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Mechanics of Deformation and Acoustic Propagation in Porous Media

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A unified treatment of the mechanics of deformation and acoustic propagation in porous media is presented, and some new results and generalizations are derived. The writer's earlier theory of deformation of porous media derived from general principles of nonequilibrium thermodynamics is applied. The fluid-solid medium is treated as a complex physical-chemical system with resultant relaxation and viscoelastic properties of a very general nature. Specific relaxation models are discussed, and the general applicability of a correspondence principle is further emphasized. The theory of acoustic propagation is extended to include anisotropic media, solid dissipation, and other relaxation effects. Some typical examples of sources of dissipation other than fluid viscosity are considered.

1. INTRODUCTION

THE purpose of the present paper is to reformulate in a more systematic manner and in a somewhat more general context the linear mechanics of fluid-saturated porous media and also to present some new results and developments with particular emphasis on viscoelastic properties and relaxation effects.

The theory finds numerous applications in a diversity of fields, including geophysics, seismology, civil engineering, and acoustics.

A linear theory of deformation of a porous elastic solid containing a viscous fluid was developed by this writer in 1941 and was applied to problems of consolidation of a foundation under a given load distribution.^{1,2} The medium was assumed to be statistically isotropic. In later years, the theory was extended to include the most general case of anisotropy for a porous elastic solid.³

The theory of deformation in a porous viscoelastic medium was developed on the basis of the thermodynamics of irreversible processes.⁴ The results included general anisotropy. It was shown that, on the basis of Onsager's relations, it is possible to extend to a viscoelastic porous medium the principle of correspondence introduced in 1954 by the writer for homogeneous solids. The principle states that the equations governing

the mechanics of porous media are formally the same for an elastic or viscoelastic system, provided that the elastic coefficients are replaced by the corresponding operators.

Equations for acoustic propagation in the elastic isotropic porous solid containing a viscous fluid were established by adding suitable inertia terms in the original theory, and the propagation of three kinds of body waves was discussed in detail.⁵ Two other publications have dealt with general solutions and stress functions in consolidation problems⁶ and with a discussion of the physical significance and of methods of measurement of the elastic coefficients.⁷

Consolidation theories have also been developed by Florin⁸ and Barenblatt and Krylov.⁹ An acoustic propagation theory has been initiated by Frenkel,¹⁰ who brought out the existence of two dilatational waves.

The importance of viscoelasticity in consolidation problems of clay has been emphasized by Tan Tjong Kie¹¹ and demonstrated in test results by Geuze and Tan Tjong Kie.¹² Special aspects of the consolidation

⁵ M. A. Biot, *J. Acoust. Soc. Am.* **28**, 168-178 (1956); **28**, 179-191 (1956).

⁶ M. A. Biot, *J. Appl. Mech.* **23**, 91-96 (1956).

⁷ M. A. Biot and D. G. Willis, *J. Appl. Mech.* **24**, 594-601 (1957).

⁸ V. A. Florin, *Theory of Soil Consolidation*, in Russian (Stroyizdat, Moscow, 1948).

⁹ G. I. Barenblatt and A. P. Krylov, *Izvest. Akad. Nauk SSSR, Tech. Sci. Div. No. 2*, 5-13 (1955).

¹⁰ J. Frenkel, *J. Phys. (U.S.S.R.)* **8**, 230 (1944).

¹¹ Tan Tjong Kie, "Investigations on the rheological properties of clay" (in Dutch with English synopsis), dissertation, Technical University, Delft, Netherlands.

¹² E. C. Geuze and Tan Tjong Kie, *The Mechanical Behavior of Clays* (Academic Press Inc., New York, 1954).

* Consultant, Shell Development Company.

¹ M. A. Biot, *J. Appl. Phys.* **12**, 155-164 (1941); **12**, 426-430 (1941).

² M. A. Biot and F. M. Clingan, *J. Appl. Phys.* **12**, 578-581 (1941); **13**, 35-40 (1942).

³ M. A. Biot, *J. Appl. Phys.* **26**, 182-185 (1955).

⁴ M. A. Biot, *J. Appl. Phys.* **27**, 459-467 (1956).

problem which do not agree with the elastic theory can be explained by the introduction of the more general thermodynamic operators of the viscoelastic theory,⁴ as further exemplified in the discussion of Secs. 5, 6, and 7 of this paper.

Sections 2 and 3 begin with a general and rigorous derivation of the stress-strain relations, which is valid for the case of an elastic porous medium with *nonuniform porosity*, i.e., for which the porosity varies from point to point. This re-emphasizes the use of the particular variables and coefficients introduced in the original paper,^{1,2} and attention is called to the fact that some of the papers written in the later sequence are formulated in the context of uniform porosity.

The derivation of Darcy's law from thermodynamic principles, which has previously been briefly outlined,^{3,4} is discussed in detail in Sec. 4.

In Secs. 5 and 6, the writer's previous thermodynamic theory of viscoelastic properties of porous media is discussed in more detail and with particular emphasis on the physical significance of the operators.

Section 7 discusses the formulation of the field equations and of their general solution in a particular case.

The theory of acoustic propagation developed previously for the isotropic elastic medium is extended to anisotropic media in Sec. 8, and particular attention is given to viscoelasticity and solid dissipation in Sec. 9. The term viscoelasticity here encompasses a vast range of physical phenomena, involving relaxation, which find their origin in physical-chemical, thermoelastic, electrical, mechanical, or other processes of the complex fluid-solid medium considered as a single system. This generality is provided by the underlying thermodynamic theory. It is also pointed out that by putting the fluid density equal to zero, we can apply all results to *thermoelastic propagation in a nonporous elastic continuum*. This is a consequence of the isomorphism between thermoelasticity and the theory of porous media.¹³ For similar reasons, the propagation equations of Sec. 9 are also applicable to a *thermoviscoelastic continuum*.

2. STRAIN ENERGY OF A POROUS ELASTIC MEDIUM

The displacement of the solid matrix is designated by the components u_x, u_y, u_z . The components of the average fluid-displacement vector are U_x, U_y, U_z . These components are defined in such a way that the volume of fluid displaced through unit areas normal to the x, y, z directions are fU_x, fU_y, fU_z , respectively, where f denotes the porosity.

We shall use the total stress components of the bulk material τ_{ij} . In earlier papers, we have used components

σ_{ij} and σ which are related to τ_{ij} by the equation¹⁴

$$\begin{aligned} \tau_{ij} &= \sigma_{ij} + \delta_{ij}\sigma, \\ \delta_{ij} &= 1, \quad i=j, \\ \delta_{ij} &= 0, \quad i \neq j. \end{aligned} \tag{2.1}$$

If we consider a unit cube of bulk material, the components σ_{ij} represent the force applied to the solid part of the faces, and σ represents the force applied to the fluid part of these faces. With p_f denoting the fluid pressure, we can write

$$\sigma = -fp_f. \tag{2.2}$$

Since we are dealing with a system which is in thermodynamic equilibrium, the fluid is at rest and p_f is constant throughout the body.¹⁵

We can define the strain energy of a porous elastic medium as the isothermal free energy of the fluid-solid system. W denotes the strain energy per unit volume. For a volume Ω bounded by a surface S , the variation of the strain energy is equal to the virtual work of the surface forces¹⁶:

$$\begin{aligned} \iiint_{\Omega} \delta W d\Omega &= \iint_S (f_x \delta u_x + f_y \delta u_y + f_z \delta u_z \\ &+ F_x \delta U_x + F_y \delta U_y + F_z \delta U_z) dS. \end{aligned} \tag{2.3}$$

In the expression $f_i dS$ and $F_i dS$, the components of the forces acting on the solid part and the fluid part of an element of surface dS are

$$\begin{aligned} f_i &= \sum^j \sigma_{ij} n_j, \\ F_i &= \sum^j \sigma \delta_{ij} n_j, \end{aligned} \tag{2.4}$$

respectively, where n_f denotes the components of the outward unit vector normal to the surface. We can express these forces in terms of τ_{ij} and p_f by introducing relations (2.1) and (2.2). Hence,

$$\begin{aligned} f_i &= \sum^j (\tau_{ij} + \delta_{ij} fp_f) n_j, \\ F_i &= \sum^j \delta_{ij} fp_f n_j. \end{aligned} \tag{2.5}$$

Introducing these expressions in Eq. (2.3) yields

$$\begin{aligned} \iiint \delta W d\Omega &= \iint \sum^{ij} \tau_{ij} n_j \delta u_i dS \\ &- p_f \iint_S (n_x \delta w_x + n_y \delta w_y + n_z \delta w_z) dS. \end{aligned} \tag{2.6}$$

¹⁴ See Eq. (25) of reference 7. In the original papers,¹ we used the total stress τ_{ij} and the fluid pressure p_f represented by the notations σ_{ij} and σ .

¹⁵ Body forces are neglected in the present derivation.

¹⁶ The boundary S can be thought of not as a physical termination of the body but as any closed surface in the body. In this way, the surface tension at a physical boundary does not have to be introduced.

¹³ M. A. Biot, J. Appl. Phys. 27, 240-253 (1956).

The vector w_i is defined as

$$w_i = f(U_i - u_i), \tag{2.7}$$

or in vector notation

$$\mathbf{w} = f(\mathbf{U} - \mathbf{u}). \tag{2.8}$$

The vector \mathbf{w} represents the flow of the fluid *relative* to the solid but measured in terms of volume per unit area of the bulk medium.

We consider now the surface integrals of Eq. (2.6). They can be transformed to volume integrals by means of Green's theorem. We write

$$\int \int_S \sum_{ij} \tau_{ij} n_j \delta u_i dS = \int \int \int_\Omega \sum_{ij} \frac{1}{\partial x_j} (\tau_{ij} \delta u_i) d\Omega. \tag{2.9}$$

The coordinates x, y, z are designated by x_i . The integrand can be transformed as

$$\sum_{ij} \frac{\partial}{\partial x_j} (\tau_{ij} \delta u_i) = \sum_{ij} \delta u_i \frac{\partial \tau_{ij}}{\partial x_j} + \sum_{ij} \tau_{ij} \frac{\delta \partial u_i}{\partial x_j}. \tag{2.10}$$

Because the total stress field is in equilibrium, the stress must satisfy the condition

$$\sum_j \frac{\partial \tau_{ij}}{\partial x_j} = 0. \tag{2.11}$$

Hence we can write

$$\sum_{ij} \frac{\partial}{\partial x_j} (\tau_{ij} \delta u_i) = \tau_{xx} \delta e_x + \tau_{yy} \delta e_y + \tau_{zz} \delta e_z + \tau_{yz} \delta \gamma_z + \tau_{zx} \delta \gamma_x + \tau_{xy} \delta \gamma_y. \tag{2.12}$$

We have put

$$\begin{aligned} e_x &= \partial u_x / \partial x, & \gamma_x &= (\partial u_y / \partial z) + (\partial u_z / \partial y) \\ e_y &= \partial u_y / \partial y, & \gamma_y &= (\partial u_z / \partial x) + (\partial u_x / \partial z) \\ e_z &= \partial u_z / \partial z, & \gamma_z &= (\partial u_x / \partial y) + (\partial u_y / \partial x). \end{aligned} \tag{2.13}$$

The components

$$e_{ij} = \begin{Bmatrix} e_x & \frac{1}{2} \gamma_z & \frac{1}{2} \gamma_y \\ \frac{1}{2} \gamma_z & e_y & \frac{1}{2} \gamma_x \\ \frac{1}{2} \gamma_y & \frac{1}{2} \gamma_x & e_z \end{Bmatrix} \tag{2.14}$$

represent the strain tensor of the porous solid.

