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Walid Ben-Ameur, José Neto

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Walid Ben-Ameur, [a] José Neto [a]

[a]*Institut TELECOM, TELECOM & Management SudParis, CNRS Samovar, 9, rue Charles Fourier, 91011 Evry, France*
{Walid.Benameur,Jose.Neto}@it-sudparis.eu

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1. Introduction

Consider a quadratic function $q : \mathbb{R}^n \to \mathbb{R}$ given by: $q(x) = x^t Q x$, with $Q \in \mathbb{R}^{n \times n}$. An unconstrained $(-1, 1)$-quadratic optimization problem can be expressed as follows:

$$(QP)\ Z^* = \min\{q(x) \mid x \in \{-1, 1\}^n\},$$

where $\{-1, 1\}^n$ denotes the set of n-dimensional vectors with entries either equal to 1 or -1. We consider here that the matrix Q is symmetric and given by its spectrum, i.e. the set of its eigenvalues and associated unit pairwise orthogonal eigenvectors.

Problem (QP) is a classical combinatorial optimization problem with many applications, e.g. in statistical physics and circuit design [2; 8; 10]. It is well-known that any (0,1)-quadratic problem expressed as: $\min\{x^t A x + c^t x \mid x \in \{0, 1\}^n\}$, $A \in \mathbb{R}^{n \times n}, c \in \mathbb{R}^n$, can be formulated in the form of problem (QP) and conversely [9; 4].

The contribution of this work is 3-fold:

- *(i)* We slightly extend the known polynomially solvable cases of (QP) to when the matrix Q has fixed rank and the number of positive diagonal entries is $\mathcal{O}(\log(n))$.
- *(ii)* We introduce a new (to our knowledge) polynomial-time algorithm for solving problem (QP) when it corresponds to such a polynomially solvable case.
- *(iii)* Preliminary experiments indicate that the proposed method may be computationally efficient. [7] .

2. Properties of optimal solutions for peculiar instances of (QP)

Let us firstly introduce some notation to be used hereafter. The eigenvalues of the matrix Q will be noted $\lambda_1(Q) \leq \lambda_2(Q) \leq \ldots \leq \lambda_n(Q)$ (or more simply $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$ when clear from the context) and the corresponding unit (in Euclidean norm) and pairwise orthogonal eigenvectors: $v_1, \ldots, v_n$. The j-th entry of the vector v_i is noted v_{ij}. Given some set of vectors $a_1, \ldots, a_q \in \mathbb{R}^n$, $q \in \mathbb{N}$, we note $Lin(a_1, \ldots, a_q)$ the subspace spanned by these vectors.
In this section we shall make the following assumptions on the matrix Q:

(i) Q has rank $p \leq n$,
(ii) Q has nonpositive diagonal entries only, and
(iii) Q is given by its set of rational eigenvalues and eigenvectors: $Q = \sum_{i=1}^{p} \lambda_i v_i v_i^t$.

Any optimal solution y^* to the problem $\min_{y \in \{-1,1\}^n} y^t Q y$ can be shown to satisfy the following implication:

$$\sum_{i=1}^{p} \lambda_i \alpha_i v_{ij} > 0 \Rightarrow y_j^* = -1 \tag{2.1}$$

And analogously:

$$\sum_{i=1}^{p} \lambda_i \alpha_i v_{ij} < 0 \Rightarrow y_j^* = 1. \tag{2.2}$$

From this simple property we can namely show that in order to find an optimal solution of problem (QP), it is sufficient to enumerate over all vectors $y \in \{-1, 1\}^n$ for which there exists a vector $\alpha \in \mathbb{R}^p$ such that $y_j = -sign(\sum_{i=1}^{p} \lambda_i \alpha_i v_{ij})$ (or equivalently $y_j = sign(\sum_{i=1}^{p} \lambda_i \alpha_i v_{ij})$, see hereafter), $\sum_{i=1}^{p} \lambda_i \alpha_i v_{ij} \neq 0, \forall j \in \{1, \ldots, n\}$, with $sign(x) = 1$ if $x > 0$ and -1 if $x < 0$. In the next section we focus on finding such a set of vectors.

