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EXISTENCE AND BLOW-UP RATE OF LARGE SOLUTIONS OF $p(x)$ -LAPLACIAN EQUATIONS WITH LARGE PERTURBATION AND GRADIENT TERMS

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Abstract. In this paper, we investigate boundary blow-up solutions of the problem

$$\begin{cases} -\Delta_{p(x)}u + f(x, u) = \rho(x, u) + K(x)|\nabla u|^{m(x)} & \text{in } \Omega, \\ u(x) \rightarrow +\infty & \text{as } d(x, \partial\Omega) \rightarrow 0, \end{cases}$$

where $\Delta_{p(x)}u = \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$ is called $p(x)$ -Laplacian. Our results extend the previous work of J. García-Melián, A. Suárez [23] from the case where $p(\cdot) \equiv 2$, without gradient term, to the case where $p(\cdot)$ is a function, with gradient term. It also extends the previous work of Y. Liang, Q.H. Zhang and C.S. Zhao [38] from the radial case in the problem to the non-radial case. The existence of boundary blow-up solutions is established and the singularity of boundary blow-up solution is also studied for several cases including when $\frac{\rho(x, u(x))}{f(x, u(x))} \rightarrow 1$ as $x \rightarrow \partial\Omega$, which means that $\rho(x, u)$ is a large perturbation. We provide an exact estimate of the pointwise different behavior of the solutions near the boundary in terms of $d(x, \partial\Omega)$. Hence, the results of this paper are new even in the canonical case $p(\cdot) \equiv 2$. In particular, we do not have the comparison principle, because we don't make the monotone assumption of nonlinear term.

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1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^N$ ($N \geq 2$) be a bounded domain with C^2 boundary $\partial\Omega$. We consider boundary blow-up solutions of the variable exponent elliptic problem as follows:

$$(P) \quad \begin{cases} -\Delta_{p(x)}u + f(x, u) = \rho(x, u) + K(x)|\nabla u|^{m(x)} & \text{in } \Omega, \\ u(x) \rightarrow +\infty & \text{as } d(x, \partial\Omega) \rightarrow 0, \end{cases}$$

where $-\Delta_{p(x)}u = -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$, with $\nabla u = (\partial_{x_1}u, \partial_{x_2}u, \dots, \partial_{x_N}u)$, is the so-called $p(x)$ -Laplacian. Throughout this paper, we denote the distance from $x \in \Omega$ to $\partial\Omega$ by $d(x) = d(x, \partial\Omega)$.

In recent years, much interest was paid to the study of various mathematical problems with variable exponent (see [1]-[2], [10], [14]-[20], [28]-[31], [35], [38], [40], [42], [45]-[46], [48]-[51], [54]-[55], [59], [61], [64]-[66], [72]-[73] and the references therein). We refer to [1], [6], [60] and [71] for the background of these problems. If $p(\cdot) \equiv p$ (a constant), (P) becomes a p -Laplacian elliptic problem. The $p(x)$ -Laplacian problems with non-constant $p(\cdot)$ are more difficult due to the non-homogeneity of $p(x)$ -Laplacian. The existence of solutions for variable exponent problems was studied in a series of papers [17], [28], [45], [55], [59], [64], [66], [72], while the regularity of the weak solutions for such equations was studied in [1], [15].

It is well known that the elliptic operator in the variable exponent equations is not homogeneous. Due to such difficulties, many times it is hard to judge whether or not results about p -Laplacian can be extended to $p(x)$ -Laplacian, and even if this can be done, it is still difficult to figure out in what forms the results should be. New methods and techniques are needed to study variable exponent equations, since the commonly known ways to study constant exponent equations fail. For example:

The first eigenvalue of $p(x)$ -Laplacian Dirichlet problem was studied in [X.L. Fan, Q.H. Zhang, D. Zhao, *Eigenvalues of $p(x)$ -Laplacian Dirichlet problem, J. Math. Anal. Appl.* 302 (2005), 306-317.], on a smooth bounded domain $\Omega \subset \mathbb{R}^N$ finding through the Rayleigh quotient that

$$\lambda_{p(\cdot)} = \inf_{u \in W_0^{1,p(\cdot)}(\Omega) \setminus \{0\}} \frac{\int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} dx}{\int_{\Omega} \frac{1}{p(x)} |u|^{p(x)} dx},$$

is zero in general, and only under some special conditions $\lambda_{p(\cdot)} > 0$ holds. On the other hand, we all know that $\lambda_p > 0$ plays a very important role in the study of p -Laplacian problems.

The function $\rho(x, u)$ in problem (P) is a perturbation term and we assume that $\rho(x, u) \sim b(x)\left(\frac{1}{d(x)}\right)^{\alpha(x)}|u|^{\theta(x)-2}u$, where $\alpha, b \in C^2(\bar{\Omega}, (0, +\infty))$, $\theta \in C^2(\bar{\Omega})$, $\theta(x) \geq 1$ which means

$$\frac{\rho(x, u)}{b(x)\left(\frac{1}{d(x)}\right)^{\alpha(x)}|u|^{\theta(x)-2}u} \rightarrow 1,$$

as $d(x) \rightarrow 0$ (so when x tends to $\partial\Omega$) and u is large enough. If $\theta(\cdot) > p(\cdot)$, we say that $\rho(x, u)$ is a higher order perturbation, whereas if $\theta(\cdot) < p(\cdot)$ it is called a lower order perturbation. Specifying for any solution $u(x)$ of (P), if $\frac{\rho(x, u)}{f(x, u)} \rightarrow 0$ as $d(x) \rightarrow 0$ we say that $\rho(x, u)$ is a small perturbation, otherwise it is called a large perturbation.

There are many papers on the boundary blow-up solutions of p -Laplacian problems (see [4], [7]-[13], [22]-[27], [33]-[34], [36]-[37], [39]-[41], [47], [52]-[53], [56]-[58], [62], [63], [67]-[69]), but most results deal with lower order perturbation and small perturbation. On the results of the boundary blow-up solutions with gradient terms we refer to [3], [5], [21], [24], [25], [39], [68], [70]. The boundary blow-up solutions of (P) have been especially investigated in the case where $p(\cdot) \equiv p$ (a constant), $K(\cdot) \equiv 0$ and $\rho(x, u) = \lambda|u|^{p-2}u$ (see [12], [22], [41], [62]). Here, generally speaking, $\rho(x, u)$ is a small perturbation having that the boundary blow-up solutions u satisfy $\frac{\lambda|u(x)|^{p-2}u(x)}{f(x, u(x))} \rightarrow 0$ as $x \rightarrow \partial\Omega$. By the localization method of [41], when $p(\cdot) \equiv 2$, $K(\cdot) \equiv 0$ and $\rho(x, u) = \lambda u$, López-Gómez studied the problem (P) in [43] while $f(x, u) = a(x)u^p$, and in [44] while $f(x, u) = f(d(x))N(u)u$ and $N(\cdot)$ is increasing for $u > 0$. It should be emphasized that we do not require the increasing monotonicity of $\rho(x, \cdot)$ and $f(x, \cdot)$.

For the existence and asymptotic behavior of boundary blow-up solutions of elliptic equations with large perturbation the results are rare. To the authors' knowledge, there are only [23], [38], [67]. Specifically, in [67], the existence and blow-up rate of boundary blow-up solutions of (P) with large perturbation are studied when $p(\cdot) \equiv p$ (a constant), $K(\cdot) \equiv 0$ and $\rho(x, u) = \lambda|u|^{\theta-2}u$. In [23], the existence and blow-up rate of boundary blow-up solutions of a system with large perturbation are studied when $p(\cdot) \equiv 2$, $K(\cdot) \equiv 0$ and all the coefficients are constant. In [38], problem (P) is examined under the assumption that it is radial.

In this paper, we discuss the existence and qualitative properties (for instance, the description of the blow-up boundary behavior) of boundary blow-up solutions of (P) in the novel setting of a general function $p(\cdot)$ and of a large perturbation $\rho(x, u)$. Our results extend the previous work [67] and [23]

from the case where p is constant, without gradient term, to the case where $p(\cdot)$ is a function, with gradient term. It also extends the previous work [38] from the radial case in the problem (P) to the non-radial case. The existence of boundary blow-up solutions is established and the singularity of boundary blow-up solution is also studied for several cases including when $\frac{\rho(x, u(x))}{f(x, u(x))} \rightarrow 1$ as $x \rightarrow \partial\Omega$, which means that $\rho(x, u)$ is a large perturbation. We provide an exact pointwise different estimate of the behavior of the solutions near the boundary in terms of $d(x, \partial\Omega)$. Hence, the results of this paper are new even in the canonical case $p(\cdot) \equiv 2$. In particular, we do not have the comparison principle in the case of problem (P).

Before stating our main results, we formulate our assumptions.

- (H₀) $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ and $\rho : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ are nonnegative continuous functions such that for any open subset $\Omega_{\#} \subset \overline{\Omega_{\#}} \subset \subset \Omega$ there are $\varepsilon > 0$ and $k_0 > \varepsilon$ with the properties that $f(x, \varepsilon) \leq \rho(x, \varepsilon)$ on $\Omega_{\#}$, $f(x, k) \geq \rho(x, k)$ on $\Omega_{\#}$ whenever $k \geq k_0$, and $\rho(x, k) \rightarrow +\infty$ as $k \rightarrow +\infty$ uniformly on $\Omega_{\#}$, where k_0 is large enough;
- (H₁) $f(x, s) \sim a(x) \left(\frac{1}{d(x)}\right)^{\beta(x)} |s|^{q(x)-2} s$ and $\rho(x, s) \sim b(x) \left(\frac{1}{d(x)}\right)^{\alpha(x)} |s|^{\theta(x)-2} s$ as $d(x) \rightarrow 0$ uniformly for $|s|$ large, where $a, b \in C^2(\overline{\Omega}, (0, +\infty))$, $\beta, q, \alpha, \theta \in C^2(\overline{\Omega})$, $1 \leq \theta(x) < q(x)$ for all $x \in \overline{\Omega}$;
- (H₂) $p(\cdot) \in C^2(\overline{\Omega})$ satisfies $1 < p^- \leq p^+$, where $p^- = \inf_{\Omega} p(x)$, $p^+ = \sup_{\Omega} p(x)$ and $\beta(x) < p(x) < q(x)$ for all $x \in \overline{\Omega}$;
- (H₃) $1 < m(\cdot) \leq p(\cdot)$ on $\overline{\Omega}$, $K : \Omega \rightarrow \mathbb{R}$ is a nonnegative continuous function satisfying $K(x) \leq C_1(d(x))^{\gamma(x)}$ as $d(x)$ is small enough, with a constant $C_1 > 0$ and a function $\gamma \in C(\overline{\Omega})$ such that

$$\gamma(x) > (s_0(x) + 1)(m(x) + 1 - p(x)) - 1, \quad \forall x \in \partial\Omega, \quad (1.1)$$

where

$$s_0(x) = \max\{s_1(x), s_2(x)\}, \quad s_1(x) = \frac{p(x) - \beta(x)}{q(x) - p(x)}, \quad s_2(x) = \frac{\alpha(x) - \beta(x)}{q(x) - \theta(x)}$$

on $\overline{\Omega}$.

We note that due to (H₃), we have $s_0(x) \geq s_1(x) > 0$ for all $x \in \overline{\Omega}$.

Now, we state our main results.

Theorem 1. *Assume (H₀)-(H₃). Then, for any $\varepsilon \in (0, \frac{1}{2})$, problem (P) has a positive boundary blow-up solution $u_{\varepsilon}(\cdot)$ satisfying*

$$\left(\frac{1}{d(x)}\right)^{(1-\varepsilon)s_0(x)} \leq u_{\varepsilon}(x) \leq \left(\frac{1}{d(x)}\right)^{(1+\varepsilon)s_0(x)},$$

on a neighborhood of $\partial\Omega$.

Theorem 2. Assume (H₀)-(H₃). If $s_0 \in C^1(\overline{\Omega})$, then problem (P) has a positive blow-up solution $u(\cdot)$ whose singularity at $\partial\Omega$ is of the type

$$t_0(x)(d(x))^{-s_0(x)},$$

i.e.,

$$\lim_{d(x) \rightarrow 0} \frac{u(x)}{t_0(x)(d(x))^{-s_0(x)}} = 1,$$

where $t = t_0(x)$ is the unique positive solution of

$$\begin{aligned} & -(s_0)^{p-1}(s_0 + 1)(p - 1)t^{p-1}(d(x))^{\xi_0 - s_0(p-1) - p} + at^{q-1} \\ & = b(d(x))^{\xi_0 - \alpha - s_0(\theta-1)}t^{\theta-1}, \quad \text{for } \xi_0 = \beta + s_0(q - 1). \end{aligned}$$

We point out two direct consequences of the stated theorems that are useful in applications.

Corollary 3. Assume (H₀)-(H₃) and that $s_2(\cdot) \leq s_1(\cdot)$ on $\overline{\Omega}$. Then (P) has a positive boundary blow-up solution $u(\cdot)$ whose singularity at $\partial\Omega$ is of the type $t_1(x)(d(x))^{-s_1(x)}$, i.e.,

$$\lim_{d(x) \rightarrow 0} \frac{u(x)}{t_1(x)(d(x))^{-s_1(x)}} = 1,$$

where $t = t_1(x)$ is the unique solution of

$$-(s_1)^{p-1}(s_1 + 1)(p - \theta)t^{p-1} + at^{q-1} = bt^{\theta-1}(d(x))^{\xi_1 - \alpha - s_1(\theta-1)},$$

with $\xi_1 = \beta + s_1(q - 1)$.

Corollary 4. Assume (H₀)-(H₃) and that $s_2(\cdot) \geq s_1(\cdot)$ on $\overline{\Omega}$. Then (P) has a positive boundary blow-up solution $u(\cdot)$ whose singularity at $\partial\Omega$ is of the type $t_2(x)(d(x))^{-s_2(x)}$, i.e.,

$$\lim_{d(x) \rightarrow 0} \frac{u(x)}{t_2(x)(d(x))^{-s_2(x)}} = 1,$$

where $t = t_2(x)$ is the unique positive solution of

$$-(s_2)^{p-1}(s_2 + 1)(p - 1)t^{p-1}(d(x))^{\xi_2 - s_2(p-1) - p} + at^{q-1} = bt^{\theta-1},$$

with $\xi_2 = \beta + s_2(q - 1)$.

The main difficulties to prove Theorems 1 and 2 are as follows: (i) the non-homogeneity of $p(x)$ -Laplacian, (ii) the gradient term contained in the equation, (iii) lack of comparison principle because we don't make the monotone assumption of nonlinear term, and (iv) the large perturbation.

