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# High-order filtered schemes for time-dependent second order HJB equations

Olivier Bokanowski\*, Athena Picarelli† and Christoph Reisinger†

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## Abstract

In this work, we present and analyse a class of “filtered” numerical schemes for second order Hamilton-Jacobi-Bellman (HJB) equations. The work follows the ideas recently introduced by Oberman and Froese [14] and applied by Bokanowski et al. [5] and Oberman and Salvador [24] to first order HJB equations. For high order approximation schemes (where “high” stands for greater than one), the inevitable loss of monotonicity prevents the use of the classical theoretical results for convergence to viscosity solutions. The work introduces a suitable local modification of these schemes by “filtering” them with a monotone scheme, such that they can be proven converge and still show an overall high order behaviour for smooth enough solutions. We give theoretical proofs of these claims and illustrate the behaviour with numerical tests.

## 1 Introduction

We consider second order Hamilton-Jacobi-Bellman (HJB) equations in  $\mathbb{R}^d$ :

$$\begin{cases} v_t + \sup_{a \in A} \mathcal{L}^a(t, x, v, D_x v, D_x^2 v) = 0 & x \in \mathbb{R}^d, t \in (0, T) \\ v(0, x) = v_0(x) & x \in \mathbb{R}^d, \end{cases} \quad (1.1)$$

where  $\mathcal{L}^a : [0, T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times \mathcal{S}^{d \times d} \rightarrow \mathbb{R}$  takes the form

$$\mathcal{L}^a(t, x, r, p, Q) = \left\{ -b(t, x, a) \cdot p - \frac{1}{2} \text{Tr}[\sigma \sigma^T(t, x, a) Q] + f(t, x, a)r + \ell(t, x, a) \right\} \quad (1.2)$$

and  $A \subset \mathbb{R}^m$  is a compact set. We also define the Hamiltonian of the system:

$$H(t, x, r, p, Q) := \sup_{a \in A} \mathcal{L}^a(t, x, r, p, Q).$$

In this article, we propose approximation schemes for (1.1) for which convergence is guaranteed in a general setting, and which exhibit high order convergence under sufficient

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regularity of the solution. As we will explain in the following, these are in a sense two conflicting goals, and we will meet them by application of a so-called “filter”, an idea introduced in [14].

The seminal work by Barles and Souganidis [4] establishes that a consistent and stable scheme converges to the viscosity solution of (1.1) if it is also monotone. That this is not simply a requirement of the proof, but can be crucial in practice, is demonstrated, e.g., by [25] (and also in Section 4.2 here). It is shown there empirically that for the uncertain volatility model from [21], perhaps the simplest non-trivial second order HJB equation there is, the (consistent and stable but) non-monotone Crank-Nicolson scheme fails to converge to the correct viscosity solution in the presence of Lipschitz initial data without higher regularity. This is in contrast to classical solutions where there is no such monotonicity requirement. As the setting of solutions which are Lipschitz in space and  $1/2$ -Hölder in time is the natural setting for HJB equations and often no higher global regularity is observed, a wide literature on monotone schemes has developed.

While Godunov’s theorem [15] limits the order of linear monotone schemes to one, the provable order of convergence for second order HJB equations under the weak assumptions above is significantly less than one. By a technique pioneered by Krylov based on “shaking the coefficients” and mollification to construct smooth sub- and/or super-solutions, [17, 18, 1, 2, 3] prove fractional convergence orders. Again contrasting this with the case of more regularity, for instance [28] prove high order convergence of discontinuous finite element approximations under a Cordes condition on the coefficients which guarantees high order Sobolev regularity for smooth enough data.

To make matters worse, in more than one dimension, in the presence of general cross-derivative terms even first order consistent monotone schemes are necessarily “non-local”, by which we mean that the length of the finite difference stencil grows relative to the mesh size as it is refined (see [16] or [27]). While standard finite difference schemes (see, e.g., [19]) are generally non-monotone unless the diffusion matrix is diagonally dominant (see [11]), schemes which are monotone by construction include semi-Lagrangian schemes [23, 10, 12] and generalised finite difference schemes [9, 8]. To utilise the second-order accuracy of standard finite differences for smooth solutions, [22] use those schemes in regions where the coefficients and controls are such that the scheme is monotone, and switches to wide stencils only if monotonicity of the standard scheme, easily checkable by the signs of the discretisation matrix, fails.

The simple idea of “filtered” schemes is to use a combination of a high order scheme and a low order monotone scheme, where the latter is known to converge *a priori* by standard results [4]. The filter ensures that the low order scheme is used locally where and when the discrepancy to the high order scheme is too large, thus ensuring at least the same convergence order as the low order scheme (see, e.g., [1, 2, 3]), but otherwise uses the high order scheme and benefits from its accuracy for smooth solutions. Such schemes have been proposed, analysed and used in [14] for the Monge-Ampere equation, in [24] for (first order) Hamilton-Jacobi equations, and in [5] for (time dependent) first order Hamilton-Jacobi-Bellmann equations. We complete this program by studying (time dependent) second order Hamilton-Jacobi-Bellmann equations as they arise from stochastic control problems.

Our results parallel the ones in [5] to prove, in the second order setting, that suitable filtered schemes converge at least of the same order as the underlying monotone scheme if

there are solutions (only) in the viscosity sense, converge of first order for classical solutions, and have higher order truncation error for sufficiently smooth solutions. Given the much more restrictive CFL condition on the time step in the second order case, we include implicit time stepping schemes in our analysis, which requires an extension of the arguments in [5] to work with the monotone implicit operators. The monotone schemes covered by the analysis include the most commonly used one-step finite difference and semi-Lagrangian schemes, while we make no assumptions on the higher order scheme beyond a higher order truncation error, and particularly allow multistep schemes such as BDF.

The rest of this article is organised as follows. In Section 2, we present the assumptions on the equations, on the monotone scheme, higher order scheme, and filter function, and give examples of the schemes which fall in this framework. Section 3 is devoted to the theoretical convergence results outlined above and their proofs. The practical convergence behaviour is illustrated in Section 4 by a non-smooth and a smooth example in one dimension, and a degenerate but smooth example in two dimensions. Section 5 offers conclusions.

## 2 Main assumptions and definition of the scheme

We consider the following classical assumptions:

- (A1)** the initial data  $v_0$  and the coefficients  $b, \sigma, f, \ell$  are bounded, Lipschitz and Hölder continuous respectively in space and time, i.e. there is a constant  $K$  such that

$$\sup_{a \in A} \left\{ \sup_{(t,x) \neq (s,y)} \frac{|\varphi(t,x,a) - \varphi(s,y,a)|}{|t-s|^{1/2} + |x-y|} + \sup_{(t,x)} |\varphi(t,x,a)| \right\} \leq K$$

for  $\varphi = v_0, b, \sigma, f, \ell$ .

Under assumption (A1), there exists a unique bounded continuous viscosity solution to (1.1). Moreover, it is possible to verify that such a solution is also Lipschitz continuous in space and 1/2-Hölder continuous in time.

In order to simplify the presentation we will focus our analysis on the one-dimensional case, but the same results hold true for general  $d$ . Let  $N \geq 1$  and let us introduce a time step  $\tau = T/N$ . We denote by  $\Delta x$  the space step. A uniform mesh in time and space is defined in one dimension by:

$$t_n = n\tau, \quad n \in \{0, \dots, N\} \quad \text{and} \quad x_i \equiv i\Delta x \quad i \in \mathbb{Z}.$$

For  $d \geq 1$ , we take  $\Delta x = \max(\Delta x_i : i = 1, \dots, d)$ . We also denote by  $\mathcal{G}_{\Delta x} := \{x_i : i \in \mathbb{Z}\}$  the space grid. We point out that the analysis does not change for a nonuniform mesh.

