Semi-global stabilization by an output feedback law from a hybrid state controller

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Abstract

This article suggests a design method of a hybrid output feedback for SISO continuous systems. We focus on continuous systems for which there exists a hybrid state feedback law. A local hybrid stabilizability and a (global) complete uniform observability are assumed to achieve the stabilization of an equilibrium with a hybrid output feedback law. This is an existence result. Moreover, assuming the existence of a robust Lyapunov function instead of a stabilizability assumption allows to design explicitely this hybrid output feedback law. This last result is illustrated for linear systems with reset saturated controls.

Key words: Hybrid systems; Output feedback; Observers

1 Introduction

In recent years, many techniques for designing a stabilizing control law for nonlinear dynamical systems have been developed. It is now possible to achieve stabilization of equilibria for a large class of models. However, due to Brockett's necessary condition for stabilizability, it is well known that some systems cannot be stabilized by a continuous controller. Some of these systems can however be stabilized with a hybrid state feedback law, i.e. a discrete/continuous controller (see e.g. Prieur and Trélat (2005), where the Brockett integrator is stabilized with a quasi optimal hybrid control). Moreover, the use of hybrid control laws may be interesting to address performance issues (see e.g. Prieur (2001)). This explains the great interest of the control community in the synthesis of hybrid control laws (see Goebel et al. (2012), Hespanha et al. (2008); Hetel et al. (2013); Fichera et al. (2013); Yuan and Wu (2014)).

The output feedback stabilization problem has also attracted the attention of numerous researchers. Indeed, employing a state feedback law is most of the cases impossible, since the sensors can only access to partial measurements of the state. Output feedback laws may be designed from a separation principle. More precisely, two tools are designed separately: a stabilizing state feedback law and an asymptotic state observer. However, if this approach is fruitful for linear systems, the separation principle does not hold in general for nonlinear systems. For instance, there exist stabilizable and observable systems for which the global asymptotic stabilization by output feedback is impossible (Mazenc et al. (1994)). Nevertheless, from weak stabilizability and observability assumptions, some semi-global results may be obtained (see e.g. Teel and Praly (1994) or (Isidori, 1995, Pages 125-172)). However, in this case the observer and the state feedback have to be jointly designed (not separately) (see also Andrieu and Praly (2009) for some global results).

The aim of this paper is to address the stabilization by hybrid output feedback law. In Teel (2010), a local separation principle is stated. However, the construction of the observer is not explicit. Here, from a hybrid state feedback controller and an observability property, an algorithm is provided to build hybrid output feedback laws which stabilize semi-globally the equilibrium plant. If moreover a robust Lyapunov function is known, the feedback law design becomes explicit.

This article is organized as follows. Section 2 introduces

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the problem together with a hybrid stabilizability and a observability assumptions. The main result is given in Section 3.1. An equivalent stabilizability assumption in terms of Lyapunov function is considered in Section 3.2. This allows to give a more explicit theorem. Section 4 explains how to prove the first theorem from the second theorem. In Section 5 technical lemmas are stated in order to construct the suggested output feedback law. An illustrative example is given in Section 6. Finally, Section 7 collects some concluding remarks.

Note that this paper is an extension of the conference paper Marx et al. (2014). It includes some missing part (the observer design), proofs and a new illustration.

Notation: Given $\lambda \in \mathbb{N}$, $\mathbb{R}_{\geq\lambda} = [\lambda, +\infty)$. Given $n \in \mathbb{N}$, $I_n \in \mathbb{R}^{n\times n}$ denotes the identity matrix, i.e. $I_n = \mathtt{diag}(1, \ldots, 1)$. $\star$ states for symmetric terms. Given $n \in \mathbb{N}$, $\mathcal{L}^{\infty}_{loc}(\mathbb{R}, \mathbb{R}^n)$ denotes the set of measurable locally bounded functions $u : \mathbb{R} \to \mathbb{R}^n$. A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ belongs to class-$\mathcal{K}$ (for short $\alpha \in \mathcal{K}$) if it is continuous, zero at zero, and strictly increasing. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ belongs to class-$\mathcal{KL}$ (for short $\beta \in \mathcal{KL}$) if it satisfies (i) for each $t \geq 0$, $\beta(., t)$ is nondecreasing and $\lim_{t\searrow 0} \beta(s, t) = 0$, and (ii) for each $s \geq 0$, $\beta(s, .)$ is nonincreasing and $\lim_{t\to\infty} \beta(s, t) = 0$.