Example. In problem (P), we let

$$f(x, s) = \begin{matrix} 1 \\ \end{matrix}$$

and $K(x) = C_1(d(x))^{\gamma(x)}$ with a positive constant C_1 , then problem (P) becomes of the typical form

$$\begin{cases} -\Delta_{p(x)}u + a(x)\left(\frac{1}{d(x)}\right)^{\beta(x)}|u|^{q(x)-2}u \\ \quad = b(x)\left(\frac{1}{d(x)}\right)^{\alpha(x)}|u|^{\theta(x)-2}u + C_1(d(x))^{\gamma(x)}|\nabla u|^{m(x)} \text{ in } \Omega, \\ u(x) \rightarrow +\infty \text{ as } d(x, \partial\Omega) \rightarrow 0, \end{cases}$$

where $q > \max(p, \theta)$. In this typical form, $f(x, s)$ and $\rho(x, s)$ are increasing with s . But Theorems 1 and 2 do not need the monotone property of nonlinear term.

The paper is organized as follows. In Section 2, we prove some lemmas that will be used in the proofs of Theorems 1 and 2. In Sections 3 and 4, we present the proofs of Theorems 1 and 2.

2. PRELIMINARIES

In order to deal with problem (P), we recall the functional spaces $L^{p(\cdot)}(\Omega)$, $W^{1,p(\cdot)}(\Omega)$ and properties of $p(x)$ -Laplacian that we will use later (see [20], [35] for more details). Let $S = \{ u : u \text{ is a measurable real-valued function in } \Omega \}$. The variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ is defined by

$$L^{p(\cdot)}(\Omega) = \left\{ u : u \in S, \int_{\Omega} |u(x)|^{p(x)} dx < +\infty \right\},$$

with the norm

$$|u|_{p(\cdot)} = \inf \left\{ \lambda > 0 : \int_{\Omega} \left| \frac{u}{\lambda} \right|^{p(x)} dx \leq 1 \right\}.$$

The space $(L^{p(\cdot)}(\Omega), |\cdot|_{p(\cdot)})$ is a separable and uniformly convex Banach space (see [20, Theorem 1.10, Theorem 1.14]). The variable exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$ is defined by

$$W^{1,p(\cdot)}(\Omega) = \{ u \in L^{p(\cdot)}(\Omega) : \nabla u \in (L^{p(\cdot)}(\Omega))^N \},$$

and is equipped with the norm

$$\|u\| = |u|_{p(\cdot)} + |\nabla u|_{p(\cdot)}, \quad \forall u \in W^{1,p(\cdot)}(\Omega).$$

We denote by $W_0^{1,p(\cdot)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$. We have that $W^{1,p(\cdot)}(\Omega)$ and $W_0^{1,p(\cdot)}(\Omega)$ are separable and uniformly convex Banach spaces (see [20, Theorem 2.1]).

We set

$$f_\#(x, u, \nabla u) = f(x, u) - \rho(x, u) - K(x)|\nabla u|^{m(x)}.$$

Definition 5. A function $u \in W_{loc}^{1,p(\cdot)}(\Omega) \cap L_{loc}^\infty(\Omega)$ is called a boundary blow-up solution of problem (P) if it satisfies

$$\int_Q |\nabla u|^{p(x)-2} \nabla u \cdot \nabla \varphi \, dx + \int_Q f_\#(x, u, \nabla u) \varphi \, dx = 0, \quad \forall \varphi \in W_0^{1,p(\cdot)}(Q) \cap L^\infty(Q),$$

for any domain $Q \subset\subset \Omega$, and $\max\{k - u, 0\} \in W_0^{1,p(\cdot)}(\Omega)$ for every positive integer k .

For any domain $Q \subset\subset \Omega$, for all $u \in W_{loc}^{1,p(\cdot)}(\Omega)$ we define the nonlinear operators

$$F, L : W_{loc}^{1,p(\cdot)}(\Omega) \rightarrow (W_0^{1,p(\cdot)}(Q) \cap L^\infty(Q))^*,$$

by

$$\begin{aligned} \langle Fu, \varphi \rangle &= \int_Q (|\nabla u|^{p(x)-2} \nabla u \cdot \nabla \varphi + f_\#(x, u, \nabla u) \varphi) \, dx, \\ \langle Lu, \varphi \rangle &= \int_Q (|\nabla u|^{p(x)-2} \nabla u \cdot \nabla \varphi + f(x, u) \varphi) \, dx, \end{aligned}$$

$\forall \varphi \in W_0^{1,p(\cdot)}(Q) \cap L^\infty(Q)$, which are globally defined provided

$$f(x, s), \rho(x, s) \leq C(1 + |s|^{p^*(x)-1}),$$

with a constant $C > 0$ and

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if } p(x) < N, \\ +\infty & \text{if } p(x) \geq N. \end{cases}$$

We have:

Lemma 6 ([17], Theorem 3.1). For any domain $Q \subset\subset \Omega$, if $f(x, \cdot)$ is nondecreasing, $q \leq p^*$, $h \in W^{1,p(\cdot)}(Q)$ and $X = h + W_0^{1,p(\cdot)}(Q)$, then $L : X \rightarrow (W_0^{1,p(\cdot)}(Q) \cap L^\infty(Q))^*$ is strictly monotone.

For any $g \in (W_0^{1,p(\cdot)}(Q) \cap L^\infty(Q))^*$, we say that $g \geq 0$ in $(W_0^{1,p(\cdot)}(Q) \cap L^\infty(Q))^*$ if

$$\langle g, \vartheta \rangle \geq 0, \quad \forall \vartheta \in W_0^{1,p(\cdot)}(Q) \cap L^\infty(Q), \quad \vartheta \geq 0 \text{ a.e. in } Q.$$

Naturally, we say that $g \leq 0$ in $(W_0^{1,p(\cdot)}(Q) \cap L^\infty(Q))^*$ if $-g \geq 0$ in $(W_0^{1,p(\cdot)}(Q) \cap L^\infty(Q))^*$.

Definition 7. Let $u \in W_{loc}^{1,p(\cdot)}(\Omega)$. If $Fu \geq 0$ (resp., $Fu \leq 0$) in $(W_0^{1,p(\cdot)}(Q) \cap L^\infty(Q))^*$, for any domain $Q \subset\subset \Omega$, then u is called a weak supersolution (resp., weak subsolution) of problem (P).

Motivated by the proof of [32, Theorem 6.5], we have the following existence result.

Lemma 8. Assume (H₀)–(H₃). Let open sets $\Omega_*, \Omega'_* \subset\subset \Omega$, $\underline{u} \in C(\Omega) \cap C^{1,\eta}(\overline{\Omega_*}) \cap C^{1,\eta}(\Omega \setminus \Omega_*)$, $\bar{u} \in C(\Omega) \cap C^{1,\eta}(\overline{\Omega'_*}) \cap C^{1,\eta}(\Omega \setminus \Omega'_*)$, both positive in Ω , $\eta \in (0, 1)$. If \underline{u} and \bar{u} are a pair of weak subsolution and supersolution of (P) in Ω such that $\underline{u}(x) \leq \bar{u}(x)$ in Ω , then there exists at least one solution $u \in C^1(\Omega)$ of (P) satisfying $\underline{u}(x) \leq u(x) \leq \bar{u}(x)$ in Ω .

Proof. Let $\Omega_n := \{x \in \Omega : d(x) > \frac{1}{n+k_0}\}$, where k_0 is a large enough integer such that $\overline{\Omega_*}, \overline{\Omega'_*} \subset \Omega_1$. Notice that, since $\partial\Omega$ is C^2 , $d(\cdot)$ is of class C^2 near $\partial\Omega$ and $\partial\Omega_n$ is C^2 . We claim that, for every $n \geq 1$, the problem

$$(P_n) \quad \begin{cases} -\Delta_{p(\cdot)}u + f_\#(x, u, \nabla u) = 0 & \text{in } \Omega_n, \\ u = \underline{u} & \text{on } \partial\Omega_n, \end{cases}$$

has a solution. To this end, we introduce the truncation operator

$$S(u) = \begin{cases} \bar{u} & \text{if } u > \bar{u}, \\ u & \text{if } \underline{u} \leq u \leq \bar{u}, \\ \underline{u} & \text{if } u < \underline{u}. \end{cases}$$

Let $\bar{U} = \bar{u} + k$ and $\underline{U} = \underline{u} - k$ in Ω_n , where $k > 0$ is a large enough constant such that $\bar{U} \geq 1, \underline{U} \leq -1$ on Ω_n , and $f_\#(x, S(\bar{u}), \nabla S(\bar{u})) + k \geq 1, f_\#(x, S(\underline{u}), \nabla S(\underline{u})) - k \leq -1$ on Ω_n . Denote

$$M = \max\{f_\#(x, S(\bar{u}), \nabla S(\bar{u})) + k, k - f_\#(x, S(\underline{u}), \nabla S(\underline{u}))\} \geq 1.$$

Thus, $M(\cdot) \in L^\infty(\Omega_n)$. For any $\lambda \in [0, 1]$, we set

$$\begin{aligned} \Phi_\lambda(x, u) &= [\lambda + (1 - \lambda)M(x)]u, \\ \Psi(x, u, \nabla u) &= S(u) - f_\#(x, S(u), \nabla S(u)). \end{aligned}$$

We consider the Dirichlet problem

$$(P_n^*)_\lambda \quad \begin{cases} -\Delta_{p(x)}u + \Phi_\lambda(x, u) = \lambda\Psi(x, u, \nabla u) & \text{in } \Omega_n, \\ u = \underline{u} & \text{on } \partial\Omega_n. \end{cases}$$

It is readily seen that \bar{u} is a supersolution of $(P_n^*)_\lambda$ for $\lambda = 1$, and \underline{u} is a subsolution of $(P_n^*)_\lambda$ for $\lambda = 1$.

First, we prove that every solution u_n of $(P_n^*)_1$ satisfies $\underline{u} \leq u_n \leq \bar{u}$. We only prove that $u_n \leq \bar{u}$ because the proof of $\underline{u} \leq u_n$ is similar. Denote

$$\Upsilon_\lambda(x, u, \nabla u) = \Phi_\lambda(x, u) - \lambda\Psi(x, u, \nabla u).$$

Since \bar{u} is a supersolution of (P), we have

$$\begin{aligned} & \int_{\Omega_n} (|\nabla \bar{u}|^{p-2} \nabla \bar{u} - |\nabla u_n|^{p-2} \nabla u_n) \cdot \nabla (u_n - \bar{u})^+ dx \\ & + \int_{\Omega_n} (\Upsilon_1(x, \bar{u}, \nabla \bar{u}) - \Upsilon_1(x, u_n, \nabla u_n))(u_n - \bar{u})^+ dx \geq 0. \end{aligned}$$

Noticing that

$$(\Upsilon_1(x, \bar{u}, \nabla \bar{u}) - \Upsilon_1(x, u_n, \nabla u_n))(u_n - \bar{u})^+ = (\bar{u} - u_n)(u_n - \bar{u})^+ \leq 0,$$

this implies

$$\int_{\Omega_n} (|\nabla \bar{u}|^{p-2} \nabla \bar{u} - |\nabla u_n|^{p-2} \nabla u_n) \cdot \nabla (u_n - \bar{u})^+ dx \geq 0.$$

Observing that in the above integral the integrand is non-positive enables us to derive that $u_n \leq \bar{u}$. A straightforward consequence of $\underline{u} \leq u_n \leq \bar{u}$ is that every solution of $(P_n^*)_1$ is also a solution of (P_n) .

Next, we prove that $(P_n^*)_1$ has a solution u_n . It is easy to check that on Ω_n there holds

$$\begin{aligned} -\Delta_{p(x)} \bar{U} + M(x) \bar{U} &\geq -\Delta_{p(x)} \bar{U} + M(x) \\ &\geq -\Delta_{p(x)} \bar{u} + f_\#(x, S(\bar{u}), \nabla S(\bar{u})) + k \geq k \geq 0, \quad (2.1) \\ -\Delta_{p(x)} \underline{U} + M(x) \underline{U} &\leq -\Delta_{p(x)} \underline{U} - M(x) \\ &\leq -\Delta_{p(x)} \underline{u} + f_\#(x, S(\underline{u}), \nabla S(\underline{u})) - k \leq -k \leq 0. \end{aligned}$$

Then by (2.1) and the definitions of \bar{U} and $S(\cdot)$, it is easy to see

$$\begin{aligned} & -\Delta_{p(x)} \bar{U} + \Phi_\lambda(x, \bar{U}) - \lambda\Psi(x, \bar{U}, \nabla \bar{U}) \\ & = (1 - \lambda)(-\Delta_{p(x)} \bar{U} + M(x) \bar{U}) + \lambda(\bar{U} - S(\bar{U})) \\ & + \lambda(-\Delta_{p(x)} \bar{U} + f_\#(x, S(\bar{U}), \nabla S(\bar{U}))) \\ & \geq \lambda(-\Delta_{p(x)} \bar{U} + f_\#(x, S(\bar{U}), \nabla S(\bar{U}))) \\ & = \lambda(-\Delta_{p(x)} \bar{u} + f_\#(x, S(\bar{u}), \nabla S(\bar{u}))) \geq 0. \end{aligned}$$

This renders that \bar{U} is a supersolution of $(P_n^*)_\lambda$ for all $\lambda \in [0, 1]$. Similarly, \underline{U} is a subsolution of $(P_n^*)_\lambda$ for all $\lambda \in [0, 1]$. As in the above proof, we can

see that every solution u of $(P_n^*)_\lambda$ (for all $\lambda \in [0, 1]$) satisfies $\underline{U} \leq u \leq \overline{U}$. For any $\zeta(\cdot) \in L^\infty(\Omega)$, we consider the following problem

$$(I) \quad \begin{cases} -\Delta_{p(x)}u + \Phi_\lambda(x, u) = \zeta(x) & \text{in } \Omega_n, \\ u = \underline{u} & \text{on } \partial\Omega_n. \end{cases}$$

On account of that $\Phi_\lambda(x, u)$ is increasing with respect to u , by Lemma 6, problem (I) has a unique solution $u \in L^\infty(\Omega_n)$ (see [19, Theorem 4.1]), which by virtue of [15, Theorem 1.1] satisfies $u \in C^{1,\eta}(\overline{\Omega}_n)$, with some $\eta \in (0, 1)$. Therefore we can construct the operator $T_\lambda : L^\infty(\Omega_n) \rightarrow C^1(\overline{\Omega}_n)$, $T_\lambda(\zeta) = u$. For any bounded set \mathcal{A} of $C^1(\overline{\Omega}_n)$, it is easy to see that $\Psi : \mathcal{A} \rightarrow L^\infty(\Omega_n)$ is bounded, thus $T_\lambda(\lambda\Psi) : \mathcal{A} \rightarrow C^{1,\eta}(\overline{\Omega}_n)$ is bounded, with some $\eta \in (0, 1)$. By the uniform $C^{1,\alpha}(\overline{\Omega}_n)$ estimates and the compactness of the embedding of $C^{1,\alpha}(\overline{\Omega}_n)$ into $C^1(\overline{\Omega}_n)$, it turns out that T_λ is a compact operator, thereby the map $G : [0, 1] \times C^1(\overline{\Omega}_n) \rightarrow C^1(\overline{\Omega}_n)$ given by

$$G(\lambda, u) := T_\lambda(\lambda\Psi(x, u, \nabla u)),$$

is compact. Since all the solutions of $u = G(\lambda, u)$ for any $\lambda \in [0, 1]$ are between \underline{U} and \overline{U} , through standard estimates, one can show that all the solutions of $u = G(\lambda, u)$ for any $\lambda \in [0, 1]$ are uniformly bounded. So the equation $u = G(\lambda, u)$ has no solution on $\partial B(0, R)$, provided $R > 0$ is sufficiently large, where $B(0, R) = \{u \in C^1(\overline{\Omega}_n) : \|u\|_{C^1(\overline{\Omega}_n)} < R, u = \underline{u} \text{ on } \partial\Omega_n\}$. The homotopy invariance property of the Leray–Schauder degree deg_{LS} yields

$$\text{deg}_{\text{LS}}(I - G(1, \cdot), B(0, R), 0) = \text{deg}_{\text{LS}}(I - G(0, \cdot), B(0, R), 0) = 1.$$

Consequently, $(P_n^*)_1$ has a solution u_n . As pointed out before, it results in a solution of (P_n) .

