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Transient response of elastic bodies connected by a thin stiff viscoelastic layer with evanescent mass

Réponse transitoire de corps élastiques liés par une mince et raide bande viscoélastique de faible masse

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Abstract

We extend the study [1] devoted to the dynamic response of a structure made up of two linearly elastic bodies connected by a thin soft adhesive layer made of a Kelvin–Voigt-type nonlinear viscoelastic material to the cases of stiff and very stiff adhesives whose mass vanishes. We use a nonlinear extension of Trotter's theory of convergence of semi-groups of operators acting on variable spaces to identify the asymptotic behavior of the mechan-ical state of the system, when some geometrical and mechanical parameters tend to their natural limits. The models we obtain describe the behavior of a structure consisting of two linearly elastic adherents perfectly bonded to a material deformable flat surface whose behavior is of the same kind as that of the genuine adhesive.

Keywords: Bonding problems; Kelvin–Voigt viscoelasticity Dynamics; Maximal monotone operators

Résumé

Nous étendons aux adhésifs durs ou très durs, dont la masse est évanescente, l'étude menée en [1] consacrée au comportement dynamique d'un assemblage de deux corps linéairement élastiques liés par une couche adhésive mince et molle constituée d'un ma-tériau viscoélastique non linéaire de type Kelvin–Voigt. Afin d'identifier le comportement asymptotique de l'état mécanique du système lorsque des paramètres mécanique et géomé-triques tendent vers leurs limites naturelles, nous utilisons une extension non linéaire de la théorie de Trotter de convergence de semi-groupes d'opérateurs agissant sur des espaces variables. Les modèles obtenus décrivent le comportement d'une structure constituée de deux adhérents élastiques parfaitement collés à une surface matérielle plate et déformable, dont le comportement est identique à celui de l'adhésif.

Mots-clés : Problèmes de collage; Viscoélasticité de Kelvin–Voigt Dynamique; Opérateurs maximaux monotones

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1. Setting the problem

We extend to the situations of high and very high stiffness the results obtained in [1] concerning the dynamics of elastic bodies connected by a thin soft viscoelastic layer. Let $\{e_1, e_2, e_3\}$ be an orthonormal basis of $\mathbb{R}^3$ assimilated to the Euclidean space. For all $\xi = (\xi_1, \xi_2, \xi_3)$ in $\mathbb{R}^3$, $\widehat{\xi}$ stands for (ξ_1, ξ_2). The space of all $(n \times n)$ symmetric matrices is denoted by $\mathbb{S}^n$ and equipped with the usual inner product and norm denoted by $\cdot$ and $|\ |$ (as in $\mathbb{R}^3$). For all η in $\mathbb{S}^3$, $\widehat{\eta}$ stands for the matrix $(\eta_{\alpha\beta})_{1\le\alpha,\beta\le 2}$ in $\mathbb{S}^2$. We study the dynamic response of a structure consisting of two adherents connected by a thin adhesive layer which is subjected to a given loading. Let Ω be a domain of $\mathbb{R}^3$ with Lipschitz-continuous boundary $\partial\Omega$. The intersection of Ω with $\{x_3 = 0\}$ is a domain S of $\mathbb{R}^2$ with a positive two-dimensional Haussdorf measure $\mathcal{H}_2(S)$. Let ε be a positive number and $\Omega_\pm := \Omega \cap \{\pm x_3 > 0\}$, then adhesive and adherents occupy $B^\varepsilon := S \times (-\varepsilon, +\varepsilon)$ and $\Omega^\varepsilon_\pm := \Omega_\pm \pm \varepsilon e_3$ respectively; we define $\Omega^\varepsilon := \Omega^\varepsilon_+ \cup \Omega^\varepsilon_-$, $S^\varepsilon_\pm := S \pm \varepsilon e_3$ and $\mathcal{O}^\varepsilon := \Omega^\varepsilon \cup B^\varepsilon \cup S^\varepsilon_+ \cup S^\varepsilon_-$. We consider a partition (Γ_0, Γ_1) of $\partial\Omega$ and, for all Γ in $\{\Gamma_0, \Gamma_1\}$, the sets $\Gamma_\pm$, $\Gamma^\varepsilon_\pm$ and Γ^ε respectively denote $\Gamma \cap \{\pm x_3 > 0\}$, $\Gamma_\pm \pm \varepsilon e_3$ and $\Gamma^\varepsilon_+ \cup \Gamma^\varepsilon_-$. Moreover, we assume that $\mathcal{H}_2(\Gamma_{0^+}) > 0$. The structure made of the adhesive and the two adherents, perfectly stuck together along $S^\varepsilon_\pm$, is clamped on Γ^ε_0 and subjected to body forces of density f^ε and to surface forces g^ε on Γ^ε_1. The adherents are modeled as linearly elastic materials with a strain energy density W^ε such that

$$W^\varepsilon(x, e) = \frac{1}{2} a^\varepsilon(x) e \cdot e, \quad a.e.\, x \in \Omega^\varepsilon, \quad \forall e \in \mathbb{S}^3 \tag{1}$$

The *thin* adhesive is assumed to be made of a homogeneous, isotropic and "viscoelastic of Kelvin–Voigt generalized type". Its strain energy density reads as $\mu\, w_I$, while its dissipation potential is denoted by $b\,\mathcal{D}$, where μ and b are positive scalars; w_I is a positive definite quadratic form on $\mathbb{S}^3$ and $\mathcal{D}$ a convex and positively homogeneous function of degree q, $1 \le q \le 2$.