Let  $S_M$  and  $S_H$  respectively denote a first order monotone scheme and a higher (higher stands for greater than one) order scheme. It is well known by Godunov's theorem that, for linear numerical schemes as those considered in this paper, the monotonicity property necessary to establish the convergence of the scheme in the framework of Barles and Souganidis [4] can only be satisfied by schemes of order one. This means that we cannot expect the scheme  $S_H$  to be monotone.

As already pointed out in the introduction, our aim will be to produce a new “almost-monotone” scheme showing a high order behaviour for which it is still possible to prove convergence using the arguments in [4]. This is obtained applying to the scheme a filter. The filter function we will use has been previously introduced in [5, 24] and is defined by

$$F(x) := \begin{cases} x & \text{if } |x| \leq 1 \\ 0 & \text{otherwise.} \end{cases} \quad (2.1)$$

Analogous theoretical results are obtained using as a filter the continuous function

$$F(x) := \begin{cases} x & |x| \leq 1 \\ 0 & |x| \geq 2 \\ -x + 2 & 1 \leq x \leq 2 \\ -x - 2 & -2 \leq x \leq -1, \end{cases}$$

as in [14]. Observe that  $|F(x)| \leq 1$  for any  $x \in \mathbb{R}$ . The filtered scheme is then defined in the following form:

$$S_F := S_M + \varepsilon \tau F \left( \frac{S_H - S_M}{\varepsilon \tau} \right),$$

where  $\varepsilon$  is given by

$$\varepsilon = \varepsilon_0(\tau + \Delta x) \quad (2.2)$$

for some constant  $\varepsilon_0 > 0$ .

We will denote the numerical solution of the scheme by  $u$  and we will write for brevity

$$u(t_n, x_i) = u_i^n.$$

## 2.1 The monotone scheme

Let us consider a monotone scheme written in the general form:

$$\begin{aligned} \mathcal{S}_M(t_n, x_i, u_i^{n+1}, [u])_i &= 0 & n = 0, \dots, N-1, & \quad i \in \mathbb{Z} \\ u_i^0 &= v_0(x_i) & & \quad i \in \mathbb{Z}, \end{aligned} \quad (2.3)$$

where  $[\cdot]$  denotes all the components of  $u$  except  $u_i^{n+1}$ . This formulation may include both explicit and implicit schemes. The scheme  $\mathcal{S}_M$  satisfies the following assumptions:

- (A2)** (a) **(Consistency)** For any smooth function  $\varphi$ , there exists a continuous function  $\mathcal{E}_M$  such that  $\mathcal{E}_M(\tau, \Delta x) \rightarrow 0$  as  $(\tau, \Delta x) \rightarrow 0$  and

$$\left| \mathcal{S}_M(t, x, \varphi(t + \tau, x), [\varphi]) - \left( \varphi_t + H(t, x, \varphi, \varphi_x, \varphi_{xx}) \right) \right| \leq \mathcal{E}_M(\tau, \Delta x).$$

- (b) **(Monotonicity)** For any bounded functions  $\phi$  and  $\psi$  such that  $\phi \leq \psi$  one has

$$\mathcal{S}_M(t, x, r, [\phi]) \geq \mathcal{S}_M(t, x, r, [\psi]).$$

- (c) **(Stability)** For any  $(\tau, \Delta x)$ , there exists a bounded solution of (2.3) with a bound independent of  $(\tau, \Delta x)$ .

We will refer to  $\mathcal{E}_M$  as the consistency error of the monotone scheme. As a consequence of Godunov's theorem, a scheme satisfying these assumptions can be at most of order one. Therefore we assume to have

$$|\mathcal{E}_M(\tau, \Delta x)| \leq C_M(\Delta x + \tau)$$

for some constant  $C_M \geq 0$  depending on the bounds on the higher order derivatives of  $\varphi$ . (In the presence of a dominant diffusion term,  $\Delta x$  can sometimes be replaced by  $\Delta x^2$ , especially in dimension 1.)

**(A2) (d) (Form of scheme)** The monotone scheme can be written in the form

$$\sup_{a \in A} \left\{ M^{a,n+1} u^{n+1} + G^{a,n}(u^n) \right\} = 0, \quad n = 0, \dots, N-1, \quad (2.4)$$

where  $M^{a,n}$  is an M-matrix for all  $a, n$  with  $\|(M^{\bar{a},n})^{-1}\| \leq K$  ( $K$  independent of  $a, n, \Delta x, \tau$ ), where

$$\phi \leq \psi \quad \Rightarrow \quad G^{a,n}(\phi) \geq G^{a,n}(\psi), \quad a \in A, \quad n = 0, \dots, N-1,$$

and there exists  $C > 0$  such that

$$|G^{a,n}(\phi) - G^{a,n}(\psi)| \leq (1 + C\tau)|\phi - \psi|, \quad a \in A, \quad n = 0, \dots, N-1.$$

**Remark 1.** *The form of the scheme in Assumption (A2)(d) is natural and satisfied by all the schemes considered in this paper, which are of the “discretise, then optimise” type (see [13]), where we discretise the linear operator in (1.2) by a monotone linear scheme for a fixed control  $a$  and then carry out the optimisation in (1.1). We only consider one-step time marching schemes of the form*

$$\begin{aligned} \frac{u_i^{n+1} - u_i^n}{\tau} + \sup_{a \in A} \left\{ \theta \left[ (L^{a,n+1} u^{n+1})_i + f(t_{n+1}, x_i, a) u_i^{n+1} + \ell(t_{n+1}, x_i, a) \right] \right. \\ \left. + (1 - \theta) \left[ (L^{a,n} u^n)_i + f(t_n, x_i, a) u_i^n + \ell(t_n, x_i, a) \right] \right\} = 0, \end{aligned}$$

for  $\theta \in [0, 1]$ , where  $L^{a,n}$  is a discretisation of the first and second order derivative terms in (1.2). It is easy to verify that if the entries of  $I + \tau(1 - \theta)L^{a,n}$  are positive, then Assumption (A2)(d) is satisfied. For the backward Euler scheme,  $\theta = 1$ ,  $G(u) = -u$  and this is unconditionally true. For  $\theta < 1$ , there is a CFL condition requiring  $\tau = \mathcal{O}(\Delta x^2)$ .

In order to define the filtered scheme, it will be useful to write the scheme in the form

$$\begin{aligned} u_i^{n+1} &= S_M(u_n)_i & n = 0, \dots, N-1, & \quad i \in \mathbb{Z} \\ u_i^0 &= v_0(x_i) & & \quad i \in \mathbb{Z}. \end{aligned}$$

If the scheme is explicit, i.e. if  $[u]$  only contains values of  $u^n$  (multi-step schemes will not be considered for the monotone scheme), this follows directly from (2.3) just making  $u^{n+1}$  explicit. In case of an implicit scheme, this formulation comes from the solution of (2.3) for  $u^{n+1}$  (by the assumptions, there exists a unique solution). In practice, the solution can be obtained by an iterative technique, such as Howard's algorithm, which is guaranteed to converge by the M-matrix property of  $M^{a,n+1}$  (see [7]).

**Remark 2.** If the scheme  $S_M$  is explicit, the monotonicity assumption automatically implies that for any functions  $\phi, \psi$  such that  $\phi \leq \psi$  one has

$$S_M(\phi) \leq S_M(\psi).$$

It is possible to show that this remains true for any implicit finite difference scheme written in the general form (2.4). For the optimal control  $\bar{a} \equiv \bar{a}(\psi)$  such that

$$\sup_{a \in A} \left\{ M^{a, n+1} S_M(\psi) - G^{a, n}(\psi) \right\} = M^{\bar{a}, n+1} S_M(\psi) - G^{\bar{a}, n}(\psi) = 0,$$

one also has for any  $\phi$

$$M^{\bar{a}, n+1} S_M(\phi) - G^{\bar{a}, n}(\phi) \leq 0,$$

such that

$$M^{\bar{a}, n+1} (S_M(\phi) - S_M(\psi)) \leq G^{\bar{a}, n}(\phi) - G^{\bar{a}, n}(\psi) \leq 0.$$

From the M-matrix property of  $M^{\bar{a}, n+1}$ ,

$$S_M(\phi) \leq S_M(\psi).$$

Hence, Assumption (A2)(d) implies monotonicity and subsumes (A2)(b).

