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About viability and minimal time crisis problem* for the Lotka-Volterra prey-predator model

Terence Bayen[†], Alain Rapaport[‡]

September 27, 2016

Abstract

We study the minimal time crisis problem (see [7]) for the classical Lotka-Volterra prey-predator model. Our aim is to maintain the system in a subset $K(\underline{x})$ for which the number of preys is above a given threshold $\underline{x}$. In the case where the viability kernel of $K(\underline{x})$ is non-empty, we provide an analytic description of this set and compute the minimal time function steering the system to this set. We compare this strategy with the minimal time crisis function that is computed numerically via a regularization method [3]. We also investigate the minimal time crisis problem numerically in the case where the viability kernel is empty.

Keywords. Optimal Control, Viability Theory, Pontryagin Maximum Principle, Lotka-Volterra system.

1 Introduction

We consider the classical Lotka-Volterra prey-predator model

$$\left\{ \begin{array}{rcl} \dot{x} & = & rx - axy, \\ \dot{y} & = & -my + bxy - cuy, \end{array} \right.$$

in the positive orthant domain $\mathcal{D} := (\mathbb{R}_+ \setminus \{0\})^2$ where x and y denote respectively the prey and predator densities, and a, b, c, m, r are five positive parameters. The additional mortality term $-uy$ represents a *killing* action (by chemical or biological means) on the predators, and the removal effort $u \in [0, \bar{u}]$ is the control variable. In the present work, we consider that one aims to *protect* the preys from the predators, playing with the control u and the objective is that the state belongs as often as possible to the subset

$$K(\underline{x}) := \{(x, y) \in \mathcal{D} \,|\, x \geq \underline{x}\},$$

where $\underline{x} > 0$ is a given threshold. We consider time varying controls $u(\cdot)$ in the set

$$\mathcal{U} := \{u : \mathbb{R}_+ \to [0, \bar{u}] \; ; \; u(\cdot) \text{ meas.}\}.$$

Differently to the uncontrolled case, the controlled dynamics can have unbounded solutions. For simplicity and without any loss of generality, we choose the coefficients a, b, c equal to 1 and we obtain the system:

$$\left\{ \begin{array}{rcl} \dot{x} & = & rx - xy, \\ \dot{y} & = & -my + xy - uy. \end{array} \right. \tag{1.1}$$

For any initial condition $(x_0, y_0) \in \mathcal{D}$ and control law $u(\cdot) \in \mathcal{U}$, we denote by $(x, y)_{x_0, y_0, u(\cdot)}(\cdot)$ the solution of (1.1). In view of proposing control policies in the intention to protect the preys, we study the three following problems.

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1. Determine the viability kernel, that is the set

$$Viab(\underline{x}) := \{(x_0, y_0) \in \mathcal{D} \text{ s.t. } \exists u(\cdot) \in \mathcal{U},\ (x,y)_{x_0,y_0,u(\cdot)}(t) \in K(\underline{x}),\ \forall t \geq 0\},$$

   and the controls that allow a trajectory to stay in $Viab(\underline{x})$,

2. Given a terminal time $T > 0$, characterize the sets

$$E_T(\underline{x}) := \{(x,y) \in \mathcal{D} \text{ s.t. } \exists u(\cdot) \in \mathcal{U},\ (x,y)_{x_0,y_0,u(\cdot)}(T) \in Viab(\underline{x})\},$$

   and the controls that allows to reach $Viab(\underline{x})$ in minimal time,

3. Study the minimal time crisis over the finite horizon $[0,T]$, which consists in minimizing the cost

$$J_T(u) := \int_0^T \mathbb{1}_{K(\underline{x})^c}(x_{x_0,y_0,u(\cdot)}(t), y_{x_0,y_0,u(\cdot)}(t))\,\mathrm{d}t \tag{1.2}$$

   where $\mathbb{1}_{K^c}$ denotes the characteristics function

$$\mathbb{1}_{K^c}(x,y) = \left| \begin{array}{ll} 1 & \text{if } (x,y) \notin K, \\ 0 & \text{if } 0 \in K. \end{array} \right.$$

For the practitioners, solutions of these three problems bring complementary information. The viability kernel $Viab(\underline{x})$ defines the *safety* subset of the set $K(\underline{x})$. Outside this domain, the sets $E_T(\underline{x})$ provides measures of the guaranteed time to reach this safety domain (and then to stay in it). When it is not possible to reach this domain within a given duration T, the minimal time crisis measures the smallest duration spent outside the set $K(\underline{x})$ over the horizon $[0,T]$ (as a trajectory may enter and leave the set $K(\underline{x})$).

2 Study of the viability kernel

Given a non-empty subset A of $\mathbb{R}^2$, we will denote by $\mathrm{Int}(A)$ its interior. For a fixed $u \in [0, \bar{u}]$, we define a function $V_u : \mathcal{D} \to \mathbb{R}$ by:

$$V_u(x,y) := x - (m+u)\ln x + y - r\ln y, \quad (x,y) \in \mathcal{D},$$

together with the number $\underline{c}(u) \in \mathbb{R}$ defined by

$$\underline{c}(u) := V_u(m+u, r) = (m+u)(1 - \ln(m+u)) + r(1 - \ln r),$$

and the equilibrium point $E^\star(u)$ for (1.1)

$$E^\star(u) := (x^\star(u), y^\star) = (m+u, r).$$

For a given number $c \geq \underline{c}(u)$, we denote by $L_u(c)$, resp. by $S_u(c)$, the level set, resp. the sub-level set of V_u defined by

$$L_u(c) := \{(x,y) \in \mathcal{D},\ V_u(x,y) = c\}, \text{ resp. } S_u(c) := \{(x,y) \in \mathcal{D},\ V_u(x,y) \leq c\}.$$

The following lemma is standard when dealing with Lotka-Volterra type systems.

Lemma 2.1. *For a constant control u, a trajectory of (1.1) belongs to a level set $L_u(c)$ with $c \geq \underline{c}(u)$. The sets $L_u(c)$ are closed curves that surround the steady state $E^\star(u)$.*

Proof. By differentiating V_u w.r.t. x and y, one finds $\partial_x V_u(x,y) = 1 - \frac{m+u}{x}$ and $\partial_x V_u(x,y) = 1 - \frac{r}{y}$ for $(x,y) \in \mathcal{D}$. If $(x(\cdot), y(\cdot))$ is a solution of (1.1) with the constant control u, a direct computation gives

$$\frac{d}{dt} V_u(x(t), y(t)) = \partial_x V_u \dot{x} + \partial_y V_u \dot{y} = 0.$$

So, any solution of (1.1) with the constant control u belongs to a level set of the function V_u. As $V_u(x,y) \to +\infty$ when $\|(x,y)\| \to +\infty$, each level set $L_u(c)$ is bounded. For a constant control u, one can check that the single equilibrium of the dynamics in $\mathcal{D}$ is $E^\star(u)$, and that the level set $L_u(\underline{c}(u))$ is the singleton $\{E^\star(u)\}$. Therefore, for any initial condition in $\mathcal{D} \setminus \{E^\star(u)\}$, the trajectory belongs to a level set $L_u(c)$ with $c > \underline{c}(u)$ (recall that $V_u(x,y) \to +\infty$ when $\|(x,y)\| \to +\infty$). As $L_u(c)$ is a compact set that does not contain any equilibrium point, the trajectory has to be periodic and thus $L_u(c)$ is a closed curve which surrounds $E^\star(u)$. □

For $u \in [0, \bar{u}]$, we define two functions $\phi_u : \mathbb{R}_+ \to \mathbb{R}$ and $\psi : \mathbb{R}_+ \to \mathbb{R}$ by

$$\phi_u(x) := x - (m+u)\ln x,\ x \in \mathbb{R}_+ \quad \text{and} \qquad \psi(y) := y - r\ln y,\ y \in \mathbb{R}.$$

Lemma 2.2. *Given $u \in [0, \bar{u}]$, on has the following properties*

- *For any $x > \phi_u(m+u)$, there exists an unique $x_u^+(x) \in (m+u, +\infty)$ and an unique $x_u^-(x) \in (0, m+u)$ such that $\phi_u(x_u^+(x)) = \phi_u(x^-(x)) = x$.*
- *If $p > \psi(r)$, the equation $\psi(y) = p$ has exactly two roots $y^-(p)$, $y^+(p)$ that satisfy $y^-(p) < r < y^+(p)$.*