Similarly, the second surface integral of Eq. (2.6) can be transformed to a volume integral:

$$- \int \int_S (n_x \delta w_x + n_y \delta w_y + n_z \delta w_z) dS = \int \int \int_\Omega \delta \zeta d\Omega, \tag{2.15}$$

where

$$\begin{aligned} \zeta &= -[(\partial w_x / \partial x) + (\partial w_y / \partial y) + (\partial w_z / \partial z)] \\ &= \text{div}[f(\mathbf{u} - \mathbf{U})], \end{aligned} \tag{2.16}$$

or

$$\zeta = -\text{div} \mathbf{w}.$$

From Eqs. (2.6), (2.9), (2.12), and (2.15), we derive

$$\begin{aligned} \int \int \int_\Omega \delta W d\Omega &= \int \int \int_\Omega (\tau_{xx} \delta e_x + \tau_{yy} \delta e_y + \tau_{zz} \delta e_z + \tau_{yz} \delta \gamma_z \\ &+ \tau_{zx} \delta \gamma_x + \tau_{xy} \delta \gamma_y + p_f \delta \zeta) d\Omega. \end{aligned} \tag{2.17}$$

Hence,

$$\begin{aligned} \delta W &= \tau_{xx} \delta e_x + \tau_{yy} \delta e_y + \tau_{zz} \delta e_z + \tau_{yz} \delta \gamma_z \\ &+ \tau_{zx} \delta \gamma_x + \tau_{xy} \delta \gamma_y + p_f \delta \zeta. \end{aligned} \tag{2.18}$$

The variable ζ was introduced by this writer in the original paper,¹ where it was designated by the symbol θ . It was defined by the same general Eq. (2.16) as valid for nonhomogeneous porosity. The same variable was also used in some later work⁴⁻⁶ in the context of a medium with uniform porosity. For uniform porosity, we can write Eq. (2.16) as

$$\zeta = f \text{div}(\mathbf{u} - \mathbf{U}). \tag{2.19}$$

This variable is obviously a measure of the amount of fluid which has flowed in and out of a given element attached to the solid frame, i.e., it represents the increment of fluid content.

The strain energy W must be a function of ζ and of the six strain components defined by Eq. (2.13). Hence, W is a function of seven variables:

$$W = W(e_x, e_y, e_z, \gamma_x, \gamma_y, \gamma_z, \zeta). \tag{2.20}$$

δW must be an exact differential. Hence,

$$\begin{aligned} \tau_{xx} &= \partial W / \partial e_x, & \tau_{yz} &= \partial W / \partial \gamma_z \\ \tau_{yy} &= \partial W / \partial e_y, & \tau_{zx} &= \partial W / \partial \gamma_x \\ \tau_{zz} &= \partial W / \partial e_z, & \tau_{xy} &= \partial W / \partial \gamma_y \\ p_f &= \partial W / \partial \zeta. \end{aligned} \tag{2.21}$$

These relations, obtained earlier by different methods,^{1,3,5,7} lead immediately to the formulation of the general stress-strain relations in a porous medium. Since W is the isothermal free energy, the stress-strain relations (2.21) include phenomena which depend on the physical chemistry of the fluid-solid system and others which are expressible by means of thermodynamic variables such as interfacial and surface tension effects.

3. LINEAR STRESS-STRAIN RELATIONS

In an isotropic medium, the strain energy is a function of four variables, the three invariants I_1, I_2, I_3 of the strain components and the fluid content ζ :

$$W = W(I_1, I_2, I_3, \zeta). \tag{3.1}$$

We are restricted here to linear relations. Equations (2.21) are valid for either linear or nonlinear properties. It is readily seen that expansion of W to the third degree leads to quadratic expressions for the stresses with eleven elastic constants. The case of nonlinear materials shall be analyzed in a forthcoming publication. In the

present analysis we shall consider only the linear relations.

For a linear material, the strain energy is quadratic, and we must include only the linear and quadratic invariants¹⁷

$$I_1 = e_x + e_y + e_z = e, \tag{3.2}$$

$$I_2 = e_y e_x + e_x e_z + e_x e_y - \frac{1}{4}(\gamma_x^2 + \gamma_y^2 + \gamma_z^2).$$

In this case, it is easier to use the invariant

$$I_2' = -4I_2 = \gamma_x^2 + \gamma_y^2 + \gamma_z^2 - 4e_y e_x - 4e_x e_z - 4e_x e_y, \tag{3.3}$$

instead of I_2 . We derive the quadratic form for W :

$$2W = He^2 + \mu I_2' - 2Ce\zeta + M\zeta^2. \tag{3.4}$$

The reason for using a negative constant $-2C$ and a factor 2 is one of convenience in later equations. Substituting this expression in the general Eq. (2.21), we obtain the stress-strain relations

$$\begin{aligned} \tau_{xx} &= He - 2\mu(e_y + e_z) - C\zeta \\ \tau_{yy} &= He - 2\mu(e_x + e_z) - C\zeta \\ \tau_{zz} &= He - 2\mu(e_x + e_y) - C\zeta \\ \tau_{yz} &= \mu\gamma_z, \quad \tau_{zx} = \mu\gamma_y \\ \tau_{xy} &= \mu\gamma_x, \quad p_f = -Ce + M\zeta. \end{aligned} \tag{3.5}$$

By putting

$$H = \lambda_c + 2\mu, \quad C = \alpha M, \quad \lambda_c = \lambda + \alpha^2 M, \tag{3.6}$$

we see that relations (3.5) become

$$\begin{aligned} \tau_{xx} &= 2\mu e_x + \lambda_c e - \alpha M \zeta \\ \tau_{yy} &= 2\mu e_y + \lambda_c e - \alpha M \zeta \\ \tau_{zz} &= 2\mu e_z + \lambda_c e - \alpha M \zeta \\ \tau_{yz} &= \mu\gamma_z, \quad \tau_{zx} = \mu\gamma_y, \quad \tau_{xy} = \mu\gamma_x \\ p_f &= -\alpha M e + M \zeta. \end{aligned} \tag{3.7}$$

In abbreviated notation, they are also written

$$\begin{aligned} \tau_{ij} &= 2\mu e_{ij} + \delta_{ij}(\lambda_c e - \alpha M \zeta) \\ p_f &= -\alpha M e + M \zeta. \end{aligned} \tag{3.8}$$

We can obtain an alternate form of Eq. (3.7) by substituting the value of ζ as a function of p_f and e :

$$\begin{aligned} \tau_{xx} + \alpha p_f &= 2\mu e_x + \lambda e \\ \tau_{yy} + \alpha p_f &= 2\mu e_y + \lambda e \\ \tau_{zz} + \alpha p_f &= 2\mu e_z + \lambda e \\ \tau_{yz} &= \mu\gamma_z, \quad \tau_{zx} = \mu\gamma_y \\ \tau_{xy} &= \mu\gamma_x, \quad \zeta = (1/M)p_f + \alpha e. \end{aligned} \tag{3.9}$$

In abbreviated notation, they are written

$$\begin{aligned} \tau_{ij} + \delta_{ij}\alpha p_f &= 2\mu e_{ij} + \delta_{ij}\lambda e \\ \zeta &= (1/M)p_f + \alpha e. \end{aligned} \tag{3.10}$$

Another useful form of the equations is obtained by using the so-called "effective stress," defined as

$$\tau_{ij}' = \tau_{ij} + \delta_{ij}p_f. \tag{3.11}$$

This represents the portion of the total stress which is in excess of the local fluid pressure.

With these effective stresses, the relations (3.9) become

$$\begin{aligned} \tau_{xx}' - (1-\alpha)p_f &= 2\mu e_x + \lambda e \\ \tau_{yy}' - (1-\alpha)p_f &= 2\mu e_y + \lambda e \\ \tau_{zz}' - (1-\alpha)p_f &= 2\mu e_z + \lambda e \\ \tau_{yz}' &= \mu\gamma_z, \quad \tau_{zx}' = \mu\gamma_y \\ \tau_{xy}' &= \mu\gamma_x, \quad \zeta = (1/M)p_f + \alpha e. \end{aligned} \tag{3.12}$$

For an incompressible fluid and an incompressible matrix material, we have shown^{1,7} that $\alpha = 1$ and $M = \infty$. In that case, the fluid pressure does not appear in relations (3.12).

In abbreviated notation, Eqs. (3.12) are written

$$\begin{aligned} \tau_{ij}' - (1-\alpha)\delta_{ij}p_f &= 2\mu e_{ij} + \delta_{ij}\lambda e \\ \zeta &= (1/M)p_f + \alpha e. \end{aligned} \tag{3.13}$$

The interest in the use of the effective stress components τ_{ij}' lies in the experimental fact that slip and failure properties of porous and granular media are dependent primarily upon the magnitude of these components alone. In this connection, a very useful viewpoint was introduced by Hubbert and Rubey,¹⁸ who pointed out that the average effective stresses can readily be determined by conditions of static equilibrium of these forces with the total weight and the "buoyancy" associated with the field p_f considered continuous. This provides a simple and practical procedure for the approximate analysis of failure in porous media when the distribution of pore pressure is considered.

The above equations and elastic coefficients were derived by a different procedure in previous work. The stress-strain relations in the form of Eq. (3.9) were obtained in 1941, and a discussion of the significance of elastic coefficients such as M and α was presented.¹ In later work,^{7,8} the equations were rewritten in the form of Eqs. (3.7) and (3.12), and methods of measurement of the elastic coefficients were further discussed.

For a "closed system," i.e., a system in which the pores are sealed, we must put $\zeta = 0$. Equation (3.7) then shows that in this case,

$$K_c = \lambda_c + \frac{2}{3}\mu = H - \frac{4}{3}\mu \tag{3.14}$$

is the bulk modulus for a closed system. On the other hand, by putting $p_f = 0$ in Eq. (3.12), we see that for this "open system,"

$$K = \lambda + \frac{2}{3}\mu \tag{3.15}$$

¹⁷ A. E. H. Love, *A Treatise on the Mathematical Theory of Elasticity* (Dover Publications, New York, 1944), 4th ed., pp. 43, 102.

¹⁸ M. King Hubbert and W. Rubey, *Bull. Geol. Soc. Am.* **70**, 115-166 (1959).

represents a bulk modulus which is the inverse of the "jacketed compressibility." This corresponds to a test where a fluid pressure is applied to a jacketed specimen, while the pore fluid is allowed to escape freely through a tube.

From the analogy between a porous medium and a thermoelastic solid,¹³ we conclude that the coefficients λ_c and λ correspond to the adiabatic and isothermal Lamé coefficients for a nonporous medium, respectively. It is of interest to examine the restrictions on the coefficients imposed by the nonnegative character of the strain energy W . We can write expression (3.4) in the form

$$2W = K_c e^2 - 2C e \zeta + M \zeta^2 + \frac{2}{3} \mu [(e_y - e_z)^2 + (e_z - e_x)^2 + (e_x - e_y)^2] + \mu (\gamma_x^2 + \gamma_y^2 + \gamma_z^2). \quad (3.16)$$

By putting $e = \zeta = 0$, we see that we must have

$$\mu \geq 0. \quad (3.17)$$

By putting

$$\gamma_x = \gamma_y = \gamma_z = 0$$

and

$$e_x = e_y = e_z,$$

we are left with

$$2W = K_c e^2 - 2C e \zeta + M \zeta^2. \quad (3.18)$$

This expression is never negative if

$$K_c \geq 0, \quad M \geq 0, \quad K_c M - C^2 \geq 0. \quad (3.19)$$

Thus, the four conditions (3.17) and (3.19) are both necessary and sufficient. We note that

$$K_c M - C^2 = (K_c - \alpha^2 M) M. \quad (3.20)$$

From Eqs. (3.14), (3.15), and (3.4), we derive

$$K_c - K = \lambda_c - \lambda = \alpha^2 M. \quad (3.21)$$

Hence,

$$K_c M - C^2 = K M. \quad (3.22)$$

Therefore, the necessary and sufficient conditions for W to be nonnegative are that

$$\mu \geq 0, \quad M \geq 0, \quad K = \lambda + \frac{2}{3} \mu \geq 0. \quad (3.23)$$

The significance and methods of measurement of the various coefficients have been discussed in detail earlier.^{1,7}

It was shown in reference 7 that α can be measured in two different ways, and that its value lies within the range $f \leq \alpha \leq 1$. We shall briefly recall some of the more important equations relating to the coefficients. The jacketed compressibility is

$$\kappa = 1/K = 1/(\lambda + \frac{2}{3} \mu). \quad (3.24)$$

The unjacketed compressibility δ is

$$\delta = (1 - \alpha) \kappa. \quad (3.25)$$

With a measurable "coefficient of fluid content" γ

TABLE I. Equivalence of symbols.