From now on, we can assume that if  $u^{n+1}$  and  $w^{n+1}$  are defined by (2.3) and if  $u^n \leq w^n$ , one has

$$u^{n+1} \leq w^{n+1}.$$

**Example 1** (Finite difference schemes). In the one-dimensional case, typical monotone finite difference schemes are based on the Euler approximation in time, the usual second order approximation of  $v_{xx}$

$$D^2 v_i := \frac{v_{i-1} - 2v_i + v_{i+1}}{\Delta x^2}$$

and the upwind scheme for approximating the drift term

$$D^{1,-} v_i := \frac{v_{i+1} + v_i}{\Delta x} \quad \text{and} \quad D^{1,+} v_i := \frac{v_i - v_{i-1}}{\Delta x}.$$

The explicit scheme is stable under the following CFL condition

$$\tau \leq \|\sigma^2\|_\infty \Delta x^2.$$

For avoiding such a restriction on the choice of the time step, we will only consider the implicit case. The scheme can therefore be defined as

$$\begin{aligned} \frac{u_i^{n+1} - u_i^n}{\tau} + \sup_{a \in A} \left\{ -\frac{1}{2} \sigma^2(t_{n+1}, x_i, a) D^2 u_i^{n+1} + b^+(t_{n+1}, x_i, a) D^{1,-} u_i^{n+1} \right. \\ \left. - b^-(t_{n+1}, x_i, a) D^{1,+} u_i^{n+1} + f(t_{n+1}, x_i, a) u_i^{n+1} - \ell(t_{n+1}, x_i, a) \right\} = 0, \end{aligned} \quad (2.5)$$

where we have denoted  $b^\pm$  the positive (resp. negative) part of  $b$ :

$$b^+(t, x, a) := \max(b(t, x, a), 0) \quad \text{and} \quad b^-(t, x, a) := \max(-b(t, x, a), 0).$$

We recall that, in multiple dimensions, standard finite difference schemes are in general non-monotone. Possible monotone schemes are semi-Lagrangian schemes [23, 10, 12], which we consider next, and generalised finite difference schemes [9, 8].

**Example 2** (Semi-Lagrangian schemes). *Semi-Lagrangian schemes (SL) are a class of schemes based on a discrete time approximation of the optimal control problem via a discrete version of the Dynamic Programming Principle, followed by a step of interpolation on the spatial grid. We consider here explicit SL schemes for equation (1.1) in the following form:*

$$\frac{u_i^{n+1} - u_i^n}{\tau} - \inf_{a \in A} \left\{ \sum_{j=1}^q \frac{[u^n](x_i + y_{k,j}^{a,+}(t_n, x_i)) - 2u_i^n + [u^n](x_i + y_{k,j}^{a,-}(t_n, x_i))}{2k^2} + f(t_n, x_i, a)u_i^n + \ell(t_n, x_i, a) \right\} = 0,$$

where we denote by  $\sigma_j$  the  $j$ -th column of  $\sigma$  for  $j = 1, \dots, q$ ,

$$y_{k,j}^{a,\pm}(t_n, x_i) := k^2 b(t_n, x_i, a) \pm \frac{k}{q} \sigma_j(t_n, x_i, a)$$

and  $[\cdot]$  stands for the interpolation on the spatial grid. These schemes are first order accurate, wide-stencil monotone schemes, i.e., we choose  $k \sim \sqrt{\Delta x}$ , where  $\Delta x$  is the mesh width of the spatial mesh, and then the CFL condition is  $\tau \leq q\Delta x$ . We refer to [23, 10] for their first introduction in the context of second order equations and to [12] for an exhaustive discussion of the main results.

## 2.2 The high order scheme

Let us consider a high order scheme written in the general form:

$$\begin{aligned} \mathcal{S}_H(t_n, x, u_i^{n+1}, [u])_i &= 0 \quad n = 0, \dots, N-1, \quad i \in \mathbb{Z} \\ u_i^0 &= v_0(x_i). \end{aligned} \tag{2.6}$$

The scheme  $\mathcal{S}_H$  satisfies the following assumption:

**(A3) (High order consistency)** For any smooth function  $\varphi$ , there exists a continuous function  $\mathcal{E}_H$  such that

$$\left| \mathcal{S}_H(t, x, \varphi(t + \tau, x), [\varphi]) - \left( \varphi_t + H(t, x, \varphi, \varphi_x, \varphi_{xx}) \right) \right| \leq \mathcal{E}_H(\tau, \Delta x)$$

and

$$\mathcal{E}_H(\tau, \Delta x) \leq C_H(\tau^p + \Delta x^q)$$

for some  $p, q > 1$ ,  $C_H \geq 0$ .

We will also write

$$\begin{aligned} u_i^{n+1} &= \mathcal{S}_H([\dots, u^{n-1}, u^n])_i \quad n = 0, \dots, N-1, \quad i \in \mathbb{Z} \\ u_i^0 &= v_0(x_i) \quad i \in \mathbb{Z}. \end{aligned}$$

Different choices for the high order scheme are available.

**Example 3** (BDF scheme). *In one spatial dimension, BDF2-like schemes can be written in the following compact form:*

$$\frac{3u_i^{n+1} - 4u_i^n + u_i^{n-1}}{2\tau} + \sup_{a \in A} \left\{ -\frac{1}{2}\sigma^2(t_{n+1}, x_i, a)D^2u_i^{n+1} + b^+(t_{n+1}, x_i, a)D^{1,-}u_i^{n+1} - b^-(t_{n+1}, x_i, a)D^{1,+}u_i^{n+1} + f(t_{n+1}, x_i, a)u_i^{n+1} + \ell(t_{n+1}, x_i, a) \right\} = 0, \quad (2.7)$$

where  $D^2u_i$  corresponds to the usual second order approximation of  $v_{xx}$ :

$$D^2v_i := \frac{v_{i-1} - 2v_i + v_{i+1}}{\Delta x^2},$$

and, denoting by  $b^\pm$  the positive (resp. negative) part of  $b$ , a BDF2 like approximation is used for the first derivative in space:

$$D^{1,-}v_i := \frac{3v_i - 4v_{i-1} + v_{i-2}}{2\Delta x} \quad \text{and} \quad D^{1,+}v_i := -\left(\frac{3v_i - 4v_{i+1} + v_{i+2}}{2\Delta x}\right).$$

The first time step ( $n = 1$ ) needs some special treatment and in this case we consider the same spatial discretisation as in (2.7) excepted that the time approximation term  $\frac{3u_i^{n+1} - 4u_i^n + u_i^{n-1}}{2\tau}$  will be replaced by  $\frac{u_i^{n+1} - u_i^n}{\tau}$  which corresponds to an implicit Euler scheme. Of course, considering the equation in a bounded domain, some modification of the scheme might also be necessary at the boundary.

This specific choice of a BDF2 approximation in space for the first order derivative is in order to devise a second order consistent and stable scheme even when the diffusion term vanishes,  $\sigma(t, x_i, a) \equiv 0$ , and that can be used hereafter in combination with a policy iteration algorithm.