Proof. One can easily check that $\lim_{x\to+\infty} \phi_u(x) = \lim_{x\to 0} \phi_u(x) = +\infty$. Moreover, by differentiating ϕ_u w.r.t. x, one finds $\phi_u'(x) = 1 - \frac{m+u}{x}$. So the function ϕ_u is decreasing from $+\infty$ down to $\phi_u(m+u)$ and increasing up to $+\infty$. Therefore, for any $x > \phi_u(m+u)$, the equation $\phi_u(z) = x$ has exactly two solutions $x_u^-(x)$, $x_u^+(x)$, with $x_u^-(x) < m+u$ and $x_u^+(x) > m+u$. Similarly, the function ψ is decreasing from $+\infty$ down to $\psi(r)$ and increasing up to $+\infty$, which provides the result. □

For $c \in \mathbb{R}$, we consider the subsets of $\mathcal{D}$, $L_u^+(c)$, $L_u^-(c)$, $S_u^+(c)$ and $S_u^-(c)$ defined by:

$$L_u^+(c) := L_u(c) \cap \{y \geq r\}, \quad L_u^-(c) := L_u(c) \cap \{y \leq r\},$$

and

$$S_u^+(c) := S_u(c) \cap \{y \geq r\}, \quad S_u^-(c) := S_u(c) \cap \{y \leq r\},$$

and let $r_- \in (0, r]$ be defined by:

$$r_- := y^-(V_0(x_{\bar{u}}^+(\underline{x}), r) - \phi_0(\underline{x})).$$

The next proposition provides a description of the viability kernel, $Viab(\underline{x})$, of $K(\underline{x})$ for (1.1).

Proposition 2.1. *One has the following characterization of the viability kernel:*

- *If $m + \bar{u} < \underline{x}$, the set $Viab(\underline{x})$ is empty.*
- *If $\bar{u} \geq \underline{x} - m$, the viability kernel is non empty and we have:*

$$Viab(\underline{x}) = S_{\bar{u}}^+(V_{\bar{u}}(\underline{x}, r)) \bigcup \big(S_0^-(V_0(x_{\bar{u}}^+(\underline{x}), r)) \cap K(\underline{x})\big),$$

and its boundary is the union of the three curves

$$\begin{aligned} B^+(\underline{x}) &:= L_{\bar{u}}^+(V_{\bar{u}}(\underline{x}, r)), \\ B^-(\underline{x}) &:= L_0^-(V_0(x_{\bar{u}}^+(\underline{x}), r)) \cap \{x \geq \underline{x}\}, \\ B^0(\underline{x}) &:= \{\underline{x}\} \times [r_-, r]. \end{aligned}$$

Proof. Let us assume that $\bar{u} < \underline{x} - m$ and let $\varepsilon > 0$ be such that $m + \underline{u} < \underline{x} - \varepsilon$. Consider a trajectory $(x(\cdot), y(\cdot))$ that stays in the set $K(\underline{x})$ for any time $t \geq 0$. As $0 \leq u(t) \leq \bar{u}$, we deduce that

$$\dot{y} = y(x - m - u) \geq y(x - \underline{x} + \varepsilon) \geq \varepsilon y,$$

using that $x(t) \geq \underline{x}$ for any time $t \geq 0$. Therefore $y(\cdot)$ is unbounded and thus there exists $t_1 > 0$ such that $y(t) > r$ for any time $t > t_1$. It follows that

$$\dot{x}(t) = x(t)(r - y(t)) < 0, \quad \forall t > t_1,$$

and there exists $t_2 > t_1$ such that $x(t_2) < \underline{x}$. So the trajectory $(x(\cdot), y(\cdot))$ must escapes the set $K(\underline{x})$, and we have a contradiction. Thus, the viability kernel $Viab(\underline{x})$ is empty.

Assume now that one has $\bar{u} \geq \underline{x} - m$. Notice first that the three curves $B^+(\underline{x})$, $B^-(\underline{x})$ and $B^0(\underline{x})$ belong to the set $K(\underline{x})$ and that their union $U(\underline{x})$ defines the boundary of a compact subset $W(\underline{x})$ of $K(\underline{x})$, which is such that

$$W(\underline{x}) = S_{\bar{u}}^+(V_{\bar{u}}(\underline{x}, r)) \bigcup \big(S_0^-(V_0(x_{\bar{u}}^+(\underline{x}), r)) \cap K(\underline{x})\big).$$

When $\bar{u} = \underline{x} - m$, the set $W(\underline{x})$ is reduced to the single point $E^\star(\bar{u})$ that is an equilibrium of (1.1) for the constant control $\bar{u}$. Thus, $W(\underline{x})$ is a viable set.

When $\bar{u} > \underline{x} - m$, we first show that for any initial condition in $U(\underline{x})$, there exists a trajectory that stays in $K(\underline{x})$ for any time $t \geq 0$. Consider an initial condition in the set $B^+(\underline{x})$. With the control $u = \bar{u}$, the corresponding solution of (1.1) remains on the level set $L_{\bar{u}}(V_{\bar{u}}(\underline{x}, r))$ which is contained in $K(\underline{x})$ as its extreme left point is $(\underline{x}, r)$. Take now an initial condition in $B^-(\underline{x})$. With the control $u = 0$, the corresponding solution of (1.1) remains on $B^-(\underline{x})$ until it reaches in finite time the boundary point $(x_{\bar{u}}^+(\underline{x}), r)$ that belongs to $B^+(\underline{x})$. From this point, we come back to the previous case. Finally, take an initial condition in $\mathrm{Int}(B^0(\underline{x}))$ (if not empty). On $\mathrm{Int}(B^0(\underline{x}))$, one has $\dot{x} > 0$ for any control as one has $r - y > 0$. Thus the trajectory enters the subset $W(\underline{x})$ and cannot evade from $K(\underline{x})$ on $\mathrm{Int}(B^0(\underline{x}))$. If the trajectory touches $B^+ \cup B^-$, we face one of the two previous case. So we conclude that $W(\underline{x})$ is a viable domain.

We now show that $W(\underline{x})$ is the largest viable domain included in $K(\underline{x})$, that is, the viability kernel $Viab(\underline{x})$, or equivalently that any trajectory with initial condition in $K(\underline{x}) \setminus W(\underline{x})$ leaves the set $K(\underline{x})$ in a finite horizon. For convenience we consider the two subsets of $K(\underline{x}) \setminus W(\underline{x})$, $C^+(\underline{x})$ and $C^-(\underline{x})$ defined by:

$$C^+(\underline{x}) = \big(K(\underline{x}) \setminus W(\underline{x})\big) \cap \{y \geq r\} \quad \text{and} \quad C^-(\underline{x}) = \big(K(\underline{x}) \setminus W(\underline{x})\big) \cap \{y \leq r\}.$$

Consider now an initial condition $(x_0, y_0) \in C^+(\underline{x})$, and let $(x(\cdot), y(\cdot))$ a solution of (1.1) starting from (x_0, y_0). One has $V_{\bar{u}}(x_0, y_0) > V_{\bar{u}}(\bar{x}, r)$, and by differentiating w.r.t t one finds:

$$\frac{d}{dt} V_{\bar{u}}(x(t), y(t)) = (y(t) - r)(\bar{u} - u(t)) \geq 0.$$

Therefore no trajectory can reach the level set $L_{\bar{u}}(V_{\bar{u}}(\bar{x}, r))$ from $K(\underline{x}) \setminus W(\underline{x})$. When $y(t) = r$, one has $x(t) > m + \bar{u}$ and thus $\dot{y}(t) = r(x(t) - m - u(t)) > 0$ as $u(t) \leq \bar{u}$. We deduce that if there exists a trajectory with an initial condition (x_0, y_0) in $C^+(\underline{x})$ that stays in $K(\underline{x})$, it has to stay in $C^+(\underline{x})$. By differentiating w.r.t t, we obtain on $C^+(\underline{x})$ that

$$\frac{d}{dt} V_0(x(t), y(t)) = -u(t)(y(t) - r) \leq 0,$$

and thus $V_0(x(t), y(t)) \leq V_0(x_0, y_0)$. It follows that the trajectory is bounded. Moreover, one has $\dot{x} \leq 0$. i.e. the function $t \mapsto x(t)$ is non-increasing and thus converges to a certain $x_\infty > 0$. By Barbalat's Lemma, $\dot{x}(t)$ converges to 0 which implies that $y(t)$ tends to r. Then $V_{\bar{u}}(x(t), y(t))$ converges to $V_{\bar{u}}(x_\infty, r) > V_{\bar{u}}(\bar{x}, r)$ which implies that $x_\infty < \bar{x}$. Thus, the trajectory necessary leaves the set $K(\underline{x})$ and we have a contradiction.

