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## Increase of Torsional Stiffness of a Prismatical Bar Due to Axial Tension

M. A. Biot

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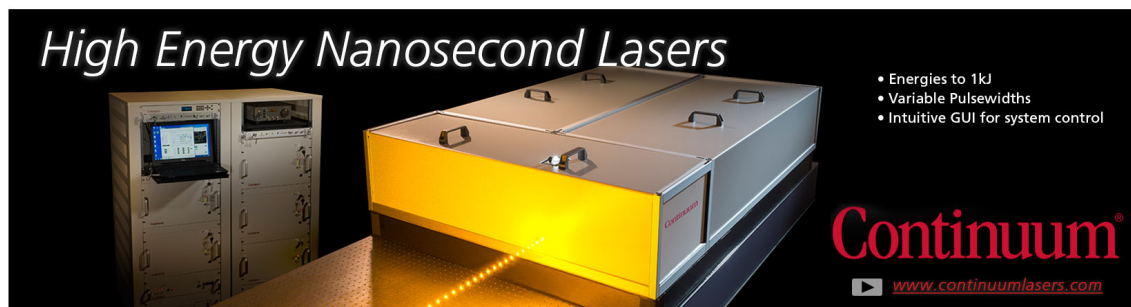
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The procedure to be followed in the transient solution of the oscillatory network is first to neglect the resistance matrix  $[R]$  and to obtain the natural frequencies  $w_r$  and the modal matrices  $(k_r)$  by the methods of the first part of this paper.

Then if the relation (155) which may be written in the form

$$(k_r)'[S](k_r)/(k_r)'[L](k_r) \gg ((k_r)'[R](k_r)/2(k_r)'[L](k_r))^2 \quad (160)$$

is satisfied the solution is given by (157) and the various attenuation constants are obtained from

(159) by direct matrix multiplication. The rest of the solution proceeds as before.

### XIII. CONCLUSION

The simplicity and power of the iterative matrix method of solution for the transient and steady-state behavior of oscillatory circuits is evident from the above discussion. If the amount of resistance is small as it is in many practical circuits this method of solution is most simple and direct. It is hoped that this presentation will give an impetus to the use of the method in dealing with this type of circuit.

## Increase of Torsional Stiffness of a Prismatical Bar Due to Axial Tension\*

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(Received March 22, 1939)

The author's theory of elasticity of the second order is being applied to calculate the increase of torsional stiffness of a prismatical bar when an axial tension is initially imposed upon it. It is found that the classical shear stress distribution is not affected by the axial stress. However, an increase of torsional stiffness is produced due to the fact that the boundary condition over the cross section contains not only the shearing stress but also the product of the axial stress by the rotations. This increase of torsional stiffness turns out to be proportional to the polar moment of inertia of the cross section with respect to its center of gravity.

**B**EFORE treating the special problem of torsion of a bar under an initial axial stress we shall first summarize the results of the general theory of elasticity for the small deformations of a body under initial stress.<sup>1</sup>

We denote by

$$\begin{matrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33}, \end{matrix} \quad (1)$$

the initial state of stress in an elastic body. These components of stress are symmetric  $S_{\mu\nu} = S_{\nu\mu}$  and they must satisfy the equilibrium conditions

$$\begin{aligned} \partial S_{11}/\partial x + \partial S_{12}/\partial y + \partial S_{13}/\partial z &= 0, \\ \partial S_{21}/\partial x + \partial S_{22}/\partial y + \partial S_{23}/\partial z &= 0, \\ \partial S_{31}/\partial x + \partial S_{32}/\partial y + \partial S_{33}/\partial z &= 0. \end{aligned} \quad (2)$$

We now assume that the material undergoes a small deformation. The deformation is associated with displacements of components  $uvw$ , rotations of components,

$$\begin{aligned} \omega_x &= \frac{1}{2}(\partial w/\partial y - \partial v/\partial z), \\ \omega_y &= \frac{1}{2}(\partial u/\partial z - \partial w/\partial x), \\ \omega_z &= \frac{1}{2}(\partial v/\partial x - \partial u/\partial y) \end{aligned} \quad (3)$$

and strain components

$$\begin{aligned} e_{11} &= \partial u/\partial x, & e_{12} &= e_{21} = \frac{1}{2}(\partial v/\partial x + \partial u/\partial y), \\ e_{22} &= \partial v/\partial y, & e_{23} &= e_{32} = \frac{1}{2}(\partial w/\partial y + \partial v/\partial z), \\ e_{33} &= \partial w/\partial z, & e_{31} &= e_{13} = \frac{1}{2}(\partial u/\partial z + \partial w/\partial x). \end{aligned} \quad (4)$$