### 2.3 The filtered scheme

The filtered scheme will be defined by

$$u_i^{n+1} = S_F([\dots, u^{n-1}, u^n])_i := S_M(u^n)_i + \varepsilon\tau F\left(\frac{S_H([\dots, u^{n-1}, u^n])_i - S_M(u^n)_i}{\varepsilon\tau}\right)$$

$$u_i^0 = v_0(x_i)$$

for any  $n = 0, \dots, N-1$  and  $i \in \mathbb{Z}$ . Here,  $F$  is given by (2.1) and  $\varepsilon$  chosen as in (2.2) with  $\varepsilon_0$  constant.

## 3 Main results

We give in this section the main theoretical results.

**Proposition 3.1.** *The solution of the filtered scheme converges locally uniformly to the unique viscosity solution to (1.1) as  $\tau, \Delta x, \varepsilon \rightarrow 0$ .*

*Proof.* Let  $u$  and  $u_M$  respectively denote the solution of the filtered and the monotone scheme. From the convergence result in [4] one has the local uniform convergence of the solution of the monotone scheme to the unique viscosity solution  $v$  to equation (1.1):

$$u_M \longrightarrow v \quad \text{locally uniformly as } (\tau, \Delta x) \rightarrow 0.$$

By the very definition of the filtered scheme one has

$$u^{n+1} - u_M^{n+1} = S_M(u^n) - S_M(u_M^n) + \varepsilon\tau F(\dots),$$

then using the monotonicity of the scheme (see Remark 2) and the fact that  $|F(x)| \leq 1$  for any  $x \in \mathbb{R}$  one obtains

$$\sup_i |u_i^{n+1} - u_{M,i}^{n+1}| \leq (1 + C\tau) \sup_i |u_i^n - u_{M,i}^n| + \varepsilon\tau \quad n = 0, \dots, N-1. \quad (3.1)$$

The last inequality may not be evident for implicit schemes.

In this case, such an estimate can be derived observing that, using the notation introduced in Remark 2, for the optimal control  $\bar{a} \equiv \bar{a}(u_M)$  such that

$$\sup_{a \in A} \left\{ M^{a,n+1} u_M^{n+1} - G^{a,n}(u_M^n) \right\} = M^{\bar{a},n+1} u_M^{n+1} - G^{\bar{a},n}(u_M^n)$$

and

$$S_M(u_M^n) = (M^{\bar{a},n+1})^{-1} G^{\bar{a},n}(u_M^n),$$

one also has for any  $\phi$

$$\sup_{a \in A} \left\{ M^{a,n+1} \phi - G^{a,n}(u_M^n) \right\} \geq M^{\bar{a},n+1} \phi - G^{\bar{a},n}(u_M^n).$$

Therefore taking  $\phi = S_M(u^n)$  (in the second step)

$$\begin{aligned} M^{\bar{a},n+1}(S_M(u^n) - S_M(u_M^n)) &= M^{\bar{a},n+1} S_M(u^n) - G^{\bar{a},n}(u^n) + G^{\bar{a},n}(u^n) - M^{\bar{a},n+1} S_M(u_M^n) \\ &\leq \sup_{a \in A} \left\{ M^{a,n+1} S_M(u^n) - G^{a,n}(u^n) \right\} + G^{\bar{a},n}(u^n) - G^{\bar{a},n}(u_M^n) \\ &= G^{\bar{a},n}(u^n) - G^{\bar{a},n}(u_M^n), \end{aligned}$$

where the last step follows from the definition of the monotone scheme applied to  $u^n$ . By the properties of the M-matrix  $M^{\bar{a},n+1}$  and Lipschitzianity of  $G$  by Assumption (A2)(d), this implies

$$(S_M(u^n) - S_M(u_M^n))_i \leq \sup_i |G^{\bar{a},n}(u^n)_i - G^{\bar{a},n}(u_M^n)_i| \leq (1 + C\tau) \sup_i |u_i^n - u_{M,i}^n| \quad \forall i.$$

Switching the roles of  $u_M$  and  $u$  and using the control  $\hat{a} \equiv \hat{a}(u)$ , one obtains similarly

$$-(1 + C\tau) \sup_i |u_i^n - u_{M,i}^n| \leq (S_M(u^n) - S_M(u_M^n))_i \quad \forall i.$$

By iterating (3.1) and recalling that  $u^0 = u_M^0$ , one has

$$\sup_{i,n} |u_i^n - u_{M,i}^n| \leq \varepsilon T \exp(CT).$$

Hence

$$\begin{aligned} |u(s, y) - v(s, y)| &\leq |u(s, y) - u_M(s, y)| + |u_M(s, y) - v(s, y)| \\ &\leq |u_M(s, y) - v(s, y)| + \varepsilon T \exp(CT) \end{aligned}$$

and the convergence follows as  $\tau$ ,  $\Delta x$  and  $\varepsilon$  go to zero.  $\square$

**Theorem 3.1.** *Let assumptions (A1)-(A3) be satisfied. Let  $u$  and  $u_M$  respectively denote the solution of the filtered and the monotone scheme and let  $v$  be the unique viscosity solution to equation (1.1). One has*

(i) (Error estimates) *If the error estimate*

$$\|u_M - v\|_\infty \leq C_1(\Delta x^p + \tau^q) \quad (3.2)$$

*holds for the monotone scheme and if  $0 \leq \varepsilon \leq C_2(\Delta x^p + \tau^q)$  for some  $C_1, C_2 \geq 0$ , one also has*

$$\|u - v\|_\infty \leq C(\Delta x^p + \tau^q)$$

*for some constant  $C \geq 0$ .*

(ii) (First order convergence for classical solutions) *If the equation admits a classical solution  $v \in C^{2,4}([0, T], \mathbb{R})$  and  $\varepsilon \leq C_0(\Delta x + \tau)$  for some  $C_0 \geq 0$  one has*

$$\|u - v\|_\infty \leq C(\Delta x + \tau)$$

*for some  $C \geq 0$ .*

(iii) (Local high order consistency) *Let the solution  $v$  be smooth in an open set  $\mathcal{O}$ , and denote  $v^n = (v(x_i, t_n))_i$ . There exists  $C_0 \geq 0$  such that if  $\varepsilon \geq C_0(\tau + \Delta x)$  and  $\tau, \Delta x$  are small enough then*

$$S_F([\dots, v^{n-1}, v^n]) = S_H([\dots, v^{n-1}, v^n]) \quad \text{and} \quad \mathcal{E}_F(\tau, \Delta x) = \mathcal{E}_H(\tau, \Delta x) \quad \text{in } \mathcal{O}.$$

*Proof.* (i) As we observed in the proof of Proposition 3.1, one has

$$\|u - u_M\|_\infty \leq T\varepsilon.$$

Then if (3.2) holds true we get

$$\|u - v\|_\infty \leq \|u - u_M\|_\infty + \|u_M - v\|_\infty \leq C_2(\Delta x^p + \tau^q) + T\varepsilon$$

and the result follows with  $C = C_1 + C_2T$  thanks to the bound on  $\varepsilon$ .