Now, as in the proof of [64, Theorem 3.3], it follows that $\{u_n\}$ has a subsequence converging to a solution u of (P). \square

3. PROOF OF THEOREM 1

Throughout the rest of the paper, we denote $\varphi(x, y) = |y|^{p(x)-2}y$ for all $(x, y) \in \Omega \times \mathbb{R}^N$.

The function $s_0(\cdot) \in C(\overline{\Omega})$ in assumption (H₃) fulfils

$$\beta + s_0(q-1) = \max\{\alpha + s_0(\theta-1), (s_0+1)(p-1) + 1\}. \quad (3.1)$$

Let $\epsilon \in (0, \frac{1}{2})$. Denoting

$$\lambda = \frac{1}{2} \left(1 + \min \left\{ \left(\frac{q-1}{p-1} \right)^-, \left(\frac{q-1}{\theta-1} \right)^- \right\} \right) > 1,$$

there are functions $s_\epsilon(\cdot), s_{-\epsilon}(\cdot) \in C^2(\overline{\Omega})$ satisfying

$$(1 + \epsilon)s_0(x) \leq s_\epsilon(x) \leq (1 + \lambda\epsilon)s_0(x), \quad (3.2)$$

$$(1 - \lambda\epsilon)s_0(x) \leq s_{-\epsilon}(x) \leq (1 - \epsilon)s_0(x). \quad (3.3)$$

This leads to

$$\beta + s_\epsilon(q - 1) > \max\{\alpha + s_\epsilon(\theta - 1), (s_\epsilon + 1)(p - 1) + 1\} \text{ on } \overline{\Omega}, \quad (3.4)$$

$$\beta + s_{-\epsilon}(q - 1) < \max\{\alpha + s_{-\epsilon}(\theta - 1), (s_{-\epsilon} + 1)(p - 1) + 1\} \text{ on } \overline{\Omega}. \quad (3.5)$$

We will prove Theorem 1 in three steps.

Step 1. Construction of a supersolution of (P). For $k > 0$ and $0 < \sigma < 2\delta$, we introduce

$$\tilde{s}_\epsilon(x) = \begin{cases} s_\epsilon(x) & \text{if } d(x) \leq \sigma, \\ \frac{1}{2}(1 + \cos \frac{d(x) - \sigma}{2\delta - \sigma} \pi) s_\epsilon(x) + \frac{1}{2}(1 - \cos \frac{d(x) - \sigma}{2\delta - \sigma} \pi) p^+ & \text{if } \sigma < d(x) < 2\delta, \\ p^+ & \text{if } 2\delta \leq d(x), \end{cases}$$

and

$$g_1(x, \sigma, \epsilon) = \begin{cases} (\frac{1}{d(x)})^{\tilde{s}_\epsilon(x)} + k & \text{if } d(x) < 3\delta, \\ (\frac{1}{3\delta})^{p^+} + k - M_\# \int_{3\delta}^{d(x)} (\frac{4\delta - t}{\delta})^{\frac{2}{p^- - 1}} dt & \text{if } 3\delta \leq d(x) < 4\delta, \\ (\frac{1}{3\delta})^{p^+} + k - M_\# \int_{3\delta}^{4\delta} (\frac{4\delta - t}{\delta})^{\frac{2}{p^- - 1}} dt & \text{if } 4\delta \leq d(x), \end{cases}$$

where $M_\# = p^+(\frac{1}{3\delta})^{p^+ + 1}$.

It holds $g_1(\cdot, \sigma, \epsilon), \tilde{s}_\epsilon(\cdot) \in C^1(\Omega)$ and the gradient of g_1 with respect to $x \in \Omega$ verifies

$$\nabla g_1(x) = -\tilde{s}_\epsilon(x) \left(\frac{1}{d(x)}\right)^{\tilde{s}_\epsilon(x) + 1} \left[\nabla d + (d(x) \ln d(x)) \frac{\nabla \tilde{s}_\epsilon}{\tilde{s}_\epsilon} \right] \text{ if } d(x) < 3\delta. \quad (3.6)$$

Setting $\varpi(x) := (\varpi^1(x), \dots, \varpi^N(x)) = -(d(x) \ln d(x)) \frac{\nabla \tilde{s}_\epsilon}{\tilde{s}_\epsilon}$, we find that $\varpi(x) \rightarrow 0$ as $d(x) \rightarrow 0$. By a direct computation involving (3.6), we obtain

$$\begin{aligned} \Delta_{p(x)} g_1 &= \operatorname{div} \varphi(x, -\tilde{s}_\epsilon(x) \left(\frac{1}{d(x)}\right)^{\tilde{s}_\epsilon(x) + 1} (\nabla d - \varpi(x))) \\ &= -\operatorname{div} \left((\tilde{s}_\epsilon(x))^{p(x) - 1} \left(\frac{1}{d(x)}\right)^{(\tilde{s}_\epsilon(x) + 1)(p(x) - 1)} \varphi(x, \nabla d - \varpi(x)) \right) \\ &= (\tilde{s}_\epsilon(x))^{p(x) - 1} (\tilde{s}_\epsilon(x) + 1) (p(x) - 1) \left(\frac{1}{d(x)}\right)^{(\tilde{s}_\epsilon(x) + 1)(p(x) - 1) + 1} \\ &\quad \times ((\nabla d) \cdot \varphi(x, \nabla d - \varpi(x)) - H(x)), \end{aligned} \quad (3.7)$$

where

$$\begin{aligned}
H(x) &= -d(x) \frac{\ln d(x)}{(\tilde{s}_\epsilon(x) + 1)(p(x) - 1)} \\
&\quad \times \varphi(x, \nabla d - \varpi(x)) \cdot \nabla((\tilde{s}_\epsilon(x) + 1)(p(x) - 1)) \\
&\quad + \frac{d(x) \varphi(x, \nabla d - \varpi(x)) \cdot \nabla((\tilde{s}_\epsilon(x))^{p(x)-1})}{(\tilde{s}_\epsilon(x) + 1)(p(x) - 1)(\tilde{s}_\epsilon(x))^{p(x)-1}} + \frac{d(x) \operatorname{div} \varphi(x, \nabla d - \varpi(x))}{(\tilde{s}_\epsilon(x) + 1)(p(x) - 1)}.
\end{aligned} \tag{3.8}$$

Note that

$$-d(x) \frac{\ln d(x)}{(p(x) - 1)(\tilde{s}_\epsilon(x) + 1)} \varphi(x, \nabla d - \varpi(x)) \cdot \nabla((\tilde{s}_\epsilon(x) + 1)(p(x) - 1)) \rightarrow 0$$

as $d(x) \rightarrow 0$. Since $\varphi(x, \nabla d - \varpi(x)) \cdot \nabla(\tilde{s}_\epsilon(x))^{p(x)-1}$ is bounded, we have

$$\frac{d(x) \varphi(x, \nabla d - \varpi(x)) \cdot \nabla(\tilde{s}_\epsilon(x))^{p(x)-1}}{(\tilde{s}_\epsilon(x) + 1)(p(x) - 1)(\tilde{s}_\epsilon(x))^{p(x)-1}} \rightarrow 0 \text{ as } d(x) \rightarrow 0.$$

Through direct computation, we get

$$\begin{aligned}
\operatorname{div} \varphi(x, \nabla d - \varpi(x)) &= \operatorname{div} [|\nabla d - \varpi(x)|^{p(x)-2} (\nabla d - \varpi(x))] \\
&= |\nabla d - \varpi(x)|^{p(x)-2} [(\nabla d - \varpi(x)) \cdot \nabla p] \ln |\nabla d - \varpi(x)| \\
&\quad + |\nabla d - \varpi(x)|^{p(x)-2} (\Delta d - \operatorname{div} \varpi(x)) \\
&\quad + (p(x) - 2) |\nabla d - \varpi(x)|^{p(x)-3} (\nabla(|\nabla d - \varpi(x)|)) \cdot (\nabla d - \varpi(x)), \tag{3.9}
\end{aligned}$$

where

$$\begin{aligned}
\operatorname{div} \varpi(x) &= \operatorname{div} \left[- (d(x) \ln d(x)) \frac{\nabla \tilde{s}_\epsilon}{\tilde{s}_\epsilon} \right] \\
&= - \frac{(\nabla d \cdot \nabla \tilde{s}_\epsilon)}{\tilde{s}_\epsilon} (1 + \ln d(x)) - (d(x) \ln d(x)) \frac{\Delta \tilde{s}_\epsilon}{\tilde{s}_\epsilon} + (d(x) \ln d(x)) \frac{|\nabla \tilde{s}_\epsilon|^2}{(\tilde{s}_\epsilon)^2},
\end{aligned}$$

and

$$\begin{aligned}
&|(p(x) - 2) |\nabla d - \varpi(x)|^{p(x)-3} (\nabla(|\nabla d - \varpi(x)|)) \cdot (\nabla d - \varpi(x))| \\
&\leq 2p(x) |\nabla d - \varpi(x)|^{p(x)-2} \sum_{i,j=1}^N (|\partial_{x_i} \partial_{x_j} d(x)| + |\partial_{x_i} \varpi^j(x)|).
\end{aligned}$$

Using that $\varpi(x) \rightarrow 0$ in \mathbb{R}^N and $d(x) \operatorname{div} \varpi(x) \rightarrow 0$ in \mathbb{R} as $d(x) \rightarrow 0$, in conjunction with the fact that $|\nabla d| = 1$ near $\partial\Omega$, we obtain

$$d(x) \operatorname{div} \varphi(x, \nabla d - \varpi(x)) \rightarrow 0 \text{ as } d(x) \rightarrow 0.$$

Thus, on the basis of (3.8) and (H_2) , we infer that $H(x) \rightarrow 0$ as $d(x) \rightarrow 0$.

We note that $1 - \epsilon \leq (\nabla d) \cdot \varphi(x, \nabla d - \varpi(x)) - H(x) \leq 1 + \epsilon$ whenever $d(x) \leq \sigma$ provided $\sigma > 0$ is sufficiently small. By (3.7), (3.4), hypothesis (H₁) and since $\min_{\bar{\Omega}} a > 0$, this yields

$$\begin{aligned} \Delta_{p(x)} g_1 &\leq (\tilde{s}_\epsilon)^{p(x)-1} (\tilde{s}_\epsilon + 1) (p-1) \left(\frac{1}{d(x)} \right)^{(\tilde{s}_\epsilon(x)+1)(p(x)-1)+1} (1 + \epsilon) \\ &< \frac{1}{4} a(x) \left(\frac{1}{d(x)} \right)^{\beta(x)} \left[\left(\frac{1}{d(x)} \right)^{\tilde{s}_\epsilon(x)} \right]^{q(x)-1} \\ &\leq \frac{1}{2} f(x, g_1) \text{ whenever } d(x) < \sigma. \end{aligned} \quad (3.10)$$

From (H₁), (H₃), and since $\min_{\bar{\Omega}} a > 0$, if $\delta > 0$ is chosen sufficiently small, we have

$$f(x, s) \geq \frac{1}{2} a(x) \left(\frac{1}{d(x)} \right)^{\beta(x)} s^{q(x)-1}, \quad (3.11)$$

$$\rho(x, s) \leq 2b(x) \left(\frac{1}{d(x)} \right)^{\alpha(x)} s^{\theta(x)-1}, \quad (3.12)$$

$$K(x) \leq C_1 \left(\frac{1}{d(x)} \right)^{-\gamma(x)}, \quad (3.13)$$

whenever $d(x) < 4\delta$ and $s \geq 0$ large. By the definition of g_1 and (3.6), we have

$$\begin{aligned} f(x, g_1) &\geq \frac{1}{2} \min_{\bar{\Omega}} a \left(\frac{1}{d(x)} \right)^{\beta(x)+s_\epsilon(x)(q(x)-1)}, \\ \rho(x, g_1) &\leq 2b(x) \left(\frac{1}{d(x)} \right)^{\alpha(x)+s_\epsilon(x)(\theta(x)-1)}, \\ K(x) |\nabla g_1|^{m(x)} &\leq C'_1 \left(\frac{1}{d(x)} \right)^{-\gamma(x)+m(x)(s_\epsilon(x)+1)}, \end{aligned}$$

whenever $d(x) < \sigma$, where $C'_1 > 0$ is a constant. From (3.4), we know that

$$\beta(x) + s_\epsilon(x)(q(x) - 1) - (\alpha(x) + s_\epsilon(x)(\theta(x) - 1)) \geq C_\epsilon > 0, \quad \forall x \in \bar{\Omega}, \quad (3.14)$$

with a constant $C_\epsilon > 0$ independent of σ . Combining (3.1) and (3.2) implies

$$\begin{aligned} &\beta(x) + s_\epsilon(x)(q(x) - 1) - (-\gamma(x) + m(x)(s_\epsilon(x) + 1)) \\ &\geq \gamma(x) - [(s_0(x) + 1)(m(x) + 1 - p(x)) - 1] \\ &+ \epsilon s_0(x)(q(x) - 1 - m(x)\lambda). \end{aligned}$$