Because of the small deformation the initial

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<sup>1</sup> M. A. Biot, "Non-Linear theory of elasticity and the linearized case for a body under initial stress," *Phil. Mag.*

27, 468 (1939). Also: "Theory of elasticity with large displacements and rotations," *Proc. Fifth International Congress for Applied Mechanics* (Cambridge, U. S. A. 1938). "Theorie de l'elasticité du second ordre," *Ann. Soc. Sci. de Bruxelles* (1939).

stresses are modified. We denote the stress increments by:

$$\begin{matrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33}. \end{matrix} \quad (5)$$

These symmetric stress components  $s_{\mu\nu} = s_{\nu\mu}$  are referred to axes which rotate locally with the material so that they are only functions of the strain. They must satisfy equilibrium conditions which will be different from the classical equations for a material initially in a free unstressed condition.

Equations expressing these conditions rigorously have been derived in the general theory. For example, one of these equations expressing equilibrium conditions in the  $x$ -direction reads

$$\begin{aligned} & \frac{\partial s_{11}}{\partial x} + \frac{\partial s_{12}}{\partial y} + \frac{\partial s_{13}}{\partial z} + (S_{11} - S_{22}) \frac{\partial \omega_z}{\partial y} - 2S_{12} \frac{\partial \omega_x}{\partial x} \\ & + S_{12} \frac{\partial \omega_x}{\partial z} - (S_{11} - S_{33}) \frac{\partial \omega_y}{\partial z} - 2S_{13} \frac{\partial \omega_y}{\partial x} \\ & - S_{13} \frac{\partial \omega_x}{\partial y} + S_{23} \left( \frac{\partial \omega_y}{\partial y} - \frac{\partial \omega_z}{\partial z} \right) + \frac{\partial S_{11}}{\partial x} (e_{22} + e_{33}) \\ & + \frac{\partial S_{12}}{\partial y} (e_{33} + e_{11}) + \frac{\partial S_{13}}{\partial z} (e_{11} + e_{22}) \\ & - \left( \frac{\partial S_{12}}{\partial x} + \frac{\partial S_{11}}{\partial y} \right) e_{12} - \left( \frac{\partial S_{13}}{\partial y} + \frac{\partial S_{12}}{\partial z} \right) e_{23} \\ & - \left( \frac{\partial S_{11}}{\partial z} + \frac{\partial S_{12}}{\partial x} \right) e_{31} = 0. \end{aligned} \quad (6)$$

We have two other similar equations. The boundary conditions can be expressed by means of the increment of force per unit area at the boundary. For example, the  $x$ -component of this boundary force increment is

$$\begin{aligned} \Delta f_x = & s_{11}\alpha + s_{12}\beta + s_{13}\gamma + [S_{13}\omega_y - S_{12}\omega_z]\alpha \\ & + [S_{23}\omega_y - S_{22}\omega_z]\beta + [S_{33}\omega_y - S_{23}\omega_z]\gamma \\ & + S_{11}[e_{22} + e_{33}]\alpha + S_{12}[e_{33} + e_{11}]\beta \\ & + S_{31}[e_{11} + e_{22}]\gamma - (S_{12}e_{12} + S_{13}e_{13})\alpha \\ & - (S_{13}e_{23} + S_{11}e_{21})\beta - (S_{11}e_{31} + S_{12}e_{32})\gamma. \end{aligned} \quad (7)$$

In these relations  $\alpha\beta\gamma$  are the directional cosines of the normal direction to the boundary before deformation.

In the particular problem that we have here in mind, the torsion of a prismatical part submitted to an axial tension, the above equations are very much simplified. Taking the  $z$  axis along the axis of the bar, the initial stresses are (Fig. 1)

$$\begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & S_{33}, \end{matrix} \quad (8)$$

where  $S_{33} = S$  is a constant representing the initial axial tension. The equilibrium equations become

$$\begin{aligned} & \frac{\partial s_{11}}{\partial x} + \frac{\partial s_{12}}{\partial y} + \frac{\partial s_{13}}{\partial z} + S \frac{\partial \omega_y}{\partial z} = 0, \\ & \frac{\partial s_{21}}{\partial x} + \frac{\partial s_{22}}{\partial y} + \frac{\partial s_{23}}{\partial z} - S \frac{\partial \omega_x}{\partial z} = 0, \end{aligned} \quad (9)$$