(ii) Let  $v$  be the exact classical solution. The consistency of the monotone scheme implies that

$$\left| \mathcal{S}_M(t, x_i, v_i^{n+1}, [v])_i \right| = \left| \mathcal{S}_M(t, x, v_i^{n+1}, [v])_i - \left( v_t + H(t, x_i, v, v_x, v_{xx}) \right) \right| \leq C_M(\Delta x + \tau).$$

If  $\mathcal{S}_M$  is explicit this automatically implies

$$\left| v_i^{n+1} - S_M(v^n)_i \right| = \left| v_i^{n+1} - S_M(v^n)_i - \tau \left( v_t + H(t, x_i, v, v_x, v_{xx}) \right) \right| \leq \tau C_M(\Delta x + \tau).$$

Let us consider an implicit one-step finite difference scheme which can be written in the following form:

$$\mathcal{S}_M(t, x, v^{n+1}, [v]) = \frac{1}{\tau} \sup_{a \in A} \left\{ M^{a, n+1} v^{n+1} + G^{a, n}(v^n) \right\}.$$

Let  $\bar{a} \in A$  be the optimal value of the control in the expression above, then

$$\tau \mathcal{S}_M(t, x, v^{n+1}, [v]) = M^{\bar{a}, n+1} v^{n+1} + G^{\bar{a}, n}(v^n)$$

and

$$S_M(v^n) = (M^{\bar{a}, n+1})^{-1} G^{\bar{a}, n}(v^n).$$

Hence

$$\begin{aligned} \left| v_i^{n+1} - S_M(v^n)_i \right| &= \left| v_i^{n+1} - (M^{\bar{a}, n+1})^{-1} G^{\bar{a}, n}(v^n)_i \right| \\ &= \left| (M^{\bar{a}, n+1})^{-1} (M^{\bar{a}, n+1} v^{n+1} + G^{\bar{a}, n}(v^n))_i \right| \\ &\leq \tau \left\| (M^{\bar{a}, n+1})^{-1} \right\| \left| \mathcal{S}_M(t, x, v_i^{n+1}, [v])_i \right| \\ &\leq \tau K C_M (\Delta x + \tau) \quad \forall i \in \mathbb{Z}. \end{aligned}$$

Therefore we obtain

$$\begin{aligned} |u_i^{n+1} - v_i^{n+1}| &= |S_F([\dots, u^{n-1}, u^n])_i - S_M(v^n)_i - v_i^{n+1} + S_M(v^n)_i| \\ &\leq |S_M(u^n)_i - S_M(v^n)_i| + \varepsilon \tau + \tau K C_M (\Delta x + \tau) \quad \forall i \in \mathbb{Z}. \end{aligned}$$

By recursion and using the bound on  $\varepsilon$  one has

$$\|u - v\|_\infty \leq T \exp(CT) (K C_M + C_0) (\tau + \Delta x).$$

(iii) One has to prove that

$$\frac{|S_H([\dots, v^{n-1}, v^n]) - S_M(v^n)|}{\varepsilon \tau} \leq 1.$$

From the consistency of the schemes one obtains:

$$\frac{|S_H([\dots, v^{n-1}, v^n]) - S_M(v^n)|}{\varepsilon \tau} \leq \frac{\mathcal{E}_M(\tau, \Delta x) + \mathcal{E}_H(\tau, \Delta x)}{\varepsilon} \leq \frac{C_M(\tau + \Delta x) + o(\tau + \Delta x)}{\varepsilon}.$$

Therefore the result follows for  $\tau, \Delta x$  small enough and  $C_0 > C_M$  big enough.  $\square$

**Remark 3.** We refer to [1, 2, 3] for theoretical results on error estimates for monotone finite-difference schemes required in (3.2). For explicit SL schemes one can prove that  $p = q = 1/4$  (see [12]).

**Remark 4.** The previous result also suggests a way to choose the constant  $\varepsilon_0$  if bounds on the derivatives of the solution are available. Indeed, in the smooth case, we aim to observe second order convergence of the scheme and, as pointed out by the proof of local high order consistency, this can be achieved by defining  $\varepsilon_0$  big enough so that

$$\varepsilon_0(\tau + \Delta x) \geq C_M(\tau + \Delta x) + o(\tau + \Delta x)$$

(where the constant  $C_M$  depends on the  $L^\infty$ -bound of the derivatives of  $v$  that appear deriving the consistency error by Taylor expansion). It follows that a good choice would be some  $\varepsilon_0 > C_M$ .

## 4 Numerical tests

Hereafter we consider the equation (1.1) in a bounded set  $[x_{\min}, x_{\max}]$  in one dimension and a rectangle in two dimensions. With the exception of Section 4.2 we will only make use of uniform meshes and we denote by  $M$  the number of space intervals, in one dimension,

$$\Delta x = \frac{x_{\max} - x_{\min}}{M}.$$

### 4.1 Mean-variance asset allocation problem

As a first example, we study the mean-variance asset allocation problem (see [29, 20, 30]) formulated by the equation

$$v_t - \inf_{a \in A} \mathcal{L}^a(x, v, v_x, v_{xx}) = 0, \quad (4.1)$$

$$v(0, x) = \left(x - \frac{\gamma}{2}\right)^2, \quad (4.2)$$

for  $t \in [0, T]$  with

$$\mathcal{L}^a(x, v, v_x, v_{xx}) = \frac{1}{2}(\sigma a x)^2 v_{xx} + (c + x(r + a\sigma\xi))v_x. \quad (4.3)$$

Here,  $x$  is the wealth of an investor who controls the fraction  $a$  to invest in a risky asset to minimise the portfolio variance under a return target. The above equation assumes a Black-Scholes model with volatility  $\sigma$  and Sharpe ratio  $\xi$ ,  $r$  the rate on a risk-free bond and  $c$  the contribution rate,  $\gamma$  a measure of the risk aversion. We use the parameters from [29], see Table 1.

$r$	$\sigma$	$\xi$	$c$	$T$	$\gamma$
0.03	0.15	0.33	0.1	20	14.47

Table 1: Parameters used in numerical experiments for mean-variance problem.

If bankruptcy ( $x < 0$ ) is not allowed, the PDE (4.1–4.2) holds on  $(0, \infty)$ . We truncate the computational domain to  $x \in [0, 5]$ . As in [26], for  $x = 5$  we fix as a Dirichlet boundary condition the solution to the equation associated with the asymptotic optimal control  $a \equiv 0$  obtained for large values of  $x$ . As in [29], the control is assumed to take values in the bounded set  $A = [0, 1.5]$ .

The boundary equation at  $x = 0$  is

$$v_t(t, 0) - cv_x(t, 0) = 0, \quad (4.4)$$

see [29] for a discussion. This is a pure transport equation and, for  $c > 0$ , 0 is an influx boundary such that no boundary condition has to be set. In the computations, we approximate (4.4) by an upwind difference from the left-most two mesh points, which leads to a monotone scheme at the boundary.

Policy iteration as in [6] can be used to solve the discrete control problem efficiently.

Figure 3 shows the approximate shape of the value function and of the optimal control.

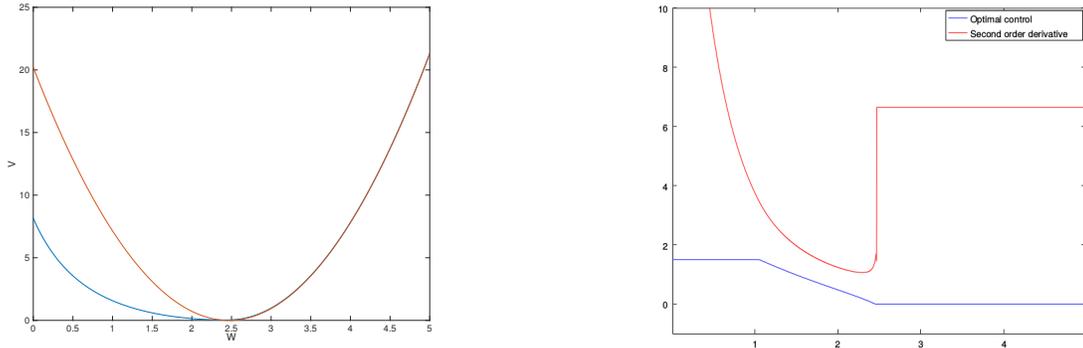


Figure 1: Mean variance asset allocation. Left: value function  $v(0, x)$  at time 0 (blue), and the asymptotic function used as right boundary value (red). Right: optimal control for  $t = 0$  and plot of  $v_{xx}(0, x)$ , a loss of regularity can be observed for  $x \sim 2.5$  corresponding to the switching point of the control.