Let $\rho > 0$, $\overline{\rho}_M > \overline{\rho}_m > 0$ and $\overline{\rho}^\varepsilon$ a measurable function. The density γ^ε of the structure is equal to $\overline{\rho}^\varepsilon$ in Ω^ε and to ρ in B^ε. Denoting by $\mathrm{Lin}(\mathbb{S}^3)$ the space of linear mappings from $\mathbb{S}^3$ into $\mathbb{S}^3$, we make the following assumptions on the data:

$$\begin{cases} \text{There exists } (f, g, a, \overline{\rho}) \text{ in } L^2(\Omega;\mathbb{R}^3) \times L^2(\Gamma_1;\mathbb{R}^3) \times L^\infty(\Omega;\mathrm{Lin}(\mathbb{S}^3)) \times L^\infty(\Omega) \text{ such that} \\ f^\varepsilon(x) = f(x \mp \varepsilon e_3) \quad a.e.\, x \in \Omega^\varepsilon_\pm, \quad f^\varepsilon(x) = 0 \quad a.e.\, x \in B^\varepsilon \\ g^\varepsilon(x) = g(x \mp \varepsilon e_3) \quad a.e.\, x \in (\Gamma_1)^\varepsilon_\pm, \quad g^\varepsilon(x) = 0 \quad a.e.\, x \in \partial S \times (-\varepsilon, \varepsilon) \\ a^\varepsilon(x) = a(x \mp \varepsilon e_3) \quad a.e.\, x \in \Omega^\varepsilon_\pm \\ \overline{\rho}^\varepsilon(x) = \overline{\rho}(x \mp \varepsilon e_3) \quad a.e.\, x \in \Omega^\varepsilon_\pm \\ \exists a_m, a_M > 0 \quad s.t. \quad a_m|e|^2 \le a(x)e \cdot e \le a_M|e|^2, \quad \forall e \in \mathbb{S}^3 \\ \exists \overline{\rho}_m, \overline{\rho}_M > 0 \quad s.t. \quad \overline{\rho}_m \le \overline{\rho}(x) \le \overline{\rho}_M, \quad a.e.\, x \in \Omega \end{cases} \tag{2}$$

Thus, the problem $(\mathcal{P}_s)$ of determining the dynamic evolution of the assembly involves a quadruplet $s := (\varepsilon, \mu, b, \rho)$ of data so that all the fields will be hereafter indexed by s. In the following, t denotes the time, $e(u)$ is the linearized strain tensor associated with the field of displacement u, and $\partial J(v)$ denotes the subdifferential at v of any lower semi-continuous convex function J, while $DJ(v)$ stands for the differential at v of any Fréchet differentiable function J. If $U^0_s = (u^0_s, v^0_s)$ is the initial state, a formulation of $(\mathcal{P}_s)$ could be

$$(\mathcal{P}_s) \quad \begin{cases} \text{Find } u_s \text{ sufficiently smooth in } \Omega \times [0,T] \text{ such that } u_s = 0 \text{ on } \Gamma^\varepsilon_0 \times (0,T] \\ \left(u_s(\cdot,0), \dfrac{\partial u_s}{\partial t}(\cdot,0)\right) = U^s_0 \text{ and there exists } \zeta \text{ in } \partial\mathcal{D}(e(\dfrac{\partial u_s}{\partial t})) \text{ satisfying:} \\ \displaystyle\int_{\mathcal{O}^\varepsilon} \gamma^\varepsilon \frac{\partial^2 u_s}{\partial t^2}\, v\, \mathrm{d}x + \int_{\Omega^\varepsilon} a^\varepsilon e(u_s) \cdot e(v)\, \mathrm{d}x + \int_{B^\varepsilon} \Big(\mu\, D w_I(e(u_s)) + b\zeta\Big) \cdot e(v)\, \mathrm{d}x = \\ \qquad\qquad = \displaystyle\int_{\mathcal{O}^\varepsilon} f^\varepsilon \cdot v\, \mathrm{d}x + \int_{\Gamma^\varepsilon_1} g^\varepsilon \cdot v\, \mathrm{d}\mathcal{H}_2 \\ \text{for all } v \text{ sufficiently smooth in } \mathcal{O}^\varepsilon \text{ and vanishing on } \Gamma^\varepsilon_0 \end{cases}$$

2. Existence and uniqueness

We assume that

$$(f, g) \in BV\big(0, T; L^2(\Omega;\mathbb{R}^3)\big) \times BV^{(2)}\big(0, T; L^2(\Gamma_1;\mathbb{R}^3)\big) \tag{$\mathbf{H}_1$}$$

where, for any Banach space X, $BV(0,T;X)$ is the subspace of $L^1(0,T;X)$ consisting of all elements whose time derivative in the sense of distributions is a bounded X-valued measure on $(0,T)$, and $BV^{(2)}(0,T;X)$ is the subspace of $BV(0,T;X)$ consisting of all elements whose time derivative in the sense of distributions belongs to $BV(0,T;X)$.