$$\frac{\partial s_{31}}{\partial x} + \frac{\partial s_{32}}{\partial y} + \frac{\partial s_{33}}{\partial z} + S \left( \frac{\partial \omega_y}{\partial x} - \frac{\partial \omega_x}{\partial y} \right) = 0$$

and the boundary conditions

$$\begin{aligned} \Delta f_x = & s_{11}\alpha + s_{12}\beta + s_{13}\gamma + S\omega_y\gamma, \\ \Delta f_y = & s_{21}\alpha + s_{22}\beta + s_{23}\gamma - S\omega_x\gamma, \\ \Delta f_z = & s_{31}\alpha + s_{32}\beta + s_{33}\gamma + S(e_{11} + e_{22})\gamma \\ & - Se_{13}\alpha - Se_{23}\beta. \end{aligned} \quad (10)$$

No approximations have been introduced in the above equations except the basic one that, the displacements are small. We shall now con-

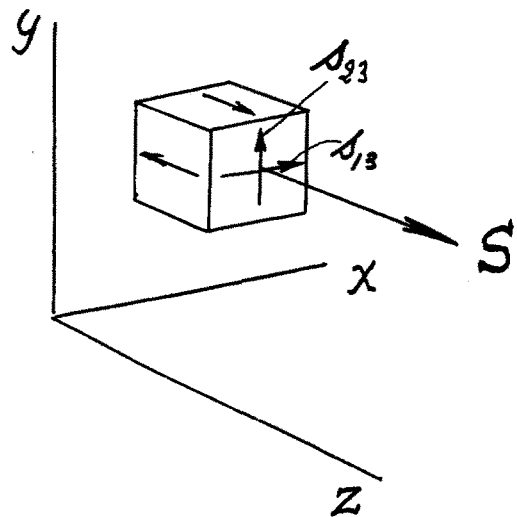


FIG. 1.

sider the relation between the strain and the stress increments  $s_{\mu\nu}$ . Here of course we have to introduce some approximations and simplifying assumptions as we do not know what these relations are for a material under an initial stress. However we will assume that these relations are the same as for the material in its natural unstressed state and if we assume this state to be isotropic, we may write

$$\begin{aligned} Ee_{11} &= s_{11} - \nu(s_{22} + s_{33}), \\ Ee_{22} &= s_{22} - \nu(s_{11} + s_{33}), \\ Ee_{33} &= s_{33} - \nu(s_{22} + s_{11}), \\ 2Ge_{12} &= s_{12}, \\ 2Ge_{23} &= s_{23}, \\ 2Ge_{31} &= s_{31}. \end{aligned} \quad (11)$$

That this is an approximation is obvious first for a theoretical reason. It was shown by the author<sup>1</sup> that the stress-strain law (11) is in contradiction with the existence of a potential energy and that the error is of the order  $S/E$ . Moreover a material which in its natural state is isotropic cannot remain such when it is stretched so that an isotropic stress-strain relation is also an approximation. Again here the error must be of the order  $S/E$ .

Keeping these remarks in mind we may solve the problem of torsion by means of Eqs. (9), (11) and the boundary conditions (10). We are going to show that the classical stress distribution of Saint Venant for the torsion of a prismatical bar is also a solution of our problem for the distribution of stress increments  $s_{\mu\nu}$ . The difference with the classical solution will appear only on the boundary of the cross section in the form of an increase in torsional stiffness. We put as in Saint Venant's solution

$$\begin{aligned} s_{11} &= s_{22} = s_{33} = s_{12} = 0 \\ u &= -\theta yz \\ v &= \theta xz \\ w &= w(x, y). \end{aligned} \quad (12)$$

We also assume that  $s_{23}$  and  $s_{31}$  are only functions of  $x$  and  $y$ .