Consider now an initial condition $(x_0, y_0) \in C^-(\underline{x})$. Similarly, one can show that no trajectory can reach the level set $L_0(V_0(x^+(\underline{x}), r))$ from $K(\underline{x}) \setminus W(\underline{x})$. It follows that a trajectory with an initial condition in $C^-(\underline{x})$ that stays in $K(\underline{x})$ has to stay in $C^-(\underline{x})$ (otherwise, it reaches $C^+(\underline{x})$ and we have shown above that its has to escape $K(\underline{x})$). As previously, one can show that a trajectory that stays in $C^-(\underline{x})$ is bounded and as $t \mapsto x(t)$ is increasing, one obtains the convergence of $x(t)$ to a certain $x_\infty > \underline{x}$. By Barbalat's Lemma, $\dot{x}(t)$ converges to 0 when $t \to +\infty$, which implies that $y(t) \to r$, and thus $x_\infty \geq x_{\bar{u}}^+(\underline{x}) > m + \bar{u}$. Therefore there exists $\varepsilon > 0$ and $t_0 > 0$ such that $\dot{y}(t) = (x(t) - m - u)y(t) > \varepsilon y(t)$ for any $t > t_0$. This gives a contradiction with the convergence of $y(t)$ to r when $t \to +\infty$. So, the trajectory has to enter $C^+(\underline{x})$ and then leaves $K(\underline{x})$. □

The viability kernel is depicted on Fig. 1 in case (ii) of Proposition 2.1 together with the three curves $B^+(\underline{x})$, $B^-(\underline{x})$,$B^0(\underline{x})$ that define its boundary. We now provide additional geometric properties of $Viab(\underline{x})$.

Property 2.1. *Suppose that $\bar{u} > \underline{x} - m$.*

(i) Consider the unique solution of (1.1) backward in time with $u = 0$ from $(x_{\bar{u}}^+(\underline{x}), r)$, and let $t' > 0$ be the first time where this trajectory intersects the axis $\{y = r\}$. Then, we have:

$$x(t') \leq \underline{x}.$$

(ii) The set $Viab(\underline{x})$ is a compact convex set.

Proof. To prove (i), suppose by contradiction that $x(t') > \underline{x}$. Denote by $(x_0(\cdot), y_0(\cdot))$ the unique solution of (1.1) with $u = 0$ and such that $(x_0(0), y_0(0)) = (x(t'), r)$. By construction, the point $(x_{\bar{u}}^+(\underline{x}), r)$ is on the graph of this curve, thus we denote by $t_0 := \inf\{t \geq 0 \ ; \ (x_0(t), y_0(t)) = (x_{\bar{u}}^+(\underline{x}), r)\}$. Now, consider the unique solution of (1.1) with $u = 1$ starting from $(x(t'), r)$. Clearly, one has $y_1(t) < y_0(t)$ for any time $t \in (0, t_0]$. But, the unique solution $(\tilde{x}_1(\cdot), \tilde{y}_1(\cdot))$ of (1.1) with $u = 1$ and starting from $(\underline{x}, r)$ passes

through the point $(x_{\bar{u}}^{+}(\underline{x}), r)$. Moreover, one also has $\tilde{y}_1(t) < y_0(t)$ for any time $t \in [0, t_0]$. To conclude, the inequality $y_1(t) < y_0(t)$ implies that there exists a time $\tau \in (0, t_0)$ such that $(x_1(\tau), y_1(\tau)) = (\tilde{x}_1(\tau), \tilde{y}_1(\tau))$. By Cauchy-Lipschitz's Theorem, both solutions $(x_1(\cdot), y_1(\cdot))$ and $(\tilde{x}_1(\cdot), \tilde{y}_1(\cdot))$ should coincide everywhere which is a contradiction as $(x_1(0), y_1(0)) \neq (\tilde{x}_1(0), \tilde{y}_1(0))$. This proves (i).

We already know that if non-empty $Viab(\underline{x})$ is a compact subset of $\mathbb{R}^2$. Now, one can easily verify that solutions of (1.1) in the plane (x, y) satisfy:

$$y < r \;\Rightarrow\; \frac{d^2y}{dx^2}_{|u=0}(x) > 0 \quad ; \quad y > r \;\Rightarrow\; \frac{d^2y}{dx^2}_{|u=1}(x) < 0,$$

which proves that $Viab(\underline{x})$ is convex. □

Remark 2.1. *(i) Whereas for $m < \underline{x}$, the property (i) is elementary (as the equilibrium point for (1.1) is (m, r) when $u = 0$), the property (2.1) shows that it remains valid whenever $m > \underline{x}$. Note also that in the latter case, then $Viab(\underline{x})$ is always non-empty as $\bar{u} > 0$.*
(ii) We know (see e.g. [1, 2]) that whenever a state is in $Viab(\underline{x})$, any control u can be taken in the interior of this set. However, one should take $u = 0$ on $B^-(\underline{x})$ and $u = 1$ on $B^+(\underline{x})$ in order to stay in this set.
(iii) If $\underline{x} > m$, then any point on the segment $[\underline{x}, \bar{u}+m] \times \{r\}$ is a steady-state point for (1.1) with a prescribed constant control whereas if $\underline{x} \leq m$, then any point on the segment $[m, \bar{u}+m] \times \{r\}$ is a steady-state point for (1.1) with a prescribed constant control.

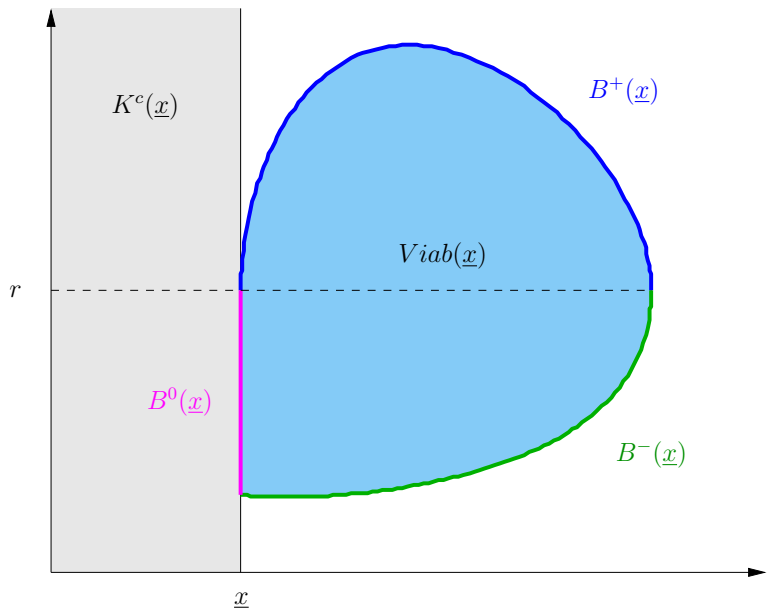

Figure 1: Viability kernel when $\bar{u} > \underline{x} - m$.

2.1 Lower bound on two consecutive crossing times

In this section, we suppose that the condition

$$\bar{u} \geq \underline{x} - m,$$

is fulfilled, i.e. that $Viab(\underline{x}) \neq \emptyset$ (see Proposition 2.1). Our aim is to show that given two consecutive crossing times $t_1 < t_2$ from K to K^c and from K^c to K such that the corresponding trajectory does not hit $Viab(\underline{x})$ at time t_1 and t_2, then $t_2 - t_1$ is minorized by a uniform constant.