Set

$$\hat{C} = \min_{\partial\Omega} \{ \gamma(x) - [(s_0(x) + 1)(m(x) + 1 - p(x)) - 1] \}.$$

Up to choosing $\epsilon > 0$ small, we may assume that

$$\hat{C} + \epsilon s_0(x)(q(x) - 1 - m(x)\lambda) \geq \frac{1}{2}\hat{C} \text{ on } \partial\Omega.$$

Now, choosing $\sigma > 0$ small, we derive

$$\gamma(x) - [(s_0(x) + 1)(m(x) + 1 - p(x)) - 1] + \epsilon s_0(x)(q(x) - 1 - m(x)\lambda) \geq \frac{1}{4}\hat{C},$$

if $d(x) < \sigma$. This results in

$$\beta(x) + s_\epsilon(x)(q(x) - 1) - (-\gamma(x) + m(x)(s_\epsilon(x) + 1)) \geq \frac{1}{4}\hat{C} \text{ if } d(x) < \sigma. \quad (3.15)$$

Using (3.15), for $\sigma > 0$ even smaller, we may assume that

$$C'_1 \left(\frac{1}{d(x)}\right)^{-\gamma(x) + m(x)(s_\epsilon(x) + 1)} \leq \frac{1}{8} \min a \left(\frac{1}{d(x)}\right)^{\beta(x) + s_\epsilon(x)(q(x) - 1)} \text{ if } d(x) < \sigma$$

and, by (3.14),

$$2b(x) \left(\frac{1}{d(x)}\right)^{\alpha(x) + s_\epsilon(x)(\theta(x) - 1)} \leq \frac{1}{8} \min a \left(\frac{1}{d(x)}\right)^{\beta(x) + s_\epsilon(x)(q(x) - 1)} \text{ if } d(x) < \sigma.$$

Summing up the previous inequalities gives

$$0 \leq \frac{1}{2}f(x, g_1) - \rho(x, g_1) - K(x)|\nabla g_1|^{m(x)} \text{ if } d(x) < \sigma. \quad (3.16)$$

Combining (3.10) and (3.16) amounts to saying that

$$\Delta_{p(x)}g_1 \leq f(x, g_1) - \rho(x, g_1) - K(x)|\nabla g_1|^{m(x)} \text{ if } d(x) < \sigma. \quad (3.17)$$

For a $\delta > 0$ small enough it is true that

$$|\varpi(x)| < \frac{1}{2}, \quad |H(x)| < \frac{1}{2} \text{ if } \sigma < d(x) < 2\delta \text{ or } 2\delta < d(x) < 3\delta.$$

Then, if $\sigma < d(x) < 2\delta$ or $2\delta < d(x) < 3\delta$, we have

$$|(\nabla d) \cdot \varphi(x, \nabla d - \varpi(x)) - H(x)| \leq |\nabla d - \varpi(x)|^{p(x) - 1} + \frac{1}{2} \leq 2^{p^+},$$

and, by (3.7), $\Delta_{p(x)}g_1$ is bounded, namely

$$|\Delta_{p(x)}g_1| \leq 2^{p^+} (\tilde{s}_\epsilon(x))^{p(x) - 1} (\tilde{s}_\epsilon(x) + 1)(p(x) - 1) \left(\frac{1}{\sigma}\right)^{(\tilde{s}_\epsilon(x) + 1)(p(x) - 1) + 1}. \quad (3.18)$$

At this point we can fix k sufficiently large such that

$$\begin{aligned} \Delta_{p(x)}g_1 &\leq \frac{1}{4}a(x) \left(\frac{1}{d(x)}\right)^{\beta(x)} k^{q(x) - 1} \leq \frac{1}{2}f(x, g_1) \\ &\leq f(x, g_1) - \rho(x, g_1) - K(x)|\nabla g_1|^{m(x)}, \end{aligned} \quad (3.19)$$

whenever $\sigma < d(x) < 2\delta$ or $2\delta < d(x) < 3\delta$. The first inequality in (3.19) follows from (3.18) and the fact that $\min_{\overline{\Omega}} a > 0$, the second inequality follows from (3.11) and the definition of g_1 , while for the last inequality we use (3.12), (3.13), the definition of g_1 , the fact that $\nabla g_1(x)$ is bounded for $\sigma < d(x) < 2\delta$ or $2\delta < d(x) < 3\delta$ (see (3.6)), and the fact that $\theta(\cdot) < q(\cdot)$ on $\overline{\Omega}$ (see (H₁)).

Through direct computation we can check that, when $3\delta < d(x) < 4\delta$, it is valid that

$$\nabla g_1(x) = -\omega \left(\frac{4\delta - d(x)}{\delta} \right)^{\frac{2}{p^- - 1}} \nabla d(x), \quad (3.20)$$

where $\omega = p^+ \left(\frac{1}{3\delta} \right)^{p^+ + 1}$, and

$$\begin{aligned} \Delta_{p(x)} g_1 &= \omega^{p(x)-1} \left(\frac{4\delta - d(x)}{\delta} \right)^{\frac{2(p(x)-1)}{p^- - 1} - 1} \left[\frac{2(p(x)-1)}{p^- - 1} \right. \\ &\quad \left. - \frac{4\delta - d(x)}{\delta} \left(\left(\ln \omega \left(\frac{4\delta - d(x)}{\delta} \right)^{\frac{2}{p^- - 1}} \right) \nabla p \cdot \nabla d + \Delta d \right) \right]. \end{aligned}$$

This shows that $|\Delta_{p(x)} g_1|$ is bounded when $3\delta < d(x) < 4\delta$, specifically

$$\begin{aligned} |\Delta_{p(x)} g_1| &\leq \omega^{p(x)-1} \left[\frac{2(p(x)-1)}{p^- - 1} \right. \\ &\quad \left. + \frac{4\delta - d}{\delta} \left(|\nabla p \cdot \nabla d| \left(|\ln \omega| + \left| \ln \left(\frac{4\delta - d}{\delta} \right)^{\frac{2}{p^- - 1}} \right| \right) + |\Delta d| \right) \right]. \end{aligned}$$

Then, in view of (3.11), (3.12), (3.13), the definition of g_1 , and (3.20), for k large, when $3\delta < d(x) < 4\delta$, we have

$$\begin{aligned} \Delta_{p(x)} g_1 &\leq \frac{1}{4} a(x) k^{q(x)-1} \leq \frac{1}{2} f(x, g_1) \\ &\leq f(x, g_1) - \rho(x, g_1) - K(x) |\nabla g_1|^{m(x)}. \end{aligned} \quad (3.21)$$

Finally, in view of (H₀), up to choosing k large enough, it holds

$$\Delta_{p(x)} g_1 = 0 \leq f(x, g_1) - \rho(x, g_1) - K(x) |\nabla g_1|^{m(x)} \text{ if } 4\delta < d(x). \quad (3.22)$$

Combining (3.17), (3.19), (3.21) and (3.22), we obtain that g_1 is a supersolution of problem (P).

Step 2. Construction of a subsolution of (P). Fix $\epsilon > 0$, $\sigma > 0$, and let

$$\phi(x) = \left(\frac{1}{d(x)} \right)^{s-\epsilon(x)}, \quad \phi_\sigma^+ = \max_{d(x)=\sigma} \phi(x),$$

and $\Omega_\sigma^+ = \{x \in \Omega : d(x) < \sigma, \phi(x) > \phi_\sigma^+\}$. We define

$$v_1(x, \sigma, \epsilon) = \begin{cases} \left(\frac{1}{d(x)}\right)^{s-\epsilon(x)} - k^* & \text{if } x \in \Omega_\sigma^+, \\ \epsilon & \text{if } x \in \Omega \setminus \Omega_\sigma^+, \end{cases}$$

where $\epsilon > 0$ is chosen by virtue of hypothesis (H₀) with $\Omega_\# = \Omega \setminus \Omega_\sigma^+$ so that $f(x, \epsilon) \leq \rho(x, \epsilon)$ for all $x \in \Omega \setminus \overline{\Omega_\sigma^+}$, and $k^* = \phi_\sigma^+ - \epsilon$. It is easy to check that $v_1 = v_1(\cdot, \sigma, \epsilon)$ is continuous on Ω , in fact locally Lipschitz, and so it belongs to $W_{loc}^{1,p}(\Omega)$.

By hypothesis (H₁), for $\tau \in (0, \frac{\epsilon}{2}]$, choosing $\sigma = \sigma(\tau) > 0$ sufficiently small, there exists a constant $M_\tau > 0$ such that

$$(1-\tau)a(x)\left(\frac{1}{d(x)}\right)^{\beta(x)}u^{q-1} - M_\tau \leq f(x, u) \leq (1+\tau)a(x)\left(\frac{1}{d(x)}\right)^{\beta(x)}u^{q-1} + M_\tau, \quad (3.23)$$

and

$$(1-\tau)b(x)\left(\frac{1}{d(x)}\right)^{\alpha(x)}u^{\theta-1} - M_\tau \leq \rho(x, u) \leq (1+\tau)b(x)\left(\frac{1}{d(x)}\right)^{\alpha(x)}u^{\theta-1} + M_\tau, \quad (3.24)$$

for all $u \geq 0, d(x) < \sigma$. Denote

$$\begin{aligned} \Omega_1 &= \{x \in \Omega : (s_{-\epsilon} + 1)(p-1) + 1 \geq \alpha + s_{-\epsilon}(\theta-1)\}, \\ \Omega_2 &= \{x \in \Omega : (s_{-\epsilon} + 1)(p-1) + 1 < \alpha + s_{-\epsilon}(\theta-1)\}. \end{aligned}$$

From (3.5), we have

$$\sup [(\beta + s_{-\epsilon}(q-1)) - ((s_{-\epsilon} + 1)(p-1) + 1)] < 0 \text{ in } \Omega_1, \quad (3.25)$$

and

$$\sup [(\beta + s_{-\epsilon}(q-1)) - (\alpha + s_{-\epsilon}(\theta-1))] < 0 \text{ in } \Omega_2. \quad (3.26)$$

As in the proof of Step 1 to derive (3.7), we obtain

$$\begin{aligned} \Delta_{p(x)}\phi &= \operatorname{div} \varphi(x, -s_{-\epsilon}(x)\left(\frac{1}{d(x)}\right)^{s_{-\epsilon}(x)+1}(\nabla d - \varpi(x))) \\ &= -\operatorname{div} \left((s_{-\epsilon}(x))^{p(x)-1} \left(\frac{1}{d(x)}\right)^{(s_{-\epsilon}(x)+1)(p(x)-1)} \varphi(x, \nabla d - \varpi(x)) \right) \\ &= (s_{-\epsilon})^{p(x)-1} (s_{-\epsilon} + 1)(p(x) - 1) \left(\frac{1}{d(x)}\right)^{(s_{-\epsilon}+1)(p(x)-1)+1} \\ &\quad \times ((\nabla d) \cdot \varphi(x, \nabla d - \varpi(x)) - H(x)), \end{aligned}$$

where $\varpi(x)$ and $H(x)$ are defined as in Step 1 with $s_{-\epsilon}$ in place of \tilde{s}_ϵ . As before one can show that $H(x) \rightarrow 0$ in \mathbb{R} as $d(x) \rightarrow 0$ (see (3.8)).

Since $|\nabla d| = 1$ near $\partial\Omega$ and $\varpi(x) \rightarrow 0$ in \mathbb{R}^N as $d(x) \rightarrow 0$, we infer that

$$1 - \epsilon \leq (\nabla d) \cdot \varphi(x, \nabla d - \varpi(x)) - H(x) \leq 1 + \epsilon \text{ if } d(x) \leq \sigma$$

provided $\sigma > 0$ is small enough. Then, according to (3.25), we have

$$\begin{aligned} \Delta_{p(x)} v_1 &\geq (1 - \epsilon)(s_{-\epsilon}(x))^{p(x)-1}(s_{-\epsilon}(x) + 1)(p(x) - 1) \\ &\quad \times \left(\frac{1}{d(x)}\right)^{(s_{-\epsilon}(x)+1)(p(x)-1)+1} \\ &\geq 3a(x) \left(\frac{1}{d(x)}\right)^{\beta(x)} \left[\left(\frac{1}{d(x)}\right)^{s_{-\epsilon}(x)}\right]^{q(x)-1}, \end{aligned} \quad (3.27)$$

for $x \in \Omega_\sigma^+$, hence, by (3.23), when $\tau > 0$ is small, we obtain

$$\begin{aligned} \Delta_{p(x)} v_1 &\geq 3a(x) \left(\frac{1}{d(x)}\right)^{\beta(x)} \left[\left(\frac{1}{d(x)}\right)^{s_{-\epsilon}(x)}\right]^{q(x)-1} \\ &\geq (1 + \tau)a(x) \left(\frac{1}{d(x)}\right)^{\beta(x)} \left[\left(\frac{1}{d(x)}\right)^{s_{-\epsilon}(x)} - k^*\right]^{q(x)-1} + M_\tau \\ &\geq f(x, v_1) \\ &\geq f(x, v_1) - \rho(x, v_1) - K(x)|\nabla v_1|^{m(x)} \text{ in } \Omega_1 \cap \Omega_\sigma^+, \end{aligned} \quad (3.28)$$

provided $\sigma > 0$ is small.