We seek u_s in the form

$$u_s = u_s^e + u_s^r \tag{3}$$

where u_s^e is the unique solution to

$$u_s^e(t) \in H^1_{\Gamma_0^\varepsilon}(\mathcal{O}^\varepsilon;\mathbb{R}^3); \quad \varphi_s(u_s^e(t),v) = L(t)(v), \quad \forall v \in H^1_{\Gamma_0^\varepsilon}(\mathcal{O}^\varepsilon;\mathbb{R}^3),\ \forall t\in[0,T] \tag{4}$$

with

$$\begin{aligned}
\varphi_s(v,v') &:= \int_{\Omega^\varepsilon} a^\varepsilon e(v)\cdot e(v')\,\mathrm{d}x + \mu\int_{B^\varepsilon} Dw_I\big(e(v)\big)\cdot e(v')\,\mathrm{d}x, \quad \forall v,v' \in H^1_{\Gamma_0^\varepsilon}(\mathcal{O}^\varepsilon;\mathbb{R}^3)\\
\Phi_s(v) &:= \varphi_s(v,v)\\
L^\varepsilon(t)(v) &:= \int_{\Gamma_1^\varepsilon} g^\varepsilon(x,t)\cdot v(x)\,\mathrm{d}\mathcal{H}_2, \quad \forall v \in H^1_{\Gamma_0^\varepsilon}(\mathcal{O}^\varepsilon;\mathbb{R}^3),\ \forall t\in[0,T]
\end{aligned} \tag{5}$$

and where $H^1_{\Gamma_0^\varepsilon}(\mathcal{O}^\varepsilon;\mathbb{R}^3)$ is the closed subspace of $H^1(\mathcal{O}^\varepsilon;\mathbb{R}^3)$ consisting of all elements with vanishing traces on Γ_0^ε. Note that this notation $H^1_g(G;\mathbb{R}^n)$ will be systematically used for any $G\subset\mathbb{R}^n$, $g\subset\partial G$ and Sobolev space $H^1(G;\mathbb{R}^n)$. As $g\mapsto u_s^e$ is linear continuous from $L^2(\Gamma_1;\mathbb{R}^3)$ into $H^1_{\Gamma_0^\varepsilon}(\mathcal{O}^\varepsilon;\mathbb{R}^3)$, we have:

$$u_s^e \in BV^{(2)}\big(0,T;H^1_{\Gamma_0^\varepsilon}(\mathcal{O}^\varepsilon;\mathbb{R}^3)\big) \tag{6}$$

The remaining part u_s^r of u_s will therefore satisfy an evolution equation governed by a maximal monotone operator A_s defined in a Hilbert space H_s of possible states with finite total mechanical (kinetic + strain) energy. The space of velocities $L^2(\mathcal{O}^\varepsilon;\mathbb{R}^3)$ is equipped with the following inner product k_s and the square of norm K_s associated with kinetic energy:

$$k_s(v,v') := \int_{\mathcal{O}^\varepsilon} \gamma^\varepsilon(x) v(x)\cdot v'(x)\,\mathrm{d}x, \quad K_s(v) := k_s(v,v), \quad \forall v,v' \in L^2(\mathcal{O}^\varepsilon;\mathbb{R}^3) \tag{7}$$

while the space of displacements, $H^1_{\Gamma_0^\varepsilon}(\mathcal{O}^\varepsilon;\mathbb{R}^3)$, is equipped with the inner product φ_s defined in (5), which is equivalent to the usual one by Korn inequality. Hence

$$H_s := H^1_{\Gamma_0^\varepsilon}(\mathcal{O}^\varepsilon;\mathbb{R}^3)\times L^2(\mathcal{O}^\varepsilon;\mathbb{R}^3) \tag{8}$$

where, for all $U=(u,v)$ and $U'=(u',v')$ in H_s, the inner product and norm are

$$(U,U')_s := \varphi_s(u,u') + k_s(v,v'), \quad |U|_s^2 := (U,U)_s \tag{9}$$

while A_s is defined by

$$\begin{cases}
D(A_s) = \left\{ U=(u,v)\in H_s;\ \begin{cases} \text{i) } v\in H^1_{\Gamma_0^\varepsilon}(\mathcal{O}^\varepsilon;\mathbb{R}^3) \\ \text{ii) } \exists\,(w,\xi)\in L^2(\mathcal{O}^\varepsilon;\mathbb{R}^3)\times\partial\mathcal{D}(e(v)) \text{ with} \\ \qquad k_s(w,v') + \varphi_s(u,v') + b\int_{B^\varepsilon}\xi\cdot e(v')\,\mathrm{d}x = 0, \quad \forall v'\in H^1_{\Gamma_0^\varepsilon}(\mathcal{O}^\varepsilon;\mathbb{R}^3) \end{cases} \right\} \\
A_sU = (-v,0) + \{(0,-w);\ w \text{ satisfies ii) of definition of } D(A_s)\}
\end{cases} \tag{10}$$

Proceeding as in [1], one has the following.

Proposition 2.1. *The operator A_s is a maximal monotone operator and, for all $\psi=(\psi^1,\psi^2)$ in H_s,*

$$\begin{cases} \overline{U}_s = (\overline{u}_s,\overline{v}_s) \text{ s.t.} \\ \overline{U}_s + A_s\overline{U}_s \ni \psi \end{cases} \quad\Leftrightarrow\quad \begin{cases} J_s(\overline{v}_s) \le J_s(v) \quad \forall v\in H^1_{\Gamma_0^\varepsilon}(\mathcal{O}^\varepsilon;\mathbb{R}^3) \\ J_s(v) := \frac12 K_s(v) - k_s(\psi^2,v) + \frac12\phi_s(v) + \varphi_s(\psi^1,v) + b\int_{B^\varepsilon}\mathcal{D}\big(e(v)\big)\,\mathrm{d}x \\ \overline{u}_s = \overline{v}_s + \psi^1 \end{cases} \tag{11}$$

Then, taking into account ($\mathbf{H}_1$), (3), (4), (6), (10), we check straightforwardly that $(\mathcal{P}_s)$ is "formally equivalent" to

$$\begin{cases} \dfrac{\mathrm{d}U_s^r}{\mathrm{d}t} + A_s U_s^r \ni F_s \\ U_s^r(0) = U_s^o - \big(u_s^e(0), 0\big) \end{cases} \tag{12}$$

where

$$F_s = \Big(-\frac{\mathrm{d}u_s^e}{\mathrm{d}t},\, f^\varepsilon/\gamma^\varepsilon\Big) \tag{13}$$

A result of [2] therefore yields Theorem 2.1.