Reference 1 (1941)	Later publications
Q	M
$\frac{\alpha}{2G\nu/(1-\nu)}$	$\frac{\alpha}{\lambda \text{ and } S}$
G	$\mu \text{ and } N$
θ	ζ
σ	$p_f \text{ and } -(\sigma/f)$

for the unjacketed test, the coefficient M is given by

$$M = 1/[\gamma + \delta - (\delta^2/\kappa)]. \quad (3.26)$$

For some types of materials, the coefficient γ can be expressed in terms of the porosity f and the fluid compressibility c as

$$\gamma = f(c - \delta). \quad (3.27)$$

These coefficients for some sandstones have been measured by Fatt.^{19,20}

Different coefficients have been used by this writer in the past. As an aid to the reader, Table I shows the equivalence of various notations used in the earlier work.

In some of the publications,^{3,5,6} stress-strain relations for uniform porosity were also written in the form

$$\begin{aligned} \sigma_{ij} &= 2N e_{ij} + \delta_{ij} (Ae + Q\epsilon) \\ \sigma &= Qe + R\epsilon, \end{aligned} \quad (3.28)$$

with

$$\epsilon = \text{div } \mathbf{U} = e - (1/f)\zeta, \quad f\sigma = -p_f. \quad (3.29)$$

These equations can be written

$$\begin{aligned} \tau_{ij} = \sigma_{ij} + \delta_{ij} \sigma &= 2N e_{ij} + \delta_{ij} [(A + 2Q + R)e \\ &\quad - (Q + R)(\zeta/f)] \\ p_f &= -(Q + R)(e/f)e + (R/f^2)\zeta. \end{aligned} \quad (3.30)$$

Comparing these equations with Eq. (3.7), we derive relations between the two sets of coefficients:

$$\begin{aligned} N &= \mu, \quad A = \lambda + M(\alpha - f)^2, \\ Q &= f(\alpha - f)M, \quad R = f^2 M. \end{aligned} \quad (3.31)$$

The coefficient H of Eq. (3.5) can be written

$$H = A + 2N + 2Q + R = \lambda_c + 2\mu, \quad (3.32)$$

which is the same as that used in the theory of acoustic propagation.⁵

For a material which is isotropic about the z axis, i.e., *transverse isotropic*, the stress-strain relations become

$$\begin{aligned} \tau_{xx} &= 2B_1 e_x + B_2(e_x + e_y) + B_3 e_z + B_6 \zeta \\ \tau_{yy} &= 2B_1 e_y + B_2(e_x + e_y) + B_3 e_z + B_6 \zeta \\ \tau_{zz} &= B_4 e_z + B_3(e_x + e_y) + B_7 \zeta \\ \tau_{yz} &= B_5 \gamma_z, \quad \tau_{zx} = B_5 \gamma_x, \quad \tau_{xy} = B_1 \gamma_z \\ p_f &= B_6(e_x + e_y) + B_7 e_z + B_8 \zeta. \end{aligned} \quad (3.33)$$

¹⁹ I. Fatt, Bull. Am. Assoc. Petrol. Geologists 42, 1924-1957 (1958).

²⁰ I. Fatt, J. Appl. Mech. 26, 296-297 (1959).

The eight elastic coefficients are chosen in order to constitute a symmetric matrix and satisfy at the same time the geometric symmetry.

For orthotropic symmetry, i.e., when the three coordinate planes are planes of elastic symmetry, the stress-strain relations become

$$\begin{aligned} \tau_{xx} &= A_{11}e_x + A_{12}e_y + A_{13}e_z + M_1\zeta \\ \tau_{yy} &= A_{12}e_x + A_{22}e_y + A_{23}e_z + M_2\zeta \\ \tau_{zz} &= A_{13}e_x + A_{23}e_y + A_{33}e_z + M_3\zeta \\ \tau_{yz} &= A_{44}\gamma_z, \quad \tau_{zx} = A_{55}\gamma_y, \quad \tau_{xy} = A_{66}\gamma_z \\ p_f &= M_1e_x + M_2e_y + M_3e_z + M\zeta. \end{aligned} \tag{3.34}$$

These equations contain thirteen elastic coefficients. Finally, in the most general case of anisotropy, the stress-strain relations are written

$$\begin{bmatrix} \tau_{xx} \\ \tau_{yy} \\ \tau_{zz} \\ \tau_{yz} \\ \tau_{zx} \\ \tau_{xy} \\ p_f \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} & A_{15} & A_{16} & M_1 \\ & A_{22} & A_{23} & A_{24} & A_{25} & A_{26} & M_2 \\ & & A_{33} & A_{34} & A_{35} & A_{36} & M_3 \\ & & & A_{44} & A_{45} & A_{46} & M_4 \\ & & & & A_{55} & A_{56} & M_5 \\ & & & & & A_{66} & M_6 \\ & & & & & & M \end{bmatrix} \begin{bmatrix} e_x \\ e_y \\ e_z \\ \gamma_z \\ \gamma_y \\ \gamma_x \\ \zeta \end{bmatrix} \tag{3.35}$$

symmetry

The matrix of twenty-eight coefficients A_{ij} , M_i , and M is symmetric about the main diagonal. The stress-strain relations (3.35) can be written in abbreviated notation by the introduction of quadruple indices for the coefficients A_{ij} and double indices for the coefficients M_i . We put

$$A_{ij\mu\nu} = A_{j\mu i\nu} = A_{\mu\nu ij}, \quad M_{ij} = M_{ji}. \tag{3.36}$$

With these coefficients, Eq. (3.35) takes the form

$$\begin{aligned} \tau_{ij} &= \sum_{\mu\nu} A_{ij\mu\nu} e_{\mu\nu} + M_{ij}\zeta \\ p_f &= \sum_{ij} M_{ij} e_{ij} + M\zeta. \end{aligned} \tag{3.37}$$

The various stress-strain relations discussed above for anisotropic media were derived earlier in equivalent form in the context of uniform porosity.³

4. DARCY'S LAW AND ITS THERMODYNAMIC FOUNDATION

In the preceding sections, we have dealt with equilibrium phenomena or thermostatics. We shall now examine an entirely different aspect of the problem—the mechanics of flow through porous media. This brings into play the thermodynamics of irreversible processes and the Onsager relations. The rate of flow of the fluid is defined by the time derivative of the volume flow vector:

$$\partial \mathbf{w} / \partial t = (\dot{w}_x, \dot{w}_y, \dot{w}_z). \tag{4.1}$$

It is possible to write a dissipation function which is

proportional to the rate of entropy production. Per unit volume of bulk material, this dissipation function is

$$D = \frac{1}{2} T_r \text{ (rate of entropy production)}, \tag{4.2}$$

where T_r = absolute temperature of the undisturbed system. It can be written as a quadratic form with the rate of volume flow as variables:

$$2D = \eta (r_{11}\dot{w}_x^2 + r_{22}\dot{w}_y^2 + r_{33}\dot{w}_z^2 + 2r_{23}\dot{w}_y\dot{w}_z + 2r_{31}\dot{w}_z\dot{w}_x + 2r_{12}\dot{w}_x\dot{w}_y). \tag{4.3}$$

The viscosity of the fluid is denoted by η .²¹

It is important to emphasize here that in applying the thermodynamics of irreversible processes, we start with a thermodynamic system under conditions of equilibrium in the initial state. The initial state in this case is chosen to be one in which no pressure gradients or gravity forces are acting on the fluid in the pores. The system is then perturbed by the application of a disequilibrium force. This force must be expressed in a form which is conjugate to the volume flow coordinate \mathbf{w} . The components of this force are

$$X_i = -(\partial p_f / \partial x_i) + \rho_f g_i, \tag{4.4}$$

where ρ_f is the mass density of the fluid, and g_i the components of the gravity acceleration. In vector form, we can write

$$\mathbf{X} = -\text{grad} p_f - \rho_f \text{grad} G. \tag{4.5}$$

The quantity G is the gravitational potential per unit mass. Because we are dealing here with a *linear theory*, we have assumed that the application of gravitational forces introduces density changes in the fluid which are "small of the first order." Therefore, the fluid density ρ_f in Eq. (4.4) can be put equal to the mass density in the initial state. Under these circumstances, the *principle of superposition* is applicable. Hubbert's analysis²² of Darcy's law introduces a total fluid potential which, under the assumption just stated, can be written

$$\phi = (p_f / \rho_f) + G. \tag{4.6}$$

With this potential, we can write Eq. (4.4) in the form

$$\mathbf{X} = -\rho_f \text{grad} \phi. \tag{4.7}$$

Applying a general procedure used earlier by this writer, we find that Onsager's principle in this case is equivalent to the relation

$$(\partial D / \partial \dot{w}_x, \partial D / \partial \dot{w}_y, \partial D / \partial \dot{w}_z) = \mathbf{X} = -\rho_f \text{grad} \phi. \tag{4.8}$$

In matrix form,

$$-(\rho_f / \eta) \begin{bmatrix} \partial \phi / \partial x \\ \partial \phi / \partial y \\ \partial \phi / \partial z \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{31} \\ r_{12} & r_{22} & r_{23} \\ r_{31} & r_{23} & r_{33} \end{bmatrix} \begin{bmatrix} \dot{w}_x \\ \dot{w}_y \\ \dot{w}_z \end{bmatrix}. \tag{4.9}$$

²¹ Attention is called to the general character of the dissipation function [Eq. (4.3)]. It is valid for slip flow or other more complex interfacial effects and does not require that η be introduced explicitly.

²² M. King Hubbert, J. Geol. 48, 785-944 (1940).

This form of the generalized Darcy's law has previously been derived⁴ for the particular case $f = \text{const}$.

The symmetric matrix

$$[r_{ij}] = \begin{bmatrix} r_{11} & r_{12} & r_{31} \\ r_{12} & r_{22} & r_{23} \\ r_{31} & r_{23} & r_{33} \end{bmatrix} \quad (4.10)$$

represents a flow resistivity, whereas its inverse

$$[r_{ij}]^{-1} = [k_{ij}] = \begin{bmatrix} k_{11} & k_{12} & k_{31} \\ k_{12} & k_{22} & k_{23} \\ k_{31} & k_{23} & k_{33} \end{bmatrix}, \quad (4.11)$$

also symmetric, represents a "permeability matrix." Introducing the latter, we can write Eq. (4.9) as

$$\begin{bmatrix} \dot{w}_x \\ \dot{w}_y \\ \dot{w}_z \end{bmatrix} = -(\rho_f/\eta) \begin{bmatrix} k_{11} & k_{12} & k_{31} \\ k_{12} & k_{22} & k_{23} \\ k_{31} & k_{23} & k_{33} \end{bmatrix} \begin{bmatrix} \partial\phi/\partial x \\ \partial\phi/\partial y \\ \partial\phi/\partial z \end{bmatrix}. \quad (4.12)$$

For the particular case of an isotropic medium,

$$\begin{aligned} k_{11} &= k_{22} = k_{33} = k \\ k_{12} &= k_{31} = k_{23} = 0, \end{aligned} \quad (4.13)$$

and Eq. (4.12) becomes

$$\partial w/\partial t = -k(\rho_f/\eta) \text{grad}\phi, \quad (4.14)$$

which is Darcy's law in the form expressed by Hubbert.²³ It can also be written

$$\partial w/\partial t = (k/\eta) \text{grad}p_f - (k/\eta)\rho_f \text{grad}G. \quad (4.15)$$

The quantity k is the usual "coefficient of permeability" of the medium. Clearly, the symmetric matrix $[k_{ij}]$ represents a generalization of this coefficient.