2 Problem statement

2.1 Hybrid state feedback law for a continuous time plant

The system under consideration is described by the following single-input single-output continuous dynamics:

$$\dot{x}_p = f_p(x_p) + g_p(x_p)u \ , \ y = h_p(x_p) \ , \tag{1}$$

where $x_p \in \mathbb{R}^{n_p}$, $y \in \mathbb{R}$, $u \in \mathcal{U} \subset \mathbb{R}$. Note that $f_p : \mathbb{R}^{n_p} \to \mathbb{R}^{n_p}$ and $g_p : \mathbb{R}^{n_p} \to \mathbb{R}^{n_p}$, $h_p : \mathbb{R}^{n_p} \to \mathbb{R}$ are $n_p + 1$ times continuously differentiable [1]. $\mathcal{U}$ can be bounded (it yields a saturated control problems). Inspired by Prieur and Trélat (2005) and Sontag (1999), the origin, which is an equilibrium point for (1), is assumed to be stabilizable by a hybrid state feedback.

Assumption 1. (Persistent Flow Stabilizability) *There exists a hybrid controller defined by* $(\mathcal{F}_c, \mathcal{J}_c, f_c, g_c, \theta_c)$, *where* $\mathcal{F}_c$ *and* $\mathcal{J}_c$ *are closed sets,* $\mathcal{F}_c \cup \mathcal{J}_c = \mathbb{R}^{n_p+n_c}$, $g_c : \mathbb{R}^{n_p+n_c} \to \mathbb{R}^{n_c}$, $f_c : \mathbb{R}^{n_p+n_c} \to \mathbb{R}^{n_c}$ *and* $\theta_c : \mathbb{R}^{n_p+n_c} \to \mathcal{U}$ *are continuous functions and a positive value* λ *in* $(0, 1)$ *such that the set* $\{0\} \times [0, 1]$ *in* $\mathbb{R}^{n_p+n_c} \times \mathbb{R}$ *is asymptotically stable for the system:*

$$\begin{cases} \dot{x}_p = f_p(x_p) + g_p(x_p)\theta_c(x_p, x_c) \\ \dot{x}_c = f_c(x_p, x_c) \\ \dot{\sigma} = 1 - \sigma \end{cases} \quad (x_p, x_c, \sigma) \in \mathcal{F}_c \times \mathbb{R}_{\geq 0} \tag{2a}$$

$$\begin{cases} x_p^+ = x_p \\ x_c^+ = g_c(x_p, x_c) \\ \sigma^+ = 0 \end{cases} \quad (x_p, x_c, \sigma) \in \mathcal{J}_c \times \mathbb{R}_{\geq\lambda} \tag{2b}$$

with basin of attraction $\mathcal{B} \times \mathbb{R}_{\geq 0}$, *where* $\mathcal{B}$ *is an open subset of* $\mathbb{R}^{n_p+n_c}$.

The sets $\mathcal{F}_c \times \mathbb{R}_{\geq 0}$ and $\mathcal{J}_c \times \mathbb{R}_{\geq\lambda}$ are called respectively the flow and jump sets associated to the continuous and discrete dynamics. The notion of solutions and of asymptotic stability discussed all along the paper are borrowed from Goebel et al. (2012).