Proposition 2.2. *Suppose that $t_1 < t_2$ are two consecutive crossing times from K to K^c and from K^c to K respectively and that $(x(t_1), y(t_1)) \notin Viab(\underline{x})$, $(x(t_2), y(t_2)) \notin Viab(\underline{x})$. Then, we have the following inequality:*

$$t_2 - t_1 \geq \frac{\ln\left(\frac{r}{r_-}\right)}{m + \bar{u}}. \tag{2.1}$$

Proof. By integrating the equation $\dot{y}(t) = y(t)(x(t) - m - u(t))$ over $[t_1, t_2]$, one deduces that:

$$(m + \bar{u})(t_2 - t_1) \geq \int_{t_1}^{t_2} (m + u(t))\ \mathrm{d}t = \int_{t_1}^{t_2} x(t)\ \mathrm{d}t - \int_{y(t_1)}^{y(t_2)} \frac{dy}{y} \geq -\int_{y(t_1)}^{y(t_2)} \frac{dy}{y} = \ln\left(\frac{r}{r_-}\right),$$

as was to be proved. □

3 Minimal time problem to reach the viability kernel

3.1 Attainability of the viability kernel

In this section, we suppose that the condition

$$\bar{u} \geq \underline{x} - m,$$

is fulfilled, i.e. that $Viab(\underline{x}) \neq \emptyset$ (see Proposition 2.1). We recall from the Viability Theory (see e.g. [1]) that the viability kernel $Viab(\underline{x})$ can be reached from outside only at its boundary in common with the boundary of $K(\underline{x})$, that is, accordingly to Proposition 2.1 at the line segment (possibly reduced to a singleton when $\bar{u} = \underline{x} - m$) $B^0(\underline{x}) = \{\underline{x}\} \times [r_-, r]$. In order to show the attainability of the target set, it is convenient to introduce the feedback control:

$$\mathbf{u}[x, y] := \begin{cases} \bar{u} & \text{if} \quad y \geq r, \\ 0 & \text{if} \quad y < r. \end{cases} \tag{3.1}$$

Given an initial condition $(x_0, y_0) \in (\mathbb{R}_+^* \times \mathbb{R}_+^*) \backslash Viab(\underline{x})$, we denote by $(x_m(\cdot), y_m(\cdot))$ the unique solution of (1.1) starting from (x_0, y_0) at time 0 and associated to the control $u_m(\cdot)$ defined by:

$$u_m(t) := \mathbf{u}[x_m(t), y_m(t)].$$

Proposition 3.1. *The feedback control* (3.1) *steers* (1.1) *from any initial condition $(x_0, y_0) \in \mathcal{D} \backslash Viab(\underline{x})$ to the viability kernel $Viab(\underline{x})$.*

Proof. First step. We show that it is enough to prove the result for any initial condition of type (x_0, r) with $x_0 > x_{\bar{u}}^+(\bar{x})$. If the initial condition (x_0, y_0) is such that $y_0 < r$, then it is enough to replace x_0 by $x_m(t_c)$ where t_c is the first time $t > 0$ such that $y_m(t_c) = r$. If $(x_m(t_c), y_m(t_c)) \in Viab(\underline{x})$, then the result is proved. Otherwise, we have $x_m(t_c) > x_{\bar{u}}^+(\bar{x})$. If now $y_0 > r$, we apply the control u_m until the first time $t_c' > 0$ such that $y_m(t_c') = r$. Then, for $t > t_c'$ (close to t_c') one has $y_m(t_c') < r$, and we conclude by the previous case.

Second step. We now show the lemma for any initial condition (x_0, r) with $x_0 > x_{\bar{u}}^+(\bar{x})$. By applying the feedback control u_m, we can define two sequences of time $(t_n)_{n \geq 0}$ and $(t_n')_{n \geq 0}$ such that:

$$y_n(t_n) = y_n(t_n') = r \quad x_n' := x_m(t_n') < \underline{x} < x_n := x_m(t_n)$$

Moreover, the trajectory is such that for any $n \in \mathbb{N}$:

$$t \in (t_n, t_n') \ \Rightarrow y_m(t) > r\ ;\ t \in (t_n', t_{n+1}) \ \Rightarrow y_m(t) < r.$$

We have $x_1 < x_0$. Indeed, consider the two solutions of (1.1), $\hat{x}_0(\cdot)$, resp. $\hat{x}_1(\cdot)$ with the control $u = 0$, resp. $u = \bar{u}$ starting from the point (x_0', r). We then have $\hat{x}_0(t) > \hat{x}_1(t)$ for $t \in (0, \hat{t}]$ where $\hat{t}$ is such that $\hat{x}_0(\hat{t}) = r$. Now, as $\hat{x}_1(\cdot)$ passes though the point (x_0, r), we deduce that $x_1 < x_0$. Now, the two solutions of (1.1) with $u = \bar{u}$ starting from (x_0, r) and (x_1, r) cannot intersect, thus we deduce that $x_1' > x_0'$. By induction, we obtain that $(x_n)_{n \geq 0}$ is decreasing and that $(x_n')_{n \geq 0}$ is increasing.

Now, integrating (1.1) on the interval (t_0, t_0'), resp. (t_0', t_1) with $u = \bar{u}$, resp. with $u = 0$ yields:

$$\begin{cases} -x_0 + (m+\bar{u})\ln(x_0) = -x_0' + (m+\bar{u})\ln(x_0'), \\ -x_1 + m\ln x_1 = -x_0' + m\ln x_0'. \end{cases}$$

Thus we obtain the relation $x_0 - x_1 - m\ln\left(\frac{x_0}{x_1}\right) + \bar{u}\ln\left(\frac{x_0'}{x_0}\right) = 0$ and by induction we get:

$$\forall n \in \mathbb{N}^*, \quad x_{n-1} - x_n - m\ln\left(\frac{x_{n-1}}{x_n}\right) + \bar{u}\ln\left(\frac{x_{n-1}'}{x_{n-1}}\right) = 0.$$

As $x_{n-1} < x_n$, we deduce that

$$x_{n-1} - x_n \geq \bar{u}\ln\left(\frac{x_{n-1}}{x_{n-1}'}\right).$$

Notice that $x_{n-1} - x_{n-1}' \geq x_{\bar{u}}^+(\underline{x})$ and $x_{n-1}' \leq \underline{x}$ for $n \geq 1$, thus $\frac{x_{n-1}}{x_{n-1}'} \geq 1 + \frac{x_{\bar{u}}^+(\bar{x})}{\underline{x}}$. It follows that

$$x_{n-1} - x_n \geq \beta,$$

where $\beta := \bar{u}\ln\left(1 + \frac{x_{\bar{u}}^+(\bar{x})}{\underline{x}}\right) > 0$. This proves that the trajectory necessary enters $Viab(\underline{x})$. □

Remark 3.1. *Following [3] we shall call the feedback* **u** *myopic. Indeed, if $a \in [0, \bar{u}]$, we know that:*

$$\frac{d}{dt}V_a(x(t), y(t)) = (a - u(t))(y(t) - r) \quad \forall t \in \mathbb{R}_+, \tag{3.2}$$

where $(x(\cdot), y(\cdot))$ is a solution of (1.1) *associated to a control function $u(\cdot) \in \mathcal{U}$. Finding an admissible control decreasing the value of V_a appears therefore to be a a good strategy in order to reach $Viab(\underline{x})$. In view of* (3.2)*, we can observe that for any admissible control u, one has*

$$\frac{d}{dt}V_a(x(t), y(t)) = (a - u_m(t))(y(t) - r) \leq \frac{d}{dt}V_a(x(t), y(t)) = (a - u(t))(y(t) - r).$$

However, there is no evidence that this strategy coincides with the minimal time strategy (that will be computed hereafter).

Our objective is now to compute an optimal control steering (1.1) in minimal time to the viability kernel $Viab(\underline{x})$. To do so, we apply the Pontryagin Maximum Principle [14] which provides necessary optimality conditions on an optimal control.

3.2 Pontryagin Maximum Principle

The minimal time control problem to reach $Viab(\underline{x})$ is defined as follows:

$$v(x_0, y_0) := \inf_{u(\cdot)\in\mathcal{U}} T_u \quad \text{s.t. } (x(T_u), y(T_u)) \in Viab(\underline{x}), \tag{3.3}$$

where $(x_u(\cdot), y_u(\cdot))$ is the unique solution of (1.1) associated to the control u and starting at some point $(x_0, y_0) \in \mathcal{D}\backslash Viab(\underline{x})$, and $v(x_0, y_0) \in [0, +\infty]$ is the *value function* associated to the problem. Recall that $Viab(\underline{x})$ can be reached from $K(\underline{x})^c$ only through the line segment $B^0(\underline{x})$. From Proposition 3.1, the set $Viab(\underline{x})$ can be reached from any initial condition $(x_0, y_0) \in \mathcal{D}$ (i.e. v is finite everywhere in $\mathcal{D}$), thus the existence of an optimal control is straightforward using Fillipov's Theorem (see [10]). We are now in position to apply the Pontryagin Maximum Principle (PMP) to derive necessary optimality conditions on problem (3.3).

