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General Fluid-Displacement Equations for Acoustic-Gravity Waves

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General equations are derived for the dynamics of a fluid under initial stress in an arbitrary potential field and perturbed from equilibrium. The motion is described in terms of the displacements of the fluid particles from their equilibrium position. A class of equations is obtained which is applicable to large displacements. Complete linearization leads to two types of equations. One type called "unmodified" corresponds to the viewpoint of the theory of elasticity. The "modified" equations representing the other type are expressed in terms of buoyancy forces. The modified equations lead to a conceptually useful analog model for internal gravity waves in a liquid. For a constant gravity field the linear equations are also applicable to large displacements. Classical examples for a constant gravity field are discussed as illustrations.

1. INTRODUCTION

THE theory of wave propagation in a fluid subject to a gravity field has been the object of analytical treatment by many authors in the classical literature.

One may distinguish acoustic-gravity waves represented by the propagation of large-scale disturbances in the atmosphere. This represents the more general case where the propagation is governed by the combined effect of compressibility and gravity.

The particular case of an incompressible fluid corresponds to pure gravity waves occurring in a heavy liquid of nonuniform density. They may be internal gravity waves if they occur in the body of the fluid and are governed by the density gradient or density discontinuity. They include surface waves as a particular case of density discontinuity.

The propagation of gravity waves in heterogeneous liquids has been treated by Love, Burnside, Rayleigh, Lamb. (Classical references will be found in Lamb's treatise,[1] p. 378.) These studies were initiated already in the late nineteenth century.

General equations for small motion of a gas about a state of equilibrium in any constant field of force have been derived by Lamb (see Lamb's treatise,[1] p. 554). Many applications have been discussed by Bjerknes[2] in the context of meteorology. In a recent book Eckart[3] has treated extensively the problems of hydrodynamics of the ocean and atmosphere. The traditional procedure is founded on Euler's equations of fluid dynamics where the motion is described by a velocity field.

The present paper is concerned with a fundamental approach of a different nature. The theory is developed from the standpoint of the fluid displacement instead of the velocity field. In this formulation the coordinates of a given fluid particle are considered as functions of the initial coordinates and the time. The equations constitute a special case of the theory of elasticity and elastic wave propagation under initial stress derived by the writer in 1940.[4] By inserting a zero value for the shear modulus the equations reduce to that of a fluid under initial stress. This reduction is *trivial and immediate*.

While the results obtained from the viewpoint of the elasticity theory are completely general there is need for a more complete treatment and discussion in the case of a fluid.

Our purpose here is to present such a treatment in rigorous and systematic form for a fluid in a nonuniform body force field.

General equations for the motion of a fluid perturbed from equilibrium are derived in Sec. 2. A class of equations is obtained which is applicable for large displacements provided the displacement gradients remain small.

By further linearization with respect to the displacements it is possible to derive two essentially different types of equations as shown in Sec. 3. In what we have called the "unmodified form" the equations embody the viewpoint of the theory of elasticity. Mathematically equivalent "modified equations" can be derived which emphasize the viewpoint of the mechanics of fluids by introducing the buoyancy forces. The modified equations are closely related to Euler's equations.

[1] H. Lamb, *Hydrodynamics* (Cambridge University Press, New York, 1932). [Reprinted by Dover Publications, New York, 1945.]
[2] V. Bjerknes *et al.*, *Physikalische Hydrodynamik* (Springer-Verlag, Berlin, 1933).
[3] C. Eckart, *Hydrodynamics of Oceans and Atmospheres* (Pergamon Press, Inc., New York, 1960).
[4] M. A. Biot, J. Appl. Phys. 11, 522 (1940).

The case of an incompressible fluid is treated in Sec. 4. It includes internal gravity waves in a nonhomogeneous liquid. A conceptually very useful analog model is derived for this case. It is shown that the motion is the same as in a fluid free of gravity provided we add elastic forces, proportional to the displacement, distributed inside the fluid and at the free surface.

Particular forms of these equations for the case of a constant gravity field are discussed in Sec. 5 leading to classical results.