Remark 1. *An important feature of the hybrid state feedback control law is that its dynamics include a timer* σ. *It implies that there exists a dwell time between two consecutive jumps and consequently it prevents the existence of Zeno solutions. In the case in which this property is not satisfied for the state feedback, a timer can be added as presented in (Cai et al., 2008, Part V, C.). Such a technique is called a* temporal regularization. *However, in this case, only semi-global practical stability is obtained.* ◦

The problem under consideration in this paper is to design a stabilizing output feedback law based on this hybrid state feedback. The design presented in this paper requires an observability property for system (1) as described in the following section.

2.2 Observability notions

Following Gauthier et al. (1992), define the C^1 mapping $\phi : \mathbb{R}^{n_p} \to \mathbb{R}^{n_p}$ as follows

$$\phi(x_p) = \Big[h_p(x_p) \ L_{f_p}h_p(x_p) \ \ldots \ L_{f_p}^{n_p-1}h_p(x_p)\Big]^\top \ , \tag{3}$$

where $L^i_{f_p}h_p(x)$ denotes the i-th Lie derivative of h_p along f_p. The observability assumption employed all along the paper can be now stated.

Assumption 2 ((Global) Complete Uniform Observability (Gauthier et al. (1992))). *System (1) is completely uniformly observable, that is*

(i) The mapping $\phi : \mathbb{R}^{n_p} \to \phi(\mathbb{R}^{n_p}) = \mathbb{R}^{n_p}$ *is a diffeomorphism;*

(ii) System (1) is observable for any input $u(t)$, *i.e. on any finite time interval* $[0, T]$, *for any measurable bounded input* $u(t)$ *defined on* $[0, T]$, *the initial state is uniquely determined on the basis of the output* $y(t)$ *and the input* $u(t)$.

[1] These mappings are sufficiently smooth so that the mapping ϕ defined in (3) is C^1 and so that the function B defined in (12) is locally Lipschitz.

Remark 2. *In Marx et al. (2014), from a weaker observability assumption, i.e. an observability property holding for just one control, a finite-time convergent observer and a hybrid state feedback controller has been used to design an output feedback law. Such a strategy does not need a persistent flow stabilizability assumption. However only a weak stability property is obtained for the closed-loop system.* ○

3 Semi-global output feedback result

3.1 First main result

Inspired by the approach of Teel and Praly (1994), from Assumptions 1 and 2, a semi-global output feedback result may be obtained.

Theorem 1 (Semi-global asymptotic stability). *Assume Assumptions 1 and 2 hold. Assume moreover that g_c satisfies that, for all (x_p, x_c) in $\mathcal{B} \cap \mathcal{J}_c$, the set*

$$\{(x_p, g_c(w, x_c)), w \in \mathbb{R}^{n_p}\} \tag{4}$$

is a compact subset of $\mathcal{B}$, then the origin of system (1) is semi-globally asymptotically stabilizable by a hybrid output feedback. In other words, for all compact sets Γ contained in $\mathcal{B}^p := \{x_p \in \mathbb{R}^{n_p}, (x_p, 0) \in \mathcal{B}\}$, there exist a C^1 function $\Psi_p : \mathbb{R}^{n_p} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and a positive real number c_x such that the set $\{0\} \times [0, 1]$ in $\mathbb{R}^{2n_p+n_c} \times [0, 1]$ is asymptotically stable for the system

$$\begin{cases} \dot{x}_p = f_p(x_p) + g_p(x_p)u \\ \dot{\hat{x}}_p = \Psi_p(\hat{x}_p, y, u) \\ \dot{x}_c = f_c(\tilde{x}_p, x_c) \\ \dot{\sigma} = 1 - \sigma \\ y = h_p(x_p) \ , \ u = \theta_c(\tilde{x}_p, x_c) \end{cases} \tag{5a}$$

$$(\tilde{x}_p, x_c, \sigma) \in \mathcal{F}_c \times \mathbb{R}_{\geq 0}$$

$$\begin{cases} x_p^+ = x_p \\ \hat{x}_p^+ = \hat{x}_p \\ x_c^+ = g_c(\tilde{x}_p, x_c) \\ \sigma^+ = 0 \end{cases} \quad (\tilde{x}_p, x_c, \sigma) \in \mathcal{J}_c \times \mathbb{R}_{\geq \lambda}. \tag{5b}$$

where $\tilde{x}_p$ is defined by [2]

$$\tilde{x}_p = \mathbf{sat}_{c_x}(\hat{x}_p) \ , \tag{6}$$

with basin of attraction containing $\Gamma \times \{0\} \times \{0\} \times \mathbb{R}_{\geq 0}$ (which is a subset of $\mathbb{R}^{n_p} \times \mathbb{R}^{n_p} \times \mathbb{R}^{n_c} \times \mathbb{R}_{\geq 0}$).