We deduce  $e_{11} = e_{22} = e_{33} = e_{12} = 0$ , hence the first four relations (11) are satisfied and the two last ones become

$$\begin{aligned} G(\partial w/\partial y + \theta x) &= s_{23} \\ G(-\theta y + \partial w/\partial x) &= s_{31}. \end{aligned} \quad (13)$$

The two first equilibrium Eqs. (9) are also satisfied because

$$\omega_x = w(\partial\omega/\partial y - \theta x), \quad \omega_y = \frac{1}{2}(-\theta y - \partial w/\partial x) \quad (14)$$

do not depend on  $z$ . The third Eq. (9) becomes

$$\frac{\partial s_{31}}{\partial x} + \frac{\partial s_{32}}{\partial y} - S\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) = 0. \quad (15)$$

Now from Eq. (13) we deduce

$$G\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) = \frac{\partial s_{31}}{\partial x} + \frac{\partial s_{23}}{\partial y}. \quad (16)$$

Hence combining (15) and (16)

$$\partial s_{31}/\partial x + \partial s_{32}/\partial y = 0. \quad (17)$$

As we know Eqs. (13) and (17) are the classical equations of Saint Venant's problem of torsion of prismatical bars. This can be immediately verified by introducing a stress function  $\psi$  such that  $s_{31} = \partial\psi/\partial y$  and  $s_{32} = -(\partial\psi/\partial x)$ . Then Eq. (17) is satisfied and the elimination of  $w$  between Eqs. (13) yields the well-known equation,

$$\partial^2\psi/\partial x^2 + \partial^2\psi/\partial y^2 = -2G\theta. \quad (18)$$

However, we still have to satisfy the boundary conditions (10). At the surface of the prismatical bar  $\gamma = 0$ . Expressing that this surface is free of forces we find the two first relations (10) to be identically satisfied and the last one becomes

$$0 = s_{31}\alpha + s_{32}\beta - Se_{13}\alpha - Se_{23}\beta. \quad (19)$$

Now because

$$2Ge_{13} = s_{13} \quad 2Ge_{23} = s_{23},$$

this condition may be written

$$s_{31}\alpha + s_{32}\beta = 0.$$

This is the same boundary condition as in Saint Venant's problem, namely that the shear stress on the cross section be tangent to the contour of the section. Eq. (18) with the boundary conditions (19) determines completely the shear stress  $s_{31}s_{32}$  and its distribution over the cross section is therefore the same as in the classical solution of Saint Venant.

We will now calculate the total torque over the cross sections. For this purpose we must use expression (10) for  $\Delta f_x$  and  $\Delta f_y$  with  $\gamma = 1$ ,  $\alpha = \beta = 0$ .

We find

$$\Delta f_x = s_{13} + S\omega_y, \quad \Delta f_y = s_{23} - S\omega_x. \quad (20)$$

From Eqs. (13) and (14)

$$\omega_x = \frac{1}{2}s_{23}/G - \theta x, \quad \omega_y = -\frac{1}{2}s_{31}/G - \theta y, \quad (21)$$

hence the force along  $x$  and  $y$  acting on the cross section are

$$\begin{aligned} \Delta f_x &= s_{13}(1 - S/2G) - S\theta y, \\ \Delta f_y &= s_{23}(1 - S/2G) + S\theta x. \end{aligned} \quad (22)$$

The torque is

$$\begin{aligned} T &= (1 - S/2G) \iint (-s_{13}y + s_{23}x) dx dy \\ &\quad + S\theta \iint (x^2 + y^2) dx dy, \end{aligned} \quad (23)$$

where the double integral is extended to the area of the cross section. The term

$$T_{SV} = \iint (-s_{13}y + s_{23}x) dx dy \quad (24)$$

represents the torque calculated by the Saint Venant theory in the assumption that the bar is initially in its natural unstressed state. In the second term appears the polar moment of inertia

$$I_0 = \iint (x^2 + y^2) dx dy$$

of the cross section with respect to the origin of the coordinates. It depends on the choice of this origin. However the position of this origin is not arbitrary because the forces acting on the cross section are only a pure torque if

$$\iint \Delta f_x dx dy = \iint \Delta f_y dx dy = 0. \quad (25)$$

Now we know that the Saint Venant shear distribution is a pure torque and therefore if we introduce expression (22) in the condition (25) the latter becomes

$$\iint y dx dy = \iint x dx dy = 0. \quad (26)$$

Hence the origin must be at the center of gravity of the cross section and

$$I_G = \iint (x^2 + y^2) dx dy$$

is the polar moment of inertia of the cross section with respect to its center of gravity.