The optimal control is at the upper bound for small  $x$ , then decreases linearly to zero, the lower bound, and then stays constant at 0.

We report in Table 2 the norms of the error and the corresponding order of convergence for the filtered BDF scheme (Section 3). For the computation of the error we use an accurate numerical solution. The scheme shows a clear second order of convergence in norm  $L^1$ . The order in the  $L^\infty$ -norm deteriorates around the critical point  $x \sim 2.5$ . At this point we can in fact notice a loss of regularity of the solution as shown by the plot of the second order derivative in space reported in Figure 3 (bottom). Table 3 shows the same computation obtained avoiding such a singularity.

$N$	$M$	Error $L^1$	Order $L^1$	Error $L^2$	Order $L^2$	Error $L^\infty$	Order $L^\infty$
40	40	2.97E-01	1.94	1.64E-01	1.98	2.24E-01	1.85
80	80	7.64E-02	1.96	4.15E-02	1.98	5.99E-02	1.90
160	160	1.95E-02	1.97	1.05E-02	1.98	1.56E-02	1.94
320	320	4.97E-03	1.97	2.68E-03	1.97	4.01E-03	1.96
640	640	1.26E-03	1.98	6.86E-04	1.97	1.02E-03	1.98
1280	1280	3.16E-04	1.99	1.77E-04	1.95	3.35E-04	1.61
2560	2560	7.93E-05	2.00	4.62E-05	1.94	1.35E-04	1.31
5120	5120	2.01E-05	1.98	1.23E-05	1.90	5.41E-05	1.31

Table 2: Mean variance asset allocation problem. Error and order of convergence for the filtered ( $\varepsilon_0 = 1$ ) BDF scheme in  $[0, 5]$ .

Figure 2 represents the convergence of the BDF scheme depending on the values of  $\varepsilon_0$ . For small values of  $\varepsilon_0$ , after some refinements the order of convergence goes to one. Applying the analysis of Remark 4 to this case, we can observe that in order to obtain the

$N$	$M$	Error $L^1$	Order $L^1$	Error $L^2$	Order $L^2$	Error $L^\infty$	Order $L^\infty$
40	40	2.85E-01	1.96	1.63E-01	1.99	2.24E-01	1.85
80	80	7.13E-02	2.00	4.07E-02	2.00	5.99E-02	1.90
160	160	1.80E-02	1.99	1.02E-02	1.99	1.56E-02	1.94
320	320	4.52E-03	1.99	2.57E-03	2.00	4.01E-03	1.96
640	640	1.12E-03	2.01	6.42E-04	2.00	1.02E-03	1.98
1280	1280	2.79E-04	2.01	1.61E-04	2.00	2.58E-04	1.98
2560	2560	6.94E-05	2.01	4.03E-05	1.99	6.53E-05	1.98
5120	5120	1.76E-05	1.98	1.03E-05	1.97	1.69E-05	1.95

Table 3: Mean variance asset allocation problem. Local error and order of convergence for the filtered ( $\varepsilon_0 = 1$ ) BDF scheme in  $[0, 5] \setminus [2.3, 2.7]$ .

second order of the scheme the safe choice of  $\varepsilon_0$ , i.e.  $\varepsilon_0 > C_M$ , would be

$$\varepsilon_0 > \frac{1}{2} \left\| \max \left( u_{tt}, (\pi + x(r + a_{\max} \sigma \xi)) u_{xx} \right) \right\|_\infty \sim 25,$$

where the derivatives were approximated from the numerical solution. However, for the number of refinements considered in Figure 5 and Tables 2 and 3 we can already observe the second order of convergence for  $\varepsilon_0 = 1$ .

## 4.2 Butterfly option pricing with uncertain volatility

We seek to approximate the price of a financial option under a Black-Scholes model with uncertain volatility [21]. In particular, we aim to find the price associated with the worst scenario for a “butterfly spread” with respect to all the possible values of the volatility  $\sigma$  in the interval  $[\sigma_{\min}, \sigma_{\max}]$ . This problem is associated with the following PDE in  $\mathbb{R}_+ \times (0, T)$  for some expiry  $T > 0$ :

$$v_t + \sup_{\sigma \in \{\sigma_{\min}, \sigma_{\max}\}} \left( -\frac{\sigma^2}{2} x^2 v_{xx} \right) - r x v_x + r v = 0, \quad (4.5)$$

$$v(0, x) = v_0(x), \quad (4.6)$$

where

$$v_0(x) := \max(x - K_1, 0) - 2 \max(x - (K_1 + K_2)/2, 0) + \max(x - K_2, 0).$$

We follow the example presented in [25] with parameters given by Table 4 on the same non-uniform grid.<sup>1</sup>

$r$	$\sigma_{\min}$	$\sigma_{\max}$	$T$	$K_1$	$K_2$
0.1	0.15	0.25	0.1	90	110

Table 4: Parameters used in numerical experiments for uncertain volatility problem.

<sup>1</sup>We are grateful to Peter Forsyth for the provision of his code.

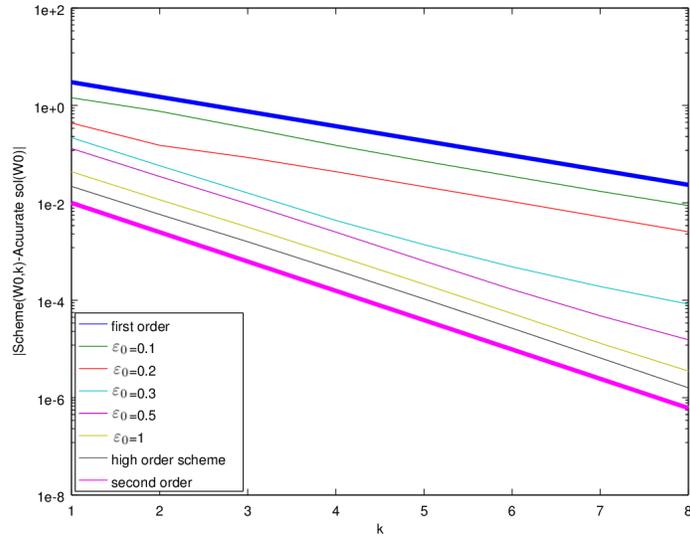


Figure 2: Mean variance asset allocation problem. Speed of convergence for the filtered BDF scheme for different values of  $\varepsilon_0$ . The error is computed at a single point  $x = 1$ .

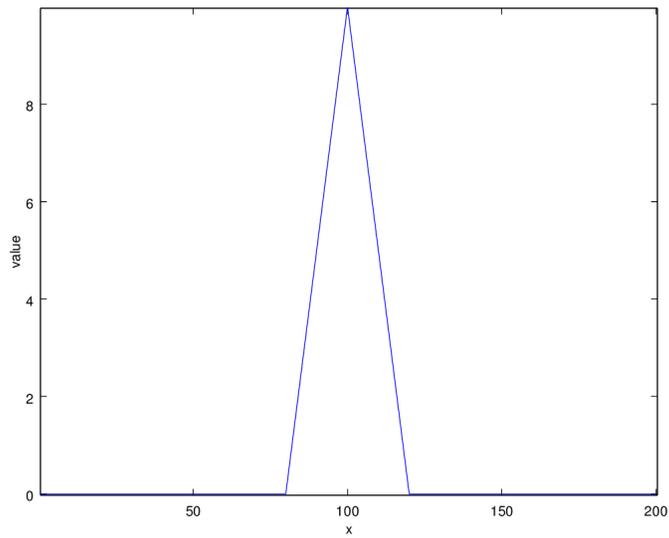


Figure 3: Payoff function for a butterfly spread with parameters given by Table 4.