Use of the displacement field equations instead of the Eulerian description has many advantages. It provides a clearer and more general physical description, and leads immediately to variational principles. These principles will be derived in the next paper.[5]

The displacement-field equations also provide a link with the theory of elasticity and a unified theory applicable to both fluids and solids. This is particularly useful in the analysis of composite fluid–solid media. The equations considered as a particular case of the theory of elasticity will be derived in a third paper.[6]

2. DYNAMICS OF A FLUID CONTINUUM UNDER INITIAL STRESS

Consider a fluid in a state of equilibrium under a body force field. This state of equilibrium will be referred to as the "initial state." The force field acting on the fluid per unit mass is represented by its components

$$X_i = X_i(x). \tag{2.1}$$

They are functions of the coordinates x, y, z designated in abbreviated form as x. The initial stress in the fluid is an isotropic stress field designated as

$$S = S(x) \tag{2.2}$$

represented by the initial fluid pressure changed in sign. The initial mass density of the fluid is a function of the coordinates

$$\rho = \rho(x). \tag{2.3}$$

No relation is assumed *a priori* between ρ and S with the implication that the fluid may be *heterogeneous*. The distribution of heterogeneity may be chosen arbitrarily. The only requirement of this initial state is that the fluid be in mechanical equilibrium, hence that the following equation be satisfied:

$$\partial S/\partial x_i + \rho(x)X(x) = 0. \tag{2.4}$$

This equilibrium condition implies a relation between the body force field and the density. In vector notation this relation is written

$$\text{curl } (\rho \mathbf{X}) = 0. \tag{2.5}$$

Let the fluid be disturbed from equilibrium by a small perturbation. A mass particle originally at the point x, y, z is displaced to a point of coordinates ξ, η, ζ. We write

$$\xi = x + u, \qquad \eta = y + v, \qquad \zeta = z + w. \tag{2.6}$$

The components of the displacement vector of the particle are u, v, w. We shall designate the coordinates ξ, η, ζ by ξ_i and the components u, v, w by u_i. Equations (2.6) then assume the abbreviated form

$$\xi_i = x_i + u_i. \tag{2.7}$$

Consider now the dynamical equations for the fluid in this perturbed condition. We shall assume here that in this perturbed motion the fluid particle behaves as a frictionless fluid. Hence the stress in the perturbed fluid remains isotropic. A fluid particle originally at point x_i has moved to the point ξ_i. The isotropic stress on the displaced particle has now become σ, its density ρ', and its acceleration is a_i. The dynamical equations for the motion of this particle are

$$\partial\sigma/\partial\xi_i + \rho'(\xi)X_i(\xi) = \rho'(\xi)a_i. \tag{2.8}$$

These equations are expressed in terms of the coordinates ξ_i as independent variables. The problem is to transform these equations so that the original coordinates x_i become the independent variables. An equivalent form of Eqs. (2.8) is

$$(\partial\sigma/\partial x_j)(\partial x_j/\partial\xi_i) + \rho' X_i = \rho' a_i. \tag{2.9}$$

The usual summation rule is assumed in the notation. The partial derivatives $\partial x_j/\partial\xi_i$ are easily expressed in terms of the inverse derivatives by the standard procedure. We write the total differentials

$$d\xi_i = (\partial\xi_i/\partial x_j)\, dx_j \tag{2.10}$$

and solve this system for dx_j,

$$dx_j = (1/J)M_{ij}\, d\xi_i; \tag{2.11}$$

hence

$$dx_j/d\xi_i = (1/J)M_{ij}. \tag{2.12}$$

The Jacobian of the transformation (2.10) is

$$J = \begin{vmatrix} \frac{\partial\xi}{\partial x} & \frac{\partial\xi}{\partial y} & \frac{\partial\xi}{\partial z} \\ \frac{\partial\eta}{\partial x} & \frac{\partial\eta}{\partial y} & \frac{\partial\eta}{\partial z} \\ \frac{\partial\zeta}{\partial x} & \frac{\partial\zeta}{\partial y} & \frac{\partial\zeta}{\partial z} \end{vmatrix}. \tag{2.13}$$

[5] M. A. Biot, Phys. Fluids (to be published).
[6] M. A. Biot, Phys. Fluids (to be published).