The design of the output feedback law which proves this theorem is based on a Lyapunov inverse theorem. However, two datas miss in the output feedback law given in Theorem 1: the positive real number c_x, the saturation level for the feedback law, and the observer dynamics Ψ_p. In order to give an explicit, in the next section, the existence of a robust Lyapunov function is assumed.

An important feature of this theorem is that an assumption needs to be imposed on the function g_c (see equation (4)). This is due to a design of a set in which the solution should stay for a suitable duration. In the particular case in which $\mathcal{B}$ is $\mathbb{R}^{n_p+n_c}$, the previous condition is trivially satisfied if g_c is such that

$$|g_c(w, x_c)| \leq \gamma(|x_c|) \ , \ \forall (w, x_c) \in \mathbb{R}^{n_p+n_c} \cap \mathcal{J}_c$$

where $\gamma \in \mathcal{K}$. Moreover, note that there is a large class of systems that satisfy such a condition. For instance, switch systems or reset systems, as the one considered in Section 6 below.

3.2 Second main result

In this section an explicit result is introduced. It is based on the following ssumption, where it is denoted $X = \left[x_p^\top \ x_c^\top \ \sigma^\top \right]^\top$.

Assumption 3 (Robust Lyapunov function). *Let $\mathcal{B}$ be an open subset of $\mathbb{R}^{n_p} \times \mathbb{R}^{n_c}$ and let denote $\mathcal{A} := \{0\} \times [0, 1] \subset \mathcal{B} \times \mathbb{R}_{\geq 0}$. There exist a hybrid controller defined by $(\mathcal{F}_c, \mathcal{J}_c, f_c, g_c, \theta_c)$, where $\mathcal{F}_c$ and $\mathcal{J}_c$ are closed sets, $\mathcal{F}_c \cup \mathcal{J}_c = \mathbb{R}^{n_p+n_c}$, $g_c : \mathbb{R}^{n_p+n_c} \to \mathbb{R}^{n_c}$, $f_c : \mathbb{R}^{n_p+n_c} \to \mathbb{R}^{n_c}$ and $\theta_c : \mathbb{R}^{n_p+n_c} \to \mathcal{U}$ are continuous functions, a positive value λ in $(0, 1)$, positive values α_1 and $\alpha_2 \in (0, 1)$ and a C^1 proper*[3] *function $V : \mathcal{B} \times \mathbb{R} \to \mathbb{R}_{\geq 0}$ satisfying $\{X \in \mathcal{B} \times \mathbb{R}, V(X) = 0\} = \mathcal{A}$. For all positive real numbers l, the level set of V defined as*

$$D_l := \{(x_p, x_c, \sigma) \in \mathcal{B} \times \mathbb{R}_{\geq 0} \ : \ V(x_p, x_c, \sigma) \leq l\} \ , \tag{7}$$

is a compact subset of $\mathcal{B} \times \mathbb{R}$.

Moreover, there exists a positive real number ε_r and an increasing C^0 function $\rho : [0, \varepsilon_r] \to \mathbb{R}_+$ with $\rho(0) = 0$ such that for all (X, e) in $D_l \times \mathbb{R}^{n_p}$ such that $|e| \leq \varepsilon_r$, the following inequalities hold.