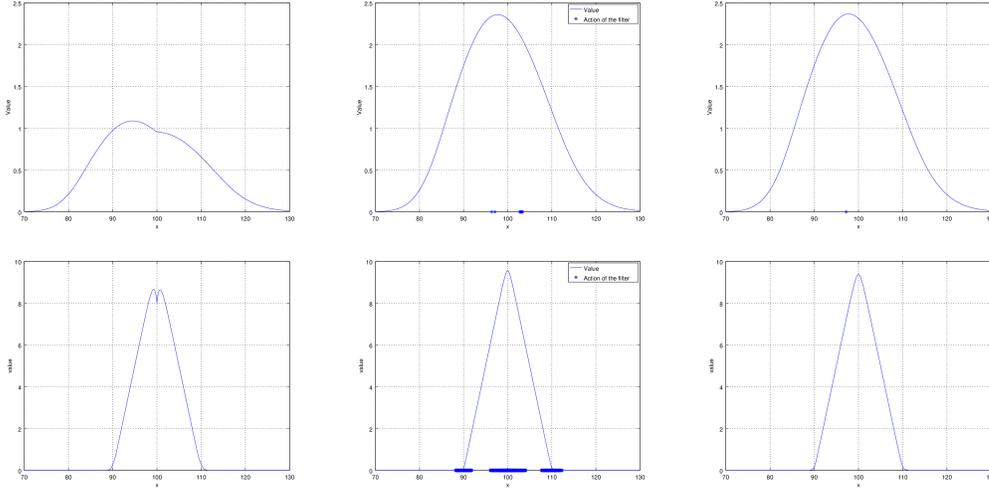


Figure 4: Butterfly option pricing with uncertain volatility. Top: Value function at the terminal time. Bottom: Value function at the first time step. Left: CN scheme, center: filtered CN scheme, right: BDF scheme.

In [25], it is shown that, in absence of a CFL condition constraining  $\tau$  to be of the order of  $\Delta x^2$ , the Crank-Nicolson (CN) scheme may not converge to the unique viscosity solution of the problem. We refer to [25] for the precise definition of the scheme used. Figure 4 shows the result obtained by the CN scheme (left) and the corresponding result obtained using the fully implicit first order finite difference scheme as a filter (center). The filter arrives at correct the scheme, but the overall order of convergence it is not greater than one (compare with the results in Table 5), due to the fact that the filter is applied in a quite wide area (Figure 4, center). In particular, the measure of the area does not diminish as the mesh is refined and therefore the contribution to the overall error from the low order scheme dominates. In Table 5 we compare the order of convergence of the filtered CN scheme with the Rannacher scheme (i.e., the fully implicit scheme replaces CN for the first two time iterations) used in [25] and the fully implicit BDF scheme defined in Section 3. Both the Rannacher scheme and BDF scheme appear to converge to the true solution with order two, in spite of their lack of monotonicity, so that convergence is not theoretically established.

Next, in order to ensure convergence, we apply the filter to the BDF scheme. The orders of convergence for different values of  $\varepsilon_0$  are shown in Figure 5. However, for this example, the unboundedness of the derivatives for  $t \downarrow 0$  and the resulting unboundedness of the constant  $C_M$  do not allow the high order of convergence in the relevant region of the domain.

$N$	$M$ (Non uniform)	CN	Filtered CN	BDF	Rannacher
25	60	-	-	-	-
50	120	0.32	1.25	2.07	0.97
100	240	0.50	0.85	1.97	1.85
200	480	0.55	0.90	1.99	1.86
400	960	0.58	1.10	1.96	1.92
800	1920	0.62	0.94	2.01	1.98
1600	3840	0.65	1.10	1.99	1.99

Table 5: Butterfly option pricing with uncertain volatility. Order of convergence for CN (which does not converge to the viscosity solution), filtered CN (with  $\varepsilon = 10(\tau + \Delta x_{\max})$ ), BDF, Rannacher (fully implicit low order scheme for the first 2 time steps).

### 4.3 An example in two dimensions

We now test the effects of the filter on a two-dimensional example,

$$v_t - \inf_{a \in A} \left( \frac{1}{2} \text{tr}[\sigma \sigma^T(a) D_x^2 v] + \ell(t, x, a) \right) = 0,$$

$$v(0, x) = 2 \sin x_1 \sin x_2,$$

where  $t \in [0, T]$ ,  $x \equiv (x_1, x_2) \in [-\pi, \pi] \times [-\pi, \pi]$  and control set and coefficients are given by

$$A = \{a \in \mathbb{R}^2 : a_1^2 + a_2^2 = 1\},$$

$$\sigma(a) = \sqrt{2} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \quad \ell(t, x, a) = (1-t) \sin x_1 \sin x_2 + (2-t)(a_1^2 \cos x_1 + a_2^2 \cos x_2).$$

This problem has exact solution

$$v(t, x) = (2-t) \sin x_1 \sin x_2.$$

This example is similar (but more complex for what concerns the solution of the optimization problem) to the one discussed in [12, Section 9.3 (B)]. In fact, because of the presence of cross-derivatives and the fact that the diffusion matrix  $(a_1, a_2)^T (a_1, a_2)$  may be not diagonally dominant, finite difference schemes are not monotone. Therefore the monotone scheme we consider here is a semi-Lagrangian scheme as defined in [12]. However, in the example discussed in [12] the problem reduces to a linear as the solution satisfies the PDE for any control (not only the optimal one) and no minimisation in  $a$  is necessary.

The monotone scheme is then given by:

$$\frac{u^{n+1} - u^n}{\tau} = \inf_{a \in \bar{A}} \left\{ \frac{[u^n](x + k\sigma(a)) - 2u^n + [u^n](x - k\sigma(a))}{2k^2} + \ell(t_n, x, a) \right\},$$

where  $k^2 = T/M$  ( $M$  the number of space steps) and the infimum is computed using the following discretisation of the controls:

$$\bar{A} = \{a_i : i = 0, \dots, P\}, \quad a_i = (\cos(i\Delta\theta), \sin(i\Delta\theta)), \quad \text{and } \Delta\theta = 2\pi/P, \quad (4.7)$$

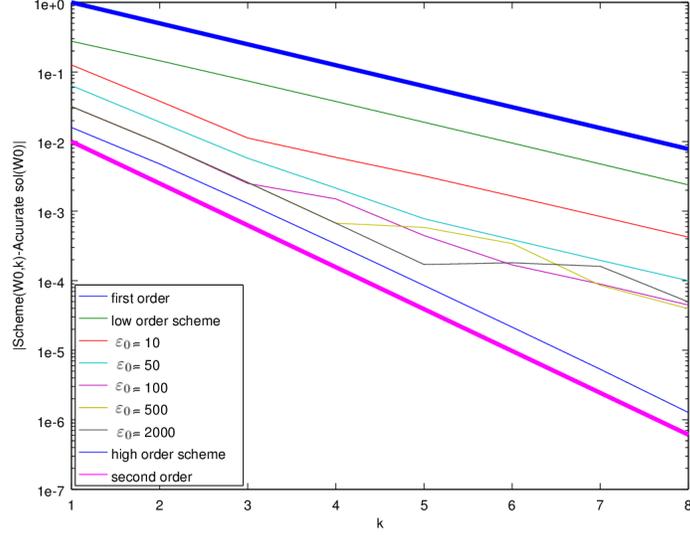


Figure 5: Butterfly option pricing with uncertain volatility. Convergence rate of the  $L^\infty$ -error obtained for  $\varepsilon = \varepsilon_0(\tau + (\Delta x_{\min}))$  and different values of  $\varepsilon_0$ , using the non uniform mesh defined in [25]. The error is computed by comparison with an accurate numerical solution.

$P$  the number of discretised control values.