The cofactors of this determinant are M_{ij}. They are the partial Jacobians with suitable signs. They contain terms which are linear and quadratic in the displacement gradients $\partial u_i/\partial x_j$. In evaluating M_{ij} we shall introduce the assumption that the displacement gradients are small quantities such that their squares and products may be neglected. This is equivalent to the condition that the *strain and rotations are small.* However, *no such restriction is imposed on the magnitude of the displacement* u_i. Under these conditions we may write

$$M_{ij} = (1 + e)\delta_{ij} - \partial u_j/\partial x_i. \tag{2.14}$$

In this expression e is the dilatation

$$e = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \frac{\partial u_i}{\partial x_i} \tag{2.15}$$

and δ_{ij} is the Kronecker symbol

$$\delta_{ij} = \begin{cases} 1 & i = j, \\ 0 & i \neq j. \end{cases} \tag{2.16}$$

Substituting the partial derivatives (2.12) into Eqs. (2.8) and taking into account the law of conservation of mass

$$\rho = \rho' J, \tag{2.17}$$

we derive

$$M_{ij}\, \partial\sigma/\partial x_j + \rho(x)X_i(\xi) = \rho(x)a_i. \tag{2.18}$$

The stress σ on the particle may be written

$$\sigma = S + s, \tag{2.19}$$

where $-s$ is the pressure increment. If it depends only on the dilatation we write

$$s = \lambda e \tag{2.20}$$

with an *incremental* bulk modulus λ. We may choose it to be either the isothermal or adiabatic modulus.

Again by assuming the deformation and rotations to be small we retain only the terms which are linear in the displacements gradients. With this approximation we may write

$$M_{ij}\frac{\partial\sigma}{\partial x_j} = \frac{\partial s}{\partial x_i} + e\frac{\partial S}{\partial x_i} - \frac{\partial u_j}{\partial x_i}\frac{\partial S}{\partial x_j} + \frac{\partial S}{\partial x_i}. \tag{2.21}$$

When substituting this expression in Eqs. (2.18) we take into account the equilibrium condition (2.4) for the initial stress field and write

$$\partial S/\partial x_i + \rho(x)X_i(\xi) = \rho(x)\Delta X_i \tag{2.22}$$

with

$$\Delta X_i = X_i(\xi) - X_i(x). \tag{2.23}$$

Equation (2.18) becomes

$$\frac{\partial s}{\partial x_i} + e\frac{\partial S}{\partial x_i} - \frac{\partial u_j}{\partial x_i}\frac{\partial S}{\partial x_j} + \rho\Delta X_i = \rho a_i. \tag{2.24}$$

The acceleration which appears on the right-hand side is

$$a_i = \partial^2 u_i/\partial t^2. \tag{2.25}$$

Hence the dynamical equations are

$$\frac{\partial s}{\partial x_i} + e\frac{\partial S}{\partial x_i} - \frac{\partial u_j}{\partial x_i}\frac{\partial S}{\partial x_j} + \rho\Delta X_i = \rho\frac{\partial^2 u_i}{\partial t^2}. \tag{2.26}$$

With suitable initial and boundary conditions they determine the displacement field u_i as a function of time. The initial mass density $\rho(x)$ and the initial stress $S(x)$ are given functions of the initial coordinates x_i.

Note that Eqs. (2.26) are not necessarily linear in the unknown displacements since ΔX_i are explicit functions of u_i.

Coriolis acceleration. When the frame of reference is rotating the acceleration a_i must include a Coriolis term. This is easily included in the above equations by introducing the antisymmetric matrix

$$[\Omega_{ij}] = \begin{bmatrix} 0 & -\Omega_z & \Omega_y \\ \Omega_z & 0 & -\Omega_x \\ -\Omega_y & \Omega_x & 0 \end{bmatrix}, \tag{2.27}$$

where $\Omega_x\Omega_y\Omega_z$ are the components of the angular velocity of the frame of reference. In this case the acceleration in Eq. (2.24) is replaced by

$$a_i = \partial^2 u_i/\partial t^2 + 2\Omega_{ij}\, \partial u_j/\partial t. \tag{2.28}$$

3. MODIFIED AND UNMODIFIED EQUATIONS

The general equations (2.24) may be written in a different form by introducing two assumptions. We assume that the body force is derived from a *potential* U, i.e.,

$$X_i = -\partial U/\partial x_i. \tag{3.1}$$

In addition we shall *linearize* ΔX_i by writing

$$\Delta X_i = \frac{\partial X_i}{\partial x_j}u_j = -\frac{\partial^2 U}{\partial x_i\, \partial x_j}u_j. \tag{3.2}$$

It is well known that the existence of a body force potential implies that in the equilibrium state the surfaces of constant density coincide with equipotential surfaces. This is readily established by combining Eqs. (2.4) and (3.1). We find

$$\frac{\partial\rho}{\partial x_j}\frac{\partial U}{\partial x_i} = \frac{\partial\rho}{\partial x_i}\frac{\partial U}{\partial x_j}. \tag{3.3}$$

This equation implies that the body force X_i and the density gradient $\partial\rho/\partial x_i$ are parallel vectors. Under these assumptions it is possible to derive dynamical equation of two essentially different types as will now be shown.