- *If $(x_p + e, x_c, \sigma) \in (\mathcal{B} \cap \mathcal{F}_c \times \mathbb{R}_{\geq 0})$*

$$\frac{\partial V}{\partial X}(X) F(X, x_p + e) \leq -\alpha_1 V(x_p, x_c, \sigma) + \rho(|e|) \tag{8}$$

- *If $(x_p + e, x_c, \sigma) \in (\mathcal{B} \cap \mathcal{J}_c \times \mathbb{R}_{\geq \lambda})$*

$$V(G(X, x_p + e)) - V(X) \leq -\alpha_2 V(x_p, x_c, \sigma) + \rho(|e|)$$

where F is defined by

$$F(X, .) := \left[(f_p(x_p) + g_p(x_p)\theta_c(., x_c))^\top \ f_c(., x_c)^\top \ 1 - \sigma \right]^\top$$

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[2] Given a positive real number c, $\mathtt{sat}_c : \mathbb{R}^n \to \mathbb{R}^n$ is the saturating vector function defined by $\mathtt{sat}_c(0) = 0$ and $\mathtt{sat}_c(x) := x \min\left\{1, \frac{c}{|x|}\right\}$, $\forall x \neq 0$.

[3] A map is called proper if inverse images of compact sets are compact.

and G is defined by $G(X,.) := \left[x_p^\top \;\; g_c(.,x_c)^\top \;\; 0\right]^\top$.

This assumption allows to obtain an explicit result.

Theorem 2 (Design of an output feedback law)**.** *Under Assumptions 2 and 3, assume that the set defined by (4) is a compact subset of* $\mathcal{B}$*. Then the set* $\{0\} \times [0,1]$ *in* $\mathbb{R}^{n_p+n_c} \times \mathbb{R}$ *is semi-globally asymptotically stabilizable. In other words, the conclusion of Theorem 1 holds. Moreover* c_x *is computed in Section 5.1 and* ψ_p *is computed in Section 5.2 from the Lyapunov function* V *together with the robustness margin* ε_r *and the positive value* λ *of Assumption 3, and from the function* ϕ *of Assumption 2.*

Let us note that it is difficult to be more explicit since the derivations of c_x and Ψ_p are quite long and require several steps. This is already the case in Teel and Praly (1994). Moreover, continuing what has been stated in Remark 1, it is crucial to have a Persistent-Flow Stabilizability in order to design explicitely our observer.

In a first step, c_x is computed in (11) in order to force the solution to remain in a compact set for a certain amount of (flow) time. The function Ψ_p is a high-gain observer which is tuned in Lemma 2. It forces the error to reach the robustness margin obtained from Assumption 3 before the solution escapes the compact set.

The explicit construction of these two data and the proof that this output feedback law is a solution to Theorem 2 is reported in Section 5.

4 Proof of Theorem 1 from Theorem 2

In order to prove Theorem 1 from Theorem 2 it is sufficient to prove that Assumption 1 implies Assumption 3. This can be obtained from an inverse Lyapunov result. First, from (Goebel et al., 2012, Corollary 7.32) there exists a positive value $\alpha \in (0,1)$ and a smooth proper function $V : \mathcal{B} \times \mathbb{R} \to \mathbb{R}_{\geq 0}$ satisfying

$$\{X \in \mathcal{B} \times \mathbb{R} \;:\; V(X) = 0\} = \mathcal{A}$$

$$\begin{aligned} &\frac{\partial V}{\partial X}(X)F(X,x_p) \leq -V(X), \\ &\qquad\qquad \forall (x_p,x_c,\sigma) \in (\mathcal{B} \cap \mathcal{F}_c) \times [0,\lambda] \\ &V(G(X,x_p)) - V(X) \leq -\alpha V(X), \\ &\qquad\qquad \forall (x_p,x_c,\sigma) \in (\mathcal{B} \cap \mathcal{J}_c) \times \mathbb{R}_{\geq 0} \end{aligned} \tag{9}$$