The solution is linear in time, and since the optimal control depends on  $x$  but not on  $t$ , this behaviour is inherited by the numerical scheme. Therefore one time step is sufficient,

$$u^N - u^0 = T \inf_{a \in \bar{A}} \left\{ \frac{[u^0](x + k\sigma(a)) - 2u^0 + [u^0](x - k\sigma(a))}{2k^2} + \ell(0, x, a) \right\}.$$

The high order scheme is a finite difference scheme based on the following naive approximation of the second order derivatives for  $v_{i,j} \equiv v(x_i, x_j)$ :

$$D_{xx}^2 v_{i,j} := \frac{v_{i+1,j} - 2v_{i,j} + v_{i-1,j}}{\Delta x^2}, \quad D_{yy}^2 v_{i,j} := \frac{v_{i,j+1} - 2v_{i,j} + v_{i,j-1}}{\Delta y^2},$$

$$D_{xy}^2 v_{i,j} := \frac{v_{i+1,j+1} - v_{i+1,j-1} + v_{i-1,j-1} - v_{i-1,j+1}}{4\Delta x \Delta y}.$$

Due to the linearity of the solution in time, we do not need BDF timestepping to obtain higher order, and use again a single backward Euler step. The scheme can be written in compact form by:

$$u^N - T \inf_{a \in \bar{A}} \left\{ a_1^2 D_{xx}^2 u^N + 2a_1 a_2 D_{xy}^2 u^N + a_2^2 D_{yy}^2 u^N + \ell(T, x, a) \right\} = u^0.$$

The optimization problem is solved by policy iteration again passing by the discretisation of the controls (4.7).

In order to find the suitable value of  $\varepsilon_0$  (see Remark 4) we estimate the constant  $C_M$  that

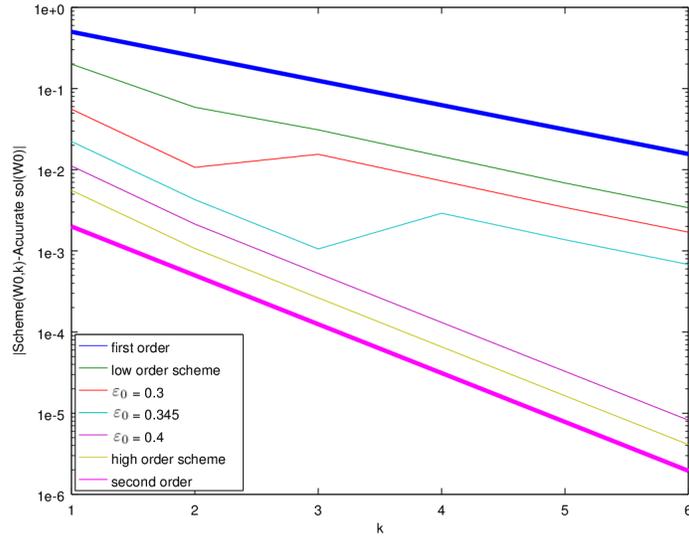


Figure 6: Order of convergence in the  $L^\infty$ -norm for the two-dimensional example with linearity in time for different values of  $\varepsilon_0$ . For the number of refinements considered here we start to observe the second order of convergence for  $\varepsilon_0 = 0.4$ , which seems consistent with the theoretical value  $4/3$  provided following the arguments of Remark 4.

appears in the truncation error of the monotone scheme. For the semi-Lagrangian scheme above (recall that  $v_{tt} = 0$ ) this is given by

$$\begin{aligned} & \frac{1}{4!} \sup_{a \in A} \left\{ a_1^4 \|v_{xxxx}\|_\infty + 4a_1 a_2^3 \|v_{xyyy}\|_\infty + 6a_1^2 a_2^2 \|v_{xxyy}\|_\infty + 4a_2 a_2^3 \|v_{xxxy}\|_\infty + a_2^4 \|v_{yyyy}\|_\infty \right\} \\ & \leq \frac{1}{4!} |2 - t| (1 + 4 + 6 + 4 + 1) = \frac{4}{3}. \end{aligned}$$

Figure 6 seems to confirm this result. Indeed, even if we did not push the number of refinements much further, we can start observing a second order of convergence already for  $\varepsilon_0 = 0.4 < 4/3$ . In Table 6 we report the results obtained for  $\varepsilon_0 = 0.4$  and Figure 7 shows the points of activity of the filter for different values of the constant  $\varepsilon_0$ .

## 5 Conclusions

Filtered schemes are designed to combine the advantages of the guaranteed convergence of low order monotone schemes and the superior accuracy – in regions where the solution is smooth – of higher order non-monotone schemes. The theoretical results in this paper confirm these properties.

In our numerical tests, the schemes delivered the accuracy of the high order scheme if the solution is smooth. For an example with a locally non-smooth solution, the filter was seen to turn a divergent higher order time stepping scheme (the Crank-Nicolson scheme)

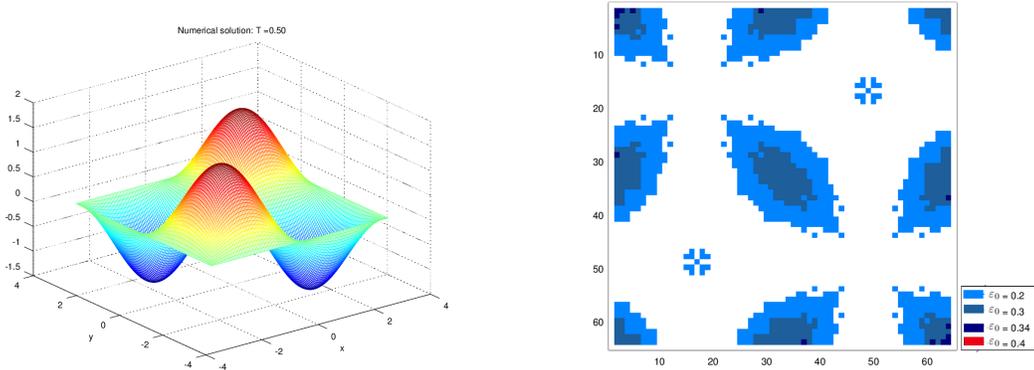


Figure 7: Two-dimensional example. Left: value function at the terminal time  $T = 0.5$ . Right: points of activity of the filter corresponding to different values of  $\varepsilon_0$ . As soon as  $\varepsilon_0 \geq 0.4$  we do not observe any use of the filter.

$N$	$M$	Error $L^1$	Order $L^1$	Error $L^2$	Order $L^2$	Error $L^\infty$	Order $L^\infty$
1	4	1.13E+00	-	9.27E-01	-	5.57E-01	-
1	8	4.16E-01	1.44	2.79E-01	1.73	1.08E-01	2.37
1	16	1.39E-01	1.58	9.61E-02	1.54	3.13E-02	1.78
1	32	3.76E-02	1.89	2.70E-02	1.83	8.82E-03	1.83
1	64	9.58E-03	1.97	7.06E-03	1.94	2.31E-03	1.93
1	128	2.51E-03	1.93	1.83E-03	1.95	6.10E-04	1.92

Table 6: Two-dimensional example. Error and order of convergence for the filtered scheme with  $\varepsilon_0 = 0.4$ . Due to the linearity with respect to time one step of Euler ( $N = 1$ ) is sufficient to ensure convergence.

into a convergent scheme, albeit only at the order of the low-order scheme. Although non-monotone high order schemes with better stability (such as the BDF2 scheme) empirically gave second order convergence, the filter reduced this order to one due to singularities of the higher order derivatives of the solution resulting in a wide application of the filter. Nonetheless, the absolute error is almost by a factor of 100 lower than for the pure low order scheme.

A possible future research direction are filters which give global high order convergence for solutions with localised singularities.

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