The expression for the torque may be written finally

$$T = (1 - S/2G)T_{SV} + I_G S\theta. \quad (27)$$

This simple expression has been obtained with the only assumption that the stress-strain rela-

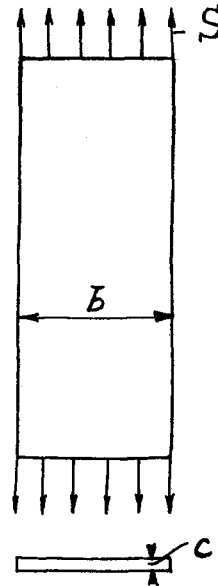


FIG. 2.

tion is the same as the classical Hooke's law for an isotropic medium. As we remarked above this is an approximation which involves an error of the order  $S/2G$ . Therefore the term  $S/2G$  in the factor  $(1 - S/2G)$  is not significant as it represents a quantity of the order of what we neglected from the start. We write, therefore

$$T = T_{SV} + I_G S\theta. \quad (28)$$

We will now discuss the application of this formula to various cases.

Consider first a circular section. In this case the Saint Venant torque is

$$T_{SV} = GI_G\theta$$

and the total torque is

$$T = I_G\theta(G + S) = GI_G\theta(1 + S/G).$$

Here the correction  $S/G$  due to the axial stress is of the order of what we neglect in the theory and for this case the correction term loses its physical significance. It would not be so, however, for sections having a low torsional rigidity in their natural state.

Consider for instance the case of a strip of rectangular cross section (Fig. 2). The thickness of the strip being  $c$  and the width  $b$  (Fig. 2). The Saint Venant torque is

$$T_{SV} = \frac{1}{3}bc^3G\theta.$$

The polar moment of inertia

$$I_G = cb^3/12.$$

When the strip is submitted to an axial tension  $S$ , applying Eq. (28), the total torque is

$$T = \frac{1}{3}bc^3G\theta + \frac{cb^3}{12}S\theta = \frac{1}{3}bc^3G \left[ 1 + \frac{1}{4} \left( \frac{b}{c} \right)^2 \frac{S}{G} \right] \theta.$$

We see that if the thickness  $c$  is small compared to the width,  $b$ , the correction  $\frac{1}{4}(b/c)^2 S/G$  can become very large.

As another example we take the case of a circular thin-walled cylinder split along a generator (Fig. 3), the radius being  $r$  and the thickness,  $c$ . The Saint Venant's torque is

$$T_{SV} = \frac{2}{3}\pi r^3 c^3 G \theta$$

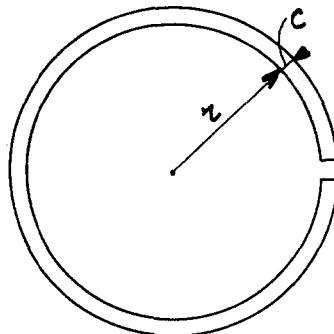


FIG. 3.

and the polar moment of inertia is

$$2\pi r^3 c = I_G.$$

The total torque when the cylinder is submitted to an axial tension  $S$  is, therefore:

$$T = \frac{2}{3}\pi r^3 c^3 G \left[ 1 + 3 \left( \frac{r}{c} \right)^2 \frac{S}{G} \right] \theta.$$

The correction in this case is  $3(r/c)^2 S/G$ . If for instance,  $S/G = 1/1000$  and  $r/c = 10$ , the increase of torsional rigidity is 30 percent.

## The Theory of Klystron Oscillations

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(Received June 23, 1939)

The principles governing the oscillations in a klystron may be divided into four groups: (1) those of space resonance within each rhumbatron, previously discussed by Hansen; (2) the equations of coupled circuits, with modifications appropriate to space-resonant systems; (3) the kinematics and dynamics of cathode-ray bunching, previously discussed by the author; (4) the constraint on phase relations between the rhumbatrons introduced by the cathode-ray bunches. It is shown here that (4) makes the oscillations assume forms very different from those familiar in free oscillations. These forms are described by general equations and then these equations are applied to three specific problems: maximizing the power output from an oscillator, minimizing the power input for an ideal regenerative amplifier, and finding how stable the frequency is against changes in cathode potential.

### I. INTRODUCTION

THE klystron, described in this Journal by R. H. Varian and S. F. Varian,<sup>1</sup> consists essentially of two space-resonant oscillators of

<sup>1</sup> Russell H. Varian and Sigurd F. Varian, "A High Frequency Oscillator and Amplifier," J. App. Phys. 10, 321 (1939).

one of the types described previously by Hansen<sup>2</sup> and Hansen and Richtmyer<sup>3</sup> and called rhumbatrons, with various accessories which include a

<sup>2</sup> W. W. Hansen, "A Type of Electrical Resonator," J. App. Phys. 9, 654 (1938).

<sup>3</sup> W. W. Hansen and R. D. Richtmyer, "On Resonators Suitable for Klystron Oscillators," J. App. Phys. 10, 189 (1939).