Theorem 2.1. *If (f, g) satisfies ($\mathbf{H}_1$) and $U_s^o \in \big(u_s^e(0), 0\big) + D(A_s)$, then (12) has a unique solution such that U_s^r belongs to $W^{1,\infty}(0, T; H_s)$ and the first line of (12) is satisfied almost everywhere in $[0, T]$. Hence, there exists a unique u_s in $W^{1,\infty}\big(0, T; H^1_{\Gamma_0^\varepsilon}(\mathcal{O}^\varepsilon; \mathbb{R}^3)\big) \cap W^{2,\infty}\big(0, T; L^2(\mathcal{O}^\varepsilon; \mathbb{R}^3)\big)$ which does satisfy*

$$\begin{cases} \exists \xi \in \partial\mathcal{D}\Big(e\Big(\dfrac{\mathrm{d}u_s}{\mathrm{d}t}\Big)\Big) \text{ such that} \\ \displaystyle\int_{\mathcal{O}^\varepsilon} \gamma^\varepsilon \frac{\mathrm{d}^2 u_s}{\mathrm{d}t^2} v \,\mathrm{d}x + \int_{\Omega^\varepsilon} a^\varepsilon e(u_s)\cdot e(v)\,\mathrm{d}x + \mu \int_{B^\varepsilon} Dw_I\big(e(u_s)\big)\cdot e(v)\,\mathrm{d}x + b\int_{B^\varepsilon} \xi\cdot e(v)\,\mathrm{d}x \\ \qquad\qquad = \displaystyle\int_{\mathcal{O}^\varepsilon} f^\varepsilon \cdot v\,\mathrm{d}x + \int_{\Gamma_1^\varepsilon} g^\varepsilon \cdot v\,\mathrm{d}\mathcal{H}_2, \quad \forall v \in H^1_{\Gamma_0^\varepsilon}(\mathcal{O}^\varepsilon; \mathbb{R}^3),\ \text{a.e. } t \in (0, T] \\ u_s(0) = u_s^0,\ \dfrac{\mathrm{d}u_s}{\mathrm{d}t}(0) = v_s^0 \end{cases} \tag{14}$$

We set

$$U_s^e = \Big(u_s^e, 0\Big), \quad U_s = U_s^r + U_s^e \tag{15}$$

3. Asymptotic behavior

Now we regard the quadruplet s of geometrical and mechanical *data* as a quadruplet of *parameters* taking values in a countable subset of $(0, +\infty)^4$ with a single cluster point $\overline{s}$ and study the asymptotic behavior of U_s in order to obtain a simplified but accurate enough model for the genuine physical situation. We will show that two different models indexed by $p \in \{1, 2\}$ appear at the limit depending on the relative behavior of ε and μ. We make the following assumptions:

$$\begin{cases} \text{i)} & \overline{s}_p \in \{0\} \times \{+\infty\}^2 \times [0, +\infty] \\ \text{ii)} & \exists\, (\overline{\mu}_p, \overline{b}_p) \in (0, +\infty)^2 \text{ s.t. } \lim\limits_{s \to \overline{s}} 2\varepsilon\Big(\dfrac{\varepsilon^{2(p-1)}}{2p-1}\mu,\ \dfrac{\varepsilon^{q(p-1)}}{1+(p-1)q} b\Big) = (\overline{\mu}_p, \overline{b}_p) \\ \text{iii)} & \lim\limits_{s \to \overline{s}} 2\varepsilon\rho = 0 \\ \text{iv)} & w_I \text{ is an even function of } x_3 \\ \text{v)} & \exists\, \varepsilon_0 > 0 \text{ s.t. } S \times (0, \varepsilon_0) \subset \Omega_+ \end{cases} \tag{$\mathbf{H}_2$}$$

3.1. A candidate for the limit behavior

This candidate could be determined by studying the asymptotic behavior of sequences with bounded total mechanical energy. Let

$$\begin{aligned} &{}^pH := {}^pH_d \times {}^pH_v \\ &{}^1H_d := \{u \in H^1(\Omega; \mathbb{R}^3);\ \widehat{u} \in H^1(S; \mathbb{R}^2)\} \\ &{}^2H_d := \{u \in H^1(\Omega; \mathbb{R}^3);\ \widehat{e(u)} = 0 \text{ in } S \text{ and } u_3 \in H^2(S)\} \\ &{}^pH_v := L^2(\Omega; \mathbb{R}^3), \quad p = 1, 2 \end{aligned} \tag{16}$$

We introduce

$${}^p\varphi(u, u') := \int_\Omega ae(u)\cdot e(u')\,\mathrm{d}x + \overline{\mu}_p \int_S Dw_I^{KL}(\widehat{e_p(u)})\cdot \widehat{e_p(u')}\,\mathrm{d}x, \quad {}^p\Phi(u) = {}^p\varphi(u, u),\ \forall u \in {}^pH_d \tag{17}$$

with $w_I^{KL}(\xi) = \mathrm{Inf}\{w_I(q); \widehat{q} = \xi\}$ for all ξ in $\mathbb{S}^2$ and $\widehat{e_1(u')} = \widehat{e(u')}$, $\widehat{e_2(u')} = \widehat{D^2 u'_3}$ for all u' in ${}^pH_d$, where D^2 stands for the second derivative in the distributional sense. We also define

$$ {}^pk(v, v') = \int_\Omega \overline{\rho} v \cdot v' \, \mathrm{d}x, \quad {}^pK(v) = {}^pk(v, v), \quad \forall v \in {}^pH_v, \quad p = 1, 2 \tag{18} $$

so that, for all $U^i = (u^i, v^i)$ in ${}^pH$, the inner product and norm are given by

$$ ((U^1, U^2))_p := {}^p\varphi(u^1, u^2) + {}^pk(v^1, v^2), \quad ||U||_p^2 := ((U, U))_p \tag{19} $$