Let l be a positive real number such that the level set D_l is a compact subset of $\mathcal{B} \times \mathbb{R}$. Consider the two functions r_1 and r_2 defined as

$$r_1(s) = \max_{|e|\leq s,\, (x_p+e,x_c,\sigma)\in D_l} \frac{\partial V(X)}{\partial X} F(X,x_p) + \frac{1}{2}V(X)$$

$$r_2(s) = \max_{|e|\leq s,\, (x_p+e,x_c,\sigma)\in D_l} V(G(X,x_p)) - \left(1 - \frac{1}{2}\alpha\right) V(X)$$

Since F, G are continuous and V is smooth, r_1 and r_2 are also continuous functions. Moreover $r_1(0) < 0$ and $r_2(0) < 0$. Therefore there exist ε_r^1 and ε_r^2 such that $r_1(s) < 0$ for all $s \leq \varepsilon_r^1$ and $r_2(s) < 0$ for all $s \leq \varepsilon_r^2$. Let $\varepsilon_r = \min(\varepsilon_r^1, \varepsilon_r^2)$. For all $|e| \leq \varepsilon_r$ and $(x_p,x_c,\sigma) \in D_l$ it yields the following:

- If (x_p+e,x_c,σ) in $(\mathcal{B} \cap \mathcal{F}_c \times \mathbb{R}_{\geq 0})$,

$$\frac{\partial V}{\partial X}(X)F(X,x_p) \leq -\frac{1}{2}V(X) \;, \forall X \in \mathcal{B} \cap \mathcal{F}_c \times \mathbb{R}_{\geq 0}$$

- If (x_p+e,x_c,σ) in $(\mathcal{B} \cap \mathcal{J}_c \times \mathbb{R}_{\geq \lambda})$

$$V(G(X,x_p)) - V(X) \leq -\frac{1}{2}\alpha V(X),\; \forall X \in \mathcal{B} \cap \mathcal{J}_c \times \mathbb{R}_{\geq \lambda}.$$

Hence this Lyapunov function is the same than the one introduced in Assumption 3 with $\alpha_1 = \frac{1}{2}$ and $\alpha_2 = \frac{\alpha}{2}$.

Consider now the increasing function $\rho : [0,\varepsilon_r] \to [0,+\infty)$ defined as follows [4].

$$\rho(s) \geq \max\left\{ \max_{(x_p+e,x_c,\sigma)\in(D_{l_4}\cap\mathcal{F}_c\times\mathbb{R}_{\geq 0}),|e|\leq s} \nu_1(X,e)\;, \max_{(x_p+e,x_c,\sigma)\in(D_{l_4}\cap\mathcal{J}_c\times\mathbb{R}_{\geq 0}),|e|\leq s} \nu_2(X,e) \right\}$$

where

$$\nu_1(X,e) = \left| \frac{\partial V}{\partial X}(X)\left(F(X,x_p+e) - F(X,x_p)\right) \right|,$$

and $\nu_2(X,e) = |V(G(X,x_p+e)) - V(G(X,x_p))|$. With this function, Assumption 3 is satisfied. This ends the proof of Assumption 3 from Assumption 1. Therefore, as soon as Theorem 2 is valid, Theorem 1 holds under Assumptions 1 and 2.

5 Construction of the output feedback law

In the next sections, we follow a similar approach to Teel and Praly (1994). We first compute a saturation level c_x, a time of existence $T_{\min}$ and a compact subset of $\mathcal{B} \times \mathbb{R}_{\geq 0}$ denoted by D_{l_4} such that, when saturating the controller with c_x, the solution starting from $\mathcal{B} \times \mathbb{R}_{\geq 0}$ remains in D_{l_4} for all time less than $T_{\min}$. Then, with $T_{\min}$ and the margin of robustness c_e from Assumption 3, we design an observer such that the error dynamics converges to 0 asymptotically and such that,for all time higher than $T_{\min}$, the error dynamics belongs to the margin of robustness. Finally, we prove the attractiveness and the stability of the closed-loop system with the output feedback law.

[4] This function is well defined due to the fact that $\mathcal{F}_c$ and $\mathcal{J}_c$ are closed subsets.