3. Determining cells in an arrangement of n hyperplanes

Let $v_1, ..., v_p \in \mathbb{R}^n$ denote p independent vectors. Let $V \in \mathbb{R}^{n \times p}$ denote the matrix whose columns correspond to the vectors $v_1, \ldots, v_p$ and V_i the i-th row of V. From this set of vectors we define n hyperplanes in $\mathbb{R}^p$: $H_j = \{\alpha \in \mathbb{R}^p \mid V_j.\alpha = 0\}$ with $j \in \{1, \ldots, n\}$. Then we can notice that there is a one-to-one correspondence between the set of vectors in $\{-1, 1\}^n$ for which there exists a vector $\alpha \in \mathbb{R}^p$ such that $y_j = sign(\sum_{i=1}^{p} \alpha_i v_{ij})$, with $\sum_{i=1}^{p} \alpha_i v_{ij} \neq 0, \forall j \in \{1, \ldots, n\}$ and the cells (i.e. the full dimensional regions) in $\mathbb{R}^p$ of the hyperplane arrangement $\mathcal{A}(H)$ that is defined by the family of hyperplanes $(H_j)_{j=1}^n$. To see this just interpret the sign vector y as the position vector of the corresponding cell c w.r.t. an orientation of

the space by the vector V_j: cell c is *above* hyperplane H_j iff $y_j > 0$ and *under* otherwise.

For a general arrangement in $\mathbb{R}^p$ that is defined by n hyperplanes (see e.g. [6; 11] for further elements on arrangements), the number of cells is upper bounded by $\sum_{i=0}^{p} \binom{n}{i}$ (which is in $\mathcal{O}(n^p)$). (For a proof we refer the reader e.g., to Lemma 1.2 in [6]). In our case, since all the hyperplanes considered contain the origin (i.e. the arrangement is *central*), this number reduces to $\mathcal{O}(n^{p-1})$ (see Section 1.7 in [6]).

We have introduced [3] a simple procedure with time complexity lying between the time complexity of the incremental algorithm [6; 5] (in $\mathcal{O}(n^{p-1})$) and that of the reverse search algorithm [1; 7] (in $\mathcal{O}(n\ LP(n,p)\ C)$ where C denotes the number of cells in $\mathcal{A}(H)$ and $LP(n,p)$ is the time needed to solve a linear program with n inequalities and p variables) in order to compute a set of vectors in $\{-1,1\}^n$ corresponding to a description of a set containing the cells of the arrangement $\mathcal{A}(H)$. Space complexity can be shown to be polynomially bounded by the ouput size. To our view the interest of the proposed method by comparison with the former ones is 2-fold:

(i) it is very easy to understand and implement,
(ii) computationally, by using proper data structures (to be specified latter) we could solve instances of the same magnitude as the ones reported in [7], without parallelization and substantially improved computation times.

The basic principle of the proposed method may be expressed as follows. Given some integer $q \in \{1,\ldots,n\}$, let $\mathcal{B}^q(H)$ denote the arrangement in the subspace $\{\alpha \in \mathbb{R}^p \mid V_q\alpha = 0\}$ that is defined by the hyperplanes (in $\mathbb{R}^{p-1}$) $\{H_j \cap H_q \mid j \in \{1,\ldots,n\} \text{ and } j \neq q\}$. Any cell of $\mathcal{B}^q(H)$ (which is a region of dimension $p-1$) corresponds to a facet of exactly two cells of the arrangement $\mathcal{A}(H)$ i.e. one on each side of the hyperplane H_q. The other cells of $\mathcal{A}(H)$ i.e. those not intersecting H_q, are cells of the arrangement $\mathcal{C}^q(H)$ in $\mathbb{R}^p$ which is defined by the $n-1$ hyperplanes $\{H_j \mid j \in \{1,\ldots,n\} \text{ and } j \neq q\}$. Since each cell of $\mathcal{A}(H)$ intersects at least one of the hyperplanes $(H_j)_{j=1}^n$, it follows that all the cells of $\mathcal{A}(H)$ can be derived from the ones of all the arrangements $\mathcal{B}^q(H)$, $q = 1,\ldots,n$. A recursive use of this argument leads to the generation of all the cells of $\mathcal{A}(H)$.

From a complexity study of the proposed method we can show the following result.

Theorem 3.1. For a fixed integer $p \geq 2$, if the matrix Q (given by its nonzero eigenvalues and associated eigenvectors) has rank at most p and $\mathcal{O}(\log(n))$ positive diagonal entries, then problem (QP) can be solved in strongly polynomial time.

4. Conclusion

We propose a new (up to our knowledge) approach for solving in polynomial time some unconstrained quadratic optimization problems. Preliminary computational results illustrate that the recursive procedure briefly presented here can be a valuable approach on some instances by comparison with a reverse search w.r.t. computation times.

Further computational studies are under work and could involve a parallelization of the code in order to deal with larger instances.

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