Let T^ε be the mapping from $L^2(\mathcal{O}^\varepsilon; \mathbb{R}^3)$ into $L^2(\Omega; \mathbb{R}^3)$ defined by

$$ (T^\varepsilon w)(x) := w(x \pm \varepsilon x_3), \quad \forall x \in \Omega_\pm \tag{20} $$

Note that if w belongs to $H^1_{\Gamma_0^\varepsilon}(\mathcal{O}^\varepsilon; \mathbb{R}^3)$ then $T^\varepsilon w$ belongs to $H^1_{\Gamma_0}(\Omega \setminus S; \mathbb{R}^3)$. For any w in $H^1(\Omega_\pm; \mathbb{R}^3)$, we denote the trace of w on S by $\gamma_S^\pm(w)$. Thus, for any w in $H^1(\Omega \setminus S; \mathbb{R}^3)$, the jump of w across S, denoted by $[\![w]\!]$, is $\gamma_S^+(w^+) - \gamma_S^-(w^-)$, $w^\pm$ being the restriction of w to $\Omega_\pm$. Moreover, for any element w of $H^1(\Omega; \mathbb{R}^3)$, its trace on S is denoted by $\gamma_S(w)$.

Lastly, for any $\eta > 0$, let $V_{KL}(B^\eta)$ be the space of Kirchhoff–Love displacements defined by:

$$ \begin{aligned} V_{KL}(B^\eta) &:= \{u \in H^1(B^\eta; \mathbb{R}^3);\, e_{i3}(u) = 0, 1 \le i \le 3\} \\ &= \{u \in H^1(B^\eta; \mathbb{R}^3);\, \exists (u^M, u^F) \in H^1(S; \mathbb{R}^2) \times H^2(S) \, s.t. \\ &\qquad \widehat{u}(\widehat{x}, x_3) = u^M(\widehat{x}) - x_3 \nabla u^F(\widehat{x}), u_3(\widehat{x}, x_3) = u^F(\widehat{x})\} \end{aligned} \tag{21} $$

We have

Lemma 3.1. *For all sequences $U_s = (u_s, v_s)$ in H_s such that $|U_s|_s^2$ is bounded, there exists ${}^pU = ({}^pu, {}^pv)$ in ${}^pH$ and a not relabeled subsequence such that*

- i) – *$T^\varepsilon u_s$ weakly converges in $H^1(\Omega \setminus S; \mathbb{R}^3)$ toward ${}^pu$,*
  - *$\frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} \widehat{u_s} \, \mathrm{d}x_3$ weakly converges in $H^1(S; \mathbb{R}^2)$ toward $\widehat{{}^pu}$,*
  - *$\frac{1}{\varepsilon^3} \int_{-\varepsilon}^{\varepsilon} x_3 \widehat{e(u_s)} \, \mathrm{d}x_3$ weakly converges in $L^2(S)$ toward $-\frac{2}{3} \widehat{D^2({}^2u_3)}$ when $p = 2$,*
  - *${}^p\Phi({}^pu) \le \underline{\lim}_{s \to \bar{s}} \Phi_s(u_s)$,*
- ii) – *$T^\varepsilon v_s$ weakly converges in $L^2(\Omega; \mathbb{R}^3)$ toward ${}^pv$,*
  - *${}^pK({}^pv) \le \underline{\lim}_{s \to \bar{s}} K_s(v_s)$.*

Proof. First, the boundedness of $\Phi_s(u_s)$ implies that there exists w in $H^1_{\Gamma_0}(\Omega \setminus S; \mathbb{R}^3)$ and a sequence ρ_s in the space $\mathcal{R}$ of rigid displacements such that $((T^\varepsilon u_s)^+, (T^\varepsilon u_s)^- + \rho_s)$ converges weakly in $H^1_{\Gamma_{0^+}}(\Omega_+; \mathbb{R}^3) \times H^1_{\Gamma_{0^-}}(\Omega_-; \mathbb{R}^3)$ toward (w^+, w^-). As $[\![T^\varepsilon u_s]\!] = \int_{-\varepsilon}^{\varepsilon} \partial_3 u_s \, \mathrm{d}x_3$, $(\gamma_S^-((T^\varepsilon u_s)^-))_3$ converges strongly in $L^2(S)$ to $(\gamma_S^+(w^+))_3$ due to the boundedness of $\Phi_s(u_s)$, which, combined with $\partial_3(u_s)_\alpha = 2e_{\alpha 3} - \partial_\alpha(u_s)_3$ and

$$ \int_{B^\varepsilon} (u_s)_3^2 \, \mathrm{d}x \le 2\varepsilon \Big(\int_S | \gamma_s^+(T^\varepsilon u_s) |^2 \, \mathrm{d}\widehat{x} + 2\varepsilon \int_{B^\varepsilon} | \partial_3(u_s)_3 |^2 \, \mathrm{d}x \Big) $$