For the case of isotropy, the dissipation function is given by

$$2D = (\eta/k)(\dot{w}_x^2 + \dot{w}_y^2 + \dot{w}_z^2). \quad (4.16)$$

It is of interest to consider the possible relationship between the permeability and the deformation of the porous medium. This aspect has been discussed earlier⁴ in connection with viscoelastic properties and in the less general context of homogeneous porosity.

The porosity matrix represents a tensor analogous to a stress. If we start with a medium initially isotropic, the permeability after deformation will be

$$k_{ij} = k\delta_{ij} + \Delta k_{ij}. \quad (4.17)$$

The permeability increments to the first order will be related to the strain components by relations analogous to the stress-strain relations as in an isotropic medium:

$$\begin{aligned} \Delta k_{11} &= 2\beta_1 e_x + \beta_2 e \\ \Delta k_{22} &= 2\beta_1 e_y + \beta_2 e \\ \Delta k_{33} &= 2\beta_1 e_z + \beta_2 e \end{aligned} \quad (4.18)$$

$$\Delta k_{23} = \beta_1 \gamma_{yz}, \quad \Delta k_{31} = \beta_1 \gamma_{yz}, \quad \Delta k_{12} = \beta_1 \gamma_{xz}.$$

²³ Equation (80) of Hubbert's paper in reference 22.

We need two constants β_1 and β_2 to define the first-order change of permeability in an initially isotropic medium. The above equation need not be restricted to the first order if β_1 and β_2 are not considered as constants but as functions of the volume change e :

$$\beta_1 = \beta_1(e), \quad \beta_2 = \beta_2(e). \quad (4.19)$$

In this way, it seems possible to express a considerable variety of porosity dependence by a close analysis of the change of geometry of the pores.

An equation such as (4.18) for anisotropic media can easily be written, as previously shown.⁴ In the case of transverse isotropy, for instance, such relations involve six coefficients.

5. THERMODYNAMICS OF VISCOELASTIC BEHAVIOR—THE CORRESPONDENCE PRINCIPLE

General stress-strain relations for isotropic and anisotropic viscoelastic media have previously been derived by this writer from the thermodynamics of irreversible processes.²⁴ They were expressed in a form which brings out the complete isomorphism between theories of elasticity and viscoelasticity. It follows from this property that equations valid for the linear theory of elasticity (with linear boundary conditions and time independent constraints) can immediately be extended to viscoelasticity by the substitution of time operators for the elastic coefficients. In order to emphasize the generality of this isomorphism, we have referred to it as the *correspondence principle* and developed in more detail its applications to various areas, such as the theory of plates, wave propagation, and dynamics.^{25,26} Certain general theorems were also derived by combining the correspondence principle and variational methods in operational form.

The validity of the correspondence principle for viscoelastic porous media is self-evident in this writer's formulation of viscoelasticity for porous media.⁴ This formulation was based on the thermodynamics of irreversible processes.

If the system is initially in thermodynamic equilibrium, we consider that the strain components, the stresses, and the change in fluid pressure represent small deviations from that state of equilibrium. In a great many cases, such deviations will be governed by linear laws, and the Onsager reciprocity relations will be valid.

Several important points should be stressed regarding the type of phenomena considered in this departure from equilibrium in order to clarify what is meant here by "viscoelasticity."

²⁴ M. A. Biot, *J. Appl. Phys.* **25**, 1385-1391 (1954).

²⁵ M. A. Biot, in *Proceedings Fourth Mid-Western Conference on Solid Mechanics*, Purdue University, 1955, pp. 94-108.

²⁶ M. A. Biot, *Deformation and Flow of Solids*, IUTAM Colloquium, Madrid, 1955, edited by R. Grammel (Springer-Verlag, Berlin, 1956), pp. 251-263.

We are considering only the local effect of the fluid pressure. The flow induced by the pressure gradient is treated as a different phenomenon. This point was discussed in Sec. 4 in connection with the thermodynamic derivation of Darcy's law and does not require further elaboration at present.

A second point is that, in formulating the viscoelastic properties, we have assumed the hidden inertia forces to be negligible. This excludes, for example, the inertia effects due to the motion of small particles representing hidden coordinates. This assumption, however, is not essential, and we shall indicate below what modifications must be introduced when such hidden inertia forces are taken into account.

A third point to be stressed refers to the extreme generality of the phenomena which are encompassed by the term viscoelasticity. Its meaning in the present context far exceeds the narrow concepts of the purely mechanical models usually associated with the word. The derivation of the stress-strain relations from thermodynamics is purely phenomenological. The two-phase fluid-solid aggregate is considered as a single thermodynamic system. This is in contrast with the procedure of dealing with the "dry" solid and the fluid as two separate entities, each with its own properties. Such artificial separation is incorrect because of the important role played by the surface forces at the fluid-solid interface in the pores. In the case of gels, the interfacial surface tension contributes significantly to the over-all rigidity, as pointed out many years ago by this writer.¹ Because of the large area of contact of fluid and solid, such interfacial effects should play an important role in porous media. In general, they are the result of certain equilibrium configurations of ions and molecules which involve electrical fields and physical-chemical interactions. Such configurations are represented by "hidden coordinates." When disturbed from equilibrium, they evolve toward a new state with a certain time delay or relaxation time. There may be a finite number or a continuous distribution of such relaxation times, as represented by a relaxation spectrum. Such effects are included in the thermodynamic treatment of viscoelastic behavior developed by the writer.⁴ Other effects involved here are exemplified by the behavior of a crystal in equilibrium with its solution. Under stress, this equilibrium is disturbed. Some areas of the crystal enter into solution, and precipitation occurs on others. The rate of deformations will depend not only on the stress but also on the rate of diffusion in the solvent, giving rise to a relaxation spectrum. Another type of phenomenon included here is the thermoelastic relaxation. This is due to differential temperatures arising in the solid and the fluid in the pores when stress is applied. Because of the thermal conductivity, such temperature differences tend to even out, but with a certain time lag. This gives rise to a thermoelastic relaxation spectrum. Attention is called to the difference between this type of thermo-

elastic dissipation and that occurring in a homogeneous solid. The latter is also included in the general thermodynamic theory and depends essentially on the strain gradient. The formulation of this case is quite different and was developed earlier.¹³ The viscoelastic effects might also result solely from certain physical properties of the fluid itself independent of any interaction with the solid. Such is the case, for instance, for the propagation of sound in water containing solution of certain salts. Equilibrium concentrations of the various molecular species in solution are sometimes sensitive to fluid pressure with an associated time lag and relaxation effect. These are but a few examples illustrating the enormous range and variety of phenomena included in the present theory.

The viscoelastic and relaxation properties which we have just discussed are obtained by replacing the elastic coefficients by operators. Applying this correspondence principle to Eq. (3.5) for the isotropic medium, we derive

$$\begin{aligned} \tau_{xx} &= H^*e - 2\mu^*(e_y + e_z) - C^*\zeta \\ \tau_{yy} &= H^*e - 2\mu^*(e_z + e_x) - C^*\zeta \\ \tau_{zz} &= H^*e - 2\mu^*(e_x + e_y) - C^*\zeta \\ \tau_{yz} &= \mu^*\gamma_x, \quad \tau_{zx} = \mu^*\gamma_y, \quad \tau_{xy} = \mu^*\gamma_z \\ p_f &= -C^*e + M^*\zeta. \end{aligned} \tag{5.1}$$

The operators are of the form

$$\begin{aligned} H^* &= p \int_0^\infty \frac{H(r)}{p+r} dr + H + pH' \\ \mu^* &= p \int_0^\infty \frac{\mu(r)}{p+r} dr + \mu + p\mu' \\ C^* &= p \int_0^\infty \frac{C(r)}{p+r} dr + C + pC' \\ M^* &= p \int_0^\infty \frac{M(r)}{p+r} dr + M + pM'. \end{aligned} \tag{5.2}$$

The operator p is the time differential

$$p = d/dt. \tag{5.3}$$

The operational Eq. (5.1) can be considered as relating Laplace transforms of stress and strain variable. For instance, if these Laplace transforms are

$$\begin{aligned} \mathcal{L}\tau_{xx} &= \int_0^\infty e^{-pt} \tau_{xx}(t) dt \\ \mathcal{L}\zeta &= \int_0^\infty e^{-p't} \zeta(t) dt, \end{aligned} \tag{5.4}$$

etc., the first Eq. (5.1) can be written

$$\mathcal{L}\tau_{xx} = H^*\mathcal{L}e - 2\mu^*\mathcal{L}(e_y + e_z) + C^*\mathcal{L}\zeta \tag{5.5}$$

or $\tau_{zz} = H^*e - 2\mu^*(e_y + e_z) + C^*\zeta$, (5.6)

by simply dropping the symbol \mathcal{L} .

If the variables are harmonic functions of time, we can write τ_{zz} and $\tau_{xx}e^{i\omega t}$, $e_x = e_x e^{i\omega t}$, etc., where τ_{zz} , e_x , etc., are complex amplitudes. Equation (5.1) then represents the relations between the complex amplitudes, when we put

$$p = i\omega \tag{5.7}$$

in the operators. This shows that putting $p=0$ yields stress-strain relations for very slow deformations, whereas $p = \infty$ corresponds to very fast deformations.

It is of interest to point out a difference between the case of homogeneous and porous media with isotropic symmetry. As we have pointed out,²⁷ the use of the correspondence principle for homogeneous isotropic media is a consequence of the geometric symmetry. On the other hand, for isotropic porous media, the symmetry of the operational matrix of Eq. (5.1) is a consequence of Onsager's relations. Hence, the correspondence principle for isotropic porous media invokes the laws of the thermodynamics of irreversible processes. The variable r in the operators represents the relaxation constants of the hidden degrees of freedom. We have shown from thermodynamics that they are real and positive.^{1,4} This, however, assumes that inertia forces are not significant in the hidden degrees of freedom. For extreme ranges of frequency, or in some exceptional cases, this assumption may not be justified. For example, a fluid may contain small air bubbles, and resonance may occur with their natural frequency of oscillation in the ultrasonic range. In such a case, the hidden degrees of freedom represented by these air bubbles give rise to complex conjugate values for r . For anisotropic media, the stress-strain relations are formally identical with those discussed in Sec. 3 for the elastic case, except for the replacement of the elastic coefficients by the operators. This was previously discussed in more detail.⁴ By the correspondence principle, the stress-strain relations (3.37) become

$$\tau_{ij} = \sum^{\mu\nu} A_{ij}^{*\mu\nu} e_{\mu\nu} + M_{ij}^* \zeta \tag{5.8}$$

$$p_j = \sum^{ij} M_{ij}^* e_{ij} + M^* \zeta.$$

The operators $A_{ij}^{*\mu\nu}$, M_{ij}^* , and M^* are of the same form as Eq. (5.2), and they are governed by the same symmetry relations (3.36) as the elastic case.

6. PHYSICAL SIGNIFICANCE OF THE OPERATORS

The physical significance of the operators can be illustrated by some simple examples which bring out the flexibility and generality of the operational represen-

²⁷ M. A. Biot, *Proceedings of the Third U. S. National Congress on Applied Mechanics* (American Society of Mechanical Engineers, New York, 1958), pp. 1-18.

tation. We shall consider, for example, the sixth of Eq. (5.1)

$$\tau_{xy} = \mu^* \gamma_z, \tag{6.1}$$

which represents a response to pure shear. This relation does not involve the pore pressure. It would govern, for example, the relation between torque and twist in a material such as clay. We assume the operator to be

$$\mu^* = \eta r p / (p + r). \tag{6.2}$$

In this case, we can write Eq. (6.1) as

$$\gamma_z = (1/\eta r + 1/\eta p) \tau_{xy}. \tag{6.3}$$

A constant stress of unit value applied at $t=0$ is represented by

$$\tau_{xy} = \mathbf{1}(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0. \end{cases} \tag{6.4}$$

By the rules of the operational calculus, the corresponding strain as a function of time is

$$\gamma_z = (1/\eta r + t/\eta) \mathbf{1}(t). \tag{6.5}$$

The quantity ηr is equivalent to an elastic shear modulus, whereas η represents a viscosity. The time dependence of γ_z is represented in Fig. 1. The response is that of a so-called Maxwell material represented by a mechanical model of a spring and dashpot in series (Fig. 2).