Recall that given a non-empty closed convex subset $K \subset \mathbb{R}^n$, $n \geq 1$, the normal cone to K at a point $x \in K$ is defined as $N_K(x) := \{p \in \mathbb{R}^n \ ; \ p\cdot(y - x) \leq 0, \ \forall y \in K\}$ where $a \cdot b$ denotes the standar scalar product of two vectors $a, b \in \mathbb{R}^n$. Let $H : \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be the Hamiltonan associated to (3.3) defined by:

$$H = H(x, y, p, q, p_0, u) = px(r - y) + qy(x - m - u) + p_0.$$

We now apply the Pontryagin Maximum Principle to (3.3). Let $u \in \mathcal{U}$ be an optimal control defined over a certain time interval $[0, T_u]$ with $T_u \geq 0$ and let $z_u := (x_u, y_u)$ be the associated solution. Then, there exists an absolutely continuous map $\lambda := (p, q) : [0, T_u] \to \mathbb{R}^2$ and $p_0 \leq 0$ such that the following conditions are satisfied:

- The pair $(\lambda(\cdot), p_0)$ is non-zero.
- The adjoint vector satisfies the adjoint equation $\dot{\lambda} = -\frac{\partial H}{\partial z}(z_u(t), \lambda(t), p_0, u(t))$ a.e. on $[0, T_u]$ that is:

$$\begin{cases} \dot{p} &= p(y-r) - qy, \\ \dot{q} &= px + q(u+m-x). \end{cases} \tag{3.4}$$

- As $Viab(\underline{x})$ is a non-empty compact convex subset of $\mathbb{R}^n$, the transversality condition can be expressed as $\lambda(T_u) \in -N_{Viab(\underline{x})}(z(T_u))$ (see e.g. [16]).
- The control u satisfies the maximization condition:

$$u(t) \in \arg\max_{0 \le \omega \le \bar{u}} H(z_u(t), \lambda(t), p_0, \omega) \quad \text{a.e. } t \in [0, T_u]. \tag{3.5}$$

An *extremal trajectory* is a triple $(z_u(\cdot), \lambda(\cdot), u(\cdot))$ satisfying (1.1)-(3.4)-(3.5). As the system is autonomous and T_u is free, the Hamiltonian is zero along any extremal trajectory. We say that the extremal is *normal* if $p_0 \neq 0$ and is *abnormal* if $p_0 = 0$. Whenever an extremal trajectory is normal, we can always suppose that $p_0 = -1$ (using that H and (3.4) are homogeneous). In view of (3.5), we define the *switching function* ϕ as

$$\phi := -qy,$$

and we obtain the following control law:

$$\begin{cases} \phi(t) > 0 & \Rightarrow \quad u(t) = \bar{u}, \\ \phi(t) < 0 & \Rightarrow \quad u(t) = 0, \\ \phi(t) = 0 & \Rightarrow \quad u(t) \in [0, \bar{u}]. \end{cases} \tag{3.6}$$

We call *switching time* (or switching point) a time t_c where the control is non-constant in any neighborhood of t_c. From (3.6), we deduce that any switching time satisfies $\phi(t_c) = 0$. A direct computation shows that we have:

$$\dot{\phi}(t) = -p(t)x(t)y(t) \quad \text{a.e. } t \in [0, T_u].$$

Let us now explicit the transversality condition. To do so, let $e_1 := (1, 0)$ and w the unit vector defined by $w := (\sin\psi, -\cos\psi)$ where $\psi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ is defined by

$$\tan\psi := \frac{(\underline{x} - m)r_-}{(r - r_-)\underline{x}}.$$

Property 3.1. *If $(x, y) \in B^0(\underline{x})$, we have:*

$$\begin{array}{lll} y \in (r_-, r) & \Rightarrow & N_{Viab(\underline{x})}(x, y) = \mathbb{R}_- \times \{0\}, \\ y = r_- & \Rightarrow & N_{Viab(\underline{x})}(x, y) = \{\alpha(\beta w - [1-\beta]e_1)\ ;\ (\alpha, \beta) \in \mathbb{R}_+ \times [0, 1]\}. \end{array}$$

Proof. The geometric computation of $N_{Viab(\underline{x})}(x, y)$ is standard using that $Viab(\underline{x})$ is convex, and that the normal cone to $Viab(\underline{x})$ at $(x, y) \in B^0(\underline{x})$ is the dual cone to the tangent cone to $Viab(\underline{x})$ at (x, y). □

The Pontryagin Maximum Principle then implies the following properties.

Proposition 3.2. *(i) An optimal control u is bang-bang i.e. it satisfies $u(t) \in \{0, \bar{u}\}$ for a.e. $t \in [0, T_u]$ and:*

$$u(t) = \frac{\bar{u}}{2}\left(1 + \text{sign}(\phi(t))\right) \quad \text{a.e. } t \in [0, T_u]. \tag{3.7}$$

(ii) The transversality condition reads as follows:

$$\begin{array}{lll} (x(T_u), y(T_u)) \in \{\underline{x}\} \times (r_-, r) & \Rightarrow & (p(T_u), q(T_u)) \in \mathbb{R}_+ \times \{0\}, \\ (x(T_u), y(T_u)) = (\underline{x}, r_-) & \Rightarrow & (p(T_u), q(T_u)) \in \{\alpha([1-\beta]e_1 - \beta w)\ ;\ (\alpha, \beta) \in \mathbb{R}_+ \times [0, 1]\}. \end{array} \tag{3.8}$$

(iii) Any extremal trajectory reaching the target at some point in $\{\underline{x}\} \times (r_-, r)$ is normal i.e. $p_0 \neq 0$.
(iv) Any switching point of an abnormal trajectory lies on the axis $\{y = r\}$.

Proof. To prove (i), suppose that $\phi = 0$ on some time interval $[t_1, t_2]$. By differentiating w.r.t the time t, we obtain $\dot{q} = q = 0$ over $[t_1, t_2]$ implying $p = 0$ over $[t_1, t_2]$. From (3.4), we deduce that the adjoint vector λ is zero over $[0, T_u]$. We thus obtain a contradiction with the PMP using $H = 0$. This proves that (3.7) holds almost everywhere.

The proof of (ii) follows from Property 3.1 and the transversality condition $\lambda(T_u) \in -N_{Viab(\underline{x})}(z(T_u))$.

To prove (iii), suppose that $p_0 = 0$. Then using that $H = 0$ and that $y(T_u) \neq r$, we find that $p(T_u) = 0$. Thus we would have $p(T_u) = q(T_u) = 0$ and then $\lambda \equiv 0$ using (3.4). This contradicts the PMP as the pair $(\lambda(\cdot), p_0)$ would be zero.

Finally, suppose that t_0 is a switching point of an abnormal trajectory i.e. $\phi(t_0) = q(t_0) = 0$. It follows that $p(t_0) \neq 0$ (otherwise the vector λ would be zero on $[0, T_u]$ and this would contradict the PMP). Now, suppose that $y(t_0) \neq r$, then we find that $p(t_0)x(t_0)(r - y(t_0)) \neq 0$ which again contradicts the PMP as one has $p_0 = 0$. Hence, we necessarily have $y(t_0) = r$. □

Remark 3.2. *From* (3.4) *and the fact that* $(\lambda(\cdot), p_0)$ *is non-zero, the mapping* $t \longmapsto (p(t), q(t))$ *is always non-zero. Following the proof of Proposition 3.2 (i), this argument allows to prove that the set of zeros of* ϕ *is isolated.*

3.3 Optimal synthesis

We first analyze the behavior of ϕ which is crucial in order to find an optimal control policy.

Lemma 3.1. *A normal extremal trajectory* $(z(\cdot), \lambda(\cdot), u(\cdot))$ *defined over* $[0, T_u]$ *satisfies the following properties:*
(i) The switching function satisfies the following ordinary differential equation (ODE):

$$\dot{\phi}(t) = \frac{y(t)(m + u(t) - x(t))}{r - y(t)}\phi(t) - \frac{y(t)}{r - y(t)}, \quad \text{a.e. } t \in [0, T_u], \tag{3.9}$$

(ii) At a time t_0 *where* $y(t_0) = r$*, we have* $\phi(t_0) \neq 0$ *and:*

$$\phi(t_0) = \frac{1}{u(t_0) + m - x(t_0)}. \tag{3.10}$$

Proof. Let us show that the set $S := \{t \in [0, T_u] \ ; \ y(t) = r\}$ is finite. If $t_0 \in S$, we have $q(t_0)y(t_0)(x(t_0) - m - u(t_0)) = 1$ which implies that $\dot{y}(t_0) \neq 0$, hence t_0 is isolated and thus S is finite. The proof of (ii) is straightforward combining $H = 0$ and $y(t_0) = r$. □

The previous Lemma implies the following proposition.