5.1 Selection of c_x and minimmal time of existence of solutions

This section differs from the strategy employed in Teel and Praly (1994). Indeed, since we have a hybrid dynamics, the solution of the closed-loop system can jump. Therefore, the computation of $T_{\min}$ becomes difficult. However, thanks to the timer dynamics, we can assert that between two jumps the solution of the closed-loop system belongs to the flow set. This allows us to compute T_{min} and thus c_x.

In the remaining part of this subsection, we consider the system defined by, for all (x_p, x_c, σ) in $(\mathcal{F}_c \times \mathbb{R}_{\geq 0}) \cup (\mathcal{J}_c \times \mathbb{R}_{\geq \lambda})$,

$$\begin{cases} \dot{x}_p = f_p(x_p) + g_p(x_p)\theta_c(\omega, x_c) \\ \dot{x}_c = f_c(\omega, x_c) \\ \dot{\sigma} = 1 - \sigma \end{cases} \quad (\omega, x_c, \sigma) \in \mathcal{F}_c \times \mathbb{R}_{\geq 0}, \tag{10a}$$

$$\begin{cases} x_p^+ = x_p \\ x_c^+ = g_c(\omega, x_c) \\ \sigma^+ = 0 \end{cases} \quad (\omega, x_c, \sigma) \in \mathcal{J}_c \times \mathbb{R}_{\geq \lambda}, \tag{10b}$$

where ω is an external perturbation function in $\mathbb{R}^{n_p}$. Such a system is not a classical hybrid system as the ones introduced in Goebel et al. (2012) since the flow and jump sets are defined with an external disturbance. Note that in the particular case in which $\omega = \tilde{x}_p$ defined in (6), the solution to system (10) by adding the dynamics of $\hat{x}_p$ is also solution to system (5). Hence, this implies the well-posedness of the closed-loop system as considered in (Goebel et al., 2012, Chapter 2).

Two cases may be distinguished to construct the sets: i) Solution to (10) does not jump; ii) Solution to (10) jumps at least one time. The first case is similar to the continuous case (and thus to Teel and Praly (1994)). The second case takes into account the hybrid behavior of the system under consideration. Under Assumption 3, let $l_1 = \max_{x_p \in \Gamma, \sigma \in [0,\lambda]} V(x_p, 0, \sigma)$. Note that D_{l_1} is a compact subset of $\mathcal{B} \times \mathbb{R}_{\geq 0}$ (see the notation employed in equation (7)). Let $l_2 > l_1$ such that $D_{l_2} \subset \mathcal{B} \times \mathbb{R}_{\geq 0}$. To deal with the jump that can occur, we consider $D_{l_2}^+ = \bigcup_{(x_p, x_c, \sigma) \in D_{l_2}} \{(x_p, g_c(w, x_c), 0), w \in \mathbb{R}^{n_p}\}$. Since it is assumed that the set defined in (4) is a compact subset of $\mathcal{B} \times \mathbb{R}_{\geq 0}$, it yields that $D_{l_2}^+$ is also a compact subset of $\mathcal{B} \times \mathbb{R}_{\geq 0}$. Let l_3 be such that D_{l_3} is a compact subset which satisfies $D_{l_2}^+ \subset D_{l_3} \subset \mathcal{B} \times \mathbb{R}_{\geq 0}$. Finally, let $l_4 > l_3$ so that D_{l_4} is a compact subset which contains D_{l_3}.

With these sets in hands, the positive real number c_x can be selected as

$$c_x = \max_{(x_p, x_c, \sigma) \in D_{l_4}} \{|x_p|\} \tag{11}$$

Let us now establish the following property for solutions to system (10) initiated from D_{l_1}.