implies the convergence in the sense of distributions of $\widehat{\gamma_S^-((T^\varepsilon u_s)^-)}$ toward $\widehat{\gamma_S^+(w^+)}$. As $\rho_s = (T^\varepsilon u_s)^- + \rho_s - (T^\varepsilon u_s)^-$ lives in a finite dimensional space, $\gamma_S^-(\rho_s)$ converges strongly in $L^2(S; \mathbb{R}^3)$ toward $\gamma_S^-(w^-) - \gamma_S^+(w^+)$ and, consequently, $\gamma_S^-(u_s^-)$ converges strongly in $L^2(S; \mathbb{R}^3)$ toward $\gamma_S^+(w^+)$. This implies that $T^\varepsilon u_s$ converges weakly in $H^1_{\Gamma_0}(\Omega \setminus S; \mathbb{R}^3)$ toward some ${}^pu$ and $[\![{}^pu]\!]$, the strong limit in $L^2(S; \mathbb{R}^3)$ of $[\![T^\varepsilon u_s]\!]$, vanishes, that is to say ${}^pu$ belongs to $H^1_{\Gamma_0}(\Omega; \mathbb{R}^3)$. Of course, the proof is simpler when $\mathcal{H}_2(\Gamma_{0^-}) > 0$!

Next, the boundedness of $\Phi_s(u_s)$ allows us to easily identify the weak limit in $H^1(S; \mathbb{R}^2)$ of

$$ \frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} \widehat{u}_s \, \mathrm{d}x_3 = \widehat{\gamma_s^+((T^\varepsilon u_s)^+)} - \frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} \int_{x_3}^{\varepsilon} \partial_3 \widehat{u}_s \, \mathrm{d}t \, \mathrm{d}x_3 $$

which implies that ${}^pu$ belongs to ${}^pH_d$. Concerning $\frac{1}{\varepsilon^3} \int_{-\varepsilon}^{\varepsilon} x_3 \widehat{e(u_s)} \, \mathrm{d}x$, we may proceed as in [3] or as follows. Let S_ε be the mapping from $H^1(B^\varepsilon; \mathbb{R}^3)$ into $H^1(B^1; \mathbb{R}^3)$ defined by $\widehat{S_\varepsilon w}(\widehat{x}, x_3) = \varepsilon \widehat{w}(\widehat{x}, \varepsilon x_3)$, $(S_\varepsilon w)_3(\widehat{x}, x_3) = w_3(\widehat{x}, \varepsilon x_3)$, for all $x = (\widehat{x}, x_3)$ in B^1. Then, the boundedness of $\Phi_s(u_s)$ implies the boundedness of $\int_{B^1} | e(\varepsilon, S_\varepsilon u_s) |^2 \, \mathrm{d}x$, where $e(\varepsilon, w) = I_\varepsilon \, e(w) \, I_\varepsilon$

and $I_\varepsilon = e_1\otimes e_1 + e_2\otimes e_2 + \frac{1}{\varepsilon}e_3\otimes e_3$. This implies that there exists $\mathfrak{u}$ in $V_{KL}(B^1)$ and ρ_s in $\mathcal{R}$ such that, up to a subsequence, $S_\varepsilon u_s + \rho_s$ weakly converges toward $\mathfrak{u}$ in $H^1(B^1;\mathbb{R}^3)$. As for all τ in $L^2(S;\mathbb{S}^2)$ one has

$$\int_S \tau(\widehat{x})\frac{1}{\varepsilon^3}\int_{-\varepsilon}^{\varepsilon} x_3\widehat{e(u_s)}\,\mathrm{d}x_3\,\mathrm{d}\widehat{x} = \int_{B^1}\tau(\widehat{x})x_3\widehat{e(S_\varepsilon u_s)}\,\mathrm{d}x$$

one deduces that $\frac{1}{\varepsilon^3}\int_{-\varepsilon}^{\varepsilon} x_3\widehat{e(u_s)}\,\mathrm{d}x_3$ converges weakly in $L^2(S;\mathbb{S}^2)$ toward

$$\int_{-1}^{+1}\widehat{e(\mathfrak{u})}\,\mathrm{d}x = \int_{-1}^{+1} x_3(\widehat{e}(\mathfrak{u}^M) - x_3 D^2(\mathfrak{u}^F))\,\mathrm{d}x_3 = -\frac{2}{3}D^2(\mathfrak{u}^F)$$

But the trace on $S+e_3$ of $(S_\varepsilon w)_3$ being equal to $(\gamma_S^+((T^\varepsilon u_s)^+)_3$, one deduces that $\mathfrak{u}^F = (\gamma_s({}^2u))_3$.

Finally, the lower bound for $\Phi_s(u_s)$ is obtained by a simple use of the Jensen inequality and a standard lower semi-continuity argument, which is the source of the term w_I^{KL}.

The point *ii*) is obvious. □

We can now define the limit evolution operator ${}^pA$ through

$$\begin{cases} D({}^pA) = \left\{ U=(u,v)\in {}^pH;\ \begin{cases} \text{i)} & v\in {}^pH_{\mathrm{d}} \\ \text{ii)} & \exists (w,\xi)\in L^2(\Omega;\mathbb{R}^3)\times \partial\mathcal{D}^{KL}(\widehat{e_p(v)}) \text{ s.t.} \\ & \qquad {}^pk(w,v') + {}^p\varphi(u,v') + \overline{b}_p\int_S \xi\cdot\widehat{e}_p(v')\,\mathrm{d}\hat{x} = 0, \quad \forall v'\in {}^pH_d \end{cases} \right\} \\ {}^pAU = (-v,0) + \{(0,-w);\ w \text{ satisfying ii)}\} \end{cases} \tag{22}$$

$\mathcal{D}^{KL}$ being defined in the same way as w_I^{KL}.