Note that $(d(x))^{s_{-\epsilon}(x)} k^* \leq 1$ for any $x \in \Omega_\sigma^+$. Then, according to (3.23), (3.24), (3.26), and (H₁), choosing $\sigma > 0$ small we have

$$\begin{aligned} &f(x, v_1) - \rho(x, v_1) \\ &\leq (1 + \tau)a(x) \left(\frac{1}{d(x)}\right)^{\beta(x)} v_1^{q-1} - (1 - \tau)b(x) \left(\frac{1}{d(x)}\right)^{\alpha(x)} v_1^{\theta-1} + 2M_\tau \\ &= (1 + \tau)a(x) \left(\frac{1}{d(x)}\right)^{\beta(x)} \left(\frac{1}{d(x)}\right)^{s_{-\epsilon}(x)(q-1)} (1 - (d(x))^{s_{-\epsilon}(x)} k^*)^{q-1} + 2M_\tau \\ &\quad - (1 - \tau)b(x) \left(\frac{1}{d(x)}\right)^{\alpha(x)} \left(\frac{1}{d(x)}\right)^{s_{-\epsilon}(x)(\theta-1)} ((1 - (d(x))^{s_{-\epsilon}(x)} k^*)^{\theta-1}) \\ &\leq \left[(1 + \tau)a(x) \left(\frac{1}{d(x)}\right)^{\beta(x)+s_{-\epsilon}(x)(q-1)} \right. \\ &\quad \left. - (1 - \tau)b(x) \left(\frac{1}{d(x)}\right)^{\alpha(x)+s_{-\epsilon}(x)(\theta-1)} \right] (1 - (d(x))^{s_{-\epsilon}(x)} k^*)^{\theta-1} + 2M_\tau \\ &\leq 2M_\tau \text{ in } \Omega_2 \cap \Omega_\sigma^+. \end{aligned} \quad (3.29)$$

Thus, (3.29) and the blow-up of $\Delta_{p(x)} v_1$ to the boundary (see (3.27)) ensure that

$$\Delta_{p(x)} v_1 \geq f(x, v_1) - \rho(x, v_1) - K(x)|\nabla v_1|^{m(x)} \text{ in } \Omega_2 \cap \Omega_\sigma^+, \quad (3.30)$$

(provided $\sigma > 0$ is small).

Combining (3.28) and (3.30), we see that

$$\Delta_{p(x)} v_1 \geq f(x, v_1) - \rho(x, v_1) - K(x) |\nabla v_1|^{m(x)} \text{ in } \Omega_\sigma^+. \quad (3.31)$$

Now, we show that v_1 is a subsolution of problem (P). For any nonnegative function $w \in C_0^1(\Omega)$ with $\text{supp } w \subset\subset \Omega$, we have

$$\begin{aligned} & \int_{\Omega} (|\nabla v_1|^{p(x)-2} \nabla v_1 \cdot \nabla w + f_{\#}(x, v_1, \nabla v_1) w) dx \\ = & \int_{\Omega_\sigma^+} (|\nabla v_1|^{p(x)-2} \nabla v_1 \cdot \nabla w + f_{\#}(x, v_1, \nabla v_1) w) dx \\ & + \int_{\Omega \setminus \Omega_\sigma^+} (|\nabla v_1|^{p(x)-2} \nabla v_1 \cdot \nabla w + f_{\#}(x, v_1, \nabla v_1) w) dx =: J_1 + J_2. \end{aligned}$$

The definition of v_1 and (3.31) imply

$$\begin{aligned} J_1 &= \int_{\Omega_\sigma^+} (|\nabla \phi|^{p(x)-2} \nabla \phi \cdot \nabla w + f_{\#}(x, v_1, \nabla v_1) w) dx \\ &= \int_{\partial(\Omega \setminus \Omega_\sigma^+)} w |\nabla \phi|^{p(x)-2} \nabla \phi \cdot n_1 dS \\ &+ \int_{\Omega_\sigma^+} (-\Delta_{p(x)} \phi + f_{\#}(x, v_1, \nabla v_1)) w dx \leq 0, \end{aligned}$$

where n_1 is the inward unit normal of $\partial(\Omega \setminus \Omega_\sigma^+)$.

Since $f(x, \varepsilon) < \rho(x, \varepsilon)$ for all $x \in \Omega \setminus \Omega_\sigma^+$, it is clear that

$$J_2 = \int_{\Omega \setminus \Omega_\sigma^+} f_{\#}(x, v_1, \nabla v_1) w dx = \int_{\Omega \setminus \Omega_\sigma^+} (f(x, \varepsilon) - \rho(x, \varepsilon)) w dx \leq 0.$$

Therefore, v_1 is a subsolution of (P).

Step 3. Existence of a solution of (P). In view of Steps 1 and 2, we can apply Lemma 8. The desired conclusion ensues.

4. PROOF OF THEOREM 2

In the sequel, we need some preliminaries. According to the definition of $s_0(\cdot)$, and hypotheses of Theorem 2, we can see that $s_0(\cdot) \in C^{1,1}(\overline{\Omega})$. Let $M = \{x \in \overline{\Omega} \mid s_1(x) = s_2(x)\}$. Since $s_1(\cdot), s_2(\cdot) \in C^2(\overline{\Omega})$, we can see that ∂M is C^2 , compact and has finite $N - 1$ dimensional Hausdorff measure, so the N dimensional measure of ∂M is zero. Since we consider the weak solution, it is in the sense of distribution, without loss of generality, we may assume that $s_0(\cdot) \in C^2(\overline{\Omega})$ when we compute the sub-super-solution of (P).

Let $t_0 = t_0(x)$ be the unique positive solution of the following equation

$$\begin{aligned} at^{q-1} &= bt^{\theta-1}(d(x))^{\xi_0-\alpha-s_0(\theta-1)} \\ &\quad + (s_0)^{p-1}(s_0+1)(p-1)t^{p-1}(d(x))^{\xi_0-s_0(p-1)-p}. \end{aligned} \quad (4.1)$$

We claim that

$$0 < \inf_{x \in \Omega} t_0(x) \leq \sup_{x \in \Omega} t_0(x) < +\infty. \quad (4.2)$$

Indeed, for every $x \in \Omega$, we have either $\xi_0(x) = \alpha(x) + s_0(x)(\theta(x) - 1)$ or $\xi_0(x) = p(x) + s_0(x)(p(x) - 1)$.

In the former case, we obtain $a(x)t_0(x)^{q(x)-\theta(x)} \geq b(x)$, whereas in the latter case, we get

$$a(t)t_0(x)^{q(x)-p(x)} \geq (s_0(x))^{p(x)-1}(s_0(x)+1)(p(x)-1).$$

In both cases, knowing that $a, b, s_0, p > 0$ on $\bar{\Omega}$, we deduce that $t_0(x) \geq c_0 > 0$ for all $x \in \Omega$, for some constant $c_0 > 0$. Finally, since $q > \max\{p, \theta\}$, we see that

$$\begin{aligned} at^{q-1} - bt^{\theta-1}(d(x))^{\xi_0-\alpha-s_0(\theta-1)} \\ - (s_0)^{p-1}(s_0+1)(p-1)t^{p-1}(d(x))^{\xi_0-s_0(p-1)-p} > 0, \end{aligned}$$

for all $x \in \Omega$, all $t > M$, for some constant $M > 0$ (independent of x). This readily implies that $\sup_{x \in \Omega} t_0(x) \leq M < +\infty$. This establishes (4.2).

Denote

$$\begin{aligned} \Theta &= a(q-1)t_0^{q-2} - (s_0)^{p-1}(s_0+1)(p-1)^2t_0^{p-2}(d(x))^{\xi_0-s_0(p-1)-p} \\ &\quad - b(\theta-1)t_0^{\theta-2}(d(x))^{\xi_0-\alpha-s_0(\theta-1)}. \end{aligned}$$

Note from (4.1), (4.2) that, for every $x \in \Omega$,

$$\begin{aligned} \Theta(x) &= \frac{q-1}{t_0} [at_0^{q-1} - (s_0)^{p-1}(s_0+1)(p-1)t_0^{p-1}(d(x))^{\xi_0-s_0(p-1)-p} \\ &\quad - bt_0^{\theta-1}(d(x))^{\xi_0-\alpha-s_0(\theta-1)}] + b(d(x))^{\xi_0-\alpha-s_0(\theta-1)}(q-\theta)t_0^{\theta-2} \quad (4.3) \\ &\quad + (s_0)^{p-1}(s_0+1)(p-1)(q-p)t_0^{p-2}(d(x))^{\xi_0-s_0(p-1)-p} \\ &= b(d(x))^{\xi_0-\alpha-s_0(\theta-1)}(q-\theta)t_0^{\theta-2} \\ &\quad + (s_0)^{p-1}(s_0+1)(p-1)(q-p)t_0^{p-2}(d(x))^{\xi_0-s_0(p-1)-p} \\ &\geq \min \left\{ \inf_{\Omega} [b(q-\theta)t_0^{\theta-2}], \inf_{\Omega} [(s_0)^{p-1}(s_0+1)(p-1)(q-p)t_0^{p-2}] \right\} > 0. \end{aligned}$$

By the implicit function theorem, we have $t_0 \in C^2(\Omega)$ and

$$\Theta \nabla t_0(x) = t_0^{\theta-1} \nabla (b(d(x))^{\xi_0-\alpha-s_0(\theta-1)}) + bt_0^{\theta-1}(d(x))^{\xi_0-\alpha-s_0(\theta-1)} (\ln t_0) \nabla \theta$$

$$\begin{aligned}
& + (s_0)^{p-1}(s_0 + 1)(p - 1)t_0^{p-1}(d(x))^{\xi_0 - s_0(p-1) - p}(\ln t_0)\nabla p \\
& + t_0^{p-1}\nabla((s_0)^{p-1}(s_0 + 1)(p - 1)(d(x))^{\xi_0 - s_0(p-1) - p}) \\
& - t_0^{q-1}\nabla a - at_0^{q-1}(\ln t_0)\nabla q.
\end{aligned} \tag{4.4}$$

Let us observe that for any function $\eta \in C(\overline{\Omega})$, $\eta \geq 0$, it is true that

$$\eta(x)(d(x))^{\eta(x)} \rightarrow 0 \text{ as } d(x) \rightarrow 0. \tag{4.5}$$

In order to prove this assertion, we may assume without loss of generality that $d(x) < 1$. Let any $\delta > 0$. When $\eta(x) < \delta$, we have

$$\eta(x)(d(x))^{\eta(x)} < \delta,$$

whereas if $\eta(x) \geq \delta$ there holds

$$\eta(x)(d(x))^{\eta(x)} \leq \eta^+(d(x))^\delta < \delta,$$

provided $d(x) < (\frac{\delta}{\eta^+ + 1})^{\frac{1}{\delta}}$, where $\eta^+ = \sup_\Omega \eta$. Thus, (4.5) is valid. From (4.5), we obtain that

$$\begin{cases} t_0^{\theta-1}b[\xi_0 - \alpha - s_0(\theta - 1)](d(x))^{\xi_0 - \alpha - s_0(\theta-1)}\nabla d(x) \rightarrow 0, \\ t_0^{p-1}(s_0)^{p-1}(s_0 + 1)(p - 1)[\xi_0 - s_0(p - 1) - p] \\ \times (d(x))^{\xi_0 - s_0(p-1) - p}\nabla d(x) \rightarrow 0, \end{cases} \tag{4.6}$$

as $d(x) \rightarrow 0$. Here, we use that $|\nabla d(x)| = 1$ near $\partial\Omega$, $\xi_0 - \alpha - s_0(\theta - 1) \geq 0$ and $\xi_0 - s_0(p - 1) - p \geq 0$. By (4.4) and (4.6), we deduce that $\Theta d(x)\nabla t_0(x) \rightarrow 0$ as $d(x) \rightarrow 0$, so

$$d(x)\nabla t_0(x) \rightarrow 0 \text{ as } d(x) \rightarrow 0, \tag{4.7}$$

due to (4.3).

Furthermore, a similar reasoning as for (4.4), based on the implicit function theorem, together with the use of (4.5), leads to the conclusion that $\Theta d(x)^2\Delta t_0 \rightarrow 0$ as $d(x) \rightarrow 0$, hence, in view of (4.3), it follows that

$$d(x)^2\Delta t_0(x) \rightarrow 0 \text{ as } d(x) \rightarrow 0. \tag{4.8}$$

Denote

$$\psi(x) = (1 + \epsilon)t_0(x)(d(x))^{-s_0(x)},$$

for some $\epsilon \in (0, 1)$. Note that

$$\nabla\psi(x) = (1 + \epsilon)\left[(d(x))^{-s_0}\nabla t_0 - (d(x))^{-s_0}t_0\left((\ln d(x))\nabla s_0 - \frac{s_0}{d(x)}\nabla d\right)\right].$$

By (H₃), we infer that

$$K(x)|\nabla\psi(x)|^{m(x)} \leq C_1(1 + \epsilon)^m(d(x))^{-m(s_0+1)+\gamma}(d(x)|\nabla t_0|)$$

$$+t_0d(x)(\ln d(x))|\nabla s_0| + s_0t_0|\nabla d|),$$

when $d(x)$ is small. Note that $d(x)|\nabla t_0| \rightarrow 0$ and $t_0d(x)(\ln d(x))|\nabla s_0| \rightarrow 0$ as $d(x) \rightarrow 0$ (see (4.7)) whereas $s_0t_0|\nabla d|$ is bounded near $\partial\Omega$. Also note that $-m(s_0 + 1) + \gamma > -\beta - s_0(q - 1)$ near $\partial\Omega$ (using (1.1) and the facts that $m > 1$ and $s_0 \geq s_1 = \frac{p-\beta}{q-p}$ (see (H₃))). We deduce that

$$K(x)|\nabla\psi(x)|^{m(x)}(d(x))^{\beta+s_0(q-1)} \rightarrow 0 \text{ as } d(x) \rightarrow 0.$$

Set $a_0(x) = a(x)t_0^{q-1}$, $b_0(x) = b(x)t_0^{\theta-1}(d(x))^{\xi_0-\alpha-s_0(\theta-1)}$, and $\tilde{b}_0(x) = (t_0s_0)^{p-1}(s_0 + 1)(p - 1)(d(x))^{\xi_0-s_0(p-1)-p}$. By definition of $t_0(x)$, we have $a_0(x) = b_0(x) + \tilde{b}_0(x)$. Note also that $\inf_{\Omega} a_0 > 0$ (see (4.2)). Since $q > \max\{\theta, p\}$ in $\bar{\Omega}$, for $\tau > 0$ small enough, we have

$$\begin{aligned} (1 - \tau)(1 + \epsilon)^{q(x)-1} - (1 + \tau)(1 + \epsilon)^{\theta(x)-1} &\geq \tilde{c}, \\ (1 - \tau)(1 + \epsilon)^{q(x)-1} - (1 + \tau)(1 + \epsilon)^{p(x)-1} &\geq \tilde{c}, \end{aligned}$$

for all $x \in \bar{\Omega}$, for some constant $\tilde{c} > 0$. Then, for $\tau > 0$ small, given any constant $M_\tau > 0$, when $\sigma > 0$ is small enough, we have

$$\begin{aligned} &(1 - \tau)(1 + \epsilon)^{q-1}a_0(x) - (1 + \tau)(1 + \epsilon)^{\theta-1}b_0(x) \\ &\quad - (1 + \tau)(1 + \epsilon)^{p-1}\tilde{b}_0(x) - 2M_\tau(d(x))^{\xi_0} \\ &\geq b_0(x)\tilde{c} + \tilde{b}_0(x)\tilde{c} - 2M_\tau((d(x))^{\xi_0}) \\ &\geq (\inf_{\Omega} a_0)\tilde{c} - 2M_\tau(d(x))^{\xi_0} > 0, \end{aligned}$$

whenever $d(x) < \sigma$. This reads as

$$\begin{aligned} &-(1 + \tau)(1 + \epsilon)^{\theta-1}b(x)t_0^{\theta-1}(d(x))^{\xi_0-\alpha-s_0(\theta-1)} + (1 - \tau)(1 + \epsilon)^{q-1}a(x)t_0^{q-1} \\ &> (1 + \tau)(1 + \epsilon)^{p-1}(t_0s_0)^{p-1}(s_0 + 1)(p - 1)(d(x))^{\xi_0-s_0(p-1)-p} \\ &\quad + 2M_\tau(d(x))^{\xi_0}, \end{aligned} \tag{4.9}$$

whenever $d(x) < \sigma$.