With the value (3.2) the dynamical equations (2.24) are written

$$\frac{\partial s}{\partial x_i} + e\frac{\partial S}{\partial x_i} - \frac{\partial u_j}{\partial x_i}\frac{\partial S}{\partial x_j} - \rho\frac{\partial^2 U}{\partial x_i\,\partial x_j}u_j = \rho a_i. \quad (3.4)$$

We shall refer to them as the *unmodified equations*. In this form the equations are the same as those obtained directly for the linearized equations of elasticity for an initially stressed continuum[4] as will be shown in another paper.[6]

Another equivalent form of Eqs. (3.4) is obtained as follows. We put

$$s = s' + \rho u_j\,\partial U/\partial x_j. \quad (3.5)$$

Substitution in Eqs. (3.4) taking into account Eqs. (2.4) and (3.1) yields

$$\frac{\partial s'}{\partial x_i} - \rho e X_i - u_j X_j\frac{\partial \rho}{\partial x_i} = \rho a_i. \quad (3.6)$$

We shall refer to these equations as the *modified equations*. By using Eq. (3.1) the variable s' is also written

$$s' = s + \rho u_j X_j. \quad (3.7)$$

The interest of the modified form lies in their physical interpretation and intuitive value.

To show this we rewrite the value (3.7) of s' by taking into account the equilibrium condition (2.4)

$$s' = s - u_j\,\partial S/\partial x_j. \quad (3.8)$$

This expression shows that s' is the increment of stress at the *fixed point* x_i.

On the other hand, let us look at the terms containing the body force in Eq. (3.6). Taking into account relations (3.3) we write

$$\rho e X_i + u_j X_j\frac{\partial \rho}{\partial x_i} = X_i\frac{\partial}{\partial x_j}(\rho u_j). \quad (3.9)$$

These terms represent the buoyancy force on the fluid in a fixed volume of space. It is directed normally to the equipotential surface. To the first order this buoyancy force and the acceleration a_i may be considered as values at the fixed point x_i.

With this interpretation Eq. (3.6) becomes intuitively self-evident as expressing Newton's law for a fluid particle at the fixed point.

Attention is called to an important property for the case of a constant gravity field. In this case ΔX_i drops out and linearization is not required. Hence for a constant gravity field the modified equations (3.6) are equivalent to Eqs. (2.24) and are therefore applicable to large displacements.

Relation to Euler's equations of fluid dynamics. Using relation (3.9) the modified equations (3.6) become

$$\frac{\partial s'}{\partial x_i} - X_i\frac{\partial}{\partial x_j}(\rho u_j) = \rho a_i. \quad (3.10)$$

This result is closely related to the equations derived from fluid dynamics. Euler's equations are

$$\rho\left(\frac{\partial v_i}{\partial t} + v_j\frac{\partial v_i}{\partial x_j}\right) = -\frac{\partial p}{\partial x_i} + \rho X_i, \quad (3.11)$$

where v_i and p denote the fluid velocity and pressure at the fixed point x_i. Taking the time derivative of Eq. (3.11) neglecting higher order terms yields

$$\rho\frac{\partial^2 v_i}{\partial t^2} = -\frac{\partial}{\partial x_i}\left(\frac{\partial p}{\partial t}\right) + X_i\frac{\partial \rho}{\partial t}. \quad (3.12)$$

The equation of conservation of mass is

$$\partial\rho/\partial t + \partial(\rho v_j)/\partial x_j = 0. \quad (3.13)$$

Hence, eliminating $\partial\rho/\partial t$ between the last two equations we obtain

$$\rho\frac{\partial^2 v_i}{\partial t^2} = -\frac{\partial}{\partial x_i}\left(\frac{\partial p}{\partial t}\right) - X_i\frac{\partial}{\partial x_j}(\rho v_j). \quad (3.14)$$

This result is identical in form with Eq. (3.10). The velocity v_i replaces the displacement u_i and $-\partial p/\partial t$ replaces s'. Dividing Eq. (3.10) by an infinitesimal time interval Δt yields Euler's equations (3.14) in the limit.

