg(Q) < n} |c(\chi_Q, f)| \leq \sum_{n \geq 1} \#\{Q : \deg Q < n\} V_n(f) = O(\sum_{n \geq 1} p^n \lambda^n) = O(1)$.

Writing $f = f - \varphi_{n+k} + \varphi_{n+k}$, we obtain $c(f, \chi_{R^n Q}) = \langle f, \chi_{R^n Q} \rangle = \langle f - \varphi_{n+k}, \chi_{R^n Q} \rangle$, if $k < \deg Q$, which implies $|c(f, \chi_{R^n Q})| \leq \|f - \varphi_{n+k}\|_\infty \leq V_{n+k}(f)$.

We deduce that $\|\Pi^n f\|_\infty \leq \sum_Q |c(f, \chi_{R^n Q})| \leq \sum_Q V_{n+\deg Q}(f)$ is bounded by

$$\sum_{k \geq 0} \#\{Q : \deg Q = k\} V_{n+k}(f) \leq \sum_{k \geq 0} p^{k+1} V_{n+k}(f) \leq C\lambda^n. \quad \square$$

A limit theorem for sums of rectangles

The case of rectangles is a special case for which the martingale method can be used to obtain a functional theorem for ergodic sums. From Theorems 1 and 8 in [4] and the previous lemma, it can be deduced:

Theorem 4.3. *Let $(R_j, j = 1, \dots, d)$ be pairwise relatively prime polynomial of degree ≥ 1 and let $(\gamma_j = \gamma_{R_j})$ be the associated family of commuting algebraic exact endomorphisms of $\mathbb{F}^{\mathbb{Z}^+}$, such that $\text{Ker}(\gamma_i) \cap \text{Ker}(\gamma_j) = \{0\}$, for $i \neq j$. Let $T_j := T_{\gamma_j}, j = 1, \dots, d$ and $T^\ell = T_1^{\ell_1} \dots T_d^{\ell_d}$. If $(D_n)_{n \geq 1}$ is an increasing sequence of rectangles and if f satisfies the regularity condition $V_n(f) = O(\lambda^n)$ with $\lambda < p^{-1}$, then the sequence $(|D_n|^{-\frac{1}{2}} \sum_{\ell \in D_n} T^\ell f)_{n \geq 1}$ satisfies a functional CLT.*

Acknowledgements This research was carried out during visits of the first author to the IRMAR at the University of Rennes 1 and of the second author to the Center for Advanced Studies in Mathematics at Ben Gurion University. The first author was partially supported by the ISF grant 1/12. The authors are grateful to their hosts for their support.

REFERENCES

- [1] Arenas-Carmona, D., Berend, D., Bergelson, V.: Ledrappier's system is almost mixing of all orders, *Ergodic Theory Dynam. Systems* 28 (2008), 339-365.
- [2] Billingsley, P.: *Convergence of probability measures*. 2d ed. John Wiley & Sons, New York (1999).
- [3] Cohen, G., Conze, J.-P.: CLT for random walks of commuting endomorphisms on compact abelian groups, to appear in *Journal of Theoretical Probability*.
- [4] Cuny C., Dedecker J. and Volný, D.: A functional CLT for fields of commuting transformations via martingale approximation, preprint 2016.
- [5] Evertse, J.-H., Schlickewei, H. P., Schmidt, W. M.: Linear equations in variables which lie in a multiplicative group. *Ann. of Math.* 155, no. 3, 807-836 (2002). doi: 10.2307/3062133
- [6] Fréchet, M., Shohat, J.: A proof of the generalized second limit theorem in the theory of probability. *Trans. Amer. Math. Soc.* 33, no. 2, 533-543 (1931). doi: 10.2307/1989421
- [7] Gordin, M. I.: Martingale-co-boundary representation for a class of stationary random fields, (Russian) *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)* 364 (2009), Veroyatnost i Statistika. 14.2, 88-108, 236; translation in *J. Math. Sci. (N. Y.)* 163 (2009), no. 4, 363-374.
- [8] Cuntz, J. and Vershik A.: C^* -algebras associated with endomorphisms and polymorphisms of compact abelian groups. *Arkiv* (2012).
- [9] Ledrappier, F.: Un champ markovien peut être d'entropie nulle et mélangeant (French). *C. R. Acad. Sci. Paris Sér. A-B.* 287, no. 7, A561-A563 (1978).
- [10] Leonov, V.P.: The use of the characteristic functional and semi-invariants in the ergodic theory of stationary processes. *Dokl. Akad. Nauk SSSR* 133, 523-526 (Russian); translated as *Soviet Math.*

- Dokl. 1, 878-881 (1960). See also: Some applications of higher semi-invariants to the theory of stationary random processes. Izdat. "Nauka", Moscow (1964) (Russian).
- [11] Leonov, V.P.: On the central limit theorem for ergodic endomorphisms of compact commutative groups (Russian). Dokl. Akad. Nauk SSSR 135, 258-261 (1960).
- [12] Schmidt, K.: Dynamical systems of algebraic origin. Progress in Mathematics 128. Birkhauser Verlag, Basel (1995).
- [13] Schmidt, K., Ward, T.: Mixing automorphisms of compact groups and a theorem of Schlickewei. Invent. Math. 111, no. 1, 69-76 (1993).

GUY COHEN,
DEPT. OF ELECTRICAL ENGINEERING,
BEN-GURION UNIVERSITY, ISRAEL

E-mail address: `guycohen@bgu.ac.il`

JEAN-PIERRE CONZE,
IRMAR, CNRS UMR 6625,
UNIVERSITY OF RENNES I, CAMPUS DE BEAULIEU, 35042 RENNES CEDEX, FRANCE

E-mail address: `conze@univ-rennes1.fr`