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► To cite this version:

Térence Bayen, Alain Rapaport. About viability and minimal time crisis problem for the Lotka-Volterra prey-predator model . 2017. hal-01369371v3

HAL Id: hal-01369371

<https://hal.science/hal-01369371v3>

Preprint submitted on 19 Apr 2017 (v3), last revised 26 Jan 2018 (v5)

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About viability and minimal time crisis problem for the Lotka-Volterra prey-predator model

Terence Bayen*, Alain Rapaport†

April 19, 2017

Abstract

In this work, we consider several approaches for the control of the classical Lotka-Volterra prey-predator model. Our aim is to maintain the system in a subset $K(\underline{x})$ for which the number of preys is above a given threshold \underline{x} . In the case where the viability kernel of $K(\underline{x})$ is non-empty, we provide an analytic description of this set and we compute an optimal feedback control for the minimum time problem to reach this set. We also provide an optimal feedback control for the so-called *time crisis problem* (see [3, 7]). We point out that for a large set of initial conditions, the duration time spent outside $K(\underline{x})$ by the solution of the time crisis problem is less than the one for the minimum time control problem.

Keywords. Optimal Control, Viability Theory, Pontryagin Maximum Principle, Lotka-Volterra system.

1 Introduction

We consider the classical Lotka-Volterra prey-predator model

$$\begin{cases} \dot{x} &= rx - axy, \\ \dot{y} &= -my + bxy - cy, \end{cases}$$

in the positive orthant domain $\mathcal{D} := (\mathbb{R}_+ \setminus \{0\})^2$. Here, x and y denote respectively the prey and predator densities, and a, b, c, m, r are five positive parameters. The additional mortality term $-uy$ represents a *killing* action (by chemical or biological means) on the predators, and the removal effort $u \in [0, \bar{u}]$ is the control variable. In the present work, we consider that one aims to *protect* the preys from the predators, playing with the control u and the objective is that the state belongs as often as possible to the subset

$$K(\underline{x}) := \{(x, y) \in \mathcal{D} ; x \geq \underline{x}\},$$

where $\underline{x} > 0$ is a given threshold. For a given $T \in \mathbb{R}_+ \cup \{+\infty\}$, we consider time varying controls $u(\cdot)$ in the set

$$\mathcal{U}_T := \{u : [0, T] \rightarrow [0, \bar{u}] ; u(\cdot) \text{ meas.}\}.$$

Differently to the uncontrolled case, the controlled dynamics can have unbounded and non-periodic solutions. For simplicity and without any loss of generality, we choose the coefficients a, b, c equal to 1 and we obtain the system:

$$\begin{cases} \dot{x} &= rx - xy, \\ \dot{y} &= -my + xy - uy. \end{cases} \quad (1.1)$$

For any initial condition $z_0 = (x_0, y_0) \in \mathcal{D}$ and control law $u(\cdot) \in \mathcal{U}_\infty$, we denote by $z_u(\cdot, z_0) = (x_u(\cdot), y_u(\cdot))$ the unique solution of (1.1) defined over \mathbb{R}_+ such that $(x(0), y(0)) = z_0$. In view of proposing control policies in the intention to protect the preys, we are interested in this paper in studying the three following problems:

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1. Determine the viability kernel of $K(\underline{x})$ for (1.1), that is, the set

$$Viab(\underline{x}) := \{z_0 \in \mathcal{D} \text{ s.t. } \exists u(\cdot) \in \mathcal{U}, z_u(t, z_0) \in K(\underline{x}), \forall t \geq 0\},$$

and the controls that allow a trajectory to stay in $Viab(\underline{x})$.

2. Determine the controls that allow to reach $Viab(\underline{x})$ in minimal time which amounts to solving the problem

$$v(z_0) := \inf_{u \in \mathcal{U}_\infty} T_u \quad \text{s.t. } z_u(T_u, z_0) \in Viab(\underline{x}). \quad (1.2)$$

3. Study the minimal time crisis problem which consists in minimizing w.r.t. u the time spent by a solution of (1.1) outside $K(\underline{x})$, that is,

$$\inf_{u \in \mathcal{U}_\infty} J_\infty(u, z_0) := \int_0^\infty \mathbb{1}_{K(\underline{x})^c}(z_u(t, z_0)) dt, \quad (1.3)$$

where $\mathbb{1}_{K(\underline{x})^c}$ denotes the characteristics function of $K(\underline{x})^c$ (*i.e.* the complementary of $K(\underline{x})$) defined by

$$\mathbb{1}_{K(\underline{x})^c}(x, y) := \begin{cases} 1 & \text{if } (x, y) \notin K(\underline{x}), \\ 0 & \text{if } (x, y) \in K(\underline{x}). \end{cases}$$

For the practitioners, solutions of these three problems bring complementary information. The viability kernel $Viab(\underline{x})$ defines the *safety* subset of the set $K(\underline{x})$. The minimum time problem to reach $Viab(\underline{x})$ provides a control policy steering the system to the viability kernel in minimal time from a given initial condition. The minimal time crisis then measures the smallest duration spent outside the set $K(\underline{x})$ over an infinite horizon (as a trajectory may enter and leave the set $K(\underline{x})$ several times). It provides an estimate of the minimal time that can be spent outside the set $K(\underline{x})$ by a solution of (1.1). The time crisis problem is interesting in particular when it is not possible to reach $Viab(\underline{x})$ over a time horizon $[0, T]$. More generally, finding solutions of this problem can be of particular interest if either the viability kernel is empty or if one is unable to compute this set. It also allows to provide a lower bound on the minimal time function $v(z_0)$ to reach the set $Viab(\underline{x})$ (if non-empty).

The minimal time crisis was introduced in the context of viability theory (see *e.g.* [1, 2]) by Doyen and Saint-Pierre (see [7]). One essential feature of this problem is to deal with the discontinuity of the integrand in (1.3). To overcome this difficulty, a regularization scheme based of the Moreau-Yosida regularization of the characteristic function of the constraint set (here $K(\underline{x})$ plays the role of the constraint set) was recently introduced in [3, 4] in the finite horizon case (*i.e.* when $T \in \mathbb{R}_+$ in (1.3)). Let us also mention [5, 6] for the study of a similar regularization scheme in the context of parabolic equations. Following [3], the hybrid maximum principle (see [9, 11, 14]) is well adapted to derive necessary optimality conditions on the time crisis problem.

This work is organized as follows:

- In section 2, we provide an analytic and exact description of the viability kernel, $Viab(\underline{x})$, of $K(\underline{x})$.
- In section 3, we compute an optimal synthesis for the minimal time problem to reach $Viab(\underline{x})$ and we show that optimal controls are bang-bang using the Pontryagin Maximum Principle (see [13]). Switching curves are computed by a numerical integration of the state-adjoint system backward in time.
- Section 4 is devoted to the study of the minimal time crisis problem. Thanks to the hybrid maximum principle and a reformulation of (1.3) over a finite time horizon, we show that an optimal control for this problem is bang-bang and we depict the switching curves numerically. Both optimal strategies for (1.2)-(1.3) are then compared. When $Viab(\underline{x})$ is empty, we show that the time crisis function is equal to $+\infty$.
- The appendix describes into details the numerical scheme that has been used to depict optimal trajectories and the switching curves for both problems (1.2)-(1.3).

One interesting byproduct of this study is to exhibit a subset of \mathcal{D} for which the corresponding optimal trajectory of (1.2) spends more time outside the set $K(\underline{x})$ than the optimal trajectory for the minimal time crisis problem. The complementary of this set contains initial conditions for which optimal trajectories of (1.2)-(1.3) are abnormal and thus both problems coincide. Optimal trajectories for both problems (1.2) and (1.3) may also enter and leave the set $K(\underline{x})$ an arbitrary large number of times (depending on the initial condition) before reaching $Viab(\underline{x})$.

2 Computation of the viability kernel

Given a non-empty subset A of \mathbb{R}^2 , we will denote by $\text{Int}(A)$ its interior and by ∂A its boundary. Next, we also denote by $\|(x, y)\|$ the euclidean norm of a vector $(x, y) \in \mathbb{R}^2$. For a fixed $u \in [0, \bar{u}]$, we define a function $V_u : \mathcal{D} \rightarrow \mathbb{R}$ by:

$$V_u(x, y) := x - (m + u) \ln x + y - r \ln y, \quad (x, y) \in \mathcal{D},$$

together with the number $\underline{c}(u) \in \mathbb{R}$ defined by

$$\underline{c}(u) := V_u(m + u, r) = (m + u)(1 - \ln(m + u)) + r(1 - \ln r),$$

and the equilibrium point $E^*(u)$ for (1.1)

$$E^*(u) := (x^*(u), y^*) = (m + u, r).$$

For a given number $c \geq \underline{c}(u)$, we denote by $L_u(c)$, resp. by $S_u(c)$, the level set, resp. the sub-level set of V_u defined by

$$L_u(c) := \{(x, y) \in \mathcal{D}, V_u(x, y) = c\}, \text{ resp. } S_u(c) := \{(x, y) \in \mathcal{D}, V_u(x, y) \leq c\}.$$

The following lemma is standard when dealing with Lotka-Volterra type systems.

Lemma 2.1. *For a constant control u , a trajectory of (1.1) belongs to a level set $L_u(c)$ with $c \geq \underline{c}(u)$. The sets $L_u(c)$ are closed curves that surround the steady state $E^*(u)$.*

Proof. By differentiating V_u w.r.t. x and y , one finds $\partial_x V_u(x, y) = 1 - \frac{m+u}{x}$ and $\partial_y V_u(x, y) = 1 - \frac{r}{y}$ for $(x, y) \in \mathcal{D}$. If $(x(\cdot), y(\cdot))$ is a solution of (1.1) with the constant control u , a direct computation gives

$$\frac{d}{dt} V_u(x(t), y(t)) = \partial_x V_u \dot{x} + \partial_y V_u \dot{y} = 0.$$

So, any solution of (1.1) with the constant control u belongs to a level set of the function V_u . As $V_u(x, y) \rightarrow +\infty$ when $\|(x, y)\| \rightarrow +\infty$, each level set $L_u(c)$ is bounded. For a constant control u , one can check that the single equilibrium of the dynamics in \mathcal{D} is $E^*(u)$, and that the level set $L_u(\underline{c}(u))$ is the singleton $\{E^*(u)\}$. Therefore, for any initial condition in $\mathcal{D} \setminus \{E^*(u)\}$, the trajectory belongs to a level set $L_u(c)$ with $c > \underline{c}(u)$ (recall that $V_u(x, y) \rightarrow +\infty$ when $\|(x, y)\| \rightarrow +\infty$). As $L_u(c)$ is a compact set that does not contain any equilibrium point, the trajectory has to be periodic and thus $L_u(c)$ is a closed curve which surrounds $E^*(u)$. \square

For $u \in [0, \bar{u}]$, we define two functions $\phi_u : \mathbb{R}_+ \rightarrow \mathbb{R}$ and $\psi : \mathbb{R}_+ \rightarrow \mathbb{R}$ by

$$\phi_u(x) := x - (m + u) \ln x, \quad x \in \mathbb{R}_+ \quad \text{and} \quad \psi(y) := y - r \ln y, \quad y \in \mathbb{R}_+.$$

Lemma 2.2. *Given $u \in [0, \bar{u}]$, on has the following properties*

- For any $x > \phi_u(m + u)$, there exists a unique $x_u^+(x) \in (m + u, +\infty)$ and an unique $x_u^-(x) \in (0, m + u)$ such that $\phi_u(x_u^+(x)) = \phi_u(x_u^-(x)) = x$.
- If $p > \psi(r)$, the equation $\psi(y) = p$ has exactly two roots $y^-(p), y^+(p)$ that satisfy $y^-(p) < r < y^+(p)$.