Another type of behavior is illustrated by the operator

$$\mu^* = a p^s, \tag{6.6}$$

with $0 < s < 1$. We derive

$$\gamma_z = (1/a p^s) \tau_{xy}, \tag{6.7}$$

and for a constant stress (6.3) applied at $t=0$,

$$\gamma_z = [t^s/a\Gamma(1+s)] \mathbf{1}(t). \tag{6.8}$$

The typical time dependence is shown in Fig. 3 (curve a). It is easily shown that the operator p^s is a particular case of the more general expression (5.2). From the

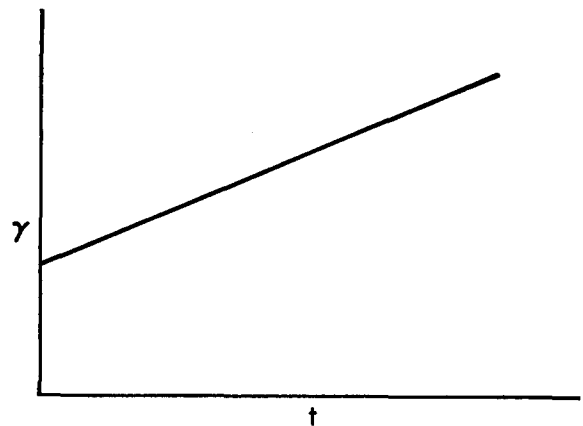


FIG. 1. Creep law represented by Eq. (6.5).

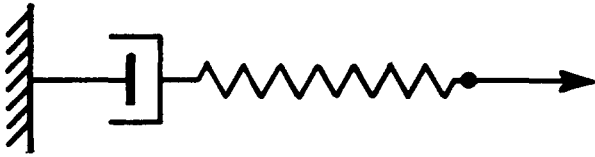


Fig. 2. Dashpot and spring in series, representing the operator (6.2) (Maxwell element).

known definite integral²⁸

$$\int_0^\infty \frac{y^{s-1}}{1+y} dy = \frac{\pi}{\sin s\pi}, \quad (6.9)$$

we derive

$$\mu^* = a p^s = a \frac{\sin s\pi}{\pi} \int_0^\infty \frac{p}{p+r} r^{s-1} dr. \quad (6.10)$$

This corresponds to a spectral distribution²⁹

$$\mu(r) = a \frac{\sin s\pi}{\pi} r^{s-1}. \quad (6.11)$$

The sum of the Eqs. (6.5) and (6.8) yields a time-dependent creep law of a very general nature which covers both the short-range fast primary creep and the long-range steady-state secondary flow (Fig. 3, curve b).

We shall now turn our attention to the creep laws which involve the pore pressure. We consider the case of isotropic stresses:

$$\tau = \tau_{xx} = \tau_{yy} = \tau_{zz}. \quad (6.12)$$

We can then write

$$\begin{aligned} \tau &= (H^* - 4\mu^*/3)e - C^*\zeta \\ p_f &= -C^*e + M^*\zeta. \end{aligned} \quad (6.13)$$

Under these conditions, we can perform a number of "thought experiments." For instance, we assume that a stress $\tau = -p_f$ and a fluid pressure p_f are applied suddenly at time zero, and we evaluate the values of e and ζ as a function of time. This corresponds to the so-called unjacketed test previously discussed, when a uniform hydrostatic pressure p_f is applied throughout the solid matrix and the fluid in the pores. In the present case of viscoelasticity, it is a thought experiment because the fluid pressure is assumed to appear instantaneously throughout the body and thereby to eliminate the additional time lag arising from the seepage of the fluid flowing through the boundary. In principle, the fluid must be thought of as being fed directly or generated inside the pores. Another possibility is to consider a test specimen which is small enough so that the time lag due to seepage is negligible relative to the time constants of relations (6.13); whereas, on the other hand, its size is still sufficient in relation to the pores for the statistical behavior to be valid.

²⁸ Ch. J. de la Vallée Poussin, *Cours d'Analyse Infinitesimale* (Gauthiers Villars, Paris, 1925), Vol. II, p. 75.

²⁹ As pointed out in an earlier paper,²⁷ this is a particular case of the problem represented by the integral equation

$$\int_0^\infty \frac{p}{p+r} f(r) dr = F(p).$$

Its solution was given by Fuoss and Kirkwood³⁰ and further discussed by Gross.³¹ The solution is

$$f(r) = (1/2\pi i) \lim_{\epsilon \rightarrow 0} [F(r+i\epsilon) - F(r-i\epsilon)].$$

³⁰ R. M. Fuoss and J. G. Kirkwood, *J. Am. Chem. Soc.* **63**, 385-394 (1941).

³¹ B. Gross, *Mathematical Structure of the Theories of Viscoelasticities* (Hermann & Cie, Paris, 1953).

neously throughout the body and thereby to eliminate the additional time lag arising from the seepage of the fluid flowing through the boundary. In principle, the fluid must be thought of as being fed directly or generated inside the pores. Another possibility is to consider a test specimen which is small enough so that the time lag due to seepage is negligible relative to the time constants of relations (6.13); whereas, on the other hand, its size is still sufficient in relation to the pores for the statistical behavior to be valid.

This unjacketed test experiment was discussed earlier for the case of an elastic matrix,⁷ and the following relations were considered:

$$\begin{aligned} e &= -\delta p_f \\ \zeta &= \gamma p_f, \end{aligned} \quad (6.14)$$

where δ is the unjacketed compressibility, and γ is a coefficient of fluid content. For a viscoelastic matrix, the coefficients δ and γ must be replaced by operators δ^* and γ^* :

$$\begin{aligned} e &= -\delta^* p_f \\ \zeta &= \gamma^* p_f. \end{aligned} \quad (6.15)$$

In some problems, it may be justified to neglect the time lags of the operators δ^* and γ^* and to assume that the relations (6.14) for the elastic matrix are applicable.

We have also pointed out that if the matrix material is constituted by elements of an elastic isotropic solid, the coefficient γ can be expressed by relation (3.27), i.e.,

$$\gamma = f(c - \delta), \quad (6.16)$$

which introduces the fluid compressibility c and the porosity f . In this case, δ represents the compressibility of the matrix material itself. If we desire to take into account the time lags in the compressibility of both fluid and solid, we write

$$\gamma^* = f(c^* - \delta^*). \quad (6.17)$$

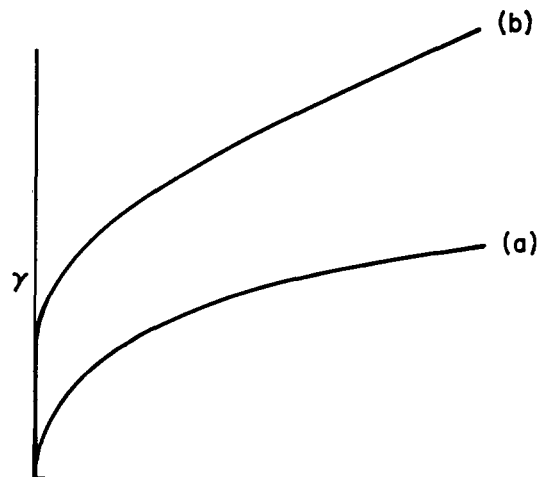


Fig. 3. (a) Creep law represented by Eq. (6.8); (b) Creep law represented by the sum of Eqs. (6.5) and (6.8).

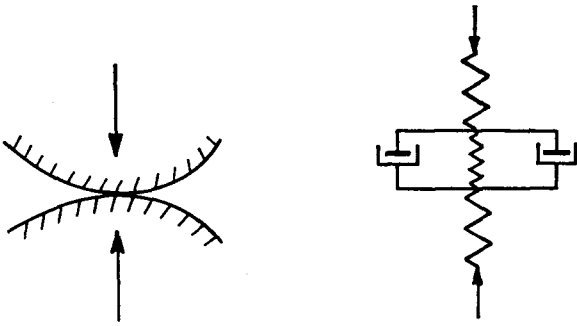


FIG. 4. Spring dashpot model representing the interaction of elasticity of the solid and fluid viscosity around areas of grain contact.

As noted above, even in the fluid alone, time lags may occur between pressure and volume. A possible cause of such effects is the relaxation of equilibrium concentrations of various chemical species in solution in the fluid. This phenomenon, sometimes referred to as *bulk viscosity*, will be reflected in the time constant contained in the operator c^* . The effect of air bubbles on damping in the fluid can also be included in the operator c^* (see Sec. 5). We should also point out an earlier remark that the resonance of such bubbles at a certain frequency can be taken into account by including the inertia forces of the hidden coordinates. In this case, the correspondence is still valid, but the form of the operators is different. Similar considerations apply to the inclusion of bulk viscosity and relaxation in the solid matrix material and can be expressed by a suitable operator.

Another thought experiment which can be considered corresponds to the jacketed compressibility. In this case, the pore pressure is assumed to be zero, and Eq. (6.13) is written

$$\begin{aligned} \tau &= (H^* - 4\mu^*/3)e - C^*\zeta \\ 0 &= -C^*e + M^*\zeta. \end{aligned} \tag{6.18}$$

Eliminating ζ , we find

$$\tau = K^*e, \tag{6.19}$$

with

$$K^* = H^* - 4\mu^*/3 - (C^{*2}/M^*). \tag{6.20}$$

The jacketed compressibility operator is

$$\kappa^* = 1/K^*. \tag{6.21}$$

The stress τ is applied to a jacketed specimen, while the pore pressure is maintained at zero throughout by some device which connects the pores directly with an outside region of zero pressure.

By the correspondence principle, it is possible to express the operators in terms of the coefficients γ^* , δ^* , and κ^* with the formulas which are valid for the elastic case.

In particular, it will often be justified to approximate the operators γ^* and δ^* by elastic coefficients:

$$\gamma^* = \gamma, \quad \delta^* = \delta. \tag{6.22}$$

From Eqs. (3.6), (3.25), and (3.26),

$$\alpha^* = 1 - \delta K^*, \tag{6.23}$$

$$M^* = 1/(\gamma + \delta - \delta^2 K^*), \tag{6.24}$$

$$C^* = \alpha^* M^* = (1 - \delta K^*)/(\gamma + \delta - \delta^2 K^*). \tag{6.25}$$

Also from Eqs. (3.6) and (3.15),

$$\lambda^* = K^* - 2\mu^*/3$$

$$H^* = \lambda^* + 2\mu^* + \alpha^{*2} M^*. \tag{6.26}$$

Hence,

$$H^* = 4\mu^*/3 + (\kappa^* + \gamma - \delta)/[\kappa^*(\gamma + \delta) - \delta^2]. \tag{6.27}$$

The operators in this case can be expressed in terms of only two operators μ^* and κ^* . They can be reduced to the spectral form [Eq. (5.2)] by application of the Fuoss-Kirkwood method or by expansion in partial fractions.

We have already pointed out that the viscoelasticity operators contain properties which are representative of solid viscosity, but which depend also on fluid-solid interaction. In such cases, the properties of the solid matrix cannot be considered separately from those of the fluid. One such type of interaction which is purely mechanical in nature is illustrated by Fig. 4. When two elastic bodies are in contact and are surrounded by a viscous fluid, a force applied in a direction normal to the area of contact will tend to squeeze the flow away from this area. Because of fluid viscosity, the fluid will not move away instantaneously. A time delay, which is exemplified by the equivalent spring dashpot model of Fig. 4, will be involved. This model corresponds to an operator of the form

$$\kappa^* = \kappa_0 + \kappa_1 r / (p + r). \tag{6.28}$$

Actually, of course, a more accurate evaluation of the interaction of fluid displacement and elasticity would yield a somewhat more complex model or a more sophisticated operator with a spectral distribution. Such effect may be more pronounced for grains containing cracks or separated by narrow gaps (Fig. 5).