4. ANALOG MODEL FOR INTERNAL GRAVITY WAVES IN A LIQUID

In an incompressible liquid we put

$$e = 0 \quad (4.1)$$

in the modified equations (3.6). They are simplified to

$$\frac{\partial s'}{\partial x_i} - u_j X_j\frac{\partial \rho}{\partial x_i} = \rho a_i. \quad (4.2)$$

Equations (4.1) and (4.2) are four equations for s' and the three displacement components u_i.

They are identical with the dynamical equations for a liquid *initially stress free* with distributed body forces acting on the fluid particles proportionally to their displacement from equilibrium.

This new physical system may be considered as an analog model for the actual fluid under initial stress.

The stress in the model is s'. It is different from

the stress in the actual fluid. The incremental stress s in the actual fluid is related to the model stress s' by Eq. (3.7),

$$s = s' - \rho u_i X_i. \tag{4.3}$$

This shows that a free surface in the actual fluid is not a free surface in the model. By putting $s = 0$ in Eq. (4.3) the following boundary condition in the model is obtained:

$$s' = \rho u_i X_i. \tag{4.4}$$

Since X_i is normal to the free surface we may write

$$s' = \rho u_n X, \tag{4.5}$$

where u_n is the normal displacement and X the normal component of the body force.

Hence at the surface of the model we must apply a normal force s' per unit area, proportional to the boundary normal displacement. If the body force is directed inward at the surface this force acts in opposition to the displacement as an elastic restoring force. In that case it is stabilizing. In the opposite case it acts as a negative elastic force and is destabilizing. This corresponds to the so-called "*Taylor instability.*"

The body force of the model can similarly be looked upon as a positive or negative elastic force applied to the fluid particles per unit volume. We may write

$$-u_i X_i \, \partial\rho/\partial x_i = -(u_n X \, \partial\rho/\partial n) n_i, \tag{4.6}$$

where n_i is the unit normal to the equipotential surface with $\partial\rho/\partial n$ equal to the normal derivative of the density along the same direction. We see that if X and $\partial\rho/\partial n$ are of the same sign, expression (4.6) represents an elastic restoring force. Hence it is stabilizing. If X and $\partial\rho/\partial n$ are of opposite sign the force is destabilizing.

If there is a surface of discontinuity for the density we may think of it as a thin layer through which the value of the density varies rapidly from ρ_1 to ρ_2. By integration across the thickness expression (4.6) becomes a force per unit area $-u_n X(\rho_2 - \rho_1) n_i$ applied to the surface of discontinuity. This includes the boundary surface which may be considered as a density discontinuity by putting $\rho_2 = 0$.

Hence if the liquid is composed of layers each of which is of constant density, the analog model is obtained by applying forces to the discontinuity surfaces including the free surface.

5. CONSTANT GRAVITY FIELD

With a vertical z axis positive upward a constant gravity field of acceleration g is represented by the components

$$X_i = (0, 0, -g). \tag{5.1}$$

The unmodified equations (3.4) become

$$\begin{aligned} \frac{\partial s}{\partial x} - \rho g \frac{\partial w}{\partial x} &= \rho \frac{\partial^2 u}{\partial t^2}, \\ \frac{\partial s}{\partial y} - \rho g \frac{\partial w}{\partial y} &= \rho \frac{\partial^2 v}{\partial t^2}, \\ \frac{\partial s}{\partial z} + \rho g e - \rho g \frac{\partial w}{\partial z} &= \rho \frac{\partial^2 w}{\partial t^2}. \end{aligned} \tag{5.2}$$

Equations of this type were also proposed earlier by the writer.[4] To these we must add Eq. (2.20) for s. By the change of variable

$$s' = s - \rho g w \tag{5.3}$$

taking into account that ρ is a function of z only equations (5.2) become

$$\begin{aligned} \frac{\partial s'}{\partial x} &= \rho \frac{\partial^2 u}{\partial t^2}, \\ \frac{\partial s'}{\partial y} &= \rho \frac{\partial^2 v}{\partial t^2}, \\ \frac{\partial s'}{\partial z} + \rho g e + w g \frac{d\rho}{dz} &= \rho \frac{\partial^2 w}{\partial t^2}. \end{aligned} \tag{5.4}$$

They correspond to the modified form (3.6) of the general case. As already pointed out, Eqs. (5.2) and (5.4) are equivalent for the case of a constant gravity field and both applicable to large displacements. Equations equivalent to Eqs. (5.4) but slightly different in form were also proposed by Eliassen and Kleinschmidt[7] using the same displacement field.