Lemma 1. (Minimal existence time of solution in D_{l_4}) *There exists $T_{\min} > 0$ such that for all ω in $\mathcal{L}^\infty_{loc}([0, +\infty); \mathbb{R}^{n_p})$ with $|\omega(t)| \leq c_x$ for all t in $[0, T_{\min}]$, and all $X^\# := (x_p^\#, x_c^\#, \sigma^\#)$ in D_{l_1}, all solutions $x(\cdot, \cdot)$ to (10) with $X(0,0) = X^\#$ and all (t,j) in $\texttt{dom}(X)$ [5] then $X(t,j) \in D_{l_4}$ for all $0 \leq t \leq T_{\min}$.*

Proof. Let $\bar{V}$ the positive real number defined by $\bar{V} = \max_{X \in D_{l_4}, |\omega| \leq c_x} \left|\frac{\partial V}{\partial X}(X) F(X, \omega)\right|$. In the remaining part of the proof, we show that Lemma 1 holds with $T_{\min}$ chosen as any positive real number satisfying

$$T_{\min} < \min\left\{-\ln(1-\lambda), \frac{l_2 - l_1}{\bar{V}}, \frac{l_4 - l_3}{\bar{V}}\right\} .$$

Let $X^\#$ be in D_{l_1} and let X be a solution to system (10) whose initial condition is $X^\#$. For all (t,j) in $\texttt{dom}(X)$. To ease the notation we denote $V(t,j) = V(X(t,j))$.

Let (t,j) in $\texttt{dom}(X)$ such that $0 \leq t \leq T_{\min}$. To prove the lemma, we need to show that $X(t,j)$ is in D_{l_4}. First of all, we show that $j \leq 1$. Indeed, assume $j \geq 2$. This implies that there exist t_0 and t_1 such that $0 \leq t_0 < t_1 \leq t$ such that $(t_0, 0)$, $(t_0, 1)$, $(t_1, 1)$, $(t_1, 2)$ are in $\texttt{dom}(X)$. Note that $\sigma(t_0, 1) = 0$ and $\sigma(t_1, 1) = \lambda$. Moreover, for all s in $[t_0, t_1]$, $(s, 1)$ is in $\texttt{dom}(X)$ and $\dot{\sigma}(s,1) = 1 - \sigma(s,1)$. Hence, integrating this equation between t_0 and t_1, we get that $t \geq t_1 - t_0 > -\ln(1-\lambda) \geq T_{\min}$. This is impossible, and therefore $j \leq 1$.

So two cases may be distinguished.

- $j = 0$ This case is illustrated by Figure 1. $j = 0$ implies that $s \in [0,t] \mapsto x(s,0)$ is a continuous mapping with $x(0,0)$ in $D_{l_1} \subset D_{l_2}$. Hence we can define t^*, the largest time in $[0,t]$ such that $x(s,0)$ is in D_{l_2} (i.e. $t^* = \max_{s \in [0,t], x(\ell,0) \in D_{l_2}, \forall \ell \in [0,s]}\{s\}$). Note that if $t^* = t$ then this implies that $x(t,0)$ is in D_{l_2}, hence the result. Assume $t^* < t$. This implies that for all s in $[0, t^*]$ we have $\dot{V}(s,0) = \frac{\partial V}{\partial X}(X(s,0)) F(X(s,0), \omega(s)) \leq \bar{V}$. This gives $V(t^*, 0) \leq \bar{V} t^* + V(0,0) \leq \bar{V} T_{\min} + l_1 < l_2$. Hence $x(t^*, 0)$ is in the interior of D_{l_2}. It yields that there exists $\varepsilon > 0$ such that $x(t^* + \varepsilon, 0)$ is in the interior of D_{l_2} which contradicts the fact that t^* is an extremum.
- $j = 1$ This case is illustrated by Figure 2. $j = 1$ implies that there exists t_0 in $[0,t]$ such that $(t_0, 0)$ and $(t_0, 1)$ are in $\texttt{dom}(X)$ and $(w(t_0,1), x_c(t_0,1))$ is in $\mathcal{J}_c$. Following the first case study, it is possible to show that $X(t_0, 0)$ is in D_{l_2}. Moreover,

[5] The definition of $\texttt{dom}$(X) is borrowed from (Goebel et al., 2012, Definition 2.3).