For $\delta > 0$ and for $\sigma \in (0, 2\delta)$ small, we introduce

$$g_2(x, \sigma, \epsilon) = \begin{cases} (1 + \epsilon)\tilde{t}_0(x)\left(\frac{1}{d(x)}\right)^{\tilde{s}_0(x)} + k & \text{if } d(x) < 3\delta, \\ C^+(\frac{1}{3\delta})^{p^+} + k - M^+ \int_{3\delta}^{d(x)} \left(\frac{4\delta-t}{\delta}\right)^{\frac{2}{p^-}-1} dt & \text{if } 3\delta \leq d(x) < 4\delta, \\ C^+(\frac{1}{3\delta})^{p^+} + k - M^+ \int_{3\delta}^{4\delta} \left(\frac{4\delta-t}{\delta}\right)^{\frac{2}{p^-}-1} dt & \text{if } 4\delta \leq d(x), \end{cases}$$

where $M^+ = C^+ p^+ (\frac{1}{3\delta})^{p^++1}$, $C^+ = (1 + \epsilon)p^+$,

$$\tilde{s}_0(x) = \begin{cases} s_0(x) & \text{if } d(x) \leq \sigma, \\ \frac{1}{2}(1 + W(x))s_0(x) + \frac{1}{2}(1 - W(x))p^+ & \text{if } \sigma < d(x) < 2\delta, \\ p^+ & \text{if } 2\delta \leq d(x) \leq 3\delta, \end{cases}$$

$$\tilde{t}_0(x) = \begin{cases} t_0(x) & \text{if } d(x) \leq \sigma, \\ \frac{1}{2}(1 + W(x))t_0(x) + \frac{1}{2}(1 - W(x))p^+ & \text{if } \sigma < d(x) < 2\delta, \\ p^+ & \text{if } 2\delta \leq d(x) \leq 3\delta, \end{cases}$$

where $W(x) = \cos \frac{d(x) - \sigma}{2\delta - \sigma} \pi$; $k = k_1 + k_2 + k_3$ and

$$k_1 = \max_{x \in (\partial\Omega)_{3\delta}} \left[\frac{4}{a(x)} 2^{p^+} ((1 + \epsilon)\tilde{t}_0\tilde{s}_0)^{p(x)-1} (\tilde{s}_0 + 1)(p(x) - 1) \times \left(\frac{1}{\sigma}\right)^{(\tilde{s}_0+1)(p-1)+1+\beta} \right]^{\frac{1}{q(x)-1}},$$

$$k_2 = \max_{3\delta \leq d(x) \leq 4\delta} \left\{ \frac{4}{a(x)} \omega^{p(x)-1} \times \left[\frac{2(p(x) - 1)}{\delta(p^- - 1)} + |\nabla p| \left| \ln \left(\omega \left(\frac{4\delta - d}{\delta} \right)^{\frac{2}{p^- - 1}} \right) \right| + |\Delta d| \right] \right\}^{\frac{1}{q(x)-1}},$$

$$k_3 \geq k_0 + C^+ p^+ \left(\frac{1}{3\delta}\right)^{p^++1} \int_{3\delta}^{4\delta} \left(\frac{4\delta - t}{\delta}\right)^{\frac{2}{p^- - 1}} dt \quad \text{with } k_0 \text{ as in (H}_0\text{)},$$

with $\omega = C^+ p^+ (\frac{1}{3\delta})^{p^++1}$. Obviously, $\tilde{s}_0(\cdot) \in C^1(\overline{(\partial\Omega)_{3\delta}})$, $\tilde{t}_0(\cdot) \in C^1((\partial\Omega)_{3\delta})$, where $(\partial\Omega)_{3\delta} = \{x \in \Omega : d(x) < 3\delta\}$, and then $g_2(\cdot, \sigma, \epsilon) \in C^1(\Omega)$.

Lemma 9. g_2 is a supersolution of (P).

Proof. We find that

$$\begin{aligned} \nabla g_2 &= -(1 + \epsilon)\tilde{t}_0(x)\tilde{s}_0(x) \left(\frac{1}{d(x)}\right)^{\tilde{s}_0(x)+1} \\ &\quad \times \left[\nabla d - \frac{d(x)}{t_0\tilde{s}_0(x)} \nabla \tilde{t}_0 + (d(x) \ln d(x)) \frac{\nabla \tilde{s}_0}{\tilde{s}_0} \right] \end{aligned}$$

when $d(x) < 3\delta$.

Denote $\varpi(x) = \frac{d(x)}{t_0\tilde{s}_0} \nabla \tilde{t}_0 - (d(x) \ln d(x)) \frac{\nabla \tilde{s}_0}{\tilde{s}_0}$. From (4.7), it follows that $|\varpi(x)| \rightarrow 0$ as $d(x) \rightarrow 0$.

A straightforward computation provides

$$\begin{aligned}
-\Delta_{p(x)}g_2 &= \operatorname{div} \left[((1 + \epsilon)\tilde{t}_0\tilde{s}_0)^{p(x)-1} \left(\frac{1}{d(x)} \right)^{(\tilde{s}_0+1)(p(x)-1)} \varphi(x, \nabla d - \varpi(x)) \right] \\
&= -((1 + \epsilon)\tilde{t}_0\tilde{s}_0)^{p(x)-1} (\tilde{s}_0 + 1)(p(x) - 1) \left(\frac{1}{d(x)} \right)^{(\tilde{s}_0+1)(p(x)-1)+1} \\
&\times [(\nabla d) \cdot \varphi(x, \nabla d - \varpi(x)) - H(x)], \tag{4.10}
\end{aligned}$$

where

$$\begin{aligned}
H(x) &= -d(x) \frac{\ln d(x)}{(\tilde{s}_0 + 1)(p(x) - 1)} \varphi(x, \nabla d - \varpi(x)) \cdot \nabla((\tilde{s}_0 + 1)(p(x) - 1)) \\
&\quad + \frac{d(x) \varphi(x, \nabla d - \varpi(x)) \cdot \nabla((1 + \epsilon)\tilde{t}_0\tilde{s}_0)^{p(x)-1}}{(\tilde{s}_0 + 1)(p(x) - 1)((1 + \epsilon)\tilde{t}_0\tilde{s}_0)^{p(x)-1}} \\
&\quad + \frac{d(x) \operatorname{div} \varphi(x, \nabla d - \varpi(x))}{(\tilde{s}_0 + 1)(p(x) - 1)}. \tag{4.11}
\end{aligned}$$

It is clear that

$$-d(x) \frac{\ln d(x)}{(p(x) - 1)(\tilde{s}_0 + 1)} \varphi(x, \nabla d - \varpi(x)) \cdot \nabla((\tilde{s}_0 + 1)(p(x) - 1)) \rightarrow 0,$$

as $d(x) \rightarrow 0$. By (4.7), we have

$$\frac{d(x) \varphi(x, \nabla d - \varpi(x)) \cdot \nabla((1 + \epsilon)\tilde{t}_0\tilde{s}_0)^{p(x)-1}}{(\tilde{s}_0 + 1)(p(x) - 1)((1 + \epsilon)\tilde{t}_0\tilde{s}_0)^{p(x)-1}} \rightarrow 0 \text{ as } d(x) \rightarrow 0.$$

As for (3.9), we have

$$\begin{aligned}
d(x) \operatorname{div} \varphi(x, \nabla d - \varpi(x)) &= d(x) \operatorname{div} [|\nabla d - \varpi(x)|^{p(x)-2} (\nabla d - \varpi(x))] \\
&= d(x) |\nabla d - \varpi(x)|^{p(x)-2} [(\nabla d - \varpi(x)) \cdot \nabla p] \ln |\nabla d - \varpi(x)| \\
&\quad + |\nabla d - \varpi(x)|^{p(x)-2} (d(x) \Delta d - d(x) \operatorname{div} \varpi(x)) \\
&\quad + d(x) (p(x) - 2) |\nabla d - \varpi(x)|^{p(x)-3} (\nabla(|\nabla d - \varpi(x)|)) \cdot (\nabla d - \varpi(x)),
\end{aligned}$$

where

$$\begin{aligned}
d(x) \operatorname{div} \varpi(x) &= d(x) \operatorname{div} \left[\frac{d(x)}{\tilde{t}_0\tilde{s}_0} \nabla \tilde{t}_0 - (d(x) \ln d(x)) \frac{\nabla \tilde{s}_0}{\tilde{s}_0} \right] \\
&= d(x) \operatorname{div} \left(\frac{d(x)}{\tilde{t}_0\tilde{s}_0} \nabla \tilde{t}_0 \right) - d(x) \frac{(\nabla d \cdot \nabla \tilde{s}_0)}{\tilde{s}_0} (1 + \ln d(x)) \\
&\quad - d(x) (d(x) \ln d(x)) \frac{\Delta \tilde{s}_0}{\tilde{s}_0} + d(x) (d(x) \ln d(x)) \frac{|\nabla \tilde{s}_0|^2}{\tilde{s}_0^2}.
\end{aligned}$$

Notice that $|\nabla \tilde{s}_0|$ and $|\Delta \tilde{s}_0|$ are bounded in $\{x \in \Omega : d(x) < \sigma \text{ or } \sigma < d(x) < 2\delta \text{ or } 2\delta < d(x) < 3\delta\}$ (using the assumption that $s_0 \in C^{1,1}(\bar{\Omega})$). By (4.8), we can see that

$$d(x) \operatorname{div} \left(\frac{d(x)}{\tilde{t}_0 \tilde{s}_0} \nabla \tilde{t}_0 \right) \rightarrow 0 \text{ as } d(x) \rightarrow 0.$$

This implies that $d(x) \operatorname{div} \varpi(x) \rightarrow 0$ as $d(x) \rightarrow 0$, so

$$d(x) \operatorname{div} \varphi(x, \nabla d - \varpi(x)) \rightarrow 0 \text{ as } d(x) \rightarrow 0,$$

which enables us to conclude that $H(x) \rightarrow 0$ as $d(x) \rightarrow 0$.

For any fixed $\tau > 0$, when $\delta > 0$ is small enough, we have

$$1 - \frac{\tau}{2} \leq (\nabla d) \cdot \varphi(x, \nabla d - \varpi(x)) - H(x) \leq 1 + \frac{\tau}{2} \text{ when } d(x) \leq 3\delta.$$

Then, from (4.10), we derive

$$\Delta_{p(x)} g_2 \leq ((1 + \epsilon) \tilde{t}_0 \tilde{s}_0)^{p(x)-1} (\tilde{s}_0 + 1) (p(x) - 1) \left(\frac{1}{d(x)} \right)^{(\tilde{s}_0+1)(p-1)+1} \left(1 + \frac{\tau}{2} \right).$$

On the other hand, in view of (H₃), (1.1), the expression of ∇g_2 , and (4.2), for $\delta > 0$ even smaller if necessary, it turns out that

$$K(x) |\nabla g_2|^m \leq \frac{\tau}{2} ((1 + \epsilon) \tilde{t}_0 \tilde{s}_0)^{p(x)-1} (\tilde{s}_0 + 1) (p(x) - 1) \left(\frac{1}{d(x)} \right)^{(\tilde{s}_0+1)(p-1)+1},$$

whenever $d(x) < 3\delta$. It follows that

$$\begin{aligned} \Delta_{p(x)} g_2 + K(x) |\nabla g_2|^{m(x)} &\leq (1 + \tau) ((1 + \epsilon) \tilde{t}_0 \tilde{s}_0)^{p(x)-1} (\tilde{s}_0 + 1) \\ &\quad \times (p(x) - 1) \left(\frac{1}{d(x)} \right)^{(\tilde{s}_0+1)(p-1)+1}, \end{aligned} \quad (4.12)$$

when $d(x) < 3\delta$.