We have previously discussed this type of viscoelasticity in connection with the effect of wall sponginess and micropores.⁴ A similar mechanism can also be responsible for the viscoelasticity represented by the operator μ^* for pure shear and can be evaluated in similar fashion.

7. FIELD EQUATIONS FOR DEFORMATION AND STRESS DISTRIBUTION

If we include the gravity force ρg_i (ρ = mass density of bulk material), the total stress field must satisfy the equilibrium equations

$$\sum_j \frac{\partial \tau_{ij}}{\partial x_j} + \rho g_i = 0, \tag{7.1}$$

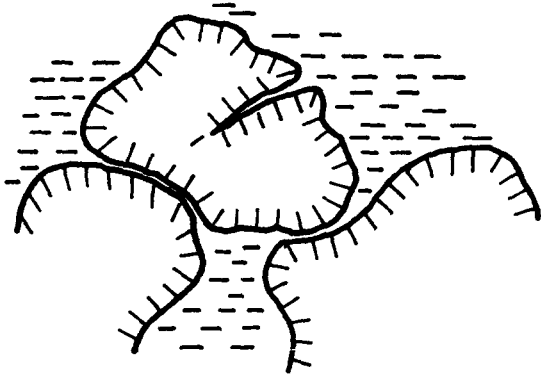


FIG. 5. Example of general grain geometry with viscoelastic behavior of the type idealized in Fig. 4.

and Darcy's law for the isotropic medium is written

$$\partial \mathbf{w} / \partial t = - (k / \eta) \text{grad} p_f + \rho_f \text{grad} G. \quad (7.2)$$

Under the assumption of linearity, as pointed out in Sec. 4, the principle of superposition is applicable. Hence, the general solution that we are seeking is the superposition of a particular solution due to gravity forces alone and of another due to other causes. Under these restrictive assumptions, we can neglect the gravity forces in the present theory and thereby isolate that part of the problem dealing with deformations resulting from causes other than gravity.

Equations (7.1) and (7.2) are therefore simplified to

$$\sum \frac{\partial \tau_{ij}}{\partial x_j} = 0 \quad (7.3)$$

and

$$\partial \mathbf{w} / \partial t = - (k / \eta) \text{grad} p_f. \quad (7.4)$$

For an elastic medium with bulk isotropy, the stress-strain relations (3.10) are

$$\begin{aligned} \tau_{ij} &= 2\mu e_{ij} + \delta_{ij}(\lambda_c e - \alpha M \zeta) \\ p_f &= -\alpha M e + M \zeta. \end{aligned} \quad (7.5)$$

In these equations, we have defined ζ and e as

$$\zeta = -\text{div} \mathbf{w} \quad \text{and} \quad e = \text{div} \mathbf{u}, \quad (7.6)$$

where \mathbf{u} is the solid displacement.

Substituting the values of τ_{ij} and p_f [Eq. (7.5)] into the Eqs. (7.3) and (7.4), we derive

$$2 \sum_{ij} \frac{\partial}{\partial x_j} (\mu e_{ij}) + \frac{\partial}{\partial x_i} (\lambda_c e - \alpha M \zeta) = 0 \quad (7.7)$$

$$\partial \mathbf{w} / \partial t = (k / \eta) \text{grad} (\alpha M e - M \zeta) = 0.$$

These are six equations for the six components of the unknown vector fields \mathbf{u} and \mathbf{w} .

We shall examine the particular case where the coefficients μ , λ_c , α , M , and k/η are constants. In this

case, Eqs. (7.7) become

$$\begin{aligned} \mu \nabla^2 \mathbf{u} + (\mu + \lambda_c) \text{grad} e - \alpha M \text{grad} \zeta &= 0 \\ \partial \mathbf{w} / \partial t &= (k / \eta) M \alpha \text{grad} e - (k M / \eta) \text{grad} \zeta. \end{aligned} \quad (7.8)$$

These equations can be written in other equivalent forms previously derived^{1,6} by the application of the divergence operator to the second Eq. (7.8). We derive

$$\begin{aligned} \mu \nabla^2 \mathbf{u} + (\mu + \lambda_c) \text{grad} e - \alpha M \text{grad} \zeta &= 0 \\ \partial \zeta / \partial t &= (k M / \eta) \nabla^2 \zeta - (k M \alpha / \eta) \nabla^2 e. \end{aligned} \quad (7.9)$$

Application of the divergence operator to the first Eq. (7.9) yields

$$(2\mu + \lambda_c) \nabla^2 e - \alpha M \nabla^2 \zeta = 0. \quad (7.10)$$

Hence, we can further transform Eqs. (7.9) to

$$\begin{aligned} \mu \nabla^2 \mathbf{u} + (\mu + \lambda_c) \text{grad} e - \alpha M \text{grad} \zeta &= 0 \\ \partial \zeta / \partial t &= (k / \eta) M_c \nabla^2 \zeta, \end{aligned} \quad (7.11)$$

with a constant M_c defined as

$$M_c = M(2\mu + \lambda_c - \alpha^2 M) / (2\mu + \lambda_c) \quad (7.12)$$

or

$$M_c = M(2\mu + \lambda) / (2\mu + \lambda_c). \quad (7.12)$$

General solutions of these equations were also derived and discussed earlier.⁶ We shall derive them hereafter in somewhat simpler form.

A convenient form of the general solution can be obtained with Eq. (7.8). The second equation indicates that \mathbf{w} must be a gradient. We can write without loss of generality

$$\mathbf{w} = \text{grad} \varphi \quad (7.13)$$

and

$$\mathbf{u} = \mathbf{u}_1 - [\alpha M / (2\mu + \lambda_c)] \text{grad} \varphi.$$

Substituting these expressions into Eq. (7.8), we obtain

$$\begin{aligned} \mu \nabla^2 \mathbf{u}_1 + (\mu + \lambda_c) \text{grad} e_1 &= 0 \\ \text{grad} \left(\frac{\partial \varphi}{\partial t} - \frac{k M_c}{\eta} \nabla^2 \varphi - \frac{k M \alpha}{\eta} e_1 \right) &= 0. \end{aligned} \quad (7.14)$$

We have put

$$e_1 = \text{div} \mathbf{u}_1. \quad (7.15)$$

The first of these equations is the classical form of Lamé's equations of the theory of elasticity. The general Papkovitch-Boussinesq solution of these equations is well known:

$$\mathbf{u}_1 = \text{grad} (\psi_0 + \mathbf{r} \cdot \psi_1) - [2(2\mu + \lambda_c) / (\mu + \lambda_c)] \psi_1, \quad (7.16)$$

where the scalar ψ_0 and the vector ψ_1 are general solutions of Laplace's equation

$$\nabla^2 \psi_0 = \nabla^2 \psi_1 = 0. \quad (7.17)$$

The vector \mathbf{r} is

$$\mathbf{r} = (x, y, z). \quad (7.18)$$

Note that we can write

$$e_1 = -[2\mu / (\mu + \lambda_c)] \text{div} \psi_1, \quad \nabla e_1 = 0. \quad (7.19)$$

We consider now the second Eq. (7.14). It implies that

$$\frac{\partial \varphi}{\partial t} - \frac{kM_c}{\eta} \nabla^2 \varphi = \frac{kM\alpha}{\eta} e_1 + C(t), \quad (7.20)$$

where $C(t)$ is a function of time independent of the coordinates. We put

$$\varphi = \psi + \int^t \left[\frac{kM\alpha}{\eta} e_1 + C(t) \right] dt. \quad (7.21)$$

Substituting in Eq. (7.20), we find

$$\partial \psi / \partial t = (k/\eta) M_c \nabla^2 \psi. \quad (7.22)$$

The general solution for w becomes

$$w = \text{grad} \psi - \frac{2k\alpha M\mu}{\eta(\mu + \lambda_c)} \int^t \text{grad} \text{div} \psi_1 dt. \quad (7.23)$$

Applying the divergence operator in this expression, we derive

$$\zeta = -\nabla^2 \psi. \quad (7.24)$$

The solution for u is

$$u = u_1 - \frac{\alpha M}{2\mu + \lambda_c} \text{grad} \psi + \frac{kM^2\alpha^2}{2\mu + \lambda_c} \int^t \text{grad} e_1 dt. \quad (7.25)$$

Because $\nabla^2 e_1 = 0$, the time integral satisfies Laplace's equation. This term can therefore be absorbed in the function ψ_0 contained in the expression for u_1 , Eq. (7.16). Hence, we can write the general solution

$$u = \text{grad}(\psi_0 + r \cdot \psi_1) - 2 \frac{2\mu + \lambda_c}{\mu + \lambda_c} \psi_1 - \frac{\alpha M}{2\mu + \lambda_c} \text{grad} \psi. \quad (7.26)$$

In these expressions, ψ satisfies the diffusion Eq. (7.22), whereas ψ_0 and ψ_1 are solutions of Laplace's equation.

For anisotropic media, the field equations are

$$\begin{aligned} \sum_j \frac{\partial}{\partial x_j} \left(\sum_{\mu\nu} A_{ij\mu\nu} e_{\mu\nu} + M_{ij} \zeta \right) &= 0 \\ - \frac{\partial}{\partial x_i} \left(\sum_j M_{ij} e_{ij} + M \zeta \right) &= \eta \sum_j r_{ij} \frac{\partial w_j}{\partial t}. \end{aligned} \quad (7.27)$$

For materials with viscoelastic properties, we have shown that the field equations are obtained from the correspondence principle.⁴ Equations (7.27) are immediately extended to viscoelastic media by replacing the elastic coefficients by operators. For example, Eq. (7.9) for the isotropic medium with uniform properties becomes

$$\begin{aligned} \mu^* \nabla^2 u + (H^* - \mu^*) \text{grad} e - C^* \text{grad} \zeta &= 0 \\ (\eta/k) (\partial \zeta / \partial t) &= M^* \nabla^2 \zeta - C^* \nabla^2 e. \end{aligned} \quad (7.28)$$

In deriving these equations from relations (7.9), we have taken into account the identities

$$\lambda_c^* + \mu^* = H^* - \mu^* \quad \text{and} \quad \alpha^* M^* = C^*. \quad (7.29)$$

As shown in earlier work,⁴ the diffusion Eq. (7.22) is converted into a generalized operational form

$$p(\eta/k) \zeta = M_c^* \nabla^2 \zeta, \quad (7.30)$$

where the operator is

$$M_c^* = (1/H^*) (H^* M^* - C^{*2}). \quad (7.31)$$

The general solution corresponding to Eqs. (7.23) and (7.26) have also been derived in operational form.⁴

In the notation of the present paper, the general solutions (7.23) and (7.26) for the viscoelastic case assume the operational form

$$w = \text{grad} \psi - \frac{2k}{\eta} \frac{C^* \mu^*}{H^* - \mu^*} \frac{1}{p} \text{grad} \text{div} \psi_1 \quad (7.32)$$

$$u = \text{grad}(\psi_0 + r \cdot \psi_1) - \frac{2H^*}{H^* - \mu^*} \psi_1 - \frac{C^*}{H^*} \text{grad} \psi,$$

and ζ is given by the same Eq. (7.24).

The functions ψ_0 and the vector ψ_1 satisfy Laplace's equation, whereas ψ is a solution of the generalized diffusion equation

$$p(\eta/k) \psi = M_c^* \nabla^2 \psi. \quad (7.33)$$

Note that ψ_0 and ψ_1 are generally functions not only of the coordinates x, y, z , but also of the operator p . This will generally be introduced by the boundary conditions which are also operational relations.