Proof. One can easily check that $\lim_{x \rightarrow +\infty} \phi_u(x) = \lim_{x \rightarrow 0} \phi_u(x) = +\infty$. Moreover, by differentiating ϕ_u w.r.t. x , one finds $\phi_u'(x) = 1 - \frac{m+u}{x}$. So the function ϕ_u is decreasing from $+\infty$ down to $\phi_u(m + u)$ and increasing up to $+\infty$. Therefore, for any $x > \phi_u(m + u)$, the equation $\phi_u(z) = x$ has exactly two solutions $x_u^-(x), x_u^+(x)$, with $x_u^-(x) < m + u$ and $x_u^+(x) > m + u$. Similarly, the function ψ is decreasing from $+\infty$ down to $\psi(r)$ and increasing up to $+\infty$, which provides the result. \square

For $c \in \mathbb{R}$, we consider the subsets of \mathcal{D} , $L_u^+(c)$, $L_u^-(c)$, $S_u^+(c)$ and $S_u^-(c)$ defined by:

$$L_u^+(c) := L_u(c) \cap \{y \geq r\}, \quad L_u^-(c) := L_u(c) \cap \{y \leq r\},$$

and

$$S_u^+(c) := S_u(c) \cap \{y \geq r\}, \quad S_u^-(c) := S_u(c) \cap \{y \leq r\},$$

and let $r_- \in (0, r]$ be defined by:

$$r_- := y^-(V_u(x_u^+(\underline{x}), r) - \phi_u(\underline{x})).$$

The next proposition provides a description of the viability kernel, $\text{Viab}(\underline{x})$, of $K(\underline{x})$ for (1.1).

Proposition 2.1. *One has the following characterization of the viability kernel:*

- *If $m + \bar{u} < \underline{x}$, the set $Viab(\underline{x})$ is empty.*
- *If $\bar{u} \geq \underline{x} - m$, the viability kernel is non-empty and we have:*

$$Viab(\underline{x}) = S_{\bar{u}}^+(V_{\bar{u}}(\underline{x}, r)) \bigcup (S_0^-(V_0(x_{\bar{u}}^+(\underline{x}), r)) \cap K(\underline{x})),$$

and its boundary is the union of the three curves

$$\begin{aligned} B^+(\underline{x}) &:= L_{\bar{u}}^+(V_{\bar{u}}(\underline{x}, r)), \\ B^-(\underline{x}) &:= L_0^-(V_0(x_{\bar{u}}^+(\underline{x}), r)) \cap \{x \geq \underline{x}\}, \\ B^0(\underline{x}) &:= \{\underline{x}\} \times [r_-, r]. \end{aligned}$$

Proof. Let us assume that $\bar{u} < \underline{x} - m$ and let $\varepsilon > 0$ be such that $m + \underline{u} < \underline{x} - \varepsilon$. Consider a trajectory $(x(\cdot), y(\cdot))$ that stays in the set $K(\underline{x})$ for any time $t \geq 0$. As $0 \leq u(t) \leq \bar{u}$, we deduce that

$$\dot{y} = y(x - m - u) \geq y(x - \underline{x} + \varepsilon) \geq \varepsilon y,$$

using that $x(t) \geq \underline{x}$ for any time $t \geq 0$. Therefore $y(\cdot)$ is unbounded and thus there exists $t_1 > 0$ such that $y(t) > r$ for any time $t > t_1$. It follows that

$$\dot{x}(t) = x(t)(r - y(t)) < 0, \quad \forall t > t_1,$$

and there exists $t_2 > t_1$ such that $x(t_2) < \underline{x}$. So the trajectory $(x(\cdot), y(\cdot))$ must escape the set $K(\underline{x})$, and we have a contradiction. Thus, the viability kernel $Viab(\underline{x})$ is empty.

Assume now that one has $\bar{u} \geq \underline{x} - m$. Notice first that the three curves $B^+(\underline{x})$, $B^-(\underline{x})$ and $B^0(\underline{x})$ belong to the set $K(\underline{x})$ and that their union $U(\underline{x})$ defines the boundary of a compact subset $W(\underline{x})$ of $K(\underline{x})$, which is such that

$$W(\underline{x}) = S_{\bar{u}}^+(V_{\bar{u}}(\underline{x}, r)) \bigcup (S_0^-(V_0(x_{\bar{u}}^+(\underline{x}), r)) \cap K(\underline{x})).$$

When $\bar{u} = \underline{x} - m$, the set $W(\underline{x})$ is reduced to the single point $E^*(\bar{u})$ that is an equilibrium of (1.1) for the constant control \bar{u} . Thus, $W(\underline{x})$ is a viable set.

When $\bar{u} > \underline{x} - m$, we first show that for any initial condition in $U(\underline{x})$, there exists a trajectory that stays in $K(\underline{x})$ for any time $t \geq 0$. Consider an initial condition in the set $B^+(\underline{x})$. With the control $u = \bar{u}$, the corresponding solution of (1.1) remains on the level set $L_{\bar{u}}(V_{\bar{u}}(\underline{x}, r))$ which is contained in $K(\underline{x})$ as its extreme left point is (\underline{x}, r) . Take now an initial condition in $B^-(\underline{x})$. With the control $u = 0$, the corresponding solution of (1.1) remains on $B^-(\underline{x})$ until it reaches in finite time the boundary point $(x_{\bar{u}}^+(\underline{x}), r)$ that belongs to $B^+(\underline{x})$. From this point, we come back to the previous case. Finally, take an initial condition in $\text{Int}(B^0(\underline{x}))$ (if not empty). On $\text{Int}(B^0(\underline{x}))$, one has $\dot{x} > 0$ for any control as one has $r - y > 0$. Thus the trajectory enters the subset $W(\underline{x})$ and cannot evade from $K(\underline{x})$ on $\text{Int}(B^0(\underline{x}))$. If the trajectory touches $B^+ \cup B^-$, we face one of the two previous case. So we conclude that $W(\underline{x})$ is a viable domain.

We now show that $W(\underline{x})$ is the largest viable domain included in $K(\underline{x})$, that is, the viability kernel $Viab(\underline{x})$, or equivalently that any trajectory with initial condition in $K(\underline{x}) \setminus W(\underline{x})$ leaves the set $K(\underline{x})$ in a finite horizon. For convenience we consider the two subsets of $K(\underline{x}) \setminus W(\underline{x})$, $C^+(\underline{x})$ and $C^-(\underline{x})$ defined by:

$$C^+(\underline{x}) = (K(\underline{x}) \setminus W(\underline{x})) \cap \{y \geq r\} \quad \text{and} \quad C^-(\underline{x}) = (K(\underline{x}) \setminus W(\underline{x})) \cap \{y \leq r\}.$$

Consider now an initial condition $(x_0, y_0) \in C^+(\underline{x})$, and let $(x(\cdot), y(\cdot))$ a solution of (1.1) starting from (x_0, y_0) . One has $V_{\bar{u}}(x_0, y_0) > V_{\bar{u}}(\bar{x}, r)$, and by differentiating w.r.t t one finds:

$$\frac{d}{dt} V_{\bar{u}}(x(t), y(t)) = (y(t) - r)(\bar{u} - u(t)) \geq 0.$$

Therefore no trajectory can reach the level set $L_{\bar{u}}(V_{\bar{u}}(\bar{x}, r))$ from $K(\underline{x}) \setminus W(\underline{x})$. When $y(t) = r$, one has $x(t) > m + \bar{u}$ and thus $\dot{y}(t) = r(x(t) - m - u(t)) > 0$ as $u(t) \leq \bar{u}$. We deduce that if there exists a trajectory

with an initial condition (x_0, y_0) in $C^+(\underline{x})$ that stays in $K(\underline{x})$, it has to stay in $C^+(\underline{x})$. By differentiating w.r.t t , we obtain on $C^+(\underline{x})$ that

$$\frac{d}{dt}V_0(x(t), y(t)) = -u(t)(y(t) - r) \leq 0,$$

and thus $V_0(x(t), y(t)) \leq V_0(x_0, y_0)$. It follows that the trajectory is bounded. Moreover, one has $\dot{x} \leq 0$. i.e. the function $t \mapsto x(t)$ is non-increasing and thus converges to a certain $x_\infty > 0$. By Barbalat's Lemma, $\dot{x}(t)$ converges to 0 which implies that $y(t)$ tends to r . Then $V_{\bar{u}}(x(t), y(t))$ converges to $V_{\bar{u}}(x_\infty, r) > V_{\bar{u}}(\bar{x}, r)$ which implies that $x_\infty < \bar{x}$. Thus, the trajectory necessary leaves the set $K(\underline{x})$ and we have a contradiction.

Consider now an initial condition $(x_0, y_0) \in C^-(\underline{x})$. Similarly, one can show that no trajectory can reach the level set $L_0(V_0(x^+(\underline{x}), r))$ from $K(\underline{x}) \setminus W(\underline{x})$. It follows that a trajectory with an initial condition in $C^-(\underline{x})$ that stays in $K(\underline{x})$ has to stay in $C^-(\underline{x})$ (otherwise, it reaches $C^+(\underline{x})$ and we have shown above that its has to escape $K(\underline{x})$). As previously, one can show that a trajectory that stays in $C^-(\underline{x})$ is bounded and as $t \mapsto x(t)$ is increasing, one obtains the convergence of $x(t)$ to a certain $x_\infty > \underline{x}$. By Barbalat's Lemma, $\dot{x}(t)$ converges to 0 when $t \rightarrow +\infty$, which implies that $y(t) \rightarrow r$, and thus $x_\infty \geq x_{\bar{u}}^+(\underline{x}) > m + \bar{u}$. Therefore there exists $\varepsilon > 0$ and $t_0 > 0$ such that $\dot{y}(t) = (x(t) - m - u)y(t) > \varepsilon y(t)$ for any $t > t_0$. This gives a contradiction with the convergence of $y(t)$ to r when $t \rightarrow +\infty$. So, the trajectory has to enter $C^+(\underline{x})$ and then leaves $K(\underline{x})$. \square

The viability kernel is depicted on Fig. 1 in case (ii) of Proposition 2.1 together with the three curves $B^+(\underline{x})$, $B^-(\underline{x})$, $B^0(\underline{x})$ that define its boundary (see the Appendix for the numerical values of the parameters). It is worth noting that $B^+(\underline{x})$ is a semi-orbit of (1.1) with $u = \bar{u}$ passing trough the point (\underline{x}, r) . Similarly, $B^-(\underline{x})$ is a semi-orbit of (1.1) with $u = 0$ passing trough the point (\underline{x}, r_-) .