Similar to the case of A_s, it can be checked easily that ${}^pA$ is maximal monotone and, more specifically, that for all $\psi = (\psi^1,\psi^2)$ in ${}^pH$:

$$\begin{cases} {}^p\overline{U} = ({}^p\overline{u},{}^p\overline{v}) \text{s.t.} \\ {}^p\overline{U} + {}^pA{}^p\overline{U} \ni \psi \end{cases} \Leftrightarrow \begin{cases} {}^pJ({}^p\overline{v}) \le {}^pJ(v) := \frac{1}{2}{}^pK(v) - {}^pk(\psi^2,v) + \frac{1}{2}{}^p\Phi(v) + \\ \qquad\qquad + {}^p\varphi(\psi^1,v) + \overline{b}_p\int_S \mathcal{D}^{KL}(\widehat{e_p(v)})\,\mathrm{d}\hat{x}, \quad \forall v\in {}^pH_d \\ {}^p\overline{u} = {}^p\overline{v} + \psi^1 \end{cases} \tag{23}$$

Consequently, the same statement as that of Theorem 2.1 is valid for the following equation, which will be shown to describe the asymptotic behavior of u_s:

$$\frac{\mathrm{d}{}^pU^r}{\mathrm{d}t} + {}^pA{}^pU^r \ni {}^pF := \left(-\frac{\mathrm{d}{}^pu^e}{\mathrm{d}t}, f/\overline{\rho}\right), \quad {}^pU^r(0) = {}^pU^{r0} \tag{24}$$

with

$$\begin{aligned} &{}^pu^e \in BV^{(2)}(0,T;{}^pH_d);\ {}^p\varphi({}^pu^e(t),u') = L(t)(u'), \quad \forall u'\in {}^pH_d, \forall t\in[0,T] \\ &L(t)(u') = \int_{\Gamma_1} g(x,t)\cdot v(x)\,\mathrm{d}\mathcal{H}_2 \end{aligned} \tag{25}$$

We set

$${}^pU^e = \left({}^pu^e, 0\right), \quad {}^pU = {}^pU^e + {}^pU^r \tag{26}$$

3.2. Convergence

As in [1], to prove the convergence of u_s toward ${}^pu = {}^pu^e + {}^pu^r$, we will use the framework of a nonlinear version of Trotter's theory of convergence of semigroups acting on *variable spaces* (see [4,5] and Appendix of [6]) because u_s^r and ${}^pu^r$ *do not inhabit the same space*. To establish the convergence of the mechanical state, we need to compare the elements of ${}^pH$ to those of H_s. We therefore define ${}^pP_s$ by:

$$(u,v)\in {}^pH \mapsto {}^pP_s(u,v) = (u_s^*, v_s^*)\in H_s \tag{27}$$

with

– $u_s^* \in H^1_{\Gamma_0^\varepsilon}(\mathcal{O}^\varepsilon;\mathbb{R}^3)$; $\varphi_s(u_s^*,u') = {}^p l_\varepsilon(u,u')$, $\forall u' \in H^1_{\Gamma_0^\varepsilon}(\mathcal{O}^\varepsilon;\mathbb{R}^3)$

$$ {}^p l_\varepsilon(u,u') = \int_\Omega a^\varepsilon e(u)\cdot e(u')\,dx + \mu \int_{B^\varepsilon} Dw_I^{KL}(\widehat{e({}^p v)})\cdot \widehat{e(u')}\,dx$$

where ${}^1v, {}^2v \in V_{KL}(B^\varepsilon)$ with $({}^1v^M, {}^1v^F) = (\gamma_S(\widehat{u}),0)$, $({}^2v^M, {}^2v^F) = (0,\gamma_S(u_3))$,
– $v_s^*(x) = v(x \mp \varepsilon x_3)$, *a.e.* $x \in \Omega_\pm^\varepsilon$, $v_s^*(x) = 0$ *a.e.* $x \in B^\varepsilon$.

Taking advantage of the *variational definition* of u_s^*, Lemma 3.1 and the classical procedure of mathematical justification of Kirchhoff–Love theory of plates (cf. [7]) imply that ${}^pP_s$ enjoys the following fundamental property.

Proposition 3.1.

i) *There exists a strictly positive constant C such that* $|{}^pP_sU|_s \le C||U||_p$, $\forall U \in {}^pH$.
ii) *When s tends to* $\bar{s}$, $|{}^pP_sU|_s$ *converges toward* $\|U\|_p$ *for all U in* ${}^pH$.

Next we state that:

$$U_s \text{ in } H_s \text{ converges in the sense of Trotter toward } U \text{ in } {}^pH \text{ if } \lim_{s\to\bar{s}} |{}^pP_sU - U_s|_s = 0 \tag{28}$$

Even if this is the right mechanical notion, it could be of interest to consider this convergence with respect to some classical conventional notions.

Proposition 3.2. *For all* $U=(u,v)$ *in* ${}^pH$, $U_s=(u_s,v_s)$ *in* H_s *converges in the sense of Trotter toward U if and only if:*

i) $T^\varepsilon u_s$ *converges strongly in* $H^1(\Omega\setminus S;\mathbb{R}^3)$ *toward u,*
ii) $\frac{1}{2\varepsilon}\int_\varepsilon^\varepsilon \widehat{u}_s\,dx_3$ *converges strongly in* $H^1(S;\mathbb{R}^2)$ *toward* $\widehat{u}$,
iii) ${}^p\Phi(u) = \lim_{s\to\bar{s}} \Phi_s(u_s)$,
iv) $T^\varepsilon v_s$ *converges strongly in* $L^2(\Omega;\mathbb{R}^2)$ *toward v,*
v) ${}^pK(v) = \lim_{s\to\bar{s}} K_s(v_s)$.