The boundary conditions are easily introduced in the solution of specific problems. Total stresses or pore pressures can be specified at certain boundaries. The condition that a boundary be impervious is introduced by putting the normal component of w at that boundary equal to zero. Also, conditions at the interface of solids of different properties are expressed by requiring that the total stresses, fluid pressure, and solid displacement be continuous at this boundary; whereas for w , the condition of continuity applies only to the normal component. Consolidation problems based on the above results for elastic and viscoelastic media have been treated by this writer^{1,4} and others.³²

8. ACOUSTIC PROPAGATION IN ISOTROPIC AND ANISOTROPIC MEDIA

Equations for acoustic propagation in a porous elastic isotropic solid containing a viscous fluid have been developed by this writer.⁵ They were obtained by adding the inertia terms to the consolidation theory. A detailed discussion was given of the propagation of body waves. These equations in slightly different form

³² G. Paria, *J. Math. and Phys.* **36**, 338-346 (1958); *G. Paria, Bull. Calcutta Math. Soc.* **50**, 71-76, 169-179 (1958).

are discussed below and extended to the anisotropic medium with an elastic matrix.

Their further extension to a medium with viscoelastic and solid dissipation properties is outlined in the next section.

Attention is called to the immediate applicability of the results presented here to acoustic propagation in a *thermoelastic continuum*. When the fluid density ρ_f is put equal to zero, the equations become identical to those of thermoelasticity, and \mathbf{w} plays the role of the entropy displacement vector. This follows from the analogy between thermoelasticity and the properties of porous media derived earlier.¹³

The following six equations in vectorial form⁵ were obtained for the six components of the displacements \mathbf{u} and \mathbf{U} :

$$\begin{aligned}
 N\nabla^2\mathbf{u} + \text{grad}[(A+N)\epsilon + Q\epsilon] \\
 &= -\frac{\partial^2}{\partial t^2}(\rho_{11}\mathbf{u} + \rho_{12}\mathbf{U}) + b\frac{\partial}{\partial t}(\mathbf{u} - \mathbf{U}) \\
 \text{grad}(Q\epsilon + R\epsilon) &= -\frac{\partial^2}{\partial t^2}(\rho_{12}\mathbf{u} + \rho_{22}\mathbf{U}) - b\frac{\partial}{\partial t}(\mathbf{u} - \mathbf{U}).
 \end{aligned}
 \tag{8.1}$$

The density parameters ρ_{11} , ρ_{12} , and ρ_{22} were discussed in the quoted paper.⁵ The coefficient b is

$$b = (\eta/k)f^2, \tag{8.2}$$

where k is the permeability coefficient of Eq. (4.14), η is the fluid viscosity, and f is the porosity. The elastic coefficients N , A , Q , R can be expressed in terms of μ , λ , α , M by the relations (3.31). Methods of measurement were analyzed in reference 7.

Expression (8.2) for b is valid for the low-frequency range, where the flow in the pores is of the Poiseuille type. For higher frequencies, a correction factor is applied to the viscosity, replacing it by ηF , where F is a complex function of the frequency which has been evaluated.⁵ Strictly speaking, a similar correction must apply to the density parameters ρ_{11} , ρ_{12} , and ρ_{22} to take into account the departure of the microvelocity field from Poiseuille flow as the frequency increases.

Equations (8.1) have recently been applied to an analysis of the effect of a discontinuity surface on the propagation.^{33,34}

Using the variable \mathbf{w} instead of \mathbf{U} , we shall discuss here a system of equations which are equivalent to Eq. (8.1) for the case of uniform porosity.

We shall first consider the low-frequency range. In this case, the components of the relative microvelocity field in the pores are determined by the vector \mathbf{w} :

$$\begin{aligned}
 v_x &= a_{11}\dot{w}_x + a_{12}\dot{w}_y + a_{13}\dot{w}_z \\
 v_y &= a_{21}\dot{w}_x + a_{22}\dot{w}_y + a_{23}\dot{w}_z \\
 v_z &= a_{31}\dot{w}_x + a_{32}\dot{w}_y + a_{33}\dot{w}_z.
 \end{aligned}
 \tag{8.3}$$

The notation w_j designates $\partial w_j / \partial t$. The coefficients a_{ij} depend on the coordinates in the pores and the pore geometry.

The kinetic energy of a unit volume of bulk material is given by

$$\begin{aligned}
 T &= \frac{1}{2}\rho_1(\dot{u}_x^2 + \dot{u}_y^2 + \dot{u}_z^2) + \frac{1}{2}\rho_f \int \int \int_{\Omega} [(\dot{u}_x + v_x)^2 \\
 &\quad + (\dot{u}_y + v_y)^2 + (\dot{u}_z + v_z)^2] d\Omega.
 \end{aligned}
 \tag{8.4}$$

In this expression, ρ_f represents the mass density of the fluid, and ρ_1 the mass of solid in the unit volume of bulk material.

The volume integral is extended to the total volume Ω of fluid in the pores. We consider the term

$$\begin{aligned}
 \frac{1}{2}\rho_f \int \int \int_{\Omega} (\dot{u}_x + v_x)^2 d\Omega \\
 &= \frac{1}{2}\rho_f \int \int \int_{\Omega} (\dot{u}_x^2 + 2\dot{u}_x v_x + v_x^2) d\Omega.
 \end{aligned}
 \tag{8.5}$$

We can write

$$\frac{1}{2}\rho_f \int \int \int_{\Omega} \dot{u}_x^2 d\Omega = \frac{1}{2}\rho_2 \dot{u}_x^2, \tag{8.6}$$

where

$$\rho_2 = f\rho_f \tag{8.7}$$

represents the mass of fluid per unit volume. Also,

$$\rho_f \int \int \int_{\Omega} \dot{u}_x v_x d\Omega = \rho_f \dot{u}_x \int \int \int_{\Omega} v_x d\Omega = \rho_f \dot{u}_x \dot{w}_x. \tag{8.8}$$

Hence,

$$\begin{aligned}
 \frac{1}{2}\rho_f \int \int \int_{\Omega} (\dot{u}_x + v_x)^2 d\Omega &= \frac{1}{2}\rho_2 \dot{u}_x^2 + \rho_f \dot{u}_x \dot{w}_x \\
 &\quad + \frac{1}{2}\rho_f \int \int \int_{\Omega} v_x^2 d\Omega.
 \end{aligned}
 \tag{8.9}$$

Expression (8.4) now becomes

$$\begin{aligned}
 T &= \frac{1}{2}\rho(\dot{u}_x^2 + \dot{u}_y^2 + \dot{u}_z^2) + \rho_f(\dot{u}_x \dot{w}_x + \dot{u}_y \dot{w}_y + \dot{u}_z \dot{w}_z) \\
 &\quad + \frac{1}{2}\rho_f \int \int \int_{\Omega} (v_x^2 + v_y^2 + v_z^2) d\Omega,
 \end{aligned}
 \tag{8.10}$$

where

$$\rho = \rho_1 + \rho_2 \tag{8.11}$$

is the total mass of bulk material per unit volume.

³³ J. Geertsma and D. C. Smit, *Geophysics* **26**, 169–181 (1961).

³⁴ H. Deresiewicz, *Bull. Seismol. Soc. Am.* **50**, 599–607 (1960); **51**, 51–59 (1961).

From relations (8.3), we derive

$$\rho_f \int \int \int_{\Omega} (v_x^2 + v_y^2 + v_z^2) d\Omega = \sum^{ij} m_{ij} \dot{w}_i \dot{w}_j, \quad (8.12)$$

with

$$m_{ij} = \rho_f \int \int \int_{\Omega} (\sum^k a_{ki} a_{kj}) d\Omega. \quad (8.13)$$

Note the reciprocal property

$$m_{ij} = m_{ji}. \quad (8.14)$$

For a medium with statistical isotropy of the micro-velocity field, the coefficients m_{ij} reduce to³⁵

$$m_{ij} = m \delta_{ij} \quad (8.15)$$

and

$$T = \frac{1}{2} \rho (\dot{u}_x^2 + \dot{u}_y^2 + \dot{u}_z^2) + \rho_f (\dot{u}_x \dot{w}_x + \dot{u}_y \dot{w}_y + \dot{u}_z \dot{w}_z) + \frac{1}{2} m (\dot{w}_x^2 + \dot{w}_y^2 + \dot{w}_z^2). \quad (8.16)$$

In order to compare with the mass parameters ρ_{11} , ρ_{12} , and ρ_{22} used in previous work,⁵ we express the kinetic energy in terms of the variables \mathbf{u} and \mathbf{U} . Substituting

$$w_i = f(U_i - u_i). \quad (8.17)$$

In the value (8.16) of T , we obtain

$$T = \frac{1}{2} \rho_{11} (\dot{u}_x^2 + \dot{u}_y^2 + \dot{u}_z^2) + \rho_{12} (\dot{u}_x \dot{U}_x + \dot{u}_y \dot{U}_y + \dot{u}_z \dot{U}_z) + \frac{1}{2} \rho_{22} (\dot{U}_x^2 + \dot{U}_y^2 + \dot{U}_z^2), \quad (8.18)$$

with

$$\rho_{11} = \rho - 2\rho_f f + m f^2, \quad \rho_{22} = m f^2, \quad \rho_{12} = \rho_f f - m f^2. \quad (8.19)$$

We put

$$\rho_1 = (1 - f)\rho_s = \rho - \rho_2, \quad \rho_2 = f\rho_f, \quad (8.20)$$

where ρ_s is the density of the solid matrix. The quantities ρ_1 and ρ_2 are the masses of solid and fluid, respectively, per unit volume of bulk material. Also writing

$$\rho_a = m f^2 - \rho_f f = m f^2 - \rho_2, \quad (8.21)$$

we see that the mass parameters become

$$\rho_{11} = \rho_1 + \rho_a, \quad \rho_{22} = \rho_2 + \rho_a, \quad \rho_{12} = -\rho_a. \quad (8.22)$$

This coincides with the relations in the quoted paper.⁵

If we now look at the forces applied to a unit volume of bulk material and consider u_i, w_i as generalized coordinates, we can apply Lagrange's equations. They are written

$$\sum^j \frac{\partial \tau_{ij}}{\partial x_j} = \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{u}_i} \right), \quad -\frac{\partial p_f}{\partial x_i} = \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{w}_i} \right) + \frac{\partial D}{\partial w_i}. \quad (8.23)$$

These are the general dynamical equations when the gravity forces are neglected. The first of these equations could have been derived directly from the linear

momentum, and the second can be interpreted as expressing the dynamics of relative motion of the fluid in a frame of reference moving with the solid. As stated in Sec. 7, this is a legitimate procedure under the assumption that the principle of superposition is applicable. Extension of the theory under less restrictive assumption will be carried out in a later paper.