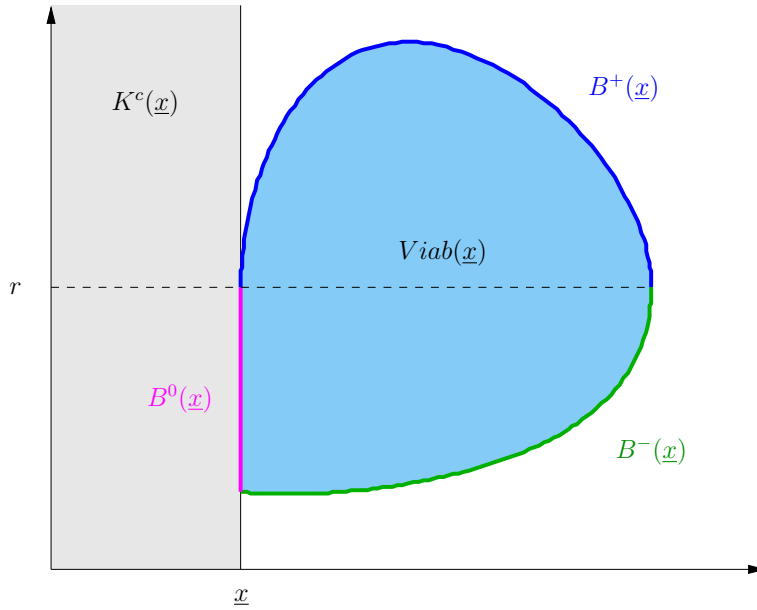


Figure 1: Viability kernel when $\bar{u} > \underline{x} - m$ (numerical values can be found in the appendix).

Next, given a solution $z(\cdot, z_0)$ of (1.1), we say that a time $t_c > 0$ is a crossing time for $z(\cdot)$ from $K(\underline{x})$ to $K(\underline{x})^c$ if the control u is left- and right-continuous at t_c , $z(t_c) \in \partial K$, and there exists $\eta > 0$ such that for any time $t \in [t_c - \eta, t_c]$, resp. $t \in (t_c, t_c + \eta]$, $z(t, z_0) \in K$, resp. $z(t, z_0) \in K^c$. Similarly, we define the notion of crossing time from $K(\underline{x})^c$ to $K(\underline{x})$.

The viability kernel of $K(\underline{x})$ enjoys the following properties.

Proposition 2.2. *Suppose that $\bar{u} > \underline{x} - m$.*

- *Consider the unique solution of (1.1) backward in time with $u = 0$ from $(x_{\bar{u}}^+(\underline{x}), r)$, and let $t' > 0$ be the first time where this trajectory intersects the axis $\{y = r\}$. Then, we have:*

$$x(t') \leq \underline{x}. \quad (2.1)$$

- The set $Viab(\underline{x})$ is a compact convex set with non-empty interior.
- Suppose that $t_1 < t_2$ are two consecutive crossing times from $K(\underline{x})$ to $K(\underline{x})^c$ and from $K(\underline{x})^c$ to $K(\underline{x})$ respectively and that $(x(t_1), y(t_1)) \notin Viab(\underline{x})$, $(x(t_2), y(t_2)) \notin Viab(\underline{x})$. Then, we have the following inequality:

$$t_2 - t_1 \geq \frac{\ln\left(\frac{r}{r_-}\right)}{m + \bar{u}}. \quad (2.2)$$

Proof. To prove the first point, suppose by contradiction that $x(t') > \underline{x}$. Denote by $(x_0(\cdot), y_0(\cdot))$ the unique solution of (1.1) with $u = 0$ and such that $(x_0(0), y_0(0)) = (x(t'), r)$. By construction, the point $(x_{\bar{u}}^+(\underline{x}), r)$ is on the graph of this curve, thus we denote by $t_0 := \inf\{t \geq 0 ; (x_0(t), y_0(t)) = (x_{\bar{u}}^+(\underline{x}), r)\}$, and let γ_0 the parametrized curve $(x_0(\cdot), y_0(\cdot))$ on the interval $[0, t_0]$.

Now, consider the unique solution $(x_1(\cdot), y_1(\cdot))$ of (1.1) with $u = \bar{u}$ starting from the point $(x(t'), r)$ and let $t_1 := \inf\{t \geq 0 ; y_1(t) = r\}$. We define γ_1 as the parametrized curve $(x_1(\cdot), y_1(\cdot))$ on the interval $[0, t_1]$. From (1.1), we know that the graph of γ_1 is below the graph of γ_0 .

To conclude, we consider the unique solution $(\tilde{x}_1(\cdot), \tilde{y}_1(\cdot))$ of (1.1) with $u = \bar{u}$ starting from (\underline{x}, r) until the first time $t'_1 > 0$ where it reaches the segment line $\{y = r\}$. It can be noticed that this curve passes through the point $(x_{\bar{u}}^+(\underline{x}), r)$ and that its graph $\tilde{\gamma}_1$ is also below the graph of γ_0 . Hence, both graphs γ_1 and $\tilde{\gamma}_1$ should intersect *i.e.* there must exist a time $\tau \in (0, t_0)$ such that $(x_1(\tau), y_1(\tau)) = (\tilde{x}_1(\tau), \tilde{y}_1(\tau))$. By Cauchy-Lipschitz's Theorem, both solutions $(x_1(\cdot), y_1(\cdot))$ and $(\tilde{x}_1(\cdot), \tilde{y}_1(\cdot))$ should coincide everywhere which is a contradiction as $(x_1(0), y_1(0)) \neq (\tilde{x}_1(0), \tilde{y}_1(0))$.

Let us now show the second point. From Proposition 2.1 and the previous point, we know that $Viab(\underline{x})$ is a compact subset of \mathbb{R}^2 with non-empty interior. Now, one can easily verify that solutions of (1.1) in the plane (x, y) satisfy:

$$\frac{d^2 y}{dx^2} \Big|_{u=0}(x) = \frac{y(r(x-m)^2 + m(r-y)^2)}{(r-y)^3 x^2} \quad \text{and} \quad \frac{d^2 y}{dx^2} \Big|_{u=\bar{u}}(x) = \frac{y(r(x-m-1)^2 + (m+1)(r-y)^2)}{(r-y)^3 x^2}.$$

It follows that for $y < r$, resp. $y > r$, one has $\frac{d^2 y}{dx^2} \Big|_{u=0}(x) > 0$, resp. $\frac{d^2 y}{dx^2} \Big|_{u=\bar{u}}(x) < 0$, which guarantees that the set $Viab(\underline{x})$ is convex.

To show the last point, we integrate the equation $\dot{y}(t) = y(t)(x(t) - m - u(t))$ over $[t_1, t_2]$, which gives:

$$(m + \bar{u})(t_2 - t_1) \geq \int_{t_1}^{t_2} (m + u(t)) \, dt = \int_{t_1}^{t_2} x(t) \, dt - \int_{y(t_1)}^{y(t_2)} \frac{dy}{y} \geq - \int_{y(t_1)}^{y(t_2)} \frac{dy}{y} \geq \ln\left(\frac{r}{r_-}\right),$$

using that $y(t_1) > r$ and $y(t_2) < r_-$. This ends the proof. \square

Remark 2.1. (i) Whereas for $m < \underline{x}$, it is clear that (2.1) holds true (as the equilibrium point for (1.1) with $u = 0$ is (m, r)), the previous proposition shows that this property remains valid whenever $m > \underline{x}$. Note also that in the latter case, the set $Viab(\underline{x})$ is always non-empty as $\bar{u} > 0$.

(ii) We know (see e.g. [1, 2]) that for a given initial state in $Viab(\underline{x})$, any control u can be chosen until that the corresponding trajectory reaches the boundary of $Viab(\underline{x})$. If $(x_0, y_0) \in B^-(\underline{x})$, resp. $(x_0, y_0) \in B^+(\underline{x})$, then only the control $u = 0$, resp. $u = \bar{u}$ is admissible in order to stay in $Viab(\underline{x})$.

(iii) If $\underline{x} > m$, then any point of the segment $[\underline{x}, \bar{u} + m] \times \{r\}$ is a steady-state point for (1.1) with a prescribed constant control whereas if $\underline{x} \leq m$, then any point of the segment $[m, \bar{u} + m] \times \{r\}$ is a steady-state point for (1.1) with a prescribed constant control.

(iv) Inequality (2.2) gives a lower bound between two consecutive crossing times and will be used in section 4.

3 Minimal time problem to reach the viability kernel

3.1 Attainability of the viability kernel

In this section, we suppose that the condition

$$\bar{u} > \underline{x} - m,$$

is fulfilled, i.e. $Viab(\underline{x}) \neq \emptyset$ and has a non-empty interior (see Proposition 2.1). We recall from the Viability Theory (see e.g. [1]) that the viability kernel $Viab(\underline{x})$ can be reached from outside only at its boundary in common with the boundary of $K(\underline{x})$, that is, accordingly to Proposition 2.1 at the line segment (possibly reduced to a singleton when $\bar{u} = \underline{x} - m$) $B^0(\underline{x}) = \{\underline{x}\} \times [r_-, r]$. In order to show the attainability of the target set, it is convenient to introduce the feedback control:

$$\mathbf{u}[x, y] := \begin{cases} \bar{u} & \text{if } y \geq r, \\ 0 & \text{if } y < r. \end{cases} \quad (3.1)$$

Given an initial condition $(x_0, y_0) \in (\mathbb{R}_+^* \times \mathbb{R}_+^*) \setminus Viab(\underline{x})$, we denote by $(x_m(\cdot), y_m(\cdot))$ the unique solution of (1.1) starting from (x_0, y_0) at time 0 and associated to the control $u_m(\cdot)$ defined by:

$$u_m(t) := \mathbf{u}[x_m(t), y_m(t)].$$

Proposition 3.1. *The feedback control (3.1) steers (1.1) from any initial condition $(x_0, y_0) \in \mathcal{D} \setminus Viab(\underline{x})$ to the viability kernel $Viab(\underline{x})$.*

Proof. First step. We show that it is enough to prove the result for any initial condition of type (x_0, r) with $x_0 > x_{\bar{u}}^+(\bar{x})$. If the initial condition (x_0, y_0) is such that $y_0 < r$, then it is enough to replace x_0 by $x_m(t_c)$ where t_c is the first time $t > 0$ such that $y_m(t_c) = r$. If $(x_m(t_c), y_m(t_c)) \in Viab(\underline{x})$, then the result is proved. Otherwise, we have $x_m(t_c) > x_{\bar{u}}^+(\bar{x})$. If now $y_0 > r$, we apply the control u_m until the first time $t'_c > 0$ such that $y_m(t'_c) = r$. Then, for $t > t'_c$ (close to t'_c) one has $y_m(t'_c) < r$, and we conclude by the previous case.

Second step. We now show the lemma for any initial condition (x_0, r) with $x_0 > x_{\bar{u}}^+(\bar{x})$. By applying the feedback control u_m , we can define two sequences of time $(t_n)_{n \geq 0}$ and $(t'_n)_{n \geq 0}$ such that:

$$y_n(t_n) = y_n(t'_n) = r \quad x'_n := x_m(t'_n) < \underline{x} < x_n := x_m(t_n)$$

Moreover, the trajectory is such that for any $n \in \mathbb{N}$:

$$t \in (t_n, t'_n) \Rightarrow y_m(t) > r; \quad t \in (t'_n, t_{n+1}) \Rightarrow y_m(t) < r.$$

We have $x_1 < x_0$. Indeed, consider the two solutions of (1.1), $\hat{x}_0(\cdot)$, resp. $\hat{x}_1(\cdot)$ with the control $u = 0$, resp. $u = \bar{u}$ starting from the point (x'_0, r) . We then have $\hat{x}_0(t) > \hat{x}_1(t)$ for $t \in (0, \hat{t}]$ where \hat{t} is such that $\hat{x}_0(\hat{t}) = r$. Now, as $\hat{x}_1(\cdot)$ passes through the point (x_0, r) , we deduce that $x_1 < x_0$. Now, the two solutions of (1.1) with $u = \bar{u}$ starting from (x_0, r) and (x_1, r) cannot intersect, thus we deduce that $x'_1 > x'_0$. By induction, we obtain that $(x_n)_{n \geq 0}$ is decreasing and that $(x'_n)_{n \geq 0}$ is increasing.