Proposition 3.3. *Let* $(z(\cdot), \lambda(\cdot), u(\cdot))$ *a normal extremal trajectory defined over* $[0, T_u]$*. Then, the following properties are satisfied:*
(i) If there exist two consecutive instants $t_2 > t_1 > 0$ *such that* $y(t_1) = y(t_2) = r$*, then the control* u *has exactly one switching time* $t_c \in (t_1, t_2)$*.*
(ii) If in addition, $x(t_1) > x(t_2)$*, resp.* $x(t_1) < x(t_2)$*, then an optimal control satisfies* $u = 0$*, resp.* $u = 1$ *on* (t_1, t_c) *and then* $u = 1$*, resp.* $u = 0$ *on* (t_c, t_2)*.*

Proof. From (3.10), the sign of $\phi(t_i)$, $i = 1, 2$ depends on the value of $x(t_i)$ w.r.t. $u(t_0) + m$. Whenever the trajectory satisfies $y(t_1) = r$ with $x(t_1) > x_{\bar{u}}^{+}(\underline{x})$, we thus have $\phi(t_1) < 0$ and thus $u = 0$. Using that $x(t_2) < m$, we deduce that $\phi(t_2) > 0$, hence the trajectory necessarily has a switching point at a time $t_c \in (t_1, t_2)$. Now, from (3.9), one has $\dot{\phi}(t_c) = -\frac{y(t_c)}{r - y(t_c)} > 0$. Thus, the only possibility for the trajectory is to switch from $u = 0$ to $u = 1$. This shows the uniqueness of t_c in (t_1, t_2). If now $x(t_1) < x_{\bar{u}}^{+}(\underline{x})$, the same argumentation shows that there exists a unique switching time from $u = 1$ to $u = 0$ in (t_1, t_2). This proves (i) and (ii). □

We denote by γ the graph of the unique solution $(\tilde{x}(\cdot), \tilde{y}(\cdot))$ of (1.1) backward in time starting from the point $(\underline{x}, r)$ associated to the feedback control (3.1). Let τ_1 be the first time where $(\tilde{x}(\cdot), \tilde{y}(\cdot))$ exits $K(\underline{x})$ and τ_2 be the first exit time of $(\tilde{x}(\cdot), \tilde{y}(\cdot))$ of the set $\{(x, y) \in \mathcal{D} \ ; \ y \leq r\}$. Finally, let γ_1 be the restriction of $(\tilde{x}(\cdot), \tilde{y}(\cdot))$ to the interval $[\tau_1, \tau_2]$.

The optimal synthesis of the problem then reads as follows (see also Fig. 2).

Theorem 3.1. *Let (x_0, y_0) be an initial condition in $\mathcal{D}\backslash Viab(\underline{x})$.*
(i) If $(x_0, y_0) \in \gamma$, then any extremal optimal trajectory steering (1.1) from (x_0, y_0) to the target set is abnormal. The corresponding control is given by u_m and switching points occur on the axis $\{y = r\}$.
(ii) If $(x_0, y_0) \notin \gamma$, then any extremal optimal trajectory steering (1.1) from (x_0, y_0) to the target set is normal. Moreover, if u denotes the optimal control, there exists $p \in \mathbb{N}^*$, $s \in \{0, 1\}$, and a sequence $(\tau_k)_{0\leq k\leq p}$ such that:

- *We have $\tau_0 = 0 < \tau_1 < \cdots < \tau_{p-1} < \tau_p = T_u$ and τ_k is a switching time of u for $1 \leq k \leq p$.*
- *The optimal control u is given by*

$$u(t) = \frac{\bar{u}}{2}(1 + (-1)^{p-k-s}) \quad t \in (\tau_k, \tau_{k+1}), \quad 0 \leq k \leq p - 1. \tag{3.11}$$

- *If $y(T_u) \in (r_-, r)$, resp. $y(T_u) = r_-$, then $s = 1$, resp. $s = 0$.*

Proof. Let us prove (i). We already know from Proposition 3.2 (iv) that any trajectory starting on the curve γ and associated to the control u_m corresponds to an abnormal extremal trajectory. We must prove that such an extremal is optimal. To do so, let us choose (x_0, y_0) on the curve γ, and let $(x(\cdot), y(\cdot), u(\cdot))$ an optimal extremal trajectory steering (1.1) from (x_0, y_0) to $Viab(\underline{x})$. Let t_0 be defined as follows:

$$t_0 := \inf\{t \geq 0 \; ; \; \exists \varepsilon > 0 \; \forall \tau \in (t, t + \varepsilon) \quad (x(\tau), y(\tau)) \notin \gamma\}.$$

Suppose by contradiction that $t_0 < T_u$. As $(x(\cdot), y(\cdot), u(\cdot))$ is extremal, (3.5) implies that t_0 is necessarily a switching time from $u = 0$ to $u = \bar{u}$ or from $u = \bar{u}$ to $u = 0$. We argue that $y(t_0) \neq r$. Indeed, otherwise, we would have a contradiction with the definition t_0 (as by definition, u_m switches on the axis $\{y = r\}$). Hence, either we have $y(t_0) < r$ or $y(t_0) > r$. Now, the fact that $y(t_0) \neq r$ implies that the extremal $(x(\cdot), y(\cdot), u(\cdot))$ is normal. Indeed, we cannot have $p(t_0) = 0$ by Cauchy-Lipschitz Theorem, but as $y(t_0) \neq r$, we obtain that $p(t_0)x(t_0)(r - y(t_0)) \neq 0$ and thus p_0 must be non-zero. Suppose for instance that $y(t_0) < r$. By construction of t_0, this point is a switching time from $u = 0$ to $u = 1$ and we necessarily have $\dot{\phi}(t_0) \geq 0$. On the other hand, we obtain from (3.9) that

$$\dot{\phi}(t_0) = -\frac{y(t_0)}{r - y(t_0)} < 0,$$

and a contradiction. If $y(t_0) > r$, we obtain a similar contradiction with the sign of $\dot{\phi}$ at time t_0. This shows that $t_0 \geq T_u$, thus we have proved that the abnormal extremal trajectory starting from (x_0, y_0) with the control u_m drives (1.1) optimally to the target.

Let us now prove (ii). We consider an extremal optimal trajectory $(x(\cdot), y(\cdot), u(\cdot))$ driving optimally (1.1) from a point $(x_0, y_0) \notin \gamma$ to $Viab(\underline{x})$. We will consider the two following cases depending if $y(T_u) \in (r_-, r)$ or $y(T_u) = r_-$.
First case: $y(T_u) \in (r_-, r)$. From Proposition 3.2 (iii), we have $p_0 \neq 0$ i.e. the trajectory is normal. Now, as $\phi(T_u) = 0$ and $\dot{\phi}(T_u) = -\frac{y(T_u)}{r-y(T_u)} < 0$, we obtain that $u = \bar{u}$ in a left neighborhood of T_u. By using Proposition 3.3 (i), we obtain that the extremal has exactly one switching time t_c between two consecutive instants $t_1 < t_2$ such that $y(t_1) = y(t_2) = r$. We thus obtain (3.11) by considering (1.1) backward in time from $t = T_u$ and by counting the number of times (denoted by $p - 1$ with $p \geq 1$) where the trajectory surrounds $Viab(\underline{x})$ before reaching (x_0, y_0). When $k = p - 1$, we obtain $u(t) = \frac{\bar{u}}{2}(1 + (-1)^{1-s})$, thus $s = 1$ as was to be proved.
Second case: $y(T_u) = r_-$. Suppose that the extremal is abnormal i.e. $p_0 = 0$. It follows that $q(T_u) > 0$. Otherwise, the transversality condition would imply $q(T_u) = 0$ and using $H = 0$ we would have $p(T_u) = 0$ and a contradiction with the PMP. We deduce that $\phi(T_u) < 0$ thus $u = 0$ in a left neighborhood of T_u. As the extremal is abnormal, switching points occur only on the axis $\{y = r\}$. This shows that we have $u = 0$ on $[t_c, T_u]$ where t_c is the last time such that $y(t_c) = r$ before reaching $B^0(\underline{x})$. Thus, by integrating backward in time (1.1) from $(\underline{x}, r_-)$, we find that $u = u_m$ and that $(x_0, y_0) \in \gamma$ which is a contradiction. We have thus proved that the extremal optimal trajectory is normal. Finally, we have two cases depending if the normal trajectory reaches $(\underline{x}, r_-)$ with $u = 0$ or $u = 1$:

- The case where we have $u = 1$ at the terminal time T_u exactly corresponds to the sub-cases where $y(T_u) \in (r_-, r)$ and thus we obtain (ii) similarly as above.