For a liquid we put $e = 0$. Hence

$$\begin{aligned} \frac{\partial s'}{\partial x} &= \rho \frac{\partial^2 u}{\partial t^2}, \\ \frac{\partial s'}{\partial y} &= \rho \frac{\partial^2 v}{\partial t^2}, \\ \frac{\partial s'}{\partial z} + w g \frac{d\rho}{dz} &= \rho \frac{\partial^2 w}{\partial t^2}. \end{aligned} \tag{5.5}$$

These equations represent the analog model. If $d\rho/dz$ is negative the body force in the model is a restoring elastic force $wg(d\rho/dz)$ proportional to the vertical displacement. At the surface of the model a restoring force is also applied. Its value per unit area is

$$s' = -\rho g w. \tag{5.6}$$

[7] A. Eliassen and E. Kleinschmitt, in *Handbuch der Physik*, edited by S. Flügge (Springer-Verlag, Berlin, 1957), Vol. 48, p. 52.

As an illustration we apply Eqs. (5.5) to the classical case of two dimensional motion in an ocean of constant depth h. We put

$$u = \frac{\partial \psi}{\partial z}, \qquad v = 0, \qquad w = -\frac{\partial \psi}{\partial x}, \tag{5.7}$$

and

$$\psi = \exp(ikx - i\omega t) f(z).$$

This solution satisfies the condition $e = 0$ of incompressibility. Substitution of the values (5.7) into Eqs. (5.5) yields two equations for s' and f. By elimination of s' we are left with a Sturm–Lionville equation for f.

$$\frac{d}{dz}\left(\rho \frac{df}{dz}\right) + \frac{\rho}{V^2}(\omega_c^2 - \omega^2) f = 0. \tag{5.8}$$

We have put

$$\omega_c^2 = -g \frac{1}{\rho} \frac{d\rho}{dz}, \qquad \frac{\omega}{k} = V. \tag{5.9}$$

The parameter V is the phase velocity along x. The boundary condition at the surface is derived from equations (5.5) and (5.6) and is written

$$\frac{g}{V^2} f = \frac{df}{dz}. \tag{5.10}$$

The other boundary condition at the rigid bottom is $f = 0$.

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Flow of a Non-Newtonian Fluid between Rotating Cylinders with Suction and Injection

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The steady flow of a certain non-Newtonian fluid in an annulus between two coaxial cylinders rotating with uniform angular velocities about the common axis is studied when there is suction at one wall and injection at the other. The stress matrix T for the non-Newtonian fluid is given by $T = -pI + \alpha_1 A_1 + \alpha_2 A_2$, where p is the pressure, I is the unit matrix, α_1, α_2 are constants and A_1, A_2 are kinematic matrices. It is found that in the case of no suction and injection, the velocity field is not affected by the presence of the non-Newtonian term $\alpha_2 A_2$, though the pressure field is affected. On the other hand, if there is suction and injection, however small, the non-Newtonian term affects the velocity field and the nature of this effects is investigated for sufficiently small suction and injection velocities.

I. INTRODUCTION

THE steady flow of a Newtonian fluid in the annulus between two coaxial cylinders rotating with uniform angular velocities about the common axis when there is suction at one wall and injection at the other has been studied by Kapur and Mallick.[1] In the present paper we study the same problem for a non-Newtonian fluid whose constitutive equation is

$$T = -pI + \alpha_1 A_1 + \alpha_2 A_2, \tag{1}$$

where T is the stress matrix $||t_{iJ}||$, I is the unit matrix $||\delta_{iJ}||$, A_1, A_2 are kinematic matrices defined by[2]

$$A_1 = ||(V_{i,J} + V_{J,i})|| = ||A_{iJ}|| = ||2d_{iJ}||, \tag{2}$$

$$A_2 = \left|\left|\frac{\partial}{\partial t} A_{iJ} + V^l A_{iJ,l} + A_{mi} V^m_{,J} + A_{mJ} V^m_{,i}\right|\right|; \tag{3}$$

p is a scalar which we identify with pressure and α_1, α_2 are constants. Here δ_{iJ} is the Kronecker set,

[1] J. N. Kapur and S. C. Mallick, *Proceedings of the Sixth Congress of Theoretical and Applied Mechanics*, India, (1960), Vol. 6, p. 125.

[2] J. L. Erickson and Rivlin, J. Ratl. Mech. Anal. 4, 323 (1955).