From (H₁), arguing as for (3.23) and (3.24), for a fixed $\tau > 0$, we can find $\sigma_0(\tau) \in (0, \delta)$ and $M_0(\tau) > 0$ such that, whenever $d(x) < \sigma_0(\tau)$ and $|s| > M_0(\tau)$, we have

$$f(x, g_2) - \rho(x, g_2) \geq (1 - \tau) a(x) \left(\frac{1}{d(x)} \right)^\beta g_2^{q-1} - (1 + \tau) b(x) \left(\frac{1}{d(x)} \right)^\alpha g_2^{\theta-1} - 2M_\tau.$$

Hence, for $d(x) < \sigma$, we have

$$\begin{aligned} &f(x, g_2) - \rho(x, g_2) \\ &\geq (1 - \tau) (1 + \epsilon)^{q-1} a(x) t_0^{q-1} \left(\frac{1}{d(x)} \right)^{\beta+s_0(q-1)} \left[1 + \frac{k(d(x))^{s_0}}{(1 + \epsilon)t_0} \right]^{q-1} \\ &\quad - (1 + \tau) (1 + \epsilon)^{\theta-1} b(x) t_0^{\theta-1} \left(\frac{1}{d(x)} \right)^{\alpha+s_0(\theta-1)} \left[1 + \frac{k(d(x))^{s_0}}{(1 + \epsilon)t_0} \right]^{\theta-1} - 2M_\tau. \end{aligned}$$

By the fact that $q > \theta$ (see (H₁)), (4.9), (4.12) and the above inequality, by choosing $\sigma > 0$ sufficiently small, for $d(x) < \sigma$, we get

$$\begin{aligned}
& f(x, g_2) - \rho(x, g_2) \\
& \geq \left(\frac{1}{d(x)}\right)^{\xi_0} \left[1 + \frac{k(d(x))^{s_0}}{(1+\epsilon)t_0}\right]^{\theta-1} \left((1-\tau)(1+\epsilon)^{q-1}a(x)t_0^{q-1} \right. \\
& \quad \left. - (1+\tau)(1+\epsilon)^{\theta-1}b(x)t_0^{\theta-1}(d(x))^{\xi_0-\alpha-s_0(\theta-1)}\right) - 2M_\tau \\
& \geq \left(\frac{1}{d(x)}\right) \left((1+\tau)(1+\epsilon)^{p-1}(t_0s_0)^{p-1}(s_0+1)(p-1)(d(x))^{\xi_0-s_0(p-1)-p} \right. \\
& \quad \left. + 2M_\tau(d(x))^{\xi_0}\right) - 2M_\tau \\
& = (1+\tau)(1+\epsilon)^{p-1}(t_0s_0)^{p-1}(s_0+1)(p-1)\left(\frac{1}{d(x)}\right)^{(s_0+1)(p-1)+1} \\
& \geq \Delta_{p(x)}g_2 + K(x)|\nabla g_2|^{m(x)}. \tag{4.13}
\end{aligned}$$

Without loss of generality, we may assume that $\delta > 0$ is small enough such that

$$|\varpi(x)| < \frac{1}{2}, \quad |H(x)| < \frac{1}{2} \quad \text{when } \sigma < d(x) < 2\delta \text{ or } 2\delta < d(x) < 3\delta,$$

which yields

$$|(\nabla d) \cdot \varphi(x, \nabla d - \varpi(x)) - H(x)| \leq 2^{p^+} \quad \text{when } \sigma < d(x) < 2\delta \text{ or } 2\delta < d(x) < 3\delta.$$

In view of (H₁) and (H₃), up to choosing $\delta > 0$ smaller if necessary, we may assume that, whenever $\sigma < d(x) < 2\delta$ or $2\delta < d(x) < 3\delta$ or $3\delta < d(x) < 4\delta$, we have

$$\frac{1}{4}a(x)g_2^{q(x)-1} \leq \frac{1}{2}f(x, g_2) \leq f(x, g_2) - \rho(x, g_2) - K(x)|\nabla g_2|^{m(x)}, \tag{4.14}$$

(arguing as for (3.16)). Note that the definition of k guarantees that we have

$$g_2(x) \geq k_1 + k_2 + k_3 \quad \text{for all } x \in \Omega. \tag{4.15}$$

When $\sigma < d(x) < 2\delta$ or $2\delta < d(x) < 3\delta$, from (4.10) we see that $|\Delta_{p(x)}g_2|$ is bounded, namely

$$|\Delta_{p(x)}g_2| \leq 2^{p^+} \left((1+\epsilon)\tilde{t}_0\tilde{s}_0\right)^{p(x)-1}(\tilde{s}_0+1)(p(x)-1)\left(\frac{1}{\sigma}\right)^{(\tilde{s}_0+1)(p(x)-1)+1}.$$

Using this relation, (4.14), and (4.15), for $\sigma < d(x) < 2\delta$ or $2\delta < d(x) < 3\delta$, we obtain that

$$\Delta_{p(x)}g_2 \leq \frac{1}{4}a(x)k_1^{q(x)-1} \leq f(x, g_2) - \rho(x, g_2) - K(x)|\nabla g_2|^{m(x)} \tag{4.16}$$

By computation, when $3\delta < d(x) < 4\delta$, we have

$$\begin{aligned} \Delta_{p(x)}g_2 &= \omega^{p(x)-1} \left(\frac{4\delta - d}{\delta} \right)^{\frac{2(p(x)-1)}{p^- - 1} - 1} \left\{ \frac{2(p(x) - 1)}{\delta(p^- - 1)} \right. \\ &\quad \left. - \frac{4\delta - d}{\delta} \left[\ln \left(\omega \left(\frac{4\delta - d}{\delta} \right)^{\frac{2}{p^- - 1}} \right) \nabla p \cdot \nabla d + \Delta d \right] \right\}, \end{aligned}$$

with $\omega = C^+ p^+ (\frac{1}{3\delta})^{p^+ + 1}$ (as before). Then $|\Delta_{p(x)}g_2|$ is bounded when $3\delta < d(x) < 4\delta$, namely

$$|\Delta_{p(x)}g_2| \leq \omega^{p(x)-1} \left[\frac{2(p(x) - 1)}{\delta(p^- - 1)} + |\nabla p| \left| \ln \left(\omega \left(\frac{4\delta - d}{\delta} \right)^{\frac{2}{p^- - 1}} \right) \right| + |\Delta d| \right].$$

When $3\delta < d(x) < 4\delta$, using (4.14), (4.15), and the definition of k_2 leads to

$$\Delta_{p(x)}g_2 \leq \frac{1}{4} a(x) k_2^{q(x)-1} \leq f(x, g_2) - \rho(x, g_2) - K(x) |\nabla g_2|^{m(x)}. \quad (4.17)$$

Finally, by (4.15) and (H₀), there holds

$$\Delta_{p(x)}g_2 = 0 \leq f(x, g_2) - \rho(x, g_2) - K(x) |\nabla g_2|^{m(x)} \text{ when } 4\delta < d(x). \quad (4.18)$$

On the basis of (4.13), (4.16), (4.17), (4.18), we conclude that g_2 is a super-solution of (P). \square

Next, we will construct a subsolution of (P). Set

$$\phi(x) = t_0(x) \left(\frac{1}{d(x)} \right)^{s_0(x)} \quad \text{and} \quad \phi_\sigma^+ = \max_{d(x)=\sigma} \phi(x). \quad (4.19)$$

Let

$$\Omega_\sigma^+ = \{x \in \Omega : d(x) < \sigma, \phi(x) > \phi_\sigma^+\}. \quad (4.20)$$

Given $\epsilon \in (0, \frac{1}{2}]$ and $\sigma > 0$, we define

$$v_2(x, \sigma, \epsilon) = \begin{cases} (1 - \epsilon) t_0(x) \left(\frac{1}{d(x)} \right)^{s_0(x)} - k^* & \text{if } x \in \Omega_\sigma^+, \\ \epsilon & \text{if } x \in \Omega \setminus \Omega_\sigma^+, \end{cases}$$

where $\epsilon > 0$ is as in (H₀) to satisfy $f(x, \epsilon) \leq \rho(x, \epsilon)$ for all $x \in \Omega \setminus \Omega_\sigma^+$, and $k^* = (1 - \epsilon) \phi_\sigma^+ - \epsilon$. It is easy to see that $v_2(\cdot, \sigma, \epsilon) \in C(\Omega)$, in fact it is locally Lipschitz on Ω and so it belongs to $W_{loc}^{1,p}(\Omega)$. We choose σ sufficiently small so that $v_2 > 0$ in Ω .

Lemma 10. v_2 is a subsolution of (P).

Proof. Proceeding as in the proof of Lemma 9, we have

$$-\Delta_{p(x)}v_2 = -((1 - \epsilon)t_0s_0)^{p(x)-1}(s_0 + 1)(p(x) - 1)\left(\frac{1}{d(x)}\right)^{(s_0+1)(p(x)-1)+1} [(\nabla d) \cdot \varphi(x, \nabla d - \varpi(x)) - H(x)],$$

where

$$\varpi(x) = \frac{d(x)}{t_0s_0} \nabla t_0 - (d(x) \ln d(x)) \frac{\nabla s_0}{s_0},$$

and

$$\begin{aligned} H(x) &= -d(x) \frac{\ln d(x)}{(s_0 + 1)(p(x) - 1)} \varphi(x, \nabla d - \varpi(x)) \cdot \nabla((s_0 + 1)(p(x) - 1)) \\ &\quad + \frac{d(x) \varphi(x, \nabla d - \varpi(x)) \cdot \nabla((1 - \epsilon)t_0s_0)^{p(x)-1}}{(s_0 + 1)(p(x) - 1)((1 - \epsilon)t_0s_0)^{p(x)-1}} \\ &\quad + \frac{d(x) \operatorname{div} \varphi(x, \nabla d - \varpi(x))}{(s_0 + 1)(p(x) - 1)}. \end{aligned}$$

From the proof of Lemma 9, we see that $\varpi(x) \rightarrow 0$ and $H(x) \rightarrow 0$ as $d(x) \rightarrow 0$. For any fixed $\tau \in (0, 1)$, when $\sigma > 0$ is small enough, we have

$$1 - \tau \leq (\nabla d) \cdot \varphi(x, \nabla d - \varpi(x)) - H(x) \leq 1 + \tau, \quad \forall x \in \Omega_\sigma^+,$$

whence

$$\begin{aligned} \Delta_{p(x)}v_2 &\geq (1 - \tau)(1 - \epsilon)^{p-1}(t_0s_0)^{p(x)-1}(s_0 + 1)(p - 1) \\ &\quad \times \left(\frac{1}{d(x)}\right)^{s_0(x)(p(x)-1)+p}, \quad \text{in } \Omega_\sigma^+. \end{aligned} \quad (4.21)$$

In view of (4.21), up to choosing $\sigma > 0$ small enough, we may assume that

$$\Delta_{p(x)}v_2 \geq 2M_\tau, \quad \forall x \in \Omega_\sigma^+. \quad (4.22)$$

From (3.23) and (3.24), we deduce

$$\begin{aligned} f(x, v_2) - \rho(x, v_2) &\leq (1 + \tau)a(x) \left(\frac{1}{d(x)}\right)^{\beta(x)} v_2^{q-1} + 2M_\tau \\ &\quad - (1 - \tau)b(x) \left(\frac{1}{d(x)}\right)^{\alpha(x)} v_2^{\theta-1}, \end{aligned} \quad (4.23)$$

when $d(x) < \sigma$, in particular if $x \in \Omega_\sigma^+$.

Let $x \in \Omega_\sigma^+$. First, assume that

$$(1 + \tau)a(x) \left(\frac{1}{d(x)}\right)^{\beta(x)} v_2^{q-\theta} - (1 - \tau)b(x) \left(\frac{1}{d(x)}\right)^{\alpha(x)} \leq 0.$$

Then, (4.22) and (4.23) enable us to write

$$f(x, v_2) - \rho(x, v_2) \leq 2M_\tau \leq \Delta_p v_2 + K(x)|\nabla v_2|^{m(x)}.$$

Second, assume that $(1 + \tau)a(x)\left(\frac{1}{d(x)}\right)^{\beta(x)}v_2^{q-\theta} - (1 - \tau)b(x)\left(\frac{1}{d(x)}\right)^{\alpha(x)} > 0$.

Then, noting in addition that $v_2 \leq (1 - \epsilon)t_0\left(\frac{1}{d(x)}\right)^{s_0(x)}$, (4.23) reads as

$$\begin{aligned} & f(x, v_2) - \rho(x, v_2) \\ & \leq v_2^{\theta-1} \left[(1 + \tau)a(x)\left(\frac{1}{d(x)}\right)^{\beta(x)}v_2^{q-\theta} - (1 - \tau)b(x)\left(\frac{1}{d(x)}\right)^{\alpha(x)} \right] + 2M_\tau \\ & \leq (1 + \tau)(1 - \epsilon)^{q-1}a(x)\left(\frac{1}{d(x)}\right)^{\beta(x)+s_0(x)(q-1)}t_0^{q-1} \\ & \quad - (1 - \tau)(1 - \epsilon)^{\theta-1}b(x)\left(\frac{1}{d(x)}\right)^{\alpha(x)+s_0(x)(\theta-1)}t_0^{\theta-1} + 2M_\tau. \end{aligned} \quad (4.24)$$

On the other hand, taking into account the fact that $q > \max\{p, \theta\}$ (see (H₁)–(H₂)), we may assume that $\tau > 0$ is chosen small so that

$$(1 + \tau)(1 - \epsilon)^{q-1} < \min\{(1 - \tau)(1 - \epsilon)^{\theta-1}, (1 - 2\tau)(1 - \epsilon)^{p-1}\} \text{ in } \bar{\Omega}. \quad (4.25)$$

Moreover, up to choosing $\sigma > 0$ smaller if necessary, the fact that $d(x) < \sigma$ guarantees that

$$\tau(1 - \epsilon)^{p-1}(t_0 s_0)^{p-1}(s_0 + 1)(p - 1)(d(x))^{\xi_0 - s_0(p-1) - p} \geq 2M_\tau(d(x))^{\xi_0}. \quad (4.26)$$

Then, according to the definition of t_0 (see (4.1)), (4.25), and (4.26), we obtain

$$\begin{aligned} & (1 + \tau)(1 - \epsilon)^{q-1}a(x)t_0^{q-1} < (1 - \tau)(1 - \epsilon)^{\theta-1}b(x)t_0^{\theta-1}(d(x))^{\xi_0 - \alpha - s_0(\theta-1)} \\ & \quad + (1 - 2\tau)(1 - \epsilon)^{p-1}(t_0 s_0)^{p-1}(s_0 + 1)(p - 1)(d(x))^{\xi_0 - s_0(p-1) - p} \\ & \leq (1 - \tau)(1 - \epsilon)^{\theta-1}b(x)t_0^{\theta-1}(d(x))^{\xi_0 - \alpha - s_0(\theta-1)} \\ & \quad + (1 - \tau)(1 - \epsilon)^{p-1}(t_0 s_0)^{p-1}(s_0 + 1)(p - 1)(d(x))^{\xi_0 - s_0(p-1) - p} \\ & \quad - 2M_\tau(d(x))^{\xi_0}. \end{aligned} \quad (4.27)$$

In view of (4.21), (4.24), and (4.27), we infer that

$$\Delta_p v_2 \geq f(x, v_2) - \rho(x, v_2) - K(x)|\nabla v_2|^{m(x)}.$$

Altogether, we have shown

$$\Delta_{p(x)} v_2 \geq f(x, v_2) - \rho(x, v_2) - K(x)|\nabla v_2|^{m(x)}, \quad \forall x \in \Omega_\sigma^+. \quad (4.28)$$

Finally, we see that

$$\Delta_{p(x)} v_2 = 0 \geq f(x, \varepsilon) - \rho(x, \varepsilon) = f(x, v_2) - \rho(x, v_2) - K(x)|\nabla v_2|^{m(x)}, \quad (4.29)$$

$\forall x \in \Omega \setminus \Omega_\sigma^+$ (see (H₀) and the definition of v_2).