- Now, suppose that we have $u = 0$ at the terminal time T_u. The trajectory necessary has a switching time on γ_1 (as it is normal). We thus obtain (3.11) by considering (1.1) backward in time from $t = T_u$ and by counting the number of times (denoted by $p - 1$ with $p \geq 1$) where the trajectory surrounds $Viab(\underline{x})$ before reaching (x_0, y_0). As $u = 0$ in a left neighborhood of T_u, we obtain $s = 0$.

□

3.4 Discussion

To highlight the optimal synthesis provided by Theorem 3.1, we provide the following remarks.

- The optimal control provided by Theorem 3.1 (ii) can be interpreted as a slight perturbation of the myopic strategy (3.1): instead of switching on the axis $\{y = r\}$, switching times are delayed and the corresponding switching points occur after the last intersection between the corresponding trajectory and the axis $\{y = r\}$ (see the switching curves in red on Fig 2).
- Abnormal trajectories are contained in the curve γ and they are the only extremal trajectories for which switching points occur on the axis $\{y = r\}$.
- Between two consecutive times $t_1 < t_2$ where a normal extremal trajectory satisfies $y(t_1) = y(t_2) = r$, the extremal has exactly one switching time.
- (iii) In Theorem 3.1, any normal trajectory reaching $B_0(\underline{x})$ in its interior satisfies:

$$p - (2j + 1) \geq 0 \quad \text{and} \quad p - 2j \geq 0 \quad \Rightarrow \quad y(\tau_{p-2j+1}) > r \quad \text{and} \quad y(\tau_{p-2j}) < r$$

  In other words, odd switching points occur below the axis $\{y = r\}$ whereas even switching points occur above the axis $\{y = r\}$.
- Any normal trajectory either reaches $B_0(\underline{x})$ with $u = \bar{u}$, or it reaches the point $(\underline{x}, r_-)$ with $u = 0$. In the latter case, an optimal trajectory switches from $u = 1$ to $u = 0$ on γ_1.

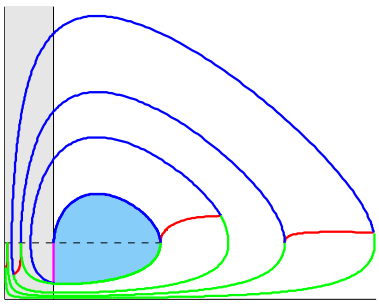

Figure 2: Optimal synthesis provided by Theorem 3.1: in red the switching curves, in green, resp. in blue the solutions of (1.1) with $u = 1$, resp. with $u = 0$.

4 The minimal time crisis problem over a finite horizon

The time crisis problem consists in studying the following optimization problem

$$\inf_{u \in \mathcal{U}} \int_0^T \mathbb{1}_{K^c}(z_u(t))\, \mathrm{d}t, \tag{4.1}$$

where $z_u := (x_u, y_u)$ is the unique solution of the Cauchy problem

$$\begin{cases} \dot{x} = x(r - y) & x(0) = x_0, \\ \dot{y} = y(x - m - u) & y(0) = y_0, \end{cases} \tag{4.2}$$

The existence of an optimal control is standard (see [3, 7]). Note that for initial conditions are taken in a compact subset C of $\mathcal{D}$, then there exists $T > 0$ such that for any initial condition $(x_0, y_0) \in C$, then (4.1) is equivalent to (1.2). To derive necessary optimality conditions on an optimal control, one can apply the hybrid maximum principle (see [11]).

4.1 Application of the hybrid maximum principle

The times t_c where a trajectory either enters or exits the set K through the set $\{x = \underline{x}\}$ are crucial in the analysis.

Definition 4.1. *(i) We say that t_c is a crossing time from K to K^c if there exists $\varepsilon > 0$ such that for any time $t \in [t_c - \varepsilon, t_c)$, resp. $t \in (t_c, t_c + \varepsilon]$ one has $z(t) \in K$, resp. $z(t) \in K^c$.*
(ii) We say that t_c is a crossing time from K^c to K if there exists $\varepsilon > 0$ such that for any time $t \in [t_c - \varepsilon, t_c)$, resp. $t \in (t_c, t_c + \varepsilon]$ one has $z(t) \in K^c$, resp. $z(t) \in K$.
(iii) We say that a crossing time t_c from K to K^c is transverse if $\langle \dot{z}(t_c), e_1 \rangle \neq 0$.

Let $H : \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be the Hamiltonian associated to (4.1) defined by:

$$H := H(x, y, p, q, p_0, u) = px(r - y) + qy(x - m - u) + p_0 \mathbb{1}_{K^c}(x, y).$$

If u is an optimal control of (4.1) and $(x_u(\cdot), y_u(\cdot))$ is the associated solution of (4.2), then the following optimality conditions are satisfied:

- There exists $p_0 \leq 0$ and a measurable function $\lambda(\cdot) := (p(\cdot), q(\cdot)) : [0, T] \to \mathbb{R}^2$ satisfying a.e. on $[0, T]$:

$$\begin{cases} \dot{p} &= p(y - r) - qy, \\ \dot{q} &= px + q(u + m - x). \end{cases} \tag{4.3}$$

- The control u satisfies the maximization condition:

$$u(t) \in \arg \max_{\omega \in [0, \bar{u}]} H(x(t), y(t), p(t), q(t), p_0, \omega) \quad \text{a.e. } t \in [0, T]. \tag{4.4}$$

- The Hamiltonian H is constant along any extremal trajectory $(x(\cdot), y(\cdot), u(\cdot), p(\cdot), q(\cdot), p_0)$ satisfying (4.2)(4.3)-(4.4).
- For each crossing time t_c from K to K^c or from K^c to K, one has $\lambda(t_c^+) - \lambda(t_c^-) \in N_K(x(t_c), y(t_c))$. Moreover, if t_c is transverse, then one has:

$$\begin{array}{lcl} x(t_c^-) \in K \text{ and } x(t_c^+) \in K^c & \Rightarrow & p(t_c^+) - p(t_c^-) = \frac{q(t_c)y(t_c)(u(t_c^+) - u(t_c^-)) - p_0}{x(t_c)(r - y(t_c))} \\ x(t_c^-) \in K^c \text{ and } x(t_c^+) \in K & \Rightarrow & p(t_c^+) - p(t_c^-) = \frac{q(t_c)y(t_c)(u(t_c^+) - u(t_c^-)) + p_0}{x(t_c)(r - y(t_c))}. \end{array} \tag{4.5}$$

- The triple $(p_0, p(\cdot), q(\cdot))$ is non-zero.
- As $(x(T), y(T)) \in \mathcal{D}$ is free, we have $p(T) = q(T) = 0$.

It is important to notice that at every transverse crossing time t_c from K to K^c or from K^c to K, the function q is continuous (thus the switching function $\phi = -qy$ is also continuous) whereas p has a jump given by (4.5) (thus $\dot{\phi} = -pxy$ is discontinuous).

Lemma 4.1. *An optimal trajectory corresponds to a normal extremal i.e. $p_0 \neq 0$.*

Proof. Suppose that $p_0 = 0$ and that the trajectory ends in K. Let t_c be the last entry time into the set K. Then at $t = t_c$, the jump condition on the adjoint vector becomes writes $p(t_c^-) = p(t_c^+)$ using that q is continuous at $t = t_c$, that $q(t_c^+) = 0$ and that $p_0 = 0$. The adjoint equation (4.3) together with $p(t_c^-) = q(t_c^-) = 0$ then implies that $p = q = 0$ on the interval $[t_e^+, t_c]$ where t_e is the last exit time from K before. Again using the jump condition on the variable p, we obtain that $p(t_e^-) = q(t_e^-) = 0$. By induction, we obtain that the adjoint vector is zero on any time interval where the associated trajectory either belongs to K or to K^c, and thus $(p(\cdot), q(\cdot)) = 0$ over $[0, T]$. This contradicts the hybrid maximum principle as $(p_0, p(\cdot), q(\cdot))$ must be non-zero. The same arguments can be used in the case where the trajectory ends in K^c. □

Without any loss of generality, we suppose that $p_0 = -1$.