Now, integrating (1.1) on the interval (t_0, t'_0) , resp. (t'_0, t_1) with $u = \bar{u}$, resp. with $u = 0$ yields:

$$\begin{cases} -x_0 + (m + \bar{u}) \ln(x_0) = -x'_0 + (m + \bar{u}) \ln(x'_0), \\ -x_1 + m \ln x_1 = -x'_0 + m \ln x'_0. \end{cases}$$

Thus we obtain the relation $x_0 - x_1 - m \ln \left(\frac{x_0}{x_1} \right) + \bar{u} \ln \left(\frac{x'_0}{x_0} \right) = 0$ and by induction we get:

$$\forall n \in \mathbb{N}^*, \quad x_{n-1} - x_n - m \ln \left(\frac{x_{n-1}}{x_n} \right) + \bar{u} \ln \left(\frac{x'_{n-1}}{x_{n-1}} \right) = 0.$$

As $x_{n-1} < x_n$, we deduce that

$$x_{n-1} - x_n \geq \bar{u} \ln \left(\frac{x_{n-1}}{x'_{n-1}} \right).$$

To conclude, we suppose by contradiction that the trajectory always stays outside the set $Viab(\underline{x})$. By noticing that $x_{n-1} - x'_{n-1} \geq x_{\bar{u}}^+(\underline{x}) - \underline{x}$ and that $x'_{n-1} \leq \underline{x}$ for $n \geq 1$, one obtains $\frac{x_{n-1}}{x'_{n-1}} \geq \frac{x_{\bar{u}}^+(\bar{x})}{\underline{x}}$ which implies

$$x_{n-1} - x_n \geq \beta,$$

where $\beta := \bar{u} \ln \left(\frac{x_{\bar{u}}^+(\bar{x})}{\underline{x}} \right) > 0$. Thus, one has for each $n \in \mathbb{N}$ $x_n \leq x_{n-1} - \beta$, therefore we obtain a contradiction and the trajectory necessary enters the set $Viab(\underline{x})$. \square

Definition 3.1. The feedback control \mathbf{u} given by (3.1) will be called *myopic*¹.

Observe indeed that along trajectories of (1.1) one has

$$\forall t \geq 0, \quad \frac{d}{dt} V_0(x(t), y(t)) = -u(t)(y(t) - r), \quad (3.2)$$

Finding a control strategy decreasing the value of V_0 appears therefore to be an efficient strategy to steer (1.1) from a given initial condition to $Viab(\underline{x})$. In view of (3.2), we obtain the following inequalities:

$$\forall t \geq 0, \quad y(t) \geq r \Rightarrow \frac{d}{dt} V_0(x(t), y(t)) \geq -\bar{u}(y(t) - r) \quad \text{and} \quad y(t) \leq r \Rightarrow \frac{d}{dt} V_0(x(t), y(t)) \geq 0. \quad (3.3)$$

Thanks to (3.3), we obtain that \mathbf{u} is the control for which the decrease of V_0 is maximal. However, there is no evidence that the feedback control \mathbf{u} corresponds to the optimal feedback control for the minimal time strategy to reach $Viab(\underline{x})$ (that control is computed in section 3.2).

Our objective is now to compute an optimal control steering (1.1) in minimal time to the viability kernel $Viab(\underline{x})$. To do so, we apply the Pontryagin Maximum Principle [13] which provides necessary optimality conditions on an optimal control.

3.2 Pontryagin Maximum Principle

The minimum time control problem to reach $Viab(\underline{x})$ from a given initial condition $z_0 = (x_0, y_0) \in \mathcal{D}$ is defined as follows:

$$v(z_0) := \inf_{u(\cdot) \in \mathcal{U}} T_u \quad \text{s.t.} \quad z_u(T_u, z_0) \in Viab(\underline{x}), \quad (3.4)$$

where $z_u(\cdot, z_0) := (x_u(\cdot), y_u(\cdot))$ is the unique solution of (1.1) associated to the control u , T_u is the first entry time of $z_u(\cdot, z_0)$ into the target set, and $v(x_0, y_0) \in [0, +\infty]$ is the *value function* associated to the problem. Recall that $Viab(\underline{x})$ can be reached from $K(\underline{x})^c$ only through the line segment $B^0(\underline{x})$. From Proposition 3.1, the set $Viab(\underline{x})$ can be reached from any initial condition $(x_0, y_0) \in \mathcal{D}$ (i.e. v is finite everywhere in \mathcal{D}), thus the existence of an optimal control is straightforward using Fillipov's Theorem (see [8]). We are now in position to apply the Pontryagin Maximum Principle (PMP) to derive necessary optimality conditions on problem (3.4).

Recall that given a non-empty closed convex subset $K \subset \mathbb{R}^2$, the normal cone to K at a point $x \in K$ is defined as $N_K(x) := \{p \in \mathbb{R}^2 ; p \cdot (y - x) \leq 0, \forall y \in K\}$ where $a \cdot b$ denotes the standard scalar product of two vectors $a, b \in \mathbb{R}^2$. Let $H : \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be the Hamiltonian associated to (3.4) defined by:

$$H = H(x, y, p, q, p_0, u) = px(r - y) + qy(x - m - u) + p_0.$$

We now apply the Pontryagin Maximum Principle to (3.4). Let $u \in \mathcal{U}$ be an optimal control defined over a certain time interval $[0, T_u]$ with $T_u \geq 0$ and let $z_u := (x_u, y_u)$ be the associated solution. Then, there exists an absolutely continuous map $\lambda := (p, q) : [0, T_u] \rightarrow \mathbb{R}^2$ and $p_0 \leq 0$ such that the following conditions are satisfied:

- The pair $(\lambda(\cdot), p_0)$ is non-zero.
- The adjoint vector satisfies the adjoint equation $\dot{\lambda}(t) = -\frac{\partial H}{\partial z}(z_u(t), \lambda(t), p_0, u(t))$ a.e. on $[0, T_u]$ that is:

$$\begin{cases} \dot{p} &= p(y - r) - qy, \\ \dot{q} &= px + q(u + m - x). \end{cases} \quad (3.5)$$

- As $Viab(\underline{x})$ is a non-empty compact convex subset of \mathbb{R}^n , the transversality condition can be expressed as $\lambda(T_u) \in -N_{Viab(\underline{x})}(z(T_u))$ (see e.g. [15]).
- The control u satisfies the maximization condition:

$$u(t) \in \arg \max_{0 \leq \omega \leq \bar{u}} H(z_u(t), \lambda(t), p_0, \omega) \quad \text{a.e. } t \in [0, T_u]. \quad (3.6)$$

¹This terminology was introduced in [3] in the case where a control policy acts separately in two components of the state domain.

An *extremal trajectory* is a triple $(z_u(\cdot), \lambda(\cdot), u(\cdot))$ satisfying (1.1)-(3.5)-(3.6). As the system is autonomous and T_u is free, the Hamiltonian is zero along any extremal trajectory. We say that the extremal is *normal* if $p_0 \neq 0$ and is *abnormal* if $p_0 = 0$. Whenever an extremal trajectory is normal, we can always suppose that $p_0 = -1$ (using that H and (3.5) are homogeneous). In view of (3.6), we define the *switching function* ϕ as

$$\phi := -qy,$$

and we obtain the following control law:

$$\begin{cases} \phi(t) > 0 & \Rightarrow u(t) = \bar{u}, \\ \phi(t) < 0 & \Rightarrow u(t) = 0, \\ \phi(t) = 0 & \Rightarrow u(t) \in [0, \bar{u}]. \end{cases} \quad (3.7)$$

We call *switching time* (or switching point) a time t_c where the control is non-constant in any neighborhood of t_c . From (4.6), we deduce that any switching time satisfies $\phi(t_c) = 0$. A direct computation shows that we have:

$$\dot{\phi}(t) = -p(t)x(t)y(t) \quad \text{a.e. } t \in [0, T_u].$$

Let us now explicit the transversality condition. To do so, let $e_1 := (1, 0)$ and w be the unit vector defined by $w := (\sin \psi, -\cos \psi)$ where $\psi \in (-\frac{\pi}{2}, \frac{\pi}{2})$ is defined by

$$\tan \psi := \frac{(\underline{x} - m)r_-}{(r - r_-)\underline{x}}.$$

Property 3.1. *If $(x, y) \in B^0(\underline{x})$, we have:*

$$\begin{aligned} y \in (r_-, r) & \Rightarrow N_{Viab(\underline{x})}(x, y) = \mathbb{R}_- \times \{0\}, \\ y = r_- & \Rightarrow N_{Viab(\underline{x})}(x, y) = \{\alpha(\beta w - [1 - \beta]e_1) ; (\alpha, \beta) \in \mathbb{R}_+ \times [0, 1]\}. \end{aligned}$$

Proof. At the boundary point (\underline{x}, r_-) of $Viab(\underline{x})$, the tangent cone is generated by the vectors $(0, 1)$ and $(\cos \psi, \sin \psi)$. The geometric computation of $N_{Viab(\underline{x})}(x, y)$ follows using that $Viab(\underline{x})$ is convex, and that the normal cone to $Viab(\underline{x})$ at $(x, y) \in B^0(\underline{x})$ is the dual cone to the tangent cone to $Viab(\underline{x})$ at (x, y) . \square

Thanks to Pontryagin' Principle, we can derive the following properties.

Proposition 3.2. *An optimal extremal trajectory $(z_u(\cdot), \lambda(\cdot), u(\cdot))$ defined over a time interval $[0, T_u]$ satisfies the following points:*

- The control u is bang-bang i.e. it satisfies $u(t) \in \{0, \bar{u}\}$ for a.e. $t \in [0, T_u]$ and:

$$u(t) = \frac{\bar{u}}{2} (1 + \text{sign}(\phi(t))) \quad \text{a.e. } t \in [0, T_u]. \quad (3.8)$$

- The transversality condition on $\lambda(\cdot)$ at time T_u reads as follows:

$$\begin{aligned} (x(T_u), y(T_u)) \in \{\underline{x}\} \times (r_-, r) & \Rightarrow (p(T_u), q(T_u)) \in \mathbb{R}_+ \times \{0\}, \\ (x(T_u), y(T_u)) = (\underline{x}, r_-) & \Rightarrow (p(T_u), q(T_u)) \in \{\alpha(-\beta w + (1 - \beta)e_1) ; (\alpha, \beta) \in \mathbb{R}_+ \times [0, 1]\}. \end{aligned} \quad (3.9)$$

- If the extremal trajectory reaches the target at some point in $\{\underline{x}\} \times (r_-, r)$, then it is normal i.e. $p_0 \neq 0$.
- If the extremal trajectory is abnormal, then any switching point lies on the axis $\{y = r\}$.

Proof. To prove the first point, suppose that $\phi = 0$ on some time interval $[t_1, t_2]$. By differentiating w.r.t the time t , we obtain $\dot{q} = q = 0$ over $[t_1, t_2]$ implying $p = 0$ over $[t_1, t_2]$. From (3.5), we deduce that the adjoint vector λ is zero over $[0, T_u]$. We thus obtain a contradiction with the PMP using $H = 0$. This proves that (3.8) holds almost everywhere.

Now, Property 3.1 together with the transversality condition $\lambda(T_u) \in -N_{Viab(\underline{x})}(z(T_u))$ straightforwardly implies (3.9).