For the isotropic medium, we substitute expressions (8.16) and (4.16) for T and D and the stress-strain relations (3.7). Equations (8.23) become

$$2 \sum^j \frac{\partial}{\partial x_j} (\mu e_{ij}) + \frac{\partial}{\partial x_i} (\lambda_c e - \alpha M \zeta) = -\frac{\partial^2}{\partial t^2} (\rho u_i + \rho_f w_i) \quad (8.24)$$

$$-\frac{\partial}{\partial x_i} (\alpha M e - M \zeta) = -\frac{\partial^2}{\partial t^2} (\rho_f u_i + m w_i) + \frac{\eta}{k} \frac{\partial w_i}{\partial t}.$$

For constant values of the parameters, these equations can be written

$$\mu \nabla^2 \mathbf{u} + (\mu + \lambda_c) \text{grad} e - \alpha M \text{grad} \zeta = -\frac{\partial^2}{\partial t^2} (\rho \mathbf{u} + \rho_f \mathbf{w}) \quad (8.25)$$

$$\text{grad} (\alpha M e - M \zeta) = -\frac{\partial^2}{\partial t^2} (\rho_f \mathbf{u} + m \mathbf{w}) + \frac{\eta}{k} \frac{\partial \mathbf{w}}{\partial t}.$$

If we multiply the second equation by f and then subtract it from the first, we obtain Eq. (8.1), in which the coefficients and the mass parameters are given by expressions (3.31) and (8.19) derived above. Putting

$$\mathbf{u} = \text{grad} \phi_1, \quad \mathbf{w} = \text{grad} \phi_2, \quad (8.26)$$

and using the constant $H = \lambda_c + 2\mu$, we obtain the equations of propagation of dilatational waves:

$$\begin{aligned} \nabla^2 (H \phi_1 + \alpha M \phi_2) &= (\partial^2 / \partial t^2) (\rho \phi_1 + \rho_f \phi_2) \\ \nabla^2 (\alpha M \phi_1 + M \phi_2) &= (\partial^2 / \partial t^2) (\rho_f \phi_1 + m \phi_2) + (\eta/k) (\partial \phi_2 / \partial t). \end{aligned} \quad (8.27)$$

When the Laplacian operator is applied, these equations can also be written in the form

$$\begin{aligned} \nabla^2 (H e - \alpha M \zeta) &= (\partial^2 / \partial t^2) (\rho e - \rho_f \zeta) \\ \nabla^2 (-\alpha M e + M \zeta) &= (\partial^2 / \partial t^2) (-\rho_f e + m \zeta) + (\eta/k) (\partial \zeta / \partial t). \end{aligned} \quad (8.28)$$

They can also be written with the constant $C = \alpha M$. A condition of "dynamic compatibility" for which a wave propagation is possible without relative motion between fluid and solid has previously been derived.⁵ We can readily obtain it in equivalent and simpler form by putting $\phi_2 = 0$ in Eq. (8.27):

$$(\alpha M / H) = (\rho_f / \rho). \quad (8.29)$$

By putting

$$\mathbf{u} = \text{Curl} \psi_1, \quad \mathbf{w} = \text{Curl} \psi_2, \quad (8.30)$$

we derive the equations of propagation of rotational

³⁵ Equation (8.15) is also valid for cubic symmetry.

waves

$$\begin{aligned} \mu \nabla^2 \psi_1 &= (\partial^2 / \partial t^2)(\rho \psi_1 + \rho_f \psi_2), \\ -(\eta/k)(\partial \psi_2 / \partial t) &= (\partial^2 / \partial t^2)(\rho_f \psi_1 + m \psi_2). \end{aligned} \tag{8.31}$$

The propagation of the dilatational and rotational waves has also been previously analyzed in detail.⁵ Equations of propagation in anisotropic media are immediately derived from the results established above. From Eqs. (8.3) and (8.10), we write for the kinetic energy the more general expression [see Eq. (8.13)]

$$\begin{aligned} T &= \frac{1}{2} \rho (\dot{u}_x^2 + \dot{u}_y^2 + \dot{u}_z^2) + \rho_f (\dot{u}_x \dot{w}_x + \dot{u}_y \dot{w}_y + \dot{u}_z \dot{w}_z) \\ &\quad + \frac{1}{2} \sum_{ij} m_{ij} \dot{w}_i \dot{w}_j. \end{aligned} \tag{8.32}$$

The dissipation function D in this general case is given by Eq. (4.3):

$$D = \frac{1}{2} \eta \sum_{ij} r_{ij} \dot{w}_i \dot{w}_j. \tag{8.33}$$

We must also use for the stress components τ_{ij} and p_f the general stress-strain relations (3.37). Introducing expressions (8.32), (8.33), and (3.37) into the dynamical Eq. (8.23), we derive

$$\begin{aligned} \sum_j \frac{\partial}{\partial x_j} (\sum_{\mu\nu} A_{ij}{}^{\mu\nu} e_{\mu\nu} + M_{ij} \zeta) &= \frac{\partial^2}{\partial t^2} (\rho u_i + \rho_f w_i), \\ -\frac{\partial}{\partial x_i} (\sum_j M_{ij} e_{ij} + M \zeta) &= \frac{\partial^2}{\partial t^2} (\rho_f u_i + \sum_j m_{ij} w_j) + \eta \sum_j r_{ij} \frac{\partial w_j}{\partial t}. \end{aligned} \tag{8.34}$$

These six equations for the unknown vector components u_i and w_i govern the propagation of waves in the general case of anisotropy. The various cases of higher symmetry, such as orthotropy and transverse isotropy, are easily derived as particular cases by the introduction of the various stress-strain relations discussed in Sec. 3.

We shall now consider the higher frequency range. As the frequency increases, a boundary layer develops where the microvelocities are out of phase. At higher frequency, this boundary layer becomes very thin. The viscous forces are then confined to this layer, and the microvelocity field in the major portion of the fluid is determined by potential flow. Similar considerations apply to the viscous forces, as shown by earlier analysis on simple models.⁵ The friction force of the fluid on the solid becomes out of phase with the relative rate of flow and exhibits a frequency dependence represented by a complex quantity.

There are several ways of approximating these effects. One approximation used by the writer for the case of isotropic media is the replacement of the viscosity η by

$$\eta^* = \eta F(p), \tag{8.35}$$

where F is a complex function of the frequency $p = i\omega$.

The function F has been evaluated numerically.⁵ A similar approximation can be introduced for the case of anisotropy by the replacement of r_{ij} by

$$r_{ij}^* = R_{ij}(p). \tag{8.36}$$

As a further refinement, the mass coefficient m for the case of isotropy can be replaced by a complex quantity

$$m^* = \rho_f D(p), \tag{8.37}$$

and for the anisotropic medium the tensor m_{ij} can be replaced by

$$m_{ij}^* = \rho_f D_{ij}(p). \tag{8.38}$$

The nature of the operators r_{ij}^* and m_{ij}^* will be discussed in more detail in a forthcoming publication.

9. WAVE PROPAGATION WITH INTERNAL DISSIPATION IN THE SOLID

The incorporation of *internal solid dissipation* in the theory of wave propagation can be accomplished without further development by the replacement of the elastic coefficients by suitable operators.

As shown in an earlier paper⁴ and in the more detailed discussion of Sec. 6, this procedure, because of its thermodynamic foundation, actually takes into account a wide variety of dissipative effects which are not restricted to the solid alone, but which are the result of complex interaction between fluid and solid of mechanical, electrical, chemical, or thermoelastic origin.

For the case of isotropy, when we introduce operators in Eq. (8.24), we find that these equations become

$$\begin{aligned} 2 \sum_j \frac{\partial}{\partial x_j} (\mu^* e_{ij}) + \frac{\partial}{\partial x_i} (\lambda_c^* e - C^* \zeta) &= \frac{\partial^2}{\partial t^2} (\rho u_i + \rho_f w_i) \\ \frac{\partial}{\partial x_i} (C^* e - M^* \zeta) &= \frac{\partial^2}{\partial t^2} (\rho_f u_i + m^* w_i) + \frac{\eta^*}{k} \frac{\partial w_i}{\partial t}. \end{aligned} \tag{9.1}$$

The operator λ_c^* is

$$\lambda_c^* = H^* - 2\mu^*. \tag{9.2}$$

In these equations, we have also replaced m by m^* and η by η^* in order to introduce the frequency dependence of these coefficients in the higher frequency range, in accordance with the discussion of the previous section. Similarly, for anisotropic media, the propagation Eqs. (8.34) are written

$$\begin{aligned} \sum_j \frac{\partial}{\partial x_j} (\sum_{\mu\nu} A_{ij}{}^{*\mu\nu} e_{\mu\nu} + M_{ij}^* \zeta) &= \frac{\partial^2}{\partial t^2} (\rho u_i + \rho_f w_i), \\ -\frac{\partial}{\partial x_i} (\sum_j M_{ij}^* e_{ij} + M^* \zeta) &= \frac{\partial^2}{\partial t^2} (\rho_f u_i + \sum_j m_{ij}^* w_j) + \eta \sum_j r_{ij}^* \frac{\partial w_j}{\partial t}. \end{aligned} \tag{9.3}$$

Again, we have introduced the frequency dependent tensor r_{ij}^* as valid for the higher frequency range. In a further refinement, m_{ij} is also replaced by m_{ij}^* .

Equations for dilatational waves in the isotropic medium of uniform properties are

$$\begin{aligned} \nabla^2(H^*e - C^*\zeta) &= (\partial^2/\partial t^2)(\rho e - \rho_f \zeta) \\ \nabla^2(-C^*\zeta + M^*\zeta) &= (\partial^2/\partial t^2)(-\rho_f + m^*\zeta) + (\eta^*/k)(\partial \zeta/\partial t). \end{aligned} \tag{9.4}$$

The physical significance of these equations can be illustrated by various types of dissipation. Consider, for instance, a purely elastic solid. In the discussion of Sec. 6, we have shown that, in this case, a dissipation can occur because of the presence of the fluid in minute cracks or in their regions adjacent to the areas of contact between the grains (Figs. 4 and 5). As we have seen [Eq. (6.27)], this effect is represented by the use of an operator of the type

$$\kappa^* = \kappa_0 + \kappa_1 r / (p + r), \tag{9.5}$$

or more generally

$$\kappa^* = \kappa_0 + \kappa_1 f(p), \tag{9.6}$$

for the jacketed compressibility. The coefficient κ_0 represents the elastic compressibility due to the elastic grains, and $\kappa_1 f(p)$ is the dissipative term corresponding to the viscoelastic effects associated with the squeezing of the fluid in the small, cracklike volumes surrounding the areas of contact. This operator κ^* is introduced in the expressions (6.24), (6.25), and (6.26), which in turn are used in the propagation Eq. (9.4).

As another example, we shall take the case of internal dissipation in the solid itself. Such a case can be represented by the operator

$$\mu^* = a p^s, \tag{9.7}$$

with $0 < s < 1$ (already considered above). We shall assume that s is small and put $p = i\omega$. Then,

$$\mu^* = a\omega^s [\cos(s\pi/2) + i \sin(s\pi/2)]. \tag{9.8}$$

Since s is small, we can write approximately

$$\mu^* = a\omega^s [1 + (i\pi/2)s]. \tag{9.9}$$

The variation of ω^s with frequency is very slow. Therefore, the imaginary part of μ^* represents a damping which varies very little within a relatively large range of frequency.

Another type of operator which exhibits the same property was proposed by this writer in an earlier paper.²⁷ We write

$$\mu^* = \int_{\epsilon}^{\infty} \frac{p}{p+r} \frac{\mu_1}{r} dr + \mu. \tag{9.10}$$

This amounts to introducing for μ^* in the general expression (5.2) a relaxation spectrum

$$\mu(r) = \begin{cases} \mu_1/r & \text{for } r > \epsilon \\ 0 & \text{for } 0 < r < \epsilon. \end{cases} \tag{9.11}$$

Performing the integration and putting $p = i\omega$ yields

$$\begin{aligned} \mu^* &= \mu_1 [\pi/2 - \tan^{-1}(\epsilon/\omega)] i \\ &\quad + \mu_1 \log(1 + \omega^2/\epsilon^2)^{1/2} + \mu. \end{aligned} \tag{9.12}$$

When ϵ is sufficiently small, the imaginary part becomes, in effect, frequency-independent over a large range.

Operators such as (9.7) and (9.10) can therefore be used to represent some of the typical features of internal friction in solids. Operators other than μ^* which may involve solid friction properties can, of course, be represented by similar expressions.

We should remember that the operational equations remain valid if the operators are of a form more general than Eq. (5.2) and incorporate terms which involve the inertia and resonance effects of hidden degrees of freedom, as previously discussed for the particular example of air bubbles.

Attention is also called to the applicability of the propagation Eq. (9.3) to the problem of *thermoviscoelasticity* in a continuum, provided that we identify w with the entropy displacement and put the fluid density ρ_f equal to zero. We are referring here to the dynamics of a homogeneous solid (without pores) in which viscoelastic and thermoelastic dissipation occur simultaneously.