4.2 Study of the problem in the case where the end-point is in K

In this part we consider the following assumption.

(H) The end-point trajectory is such that $(x(T), y(T)) \in K$.

Remark 4.1. *Suppose for instance that $Viab(\underline{x}) \neq \emptyset$ and that (x_0, y_0) belongs to a compact set C of $\mathcal{D}$. If T is large enough, it can be expected that for any optimal solution of (4.1) starting in C, there exists $T_e \leq T$ such that for any time $t \geq T_e$, then $(x(t), y(t)) \in Viab(\underline{x})$. We know that for $t \geq T_e$, the trajectory is in the viability kernel of K, thus there exists a control u such that the associated solution $(x(\cdot), y(\cdot))$ satisfies $(x(t), y(t)) \in K$ for any time $t \geq T_e$. Hence, we obtain that $\mathbb{1}_K(x(t), y(t)) = 0$ for $t \geq T_e$.*

The previous assumptions imply that the Hamiltonian is zero along any extremal trajectory as we have:

$$H = p_0 \mathbb{1}_{K^c}(x_u(T), y_u(T)) = 0.$$

Lemma 4.2. *If t_c is the last entry time from $(x(\cdot), y(\cdot))$ into K, one has $u = 1$ in a left neighborhood of t_c.*

Proof. Using that $p = q = 0$ over $[t_c^+, T]$, that q is continuous at $t = t_c$, and that $p_0 = 0$ we find that $p(t_c^-) = \frac{1}{x(t_c)(r-y(t_c))} > 0$. This implies that $\dot{\phi}(t_c^-) = -p(t_c^-)x(t_c)y(t_c) < 0$. Thus, as $\phi(t_c) = 0$, we must have $\phi > 0$ in a left neighborhood of t_c and thus we have $u = 1$ for $t < t_c$, t close to t_c. □

Next, we define $T_e(u)$ as the first entry time of the trajectory into the set $Viab(\underline{x})$. More precisely, we consider the following definition.

Definition 4.2. *If an optimal trajectory enters $Viab(\underline{x})$ before $t = T$, then we set*

$$T_e(u) := \inf\{t \in [0, T] \ (x(t), y(t)) \in B_0(\underline{x})\}.$$

Otherwise, we set $T_e(u) = +\infty$.

Lemma 4.3. *Let u an optimal control and $(x(\cdot), y(\cdot))$ the associated trajectory.*
(i) At any time t where $(x(t), y(t)) \notin \partial K$ and $y(t) \neq r$, one has:

$$\dot{\phi}(t) = \frac{y(t)(x(t) - m - u(t))}{y(t) - r}\phi(t) - \frac{y(t)}{r - y(t)}\mathbb{1}_{K^c}(x(t), y(t)).$$

(ii) Any switching time for the control u in the set $\{y > r\} \cap K^c$, resp. $\{y < r\} \cap K^c$ is from $u = 0$ to $u = 1$, resp. from $u = 1$ to $u = 0$.
(iii) At any time $t_0 < T_e(u)$ such that $y(t_0) = r$, $(x(t_0), y(t_0)) \in K$, and $y < r$ in a left neighborhood of t_0, then one has $\phi(t_0) = 0$.
(iv) At any time t_0 such that $y(t_0) = r$ and $(x(t_0), y(t_0)) \in K^c$ with $y > r$ in a left neighborhood of t_0, then one has $\phi(t_0) > 0$.

Proof. The proof of (i)-(ii) is as in Lemma 3.1 and Proposition 3.3. To prove (iii), suppose that $(x(t), y(t)) \in K$. We find that $q(t_0)y(t_0)(x(t_0) - m - u(t_0)) = 0$. If $q(t_0) > 0$, then we have $u(t_0) = 0$ and $x(t_0) = m$ hence the trajectory is at the stable steady-state (r, m) which is a contradiction. Similary, if $q(t_0) < 0$, then $u(t_0) = \bar{u}$ and $x(t_0) = m + \bar{u}$ and the trajectory has reached the stable steady-state $(r, m + \bar{u})$ which is also contradiction. Hence, we must have $q(t_0) = 0$. Let us now prove (iv). We find that $q(t_0)y(t_0)(x(t_0) - m - u(t_0)) = 1$. As $y > r$ in a left neighborhood of t_0, one has $\dot{y}(t_0) \leq 0$ which implies $x(t_0) - m - u(t_0) \leq 0$ and thus we must have $q(t_0) < 0$ i.e. $\phi(t_0) > 0$ as was to be proved.

□

We deduce the following property.

Corollary 4.1. *If* (H) *is satisfied, the strategy defined by u_m is not an optimal solution of the crisis problem.*

Lemma 4.4. *Suppose that there exists a time interval $[t_1, t_2] \subset [0, T]$ such that:*

- *For any time $t \in [t_1, t_2]$, $(x(t), y(t)) \in K$*
- *For any time $t \in [t_1, t_2]$, $p(t) = q(t) = 0$.*
- *The time t_2 is a transverse crossing time from K to K^c.*

Then, one has $u(t) = \bar{u}$ for $t > t_2$ close to t_2.

Proof. The jump condition on p at time t_2 writes $p(t_2^+) = \frac{1}{r - y(t_2)} < 0$. Thus, $\dot{\phi}(t_2^+) = -p(t_2^+)x(t_2)y(t_2) > 0$ and using that $\phi(t_2^+) = \phi(t_2^-) = 0$ one finds $\phi > 0$ for $t > t_2$ close to t_2 as was to be proved. □

4.3 Comparison between the time crisis function and the minimal time function

We suppose in this part that $Viab(\underline{x}) \neq \emptyset$. For $(x_0, y_0) \in \mathcal{D}$, let

$$\theta_1(x_0, y_0) := \inf_{u(\cdot)\in\mathcal{U}} \int_0^{+\infty} \mathbb{1}_{K(\underline{x})^c}(x_u(t), y_u(t))\ \mathrm{d}t$$
$$\theta_2(x_0, y_0) := \inf_{u(\cdot)\in\mathcal{U}} \int_0^{+\infty} \mathbb{1}_{Viab(\underline{x})^c}(x_u(t), y_u(t)).\ \mathrm{d}t$$

From the inclusion $Viab(\underline{x}) \subset K(\underline{x})$, one easily deduces that:

$$\theta_1(x_0, y_0) \leq \theta_2(x_0, y_0).$$

Now, as $Viab(\underline{x})$ is viable, Proposition 4.1 of [7] implies that $\theta_2(x_0, y_0) = v(x_0, y_0)$. Hence, we obtain the inequality $\theta_1(x_0, y_0) \leq v(x_0, y_0)$, $(x_0, y_0) \in \mathcal{D}$.

Proposition 4.1. *For any $(x_0, y_0) \in \gamma\backslash Viab(\underline{x})$, one has:*

$$\theta_1(x_0, y_0) < v(x_0, y_0). \tag{4.6}$$

Proof. First, recall that the control u_m drives (4.2) in minimal time from (x_0, y_0) to the set $B_0(\underline{x})$ i.e. $v(x_0, y_0) = T(u_m)$. Using the control u_m, we deduce that $\theta_1(x_0, y_0) < +\infty$. Hence, there exists $T > 0$ (large enough) such that (4.1) is equivalent to

$$\inf_{u(\cdot)\in\mathcal{U}} \int_0^T \mathbb{1}_{K(\underline{x})^c}(x_u(t), y_u(t))\ \mathrm{d}t,$$

By the previous results, we know that for a fixed $T > 0$, the control u_m is not an optimal solution for θ_1, thus

$$\theta_1(x_0, y_0) < \int_0^{+\infty} \mathbb{1}_{K(\underline{x})^c}(x_{u_m}(t), y_{u_m}(t))\ \mathrm{d}t \leq \int_0^{+\infty} \mathbb{1}_{Viab(\underline{x})^c}(x_{u_m}(t), y_{u_m}(t))\ \mathrm{d}t = v(x_0, y_0).$$

□

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