Let us show the third point. Suppose by contradiction that $p_0 = 0$. Using that $H = 0$ and that $y(T_u) \neq r$, one obtains $p(T_u) = 0$. Thus we would have $p(T_u) = q(T_u) = 0$ and then $\lambda \equiv 0$ using (3.5). This contradicts the PMP as the pair $(\lambda(\cdot), p_0)$ would be zero.

Finally, suppose the extremal is abnormal and let t_0 be a switching point implying $\phi(t_0) = q(t_0) = 0$. It follows that $p(t_0) \neq 0$ (otherwise the vector λ would be zero on $[0, T_u]$ and this would contradict the PMP). Now, suppose that $y(t_0) \neq r$, then we find that $p(t_0)x(t_0)(r - y(t_0)) \neq 0$ which again contradicts the PMP as one has $p_0 = 0$. Hence, we necessarily have $y(t_0) = r$. \square

Remark 3.1. From (3.5) and the fact that $(\lambda(\cdot), p_0)$ is non-zero, the mapping $t \mapsto (p(t), q(t))$ is always non-zero. Using a similar argument as in the proof of the first point of Proposition 3.2, one can prove that the set of zeros of ϕ is isolated.

3.3 Optimal synthesis

We first analyze the behavior of ϕ which is crucial in order to find an optimal control policy.

Lemma 3.1. A normal extremal trajectory $(z(\cdot), \lambda(\cdot), u(\cdot))$ defined over $[0, T_u]$ satisfies the following properties:

- The switching function satisfies the ordinary differential equation (ODE):

$$\dot{\phi}(t) = \frac{y(t)(m + u(t) - x(t))}{r - y(t)}\phi(t) - \frac{y(t)}{r - y(t)}, \quad \text{a.e. } t \in [0, T_u], \quad (3.10)$$

- At a time t_0 where $y(t_0) = r$, we have $\phi(t_0) \neq 0$ and:

$$\phi(t_0) = \frac{1}{u(t_0) + m - x(t_0)}. \quad (3.11)$$

Proof. Let us first show that the set $S := \{t \in [0, T_u] ; y(t) = r\}$ is finite. If $t_0 \in S$, we have $q(t_0)y(t_0)(x(t_0) - m - u(t_0)) = 1$ which implies that $\dot{y}(t_0) \neq 0$, hence t_0 is isolated and thus S is finite. Using that $\phi = -qy$, $\dot{\phi} = -pxy$, and that $H = 0$, we get that (3.10) holds a.e. The proof of the second point is straightforward combining $H = 0$ and $y(t_0) = r$. \square

The previous Lemma implies the following proposition.

Proposition 3.3. Let $(z(\cdot), \lambda(\cdot), u(\cdot))$ a normal extremal trajectory defined over $[0, T_u]$. Then, one has:

- If there exist two consecutive instants $t_2 > t_1 > 0$ such that $y(t_1) = y(t_2) = r$, then the control u has exactly one switching time $t_c \in (t_1, t_2)$.
- If in addition, $x(t_1) > x(t_2)$, resp. $x(t_1) < x(t_2)$, then an optimal control satisfies $u = 0$, resp. $u = \bar{u}$ on (t_1, t_c) and then $u = \bar{u}$, resp. $u = 0$ on (t_c, t_2) .

Proof. From (3.11), the sign of $\phi(t_i)$, $i = 1, 2$ depends on the value of $x(t_i)$ w.r.t. $u(t_0) + m$. Whenever the trajectory satisfies $y(t_1) = r$ with $x(t_1) > x_u^+(\underline{x})$, we thus have $\phi(t_1) < 0$ implying $u = 0$. Using that $x(t_2) < m$, we deduce that $\phi(t_2) > 0$, hence the trajectory necessarily has a switching point at some time $t_c \in (t_1, t_2)$. Now, from (3.10), one has $\dot{\phi}(t_c) = -\frac{y(t_c)}{r - y(t_c)} > 0$. Thus, the only possibility for the trajectory is to switch from $u = 0$ to $u = \bar{u}$. This shows the uniqueness of t_c in (t_1, t_2) . If now $x(t_1) < x_u^+(\underline{x})$, the same argumentation shows that there exists a unique switching time from $u = \bar{u}$ to $u = 0$ in (t_1, t_2) . This ends the proof of the proposition. \square

We denote by γ the graph of the unique solution $(\tilde{x}(\cdot), \tilde{y}(\cdot))$ of (1.1) backward in time starting from the point (\underline{x}, r) associated to the feedback control (3.1). Let τ_1 be the first time where $(\tilde{x}(\cdot), \tilde{y}(\cdot))$ exits $K(\underline{x})$ and $\tau_2 > \tau_1$ be the first exit time of $(\tilde{x}(\cdot), \tilde{y}(\cdot))$ of the set $\{(x, y) \in \mathcal{D} ; y \leq r\}$. Finally, let γ_1 be the restriction of $(\tilde{x}(\cdot), \tilde{y}(\cdot))$ to the interval $[\tau_1, \tau_2]$. The optimal synthesis of the problem then reads as follows (see also Fig. 2).

Theorem 3.1. Let (x_0, y_0) be an initial condition in $\mathcal{D} \setminus \text{Viab}(\underline{x})$.

- If $(x_0, y_0) \in \gamma$, then any optimal trajectory steering (1.1) from (x_0, y_0) to the target set is abnormal. The corresponding control is given by u_m and switching points occur on the axis $\{y = r\}$.

- If $(x_0, y_0) \notin \gamma$, then any optimal trajectory steering (1.1) from (x_0, y_0) to the target set is normal. Moreover, if u denotes the optimal control, there exists $p \in \mathbb{N}^*$, $s \in \{0, 1\}$, and a sequence of times $(\tau_k)_{0 \leq k \leq p}$ such that:

- We have $\tau_0 = 0 < \tau_1 < \dots < \tau_{p-1} < \tau_p = T_u$ and τ_k is a switching time of u for $1 \leq k \leq p$.
- The optimal control u is given by

$$u(t) = \frac{\bar{u}}{2}(1 + (-1)^{p-k-s}) \quad t \in (\tau_k, \tau_{k+1}), \quad 0 \leq k \leq p-1. \quad (3.12)$$

- If $y(T_u) \in (r_-, r)$, resp. $y(T_u) = r_-$, then $s = 1$, resp. $s = 0$.

Proof. Let us prove the first point. We already know from Proposition 3.2 that any trajectory starting on the curve γ and associated to the control u_m corresponds to an abnormal extremal trajectory. We must prove that such an extremal is optimal. To do so, let us choose (x_0, y_0) on the curve γ , and let $(x(\cdot), y(\cdot), \lambda(\cdot), u(\cdot))$ an optimal extremal trajectory steering (1.1) from (x_0, y_0) to $Viab(\underline{x})$. Let t_0 be defined as follows:

$$t_0 := \inf\{t \geq 0; \exists \varepsilon > 0 \forall \tau \in (t, t + \varepsilon) \quad (x(\tau), y(\tau)) \notin \gamma\}.$$

Suppose by contradiction that $t_0 < T_u$. As $(x(\cdot), y(\cdot), u(\cdot))$ is extremal, (3.6) implies that t_0 is necessarily a switching time from $u = 0$ to $u = \bar{u}$ or from $u = \bar{u}$ to $u = 0$. We argue that $y(t_0) \neq r$. Indeed, otherwise, we would have a contradiction with the definition of t_0 (as by definition, u_m switches on the axis $\{y = r\}$). Hence, either we have $y(t_0) < r$ or $y(t_0) > r$. Now, the fact that $y(t_0) \neq r$ implies that the extremal trajectory is a normal one. Indeed, we cannot have $p(t_0) = 0$ by Cauchy-Lipschitz's Theorem, but as $y(t_0) \neq r$, we obtain that $p(t_0)x(t_0)(r - y(t_0)) \neq 0$ and thus p_0 must be non-zero. Suppose for instance that $y(t_0) < r$. By construction of t_0 , this point is a switching time from $u = 0$ to $u = \bar{u}$ and we necessarily have $\dot{\phi}(t_0) \geq 0$. On the other hand, we obtain from (3.10) that

$$\dot{\phi}(t_0) = -\frac{y(t_0)}{r - y(t_0)} < 0,$$

and a contradiction. If $y(t_0) > r$, we obtain a similar contradiction with the sign of $\dot{\phi}$ at time t_0 . This shows that $t_0 \geq T_u$, thus we have proved that the abnormal extremal trajectory starting from (x_0, y_0) with the control u_m drives (1.1) optimally to the target.

Let us prove the second point. The first two points follow from Proposition 3.2 and from the fact that the number of switching times of an optimal control is finite. Given an extremal trajectory $(x(\cdot), y(\cdot), \lambda(\cdot), u(\cdot))$ driving optimally (1.1) from a point $(x_0, y_0) \notin \gamma$ to $Viab(\underline{x})$, we consider two cases depending if $y(T_u) \in (r_-, r)$ or $y(T_u) = r_-$.

First case: $y(T_u) \in (r_-, r)$. From Proposition 3.2, we have $p_0 \neq 0$ i.e. the trajectory is normal. Now, as $\phi(T_u) = 0$ and $\dot{\phi}(T_u) = -\frac{y(T_u)}{r - y(T_u)} < 0$, we obtain that $u = \bar{u}$ in a left neighborhood of T_u . By using Proposition 3.3, we obtain that the extremal has exactly one switching time t_c between two consecutive instants $t_1 < t_2$ such that $y(t_1) = y(t_2) = r$. We thus obtain (3.12) by considering (1.1) backward in time from $t = T_u$ and by counting the number of times (denoted by $p - 1$ with $p \geq 1$) where the trajectory surrounds $Viab(\underline{x})$ before reaching (x_0, y_0) . When $k = p - 1$, we obtain $u(t) = \frac{\bar{u}}{2}(1 + (-1)^{1-s})$, thus $s = 1$ as was to be proved.

Second case: $y(T_u) = r_-$. Suppose that the extremal is abnormal i.e. $p_0 = 0$. It follows that $q(T_u) > 0$. Otherwise, the transversality condition would imply $q(T_u) = 0$ and using $H = 0$ we would have $p(T_u) = 0$ and a contradiction with the PMP. We deduce that $\phi(T_u) < 0$ thus $u = 0$ in a left neighborhood of T_u . As the extremal is abnormal, switching points occur only on the axis $\{y = r\}$. This shows that we have $u = 0$ on $[t_c, T_u]$ where t_c is the last time such that $y(t_c) = r$ before reaching $B^0(\underline{x})$. Thus, by integrating backward in time (1.1) from (\underline{x}, r_-) , we find that $u = u_m$ and that $(x_0, y_0) \in \gamma$ which is a contradiction. We have thus proved that the extremal optimal trajectory is normal. Finally, we have two cases depending if the optimal trajectory reaches (\underline{x}, r_-) with either the control $u = 0$ or $u = \bar{u}$:

- The case where we have $u = \bar{u}$ at the terminal time T_u is similar to the first case $y(T_u) \in (r_-, r)$ above. Thus, the conclusion is obtained similarly as above.
- Now, suppose that we have $u = 0$ at the terminal time T_u . The trajectory necessary has a switching time on γ_1 (as it is normal). We thus obtain (3.12) by considering (1.1) backward in time from $t = T_u$ and by counting the number of times (denoted by $p - 1$ with $p \geq 1$) where the trajectory surrounds $Viab(\underline{x})$ before reaching (x_0, y_0) . As $u = 0$ in a left neighborhood of T_u , we obtain $s = 0$.