Lastly, we conclude by using a suitable nonlinear version (see [5,6]) of Trotter's theory of convergence of semigroups, where it suffices to make an additional assumption (**H**$_3$) about the initial states and to establish the following "static" result.

Proposition 3.3. *We have*

i) $\forall \psi \in {}^pH$, $\lim_{s\to\bar{s}} |{}^pP_s(I+{}^pA)^{-1}\psi - (I+A_s)^{-1}{}^pP_s\psi|_s = 0$,
ii) $\lim_{s\to\bar{s}} |{}^pP_s{}^pU^e(t) - U_s^e(t)|_s = 0$ *uniformly on* $[0,T]$,
iii) $\lim_{s\to\bar{s}} \int_0^T |{}^pP_s{}^pF(t) - F_s(t)|_s\,dt = 0$.

As regards point *i*), we use the same strategy as in [1] by due account of (11) and (23). Taking advantage of Lemma 3.1 and the variational definition of P_s, we obtain that a subsequence of minimizers of $\widetilde{J}_s$ defined by

$$\widetilde{J}_s(v) = \frac{1}{2}K_s(v) - k_p(\psi^2, T^\varepsilon v) + \frac{1}{2}\Phi_s(v) + \varphi_s(\psi_s^{1^*}, v) + b\int_{B^\varepsilon} \mathcal{D}(e(v))\,dx \tag{29}$$

converges to an element $\overline{v}$ in H_d satisfying ${}^pJ(\overline{v}) \le \underline{\lim}_{s\to\bar{s}} J_s(v_s)$. Indeed, $\overline{v}$ is the unique minimizer of ${}^pJ$ because, due to Proposition 3.1, for all w in ${}^pH_d$, one has $\lim_{s\to\bar{s}} J_s(w_s^*) = {}^pJ(w)$. Similar arguments as those of [1] establish *ii*) and *iii*). Thus, we deduce the convergence uniformly on $[0,T]$ in the sense of Trotter of the solution to (12) toward that to (24) with ${}^pU^{r0} := {}^pU^0 - {}^pU^e(0)$ and the additional conditions of convergence and compatibility between the initial state and loading:

$$\exists {}^pU^0 \in {}^pU^e(0) + D({}^pA);\quad U_s^0 \in U_s^e(0) + D(A_s) \text{ and } \lim_{s\to\bar{s}} |{}^pP_s{}^pU^0 - U_s^0|_s = 0 \tag{$\mathbf{H}_3$}$$

This can be rephrased in a more explicit way with respect to $(\mathcal{P}_s)$:

Theorem 3.1. *The solution to*

$$\frac{\mathrm{d}U_s}{\mathrm{d}t} + A_s(U_s - U_s^e) \ni (0,\, f^\varepsilon/\overline{\rho}^\varepsilon), \quad U_s(0) = U_s^0 \tag{30}$$

converges toward the solution to

$$\frac{\mathrm{d}^p U}{\mathrm{d}t} + {}^pA({}^pU - {}^pU^e) \ni (0,\, f/\bar{\rho}), \quad {}^pU(0) = {}^pU^0 \tag{31}$$

in the sense $\lim_{s\to\bar{s}} |{}^pP_s\,{}^pU(t) - U_s(t)|_s = 0$, $\lim_{s\to\bar{s}} |U_s(t)|_s = \|{}^pU(t)\|_p$ *uniformly on* $[0, T]$.

4. Concluding remarks

It is worthwhile to write (31) in a variational form:

$\exists\, \xi \in \partial\mathcal{D}^{KL}\Big(\widehat{e_p(v)}\Big)$ such that

$$\int_\Omega \overline{\rho}\,\frac{\mathrm{d}^2\,{}^pu}{\mathrm{d}t^2}\cdot\varphi\,\mathrm{d}x + \int_\Omega ae({}^pu)\cdot e(\varphi)\,\mathrm{d}x + \overline{\mu}_p\int_S Dw_I^{KL}\big(\widehat{e_p({}^pu)}\big)\cdot\big(\widehat{e_p(\varphi)}\big)\,\mathrm{d}\hat{x} + \overline{b}_p\int_S \xi\cdot\big(\widehat{e_p(\varphi)}\big)\,\mathrm{d}\hat{x}$$
$$= \int_\Omega f\cdot\varphi\,\mathrm{d}x + \int_{\Gamma_1} g\cdot\varphi\,\mathrm{d}\mathcal{H}_2, \quad \forall\varphi \in {}^pH_d$$

where ${}^pH_d$ and $\widehat{e_p(\cdot)}$ are defined in (16) and (17), respectively. Hence, the limit behavior describes the dynamic response to the real loads (f, g) of a structure consisting of two linearly elastic adherents occupying $\Omega_\pm$, which are perfectly bonded to a material deformable flat surface whose behavior is of the same kind as the genuine adhesive (*i.e.* non-linear viscoelasticity of Kelvin–Voigt generalized type). Moreover, the mass of the adhesive being evanescent, there is no inertial term in the interface condition. The case $p = 1$ corresponds to membrane deformations, whereas the case $p = 2$ corresponds to flexural deformations.

The Proposition 3.3 covers the static situation which has been considered in [8]. Our limit interface condition agrees with the one of [3], which studied a resembling problem (the adherents occupying the complementary of B^ε in a fixed domain Ω).

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