In view of (4.28) and (4.29), similar to the proof of Theorem 1, the proof is complete. \square

Proof of Theorem 2. Consider the functions g_2 and v_2 (see Lemmas 9 and 10) corresponding to the choice $\epsilon = \frac{1}{2}$. We will establish the existence of a sequence $\{\underline{u}_n\}_{n \geq 2}$ of subsolutions of (P) and of a sequence $\{\bar{u}_n\}_{n \geq 2}$ of supersolutions of (P), which fulfill the following properties:

$$0 < v_2 \leq \underline{u}_n \leq \bar{u}_n \leq g_2 \quad \text{in } \Omega, \quad (4.30)$$

and, for every $n \geq 2$, there are uniformly bounded $k_n, k_n^* > 0$ such that

$$\begin{aligned} \underline{u}_n(x) &= \left(1 - \frac{1}{n+1}\right) \frac{t_0(x)}{(d(x))^{s_0(x)}} - k_n^* \quad \text{and} \\ \bar{u}_n(x) &= \left(1 + \frac{1}{n+1}\right) \frac{t_0(x)}{(d(x))^{s_0(x)}} + k_n, \end{aligned} \quad (4.31)$$

whenever $d(x) < \sigma_\#$ for some $\sigma_\# \in (0, 1)$, and

$$\underline{u}_n, \bar{u}_n \in C(\Omega) \cap C^{1, \alpha_0}(\overline{\Omega_\#}) \cap C^{1, \alpha_0}(\Omega \setminus \Omega_\#), \quad (4.32)$$

for some $\alpha_0 \in (0, 1)$ and some subdomain $\Omega_\# \subset \subset \Omega$. Note that, once we obtain the existence of sequences $\{\underline{u}_n\}$ and $\{\bar{u}_n\}$ fulfilling (4.30)–(4.32), invoking Lemma 8, we deduce the existence of a sequence $\{u_n\}$ of solutions of (P), satisfying $\underline{u}_n \leq u_n \leq \bar{u}_n$ for all $n \geq 2$. Then as in the proof of [64, Theorem 3.3] we can show that $\{u_n\}$ has a subsequence converging to a large solution u of (P). Moreover, (4.31) ensures that

$$\lim_{d(x) \rightarrow 0} \frac{u(x)}{t_0(x) \left(\frac{1}{d(x)}\right)^{s_0(x)}} = 1.$$

Consequently, it remains to show the existence of sequences $\{\underline{u}_n\}$ and $\{\bar{u}_n\}$ fulfilling (4.30)–(4.32).

Step 1: construction of the supersolution \bar{u}_n . Set

$$\psi_n(x) = \left(1 + \frac{1}{n+1}\right) t_0(x) \left(\frac{1}{d(x)}\right)^{s_0(x)} \quad \text{and} \quad \psi_{n\sigma}^+ = \max_{d(x)=\sigma} \psi_n(x).$$

Let $\Omega_\sigma^* = \{x \in \Omega : d(x) < \sigma, \psi_n(x) > \psi_{n\sigma}^+\}$, which is independent of n . Indeed, notice from the definitions of ϕ and ϕ_σ^+ in (4.19) that $\psi_n(x) = \left(1 + \frac{1}{n+1}\right) \phi(x)$, $\psi_{n\sigma}^+ = \left(1 + \frac{1}{n+1}\right) \phi_\sigma^+$, and so $\Omega_\sigma^+ = \Omega_\sigma^*$. By the definition of v_2 it is clear that

$$v_2 \in C^{1, \alpha_0}(\overline{\Omega \setminus \Omega_\sigma^*}), \quad (4.33)$$

with any $\alpha_0 \in (0, 1)$. It is easy to see that $A_* := v_2(x)$ and $A^* := g_2(x)$ are constant functions on $\partial\Omega_\sigma^* \setminus \partial\Omega$ for a sufficiently small $\sigma > 0$.

We consider the following Dirichlet problem

$$(II) \quad \begin{cases} -\Delta_{p(x)}v + f_\#(x, v, \nabla v) = 0 & \text{in } \Omega \setminus \Omega_\sigma^*, \\ v(x) = A^* & \text{on } \partial\Omega_\sigma^* \setminus \partial\Omega. \end{cases}$$

Note from Lemmas 10 and 9 that v_2 and g_2 are subsolution and supersolution of (II), respectively. The proof of Lemma 8 (taking (4.33) into account) implies that problem (II) has a solution $\phi_{A^*}(x)$ satisfying $v_2(x) \leq \phi_{A^*}(x) \leq g_2(x)$ for all $x \in \Omega \setminus \Omega_\sigma^*$. We define

$$\bar{u}_n(x) = \begin{cases} (1 + \frac{1}{n+1})t_0(x)(d(x))^{-s_0(x)} + k_n & \text{if } x \in \Omega_\sigma^*, \\ \phi_{A^*}(x) & \text{if } x \in \Omega \setminus \Omega_\sigma^*, \end{cases}$$

where $k_n := A^* - \psi_{n\sigma}^+ > 0$. Obviously, $\{k_n\}$ is bounded.

Since $\bar{u}_n|_{\partial(\Omega \setminus \Omega_\sigma^*)} = g_2|_{\partial(\Omega \setminus \Omega_\sigma^*)} = A^*$ and $\bar{u}_n \leq g_2$ in $\Omega \setminus \Omega_\sigma^*$, it turns out

$$-\frac{\partial}{\partial n_1}\bar{u}_n(x) \geq -\frac{\partial}{\partial n_1}g_2(x) \text{ on } \partial(\Omega \setminus \Omega_\sigma^*),$$

and, by the definition of \bar{u}_n on Ω_σ^* and the fact that $\partial(\Omega \setminus \Omega_\sigma^*)$ is characterized (locally) by the equation $\psi_n(x) = \psi_{n\sigma}^+$, we derive

$$\frac{\partial}{\partial n_2}\bar{u}_n(x) = (1 + \frac{1}{n+1})|\nabla(t_0(d(x))^{-s_0})| \text{ on } \partial(\Omega \setminus \Omega_\sigma^*),$$

where n_1 is the inward unit normal vector of the boundary $\partial(\Omega \setminus \Omega_\sigma^*)$ of $\Omega \setminus \Omega_\sigma^*$ and $n_2 = -n_1$. Then, using (4.31), for every $x \in \partial(\Omega \setminus \Omega_\sigma^*)$, we have

$$\begin{aligned} -\frac{\partial}{\partial n_1}\bar{u}_n(x) &\geq -\frac{\partial}{\partial n_1}g_2(x) = \left(1 + \frac{1}{2}\right)|\nabla(t_0(d(x))^{-s_0})| \\ &> \left(1 + \frac{1}{n+1}\right)|\nabla(t_0(d(x))^{-s_0})| = \frac{\partial}{\partial n_2}\bar{u}_n(x). \end{aligned} \quad (4.34)$$

Denote $d_1(x) = d(x, \partial(\Omega \setminus \Omega_\sigma^*))$, and for every positive integer ℓ consider on Ω the function $\xi_\ell(x) = \max\{0, 1 - \ell d_1(x)\}$. Then for any nonnegative function $w \in C_0^1(\Omega)$ with $\text{supp } w \subset\subset \Omega$, we can write

$$\begin{aligned} &\int_{\Omega} (|\nabla\bar{u}_n|^{p(x)-2}\nabla\bar{u}_n \cdot \nabla w + f_\#(x, \bar{u}_n, \nabla\bar{u}_n)w) dx \\ &= \int_{\Omega \setminus \Omega_\sigma^*} (|\nabla\bar{u}_n|^{p(x)-2}\nabla\bar{u}_n \cdot \nabla(\xi_\ell w) + f_\#(x, \bar{u}_n, \nabla\bar{u}_n)\xi_\ell w) dx \\ &+ \int_{\Omega \setminus \Omega_\sigma^*} (|\nabla\bar{u}_n|^{p(x)-2}\nabla\bar{u}_n \cdot \nabla((1 - \xi_\ell)w) + f_\#(x, \bar{u}_n, \nabla\bar{u}_n)(1 - \xi_\ell)w) dx \end{aligned} \quad (4.35)$$

$$\begin{aligned}
& + \int_{\Omega_\sigma^*} (|\nabla \bar{u}_n|^{p(x)-2} \nabla \bar{u}_n \cdot \nabla (\xi_\ell w) + f_\#(x, \bar{u}_n, \nabla \bar{u}_n) \xi_\ell w) dx \\
& + \int_{\Omega_\sigma^*} (|\nabla \bar{u}_n|^{p(x)-2} \nabla \bar{u}_n \cdot \nabla ((1 - \xi_\ell)w) + f_\#(x, \bar{u}_n, \nabla \bar{u}_n) (1 - \xi_\ell)w) dx,
\end{aligned}$$

for all $\ell \geq 1$. The proof of Lemma 9 shows that, for $\sigma > 0$ chosen small enough, we have

$$-\Delta_p \bar{u}_n + f_\#(x, \bar{u}_n, \nabla \bar{u}_n) \geq 0 \quad \text{in } \Omega_\sigma^*.$$

Since $(1 - \xi_\ell)w|_{\Omega_\sigma^*}$ has a compact support in Ω_σ^* , it follows that

$$\int_{\Omega_\sigma^*} (|\nabla \bar{u}_n|^{p(x)-2} \nabla \bar{u}_n \cdot \nabla ((1 - \xi_\ell(x))w) + f_\#(x, \bar{u}_n, \nabla \bar{u}_n) (1 - \xi_\ell(x))w) dx \geq 0, \quad (4.36)$$

for all $\ell \geq 1$. The facts that \bar{u}_n is a weak solution of (II) and that $(1 - \xi_\ell)w|_{\Omega \setminus \Omega_\sigma^*}$ has a compact support in $\Omega \setminus \Omega_\sigma^*$ imply

$$\int_{\Omega \setminus \Omega_\sigma^*} (|\nabla \bar{u}_n|^{p(x)-2} \nabla \bar{u}_n \cdot \nabla (1 - \xi_\ell(x))w + f_\#(x, \bar{u}_n, \nabla \bar{u}_n) (1 - \xi_\ell(x))w) dx = 0, \quad (4.37)$$

for all $\ell \geq 1$. Moreover, the dominated convergence theorem yields

$$\lim_{\ell \rightarrow \infty} \left(\int_{\Omega \setminus \Omega_\sigma^*} f_\#(x, \bar{u}_n, \nabla \bar{u}_n) \xi_\ell w dx + \int_{\Omega_\sigma^*} f_\#(x, \bar{u}_n, \nabla \bar{u}_n) \xi_\ell w dx \right) = 0. \quad (4.38)$$

By (4.34)–(4.38) and the Green's formula, we obtain that

$$\begin{aligned}
& \int_{\Omega} (|\nabla \bar{u}_n|^{p(x)-2} \nabla \bar{u}_n \cdot \nabla w + f_\#(x, \bar{u}_n, \nabla \bar{u}_n) w) dx \\
& \geq \lim_{\ell \rightarrow \infty} \int_{\Omega \setminus \Omega_\sigma^*} |\nabla \bar{u}_n|^{p(x)-2} \nabla \bar{u}_n \cdot \nabla (\xi_\ell w) dx \\
& \quad + \lim_{\ell \rightarrow \infty} \int_{\Omega_\sigma^*} |\nabla \bar{u}_n|^{p(x)-2} \nabla \bar{u}_n \cdot \nabla (\xi_\ell w) dx \\
& = \int_{\partial(\Omega \setminus \Omega_\sigma^*)} \left[-|\nabla \bar{u}_n|^{p(x)-2} \frac{\partial}{\partial n_1} \bar{u}_n - |\nabla \bar{u}_n|^{p(x)-2} \frac{\partial}{\partial n_2} \bar{u}_n \right] w dS \geq 0.
\end{aligned}$$

Therefore, \bar{u}_n is a supersolution of (P). Moreover, the construction of \bar{u}_n and the regularity theory (see [15]) guarantee that \bar{u}_n satisfies (4.31) and (4.32).

Step 2: construction of the subsolution \underline{u}_n . Set

$$\phi_n(x) = \left(1 - \frac{1}{n+1}\right) t_0(x) \left(\frac{1}{d(x)}\right)^{s_0(x)}, \quad \phi_{n\sigma}^+ = \max_{d(x)=\sigma} \phi(x),$$

and recall that $\Omega_\sigma^+ = \{x \in \Omega : d(x) < \sigma, \phi(x) > \phi_\sigma^+\}$, with $\sigma \in (0, 1)$, where ϕ and ϕ_σ^+ are defined in (4.19). It is seen that $\phi_n(x) = \left(1 - \frac{1}{n+1}\right)\phi(x)$ and $\phi_{n\sigma}^+ = \left(1 - \frac{1}{n+1}\right)\phi_\sigma^+$. We have that $A_* := v_2|_{\partial(\Omega \setminus \Omega_\sigma^+)}$ is a constant function on $\partial(\Omega \setminus \Omega_\sigma^+)$. Note that v_2 a subsolution (see Lemma 10) and \bar{u}_n is a supersolution (see Step 1) of the Dirichlet problem

$$(III) \quad \begin{cases} -\Delta_{p(x)}v + f_\#(x, v, \nabla v) = 0 & \text{in } \Omega \setminus \Omega_\sigma^+, \\ v(x) = A_* & \text{on } \partial\Omega_\sigma^+ \setminus \partial\Omega. \end{cases}$$

Arguing as in the proof of Lemma 8, we find a weak solution $\tilde{\phi}_{A_*}$ of (III) such that

$$v_2 \leq \tilde{\phi}_{A_*} \leq \bar{u}_n \text{ in } \Omega \setminus \Omega_\sigma^+.$$

We define

$$\underline{u}_n(x) = \begin{cases} \left(1 - \frac{1}{n+1}\right)t_0(x)(d(x))^{-s_0(x)} - k_n^* & \text{if } x \in \Omega_\sigma^+, \\ \phi_{A_*}(x) & \text{if } x \in \Omega \setminus \Omega_\sigma^+, \end{cases}$$

with the constant $k_n^* = \phi_{n\sigma}^+ - A_* > 0$. Obviously, $\{k_n^*\}$ is bounded. Note also that, by the construction of \bar{u}_n and the regularity theory (see [15]), we have $\bar{u}_n \in C^{1,\beta}(\Omega_\sigma^+) \cap C^{1,\beta}(\overline{\Omega \setminus \Omega_\sigma^+})$ for some $\beta \in (0, 1)$. Arguing exactly as in Step 1, we can check that \underline{u}_n is a subsolution of problem (P) satisfying (4.31) and (4.32). This completes Step 2 and thus the proof of the theorem. \square

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