□

3.4 Discussion

To highlight the optimal synthesis provided by Theorem 3.1, we provide the following remarks.

- When the initial condition z_0 is apart from the target, optimal trajectories must surround the target set a number of times increasing w.r.t. $\|z_0\|$.
- The optimal control provided by Theorem 3.1 (ii) can be interpreted as a slight perturbation of the myopic strategy (3.1): instead of switching on the axis $\{y = r\}$, switching times are delayed and the corresponding switching points occur after the last intersection between the corresponding trajectory and the axis $\{y = r\}$ (see the switching curves in red on Fig 2).
- Abnormal trajectories are contained in the curve γ and they are the only extremal trajectories for which switching points occur on the axis $\{y = r\}$.
- Between two consecutive times $t_1 < t_2$ for which a normal extremal trajectory satisfies $y(t_1) = y(t_2) = r$, the extremal has exactly one switching time.
- In Theorem 3.1, any normal trajectory reaching $B_0(\underline{x})$ in its interior satisfies:

$$p - (2j + 1) \geq 0 \quad \text{and} \quad p - 2j \geq 0 \quad \Rightarrow \quad y(\tau_{p-2j+1}) > r \quad \text{and} \quad y(\tau_{p-2j}) < r.$$

- Any normal trajectory either reaches $B_0(\underline{x})$ with $u = \bar{u}$, or it reaches the point (\underline{x}, r_-) with $u = 0$. In the latter case, an optimal trajectory switches from $u = \bar{u}$ to $u = 0$ on γ_1 .

Optimal trajectories are depicted on Fig. 2 (see the Appendix for more details on the numerical simulations).

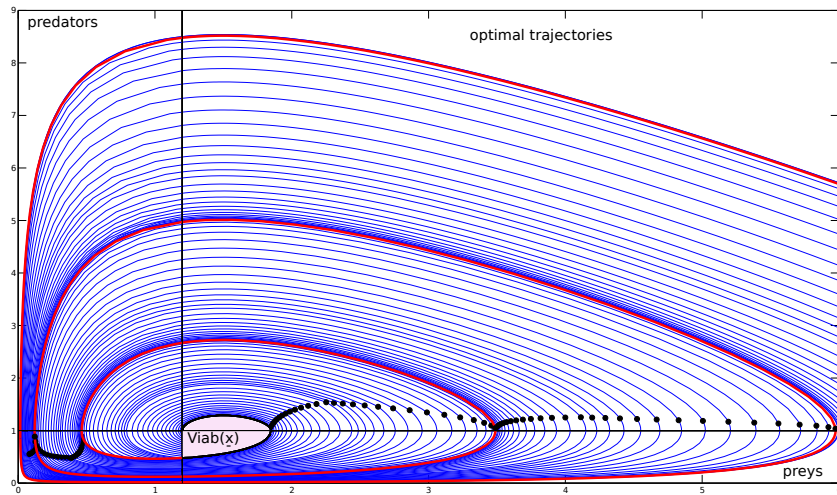


Figure 2: Optimal synthesis for (3.4) provided by Theorem 3.1 (see Appendix for the numerical values). In blue, optimal trajectories reaching the target set at $B_0(\underline{x})$. In red, the curve γ that is the only abnormal optimal trajectory reaching $B_0(\underline{x})$. Trajectories reach $B_0(\underline{x})$ with $u = \bar{u}$ and switch from 0 to \bar{u} or from \bar{u} to 0 at every dot point (in black) that represent the switching curves.

4 The minimal time crisis problem

In this section, we study the minimal time crisis problem

$$\theta(z_0) := \inf_{u \in \mathcal{U}_\infty} \int_0^\infty \mathbb{1}_{K(\underline{x})^c}(z_u(t, z_0)) \, dt, \quad (4.1)$$

when the viability kernel $Viab(\underline{x})$ has a non-empty interior, *i.e.* we suppose that $\bar{u} > \underline{x} - m$. The existence of an optimal control for (4.1) is standard (see [3, 7]).

4.1 Transformation of (4.1) into a finite horizon problem

As $\bar{u} > \underline{x} - m$, we know from Proposition 2.1 that the viability kernel is non-empty. Moreover, we have also proved that it can be reached from any initial condition $(x_0, y_0) \in \mathcal{D}$ (see Proposition 3.1). Thus, similarly as for (4.1), the problem

$$\theta(z_0) := \inf_{T \geq 0, u \in \mathcal{U}_T} \int_0^T \mathbb{1}_{K(\underline{x})^c}(z_u(t, z_0)) \, dt, \quad \text{s.t. } z_u(T, z_0) \in Viab(\underline{x}), \quad (4.2)$$

has a solution. The next proposition relates θ and $\hat{\theta}$.

Proposition 4.1. *For any $z_0 \in \mathcal{D}$, one has*

$$\theta(z_0) = \hat{\theta}(z_0). \quad (4.3)$$

Proof. As θ and $\hat{\theta}$ are zero in $Viab(\underline{x})$, we may suppose that $z_0 \notin Viab(\underline{x})$. We know from [7] that $\theta(z_0) \leq v(z_0)$ (recall that v denotes the minimal time function to reach $Viab(\underline{x})$) and that $v(z_0) < +\infty$, thus $\theta(z_0) < +\infty$. Let u^* be an optimal control for $\theta(z_0)$ and $z^*(\cdot, z_0)$ the associated solution starting from z_0 . Define a time $\tau(z_0) \in \mathbb{R}_+ \cup \{+\infty\}$ by:

$$\hat{\tau}(z_0) = \sup\{t \geq 0 ; z^*(t, z_0) \in K(\underline{x})^c\},$$

and suppose by contradiction that $\tau(z_0) = +\infty$. As $\theta(z_0) < +\infty$, there exists $t_0 \geq 0$ such that $z^*(t_0, z_0) \in K$. Now, as $\tau(z_0) = +\infty$, there exists $t_1 \geq t_0$ such that t_1 is a crossing time from $K(\underline{x})$ to $K(\underline{x})^c$. We now define t_2 as the first entry time $t > t_1$ of $z^*(\cdot, z_0)$ from $K(\underline{x})^c$ into $K(\underline{x})$ (t_2 exists as $\theta(z_0) < +\infty$). From Proposition 2.2, we deduce that $t_2 - t_1 \geq \delta > 0$ where $\delta := \ln\left(\frac{r}{r_-}\right) / (m + \bar{u})$. If we now repeat this argument an arbitrarily number of times using that $\tau(z_0) = +\infty$, we obtain $\theta(z_0) = +\infty$ and thus we have a contradiction. Therefore, we necessarily have $\tau(z_0) < +\infty$ which implies that $z^*(t, z_0) \in K$ for any time $t \geq \tau(z_0)$ *i.e.* $z^*(\tau(z_0), z_0) \in Viab(\underline{x})$. It follows that

$$\theta(z_0) = \int_0^{\tau(z_0)} \mathbb{1}_{K(\underline{x})^c}(z^*(t, z_0)) \, dt \geq \hat{\theta}(z_0),$$

using that $z^*(\tau(z_0), z_0) \in Viab(\underline{x})$. On the other hand, let $(\hat{T}, \hat{u}(\cdot)) \in \mathbb{R}_+ \times \mathcal{U}_{\hat{T}}$ an optimal pair for $\hat{\theta}(z_0)$. If $\hat{z}(\cdot, z_0)$ denotes the associated trajectory, we then have $\hat{z}(\hat{T}, z_0) \in Viab(\underline{x})$, hence we can extend \hat{u} to a control function $\bar{u} \in \mathcal{U}_\infty$ such that the associated trajectory $\bar{z}(\cdot, z_0)$ satisfies $\bar{z}(t, z_0) \in Viab(\underline{x})$ for any time $t \geq \hat{T}$. We thus have

$$\hat{\theta}(z_0) = \int_0^{\hat{T}} \mathbb{1}_{K(\underline{x})^c}(\hat{z}(t, z_0)) \, dt = \int_0^{+\infty} \mathbb{1}_{K(\underline{x})^c}(\bar{z}(t, z_0)) \, dt \geq \theta(z_0),$$

and the conclusion follows. \square

Hence, we are in position to use the hybrid maximum principle (see [9]) on (4.1) (formulated as a finite horizon problem, thanks to (4.3)).

Remark 4.1. *The reformulation of (4.1) into (4.2) can be proved in the same way in the more general context where (1.1) is replaced by a dynamics $(x, u) \in \mathbb{R}^n \times \mathbb{R}^m \mapsto f(x, u)$ and $K(\underline{x})$ is replaced by a non-empty closed subset of \mathbb{R}^n . In addition, f should satisfy the standard regularity assumptions of optimal control theory (see *e.g.* [10]) together with the following hypotheses :*

- *The viability kernel is non-empty and reachable from any initial condition.*
- *There exists $\delta > 0$ such that for any pair of consecutive crossing time $t_1 < t_2$ from K to K^c and from K^c to K one has $t_2 - t_1 \geq \delta$.*

4.2 Application of the hybrid maximum principle

Following [3], we will bypass the discontinuity of the integrand $\mathbf{1}_{K(\underline{x})^c}$ by considering the partition of the state space \mathcal{D} as $K(\underline{x}) \cup K(\underline{x})^c$ that will allow us to apply the hybrid maximum principle to derive necessary optimality conditions. Let $H : \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be the Hamiltonian associated to (4.2) defined by:

$$H := H(x, y, p, q, p_0, u) = px(r - y) + qy(x - m - u) + p_0 \mathbf{1}_{K(\underline{x})^c}(x, y).$$

If u is an optimal control of (4.2) and $(x_u(\cdot), y_u(\cdot))$ is the associated solution of (1.1), then the following optimality conditions are satisfied:

- There exists $T \geq 0$, $p_0 \leq 0$ and a measurable function $\lambda(\cdot) := (p(\cdot), q(\cdot)) : [0, T] \rightarrow \mathbb{R}^2$ satisfying a.e. on $[0, T]$:

$$\begin{cases} \dot{p} &= p(y - r) - qy, \\ \dot{q} &= px + q(u + m - x). \end{cases} \quad (4.4)$$

- The control u satisfies the maximization condition:

$$u(t) \in \arg \max_{\omega \in [0, \bar{u}]} H(x(t), y(t), p(t), q(t), p_0, \omega) \quad \text{a.e. } t \in [0, T]. \quad (4.5)$$

- The Hamiltonian H is constant equal to zero along any extremal trajectory $(x(\cdot), y(\cdot), u(\cdot), p(\cdot), q(\cdot), p_0)$ satisfying (1.1)-(4.4)-(4.5) (recall that the terminal time is free).
- For each crossing time t_c from $K(\underline{x})$ to $K(\underline{x})^c$ or from $K(\underline{x})^c$ to $K(\underline{x})$, one has

$$\lambda(t_c^+) - \lambda(t_c^-) \in N_{K(\underline{x})}(x(t_c), y(t_c)).$$

- The triple $(p_0, p(\cdot), q(\cdot))$ is non-zero.
- As $(x(T), y(T)) \in \text{Viab}(\underline{x})$, we obtain $\lambda(T) \in -N_{\text{Viab}(\underline{x})}(x(T), y(T))$ i.e. $\lambda(T)$ satisfies (3.9).

As for the minimal time control problem, the switching function $\phi := -qy$ provides the control law:

$$\begin{cases} \phi(t) > 0 & \Rightarrow u(t) = \bar{u}, \\ \phi(t) < 0 & \Rightarrow u(t) = 0, \\ \phi(t) = 0 & \Rightarrow u(t) \in [0, \bar{u}]. \end{cases} \quad (4.6)$$

Moreover, the adjoint equation implies $\dot{\phi} = -pxy$. Next, we say that a crossing time t_c is *transverse* if one has $\langle \dot{z}(t_c, z_0), e_1 \rangle \neq 0$. In other words, a transverse crossing time t_c is such that the trajectory does not hit the boundary of $K(\underline{x})$ tangentially while crossing $K(\underline{x})$.

Lemma 4.1. *Given a solution $z(\cdot, z_0)$ of (1.1), any crossing time t_c of $z(\cdot, z_0)$ such that $z(t_c, z_0) \notin \text{Viab}(\underline{x})$ is transverse.*

Proof. Suppose that a solution $z(\cdot, z_0) = (x(\cdot), y(\cdot))$ of (1.1) hits the boundary of $K(\underline{x})$ at some point (\underline{x}, y) tangentially. Then, we must have $\langle \dot{z}(t_c, z_0), e_1 \rangle = 0$ which implies that $y(t_c) = r$. Hence, we obtain that $(x(t_c), y(t_c)) = (\underline{x}, r) \in \text{Viab}(\underline{x})$ which contradicts the hypothesis of the lemma. Hence, any crossing time is transverse. \square

Thanks to this lemma, we can write the jump condition on the adjoint vector as follows (see [3]). Let t_c be a crossing time. Then, one has:

$$\begin{aligned} x(t_c^-) \in K(\underline{x}) \text{ and } x(t_c^+) \in K(\underline{x})^c & \Rightarrow p(t_c^+) - p(t_c^-) = \frac{q(t_c)y(t_c)(u(t_c^+) - u(t_c^-)) - p_0}{x(t_c)(r - y(t_c))} \\ x(t_c^-) \in K(\underline{x})^c \text{ and } x(t_c^+) \in K(\underline{x}) & \Rightarrow p(t_c^+) - p(t_c^-) = \frac{q(t_c)y(t_c)(u(t_c^+) - u(t_c^-)) + p_0}{x(t_c)(r - y(t_c))}. \end{aligned} \quad (4.7)$$

In particular, the function q is continuous over $[0, T]$ whereas p is piecewise absolutely continuous.

Lemma 4.2. (i) If an extremal trajectory $(x(\cdot), y(\cdot), u(\cdot), p(\cdot), q(\cdot), p_0)$ is abnormal (i.e. $p_0 = 0$), then the adjoint vector $(p(\cdot), q(\cdot))$ is absolutely continuous.

(ii) If an extremal trajectory $(x(\cdot), y(\cdot), u(\cdot), p(\cdot), q(\cdot), p_0)$ is normal, then (4.7) is equivalent to

$$\begin{aligned} x(t_c^-) \in K(\underline{x}) \text{ and } x(t_c^+) \in K(\underline{x})^c &\Rightarrow p(t_c^+) - p(t_c^-) = \frac{1}{x(t_c)(r-y(t_c))} \\ x(t_c^-) \in K(\underline{x})^c \text{ and } x(t_c^+) \in K(\underline{x}) &\Rightarrow p(t_c^+) - p(t_c^-) = \frac{-1}{x(t_c)(r-y(t_c))}. \end{aligned} \quad (4.8)$$

Proof. Let us first show that $q(t_c)y(t_c)(u(t_c^+) - u(t_c^-))$ is zero at any crossing time. The result is obvious if $q(t_c) = 0$. Now, if $q(t_c) < 0$, then $\phi > 0$ in a neighborhood of t_c , thus $u = \bar{u}$ in a neighborhood of t_c (recall that ϕ is continuous) so that $u(t_c^+) - u(t_c^-) = 0$. The same conclusion follows if $q(t_c) > 0$. Using that $q(t_c)y(t_c)(u(t_c^+) - u(t_c^-)) = 0$, one obtains straightforwardly (i) and (ii). \square

4.3 Qualitative properties of optimal controls

The next proposition provides optimality results for problem (4.1).

Proposition 4.2. Consider an optimal solution of problem (4.1).

- If $(x_0, y_0) \in \gamma$, then the optimal trajectory is abnormal and the optimal control is given by u_m . Switching points occur on the axis $\{y = r\}$.
- If $(x_0, y_0) \notin \gamma$, then the optimal trajectory is normal. Moreover, the following properties hold true:
 - (i) If τ is the last instant for which $y(\tau) = r$, then one has $u = \bar{u}$ over $[\tau, T]$.
 - (ii) Any switching point in $K(\underline{x})^c \cap \{y > r\}$ is from $u = 0$ to $u = \bar{u}$.
 - (iii) Any switching point in $K(\underline{x})^c \cap \{y < r\}$ is from $u = \bar{u}$ to $u = 0$.
 - (iv) If a switching point occurs in $K(\underline{x})$ at an instant t_s , then we must have $y(t_s) = r$.

Proof. First, notice that $\mathbb{1}_{K_c}$ is zero in $K(\underline{x})$, hence any switching time t_s that occurs in the set $K(\underline{x})$ necessarily satisfies $y(t_s) = r$.

To prove the first point, we suppose by contradiction (as in the proof of the first point in Theorem 3.4) that the trajectory starting from $(x_0, y_0) \in \gamma$ contains a switching point t_s such that $q(t_s) = 0$ and $y(t_s) \neq r$. We may suppose that t_s is the first one satisfying $y(t_s) \neq r$. Hence, the trajectory is normal (otherwise, the condition $H = 0$, $q(t_s) = 0$ and $y(t_s) = r$ would imply a contradiction). Finally, suppose that $y(t_s) > r$. Thus, t_s is a switching point from $u = \bar{u}$ to $u = 0$, i.e. $\phi(t_s) = 0$ implying also $\dot{\phi}(t_s) \leq 0$. From (3.10) (which remains valid in $K(\underline{x})^c$) we deduce that $\dot{\phi}(t_s) < 0$ which is a contradiction. If now $y(t_s) < r$, then t_s is by construction a switching point from $u = 0$ to $u = \bar{u}$, and we obtain a similar contradiction. Hence, we deduce that the optimal control is u_m and that the corresponding trajectory is abnormal.

To prove the second point, we use the transversality condition (3.9) which implies that either $p(T) > 0$ and $q(T) = 0$, thus $\dot{\phi}(T^-) \leq 0$ (when $y(t) \in (r_-, r)$) or $q(T) < 0$ (when $y(t) = r_-$). It follows that $\phi > 0$ in a left neighborhood of T and $u = \bar{u}$. Now, thanks to (3.10), we obtain that any switching point in $K(\underline{x})^c \cap \{y > r\}$, resp. $K(\underline{x})^c \cap \{y < r\}$ is from $u = 0$ to $u = \bar{u}$, resp. from $u = \bar{u}$ to $u = 0$. Hence, we necessarily have $u = \bar{u}$ over $[0, \tau]$ which ends the proof of the second point. \square

We deduce the following result.

Corollary 4.1. Let $(x(\cdot), y(\cdot))$ be a normal extremal trajectory defined over a time interval $[t_0, t_2]$ such that :

- At time t_0 , one has $y(t_0) = r$, $(x(t_0), y(t_0)) \in K(\underline{x}) \setminus \text{Viab}(\underline{x})$, and t_0 is a switching point from $u = 0$ to $u = \bar{u}$.
- There exists $t_1 \in (t_0, t_2)$ such that $x(t_1) = x(t_2) = \underline{x}$ with $(x(t_2), y(t_2)) \notin \text{Viab}(\underline{x})$.
- The set of points for which an optimal control has switching points on the axis $\{y = r\}$ only is γ .

Then, the trajectory has exactly one switching time $t_s \in (t_1, t_2)$ from $u = \bar{u}$ to $u = 0$ such that $y(t_s) < r$.

Proof. As t_0 is a switching point such that $(x(t_0), y(t_0)) \in K(\underline{x}) \setminus Viab(\underline{x})$, we necessarily have $u = \bar{u}$ over $[t_0, t_1]$. Now, using that q is continuous and no switching point occurs in the interval $(t_0, t_1]$, we must have $q(t_1) < 0$ thus $\phi(t_1) > 0$, and consequently one has $\phi > 0$ in a right neighborhood of t_1 . From Proposition 4.2, the trajectory cannot switch from $u = \bar{u}$ to $u = 0$ in the set $K(\underline{x})^c \cap \{y > r\}$. \square

Optimal trajectories are depicted on Fig. 3 (see the Appendix for more details on the numerical simulations). Switching points are represented in black. Switching curves consist of the collection of these points. The main features of optimal trajectories are as follows:

- Switching points in $K(\underline{x}) \setminus Viab(\underline{x})$ only occur on the axis $\{y = r\}$.
- Between two consecutive crossing time from K to K^c and from K^c to K , there exists at least one switching point in $K(\underline{x})$. Notice that the terminal time (*i.e.* the first time where the optimal trajectories enters $Viab(\underline{x})$) is not a crossing time.

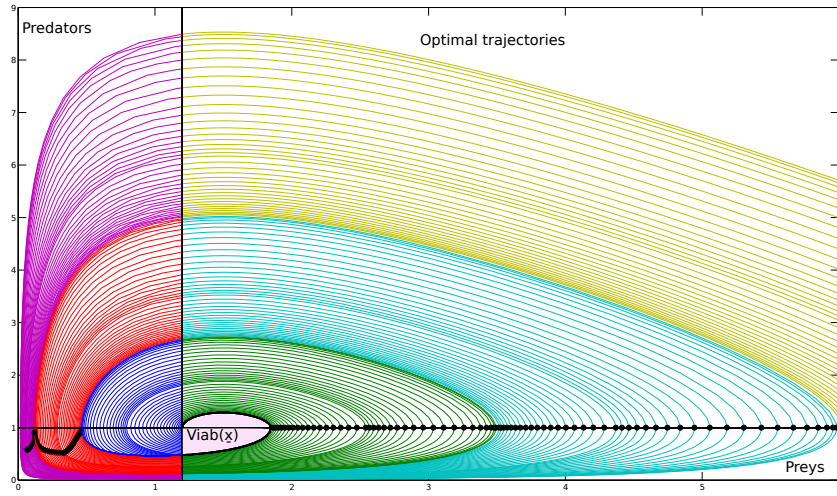


Figure 3: Optimal trajectories for problem (4.1) (see Appendix for the numerical values). At each crossing time, the color is changing. Switching points are depicted by the dot points in black and, in the sub-domain $K(\underline{x}) \setminus Viab(\underline{x})$, they occur on the axis $\{y = r\}$.

Define now a subset $\mathcal{E} \subset \mathcal{D}$ containing all the points of \mathcal{D} that can reach $Viab(\underline{x})$ with the constant control $u = \bar{u}$ and such that the corresponding optimal trajectory does not contain any switching point. Moreover, let \mathcal{F} be defined by:

$$\mathcal{F} := \mathcal{E} \cup \gamma.$$

Corollary 4.2. *Given an initial condition $z_0 = (x_0, y_0) \in \mathcal{D}$, we have the two following cases:*

- If $z_0 \in \mathcal{F}$, optimal solutions for problems (3.4) and (4.1) coincide.
- If $z_0 \in \mathcal{D} \setminus \mathcal{F}$, then one has

$$\theta(z_0) < J_\infty(u^*, z_0),$$

where u^* denotes an optimal control for (3.4).

Proof. The proof of the first point is immediate from Theorem 3.1 and Proposition 4.2. Take now an initial condition $z_0 = (x_0, y_0) \in \mathcal{D} \setminus \mathcal{F}$ and let u^* be an optimal control for (3.4). As $u^* \in \mathcal{U}_\infty$ is admissible for (4.1), we have $\theta(z_0) \leq J_\infty(u^*, z_0)$. If we have $\theta(z_0) = J_\infty(u^*, z_0)$, then u^* is necessarily an optimal control for (4.1). The switching points in $K(\underline{x})$ of the associated trajectory occur on the axis $\{y = r\}$ only from Theorem 4.2. As $z_0 \in \mathcal{D} \setminus \mathcal{F}$ the trajectory necessarily has at least one switching point in the set $K(\underline{x})$ from Theorem 3.4 at some time t_c . At this time, we have $y(t_c) > r$ (see Theorem 3.1) which gives a contradiction. \square

4.4 Discussion when $Viab(\underline{x}) = \emptyset$

When the viability kernel is empty, the time crisis function is equal to $+\infty$, as shows the next Proposition.

Proposition 4.3. *Suppose that $m + \bar{u} < \underline{x}$. Then, one has*

$$\forall z_0 \in \mathcal{D}, \quad \theta(z_0) = +\infty.$$

Proof. Recall that $m + \bar{u} < \underline{x}$ implies that $Viab(\underline{x}) = \emptyset$. Suppose by contradiction that there exists $z_0 \in \mathcal{D}$ such that $\theta(z_0) < +\infty$ and let $(x(\cdot), y(\cdot), u(\cdot))$ be the corresponding optimal trajectory. Then, $x(\cdot)$ has an infinite number of crossing times. Otherwise, either $x(\cdot)$ remains in K after a certain time $T \geq 0$ implying a contradiction with $Viab(\underline{x}) = \emptyset$, or it remains in K^c after a certain time $T \geq 0$ implying a contradiction with $\theta(z_0) < +\infty$. Without any loss of generality, we can thus suppose that there exists two sequences of times (t_n^1) and (t_n^2) satisfying:

- both sequences (t_n^1) and (t_n^2) are increasing and $t_n^1 < t_n^2$ for any $n \in \mathbb{N}$,
- for any $n \in \mathbb{N}$, t_n^1 , resp. t_n^2 is a crossing time from K to K^c , resp. from K^c to K ,
- for any time $t \in (t_n^1, t_n^2)$, one has $(x(t), y(t)) \in K^c$.

As $m + \bar{u} < \underline{x}$, there exists $\varepsilon > 0$ such that $m + \bar{u} + \varepsilon < \underline{x}$. Let us now integrate (1.1) over $[t_n^1, t_n^2]$. Since $u(t) \leq \bar{u}$ for any time t , one has:

$$\int_{y_n^1}^{y_n^2} \frac{dy}{y} = \int_{t_n^1}^{t_n^2} (x(t) - m - u(t)) dt \geq \int_{t_n^1}^{t_n^2} (x(t) - m - \bar{u}) dt > \int_{t_n^1}^{t_n^2} (x(t) - \underline{x} + \varepsilon) dt,$$

where $y_n^1 := y(t_n^1)$ and $y_n^2 := y(t_n^2)$ (recall from (1.1) that $y_n^1 > r > y_n^2$). Thus, we find that

$$\varepsilon(t_n^2 - t_n^1) + \ln\left(\frac{y_n^1}{y_n^2}\right) < \int_{t_n^1}^{t_n^2} (\underline{x} - x(t)) dt. \quad (4.9)$$

As $\theta(z_0) < +\infty$, the series $\sum (t_n^2 - t_n^1)$ converges. Hence, the sequence $(t_n^2 - t_n^1)$ goes to zero when n goes to infinity. It follows that $\ln\left(\frac{y_n^1}{y_n^2}\right)$ goes to 0, and thus $y_n^1 \sim y_n^2$ when $n \rightarrow +\infty$ i.e. $y_n^1 = y_n^2 + o(y_n^2)$. Since, $y(t) \in [y_n^2, y_n^1]$ for any $n \in \mathbb{N}$ and $t \in [t_n^1, t_n^2]$, we deduce that there exists a constant $C \geq 0$ such that

$$\forall n \in \mathbb{N}, \forall t \in [t_n^1, t_n^2], \quad 0 < y(t) \leq C.$$

It follows that

$$\forall n \in \mathbb{N}, \forall t \in [t_n^1, t_n^2], \quad |\dot{x}(t)| \leq x(t)(r + y(t)) \leq \underline{x}(r + C).$$

Hence, if we denote by $A := \underline{x}(r + C)$, the mean value Theorem implies that

$$\forall t \in [t_n^1, t_n^2], \quad |x(t) - \underline{x}| \leq A|t - t_n^1|.$$

Thus, one has $\sup_{t \in [t_n^1, t_n^2]} |x(t) - \underline{x}| \rightarrow 0$ when n goes to infinity. From (4.9), one deduces that

$$\varepsilon(t_n^2 - t_n^1) < \int_{t_n^1}^{t_n^2} (\underline{x} - x(t)) dt = o(|t_n^2 - t_n^1|),$$

which is a contradiction. This concludes the proof. \square

Remark 4.2. *This result shows that an optimal trajectory of the time crisis function starting from $z_0 \in \mathcal{D}$ cannot have an infinite number of crossing times in such a way that $\theta(z_0) < +\infty$. Hence, there is no chattering phenomenon² at the boundary of $K(\underline{x})$ in such a way that θ is finite.*

²In general, this terminology is used when an optimal control has an infinite number of switching points, see Fuller's example [16].

In the case where $m + \bar{u} < \underline{x}$ *i.e.* when the viability kernel of $K(\underline{x})$, $Viab(\underline{x})$, is empty, one should replace Problem (1.3) by

$$\inf_{u(\cdot) \in \mathcal{U}_T} \int_0^T \mathbb{1}_{K^c}(z_u(t, z_0)) \, dt,$$

where $T > 0$ is an arbitrary fixed time. Optimal controls for this problem can then be characterized in the same way as for Problem (4.2) using the hybrid maximum principle and integrating backward in time the state-adjoint system from a final condition $z_u(T)$. Notice that $z_u(T)$ is free implying that the adjoint vector $\lambda(\cdot)$ necessarily satisfies $\lambda(T) = 0$ at the terminal time. The application of the hybrid maximum principle thus leads to a similar study as in section 4.3, however we have not detailed this analysis for brevity.

5 Conclusion

In this work, we have detailed an exact computation of a viability kernel (which is usually not straightforward), and we have characterized optimal controls for both the minimum time control problem to reach this set and the minimal time crisis problem. Thanks to the Pontryagin and hybrid principles, we have characterized a subset of the state space such that any optimal trajectory steering an initial condition from this set to the viability kernel will spend more time outside $K(\underline{x})$ than the optimal trajectory for the minimal time crisis problem. In terms of state constraints (here the state constraint is expressed as $z_u(t, z_0) \in K(\underline{x})$), the feedback control for the minimum time problem guarantees a lower state constraint violation than the one for the minimal time problem to reach $Viab(\underline{x})$. Therefore, the time crisis function seems to be an interesting alternative to the strategy which consists in steering a system in minimal time to the viability kernel.

6 Acknowledgments

The authors thank the FMJH Program Gaspard Monge in optimization and operation research, the ECOS-SUD project C13E03, and the Matham-SUD project SOCDE.

7 Appendix

7.1 Numerical simulations

The numerical values that have been used to perform Fig. 1 2, 3 are given in Table 1.

r	m	\bar{u}	\bar{x}
1	1	0.5	1.2

Table 1: Numerical values for Fig. 1 2, 3.

Numerical simulations for obtaining Fig. 2, 3 have been conducted as follows. Given a terminal time $T > 0$, we consider the state-adjoint system backward in time:

$$\begin{cases} \dot{x} &= -x(r - y), \\ \dot{y} &= -y(x - m - u), \\ \dot{p} &= -p(y - r) + qy, \\ \dot{q} &= -px - q(u + m - x), \end{cases} \quad (7.1)$$

over $[0, T]$ together with the control law $u(t) = \text{sign}(-q(t))$ for a.e. $t \in [0, T]$ issued from (4.5). The initialization makes use of the transversality condition (3.9) and is explained below.

Simulation of extremal trajectories for the minimum time problem (3.4). Let $y_0 \in [r_-, r)$.

First case. If $y_0 \in (r_-, r)$, then $q(0) = 0$ and $p(0) = \frac{1}{\underline{x}(r - y_0)}$ (thanks to $H = 0$). Thus, (7.1) is initialized by the quadruple:

$$\left(\underline{x}, y_0, \frac{1}{\underline{x}(r - y_0)}, 0 \right). \quad (7.2)$$

Second case. If $y_0 = r_-$, then there exists $\alpha \geq 0$ and $\beta \in [0, 1]$ such that:

$$(p(0), q(0)) = \alpha(1 - \beta(1 + w_1), -\beta w_2) \quad \text{and} \quad \alpha = \frac{1}{\underline{x}(r - r_-)(1 - \beta(1 + w_1)) - \beta w_2 r_- (\underline{x} - m)}, \quad (7.3)$$

using that the Hamiltonian is zero along any extremal trajectory. The system (7.1) is then initialized by the quadruple:

$$(\underline{x}, r_-, p(0), q(0)), \quad (7.4)$$

with $(p(0), q(0))$ and α given by (7.3). Notice that in this case, the value of $\beta \in [0, 1]$ is a parameter (as $K(\underline{x})$ is non-smooth at (\underline{x}, r_-) , there exist infinitely many extremal trajectories arising from (\underline{x}, r_-)).

Simulation of extremal trajectories for the minimum time problem (4.1).

The initialization of (7.1) is the same as for problem (3.4). In addition, the equation (7.1) remains valid as long as the trajectory belongs to the set $K(\underline{x})^c$ or to the interior of $K(\underline{x})$. We thus impose the following condition:

$$\begin{aligned} (x(t_c^-), y(t_c^-)) \in K^c \quad \text{and} \quad (x(t_c^+), y(t_c^+)) \in K &\Rightarrow p(t_c^+) = p(t_c^-) + \frac{1}{\underline{x}(r - y(t_c))} \\ (x(t_c^-), y(t_c^-)) \in K \quad \text{and} \quad (x(t_c^+), y(t_c^+)) \in K^c &\Rightarrow p(t_c^+) = p(t_c^-) - \frac{1}{\underline{x}(r - y(t_c))}, \end{aligned} \quad (7.5)$$

at each crossing time t_c (recall that according to the hybrid principle applied on problem (4.5), only p is discontinuous).

Finally, the plot of optimal trajectories for (3.4) or (4.1) goes as follows:

- Choose $N \in \mathbb{N}^*$ and let $y_0^k := r_- + \frac{(k-1)}{N}(r - r_-)$ for $k = 1, \dots, N$.
- If $k = 1$, then choose $\beta_i = \frac{(i-1)}{N}$, $i=1, \dots, N$ and (7.1) is initialized by (7.4) with β_i in place of β .
- If $k > 1$, then (7.1) is initialized by (7.2) with y_0^k in place of y_0 .

For both problems, the numerical integration of (7.1) is stopped when $t = T$ (the real T is chosen sufficiently large). Any zero of the switching function ϕ (or equivalently q) during the numerical integration is marked by a dot point in black. These points correspond to switching points in the state space and their union form the switching curves (*i.e.* the loci where the control either switches from $u = 0$ to $u = \bar{u}$ or from $u = \bar{u}$ to $u = 0$).

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