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Summary We rigorously derive a theory of linearly piezoelectric plates by studying the limit behavior of a three-dimensional flat body as its thickness tends to zero. We here only present the static case.

INTRODUCTION

The interest of an efficient modeling of piezoelectric plates lies in the fact that a major technological application of piezoelectric effects is the control of vibrations of structures through very thin plates. We can find in the literature many derivations of modeling of piezoelectric plates (see for instance the introduction of [1]). Here, we choose to extend to the linearly piezoelectric case the mathematical derivation of the linearly elastic behavior of a plate as the limit behavior of a three-dimensional solid whose thickness tends to zero.

THE STATIC CASE

The reference configuration of a linearly piezoelectric thin plate is the closure in $\mathbb{R}^3$ of the set $\Omega^\varepsilon = \omega \times (-\varepsilon, \varepsilon)$, where ε is a small positive parameter and ω a bounded domain of $\mathbb{R}^2$ with a Lipschitz boundary. Let $(\Gamma^\varepsilon_{mD}, \Gamma^\varepsilon_{mN})$, $(\Gamma^\varepsilon_{eD}, \Gamma^\varepsilon_{eN})$ two suitable partitions of the boundary of Ω^ε; the plate is, on one hand, clamped along Γ^ε_{mD} and at an electric potential φ^ε_0 on Γ^ε_{eD} and, on the other hand, subjected to body forces and electrical loadings in Ω^ε and to surface forces and electric loadings on Γ^ε_{mD} and Γ^ε_{eD}. We note n^ε the outward unit normal to $\partial\Omega^\varepsilon$ and assume that $\Gamma^\varepsilon_{mD} = \gamma_0 \times (-\varepsilon, \varepsilon)$, with $\gamma_0 \subset \partial\omega$. Then the equations determining the electromechanical state $s^\varepsilon = (u^\varepsilon, \varphi^\varepsilon)$ at equilibrium are :

$$\begin{cases} \text{div } \sigma^\varepsilon + f^\varepsilon = 0 \text{ in } \Omega^\varepsilon,\ \sigma^\varepsilon n^\varepsilon = g^\varepsilon \text{ on } \Gamma^\varepsilon_{mN},\ u^\varepsilon = 0 \text{ on } \Gamma^\varepsilon_{mD}, \\ \text{div } D^\varepsilon + F^\varepsilon = 0 \text{ in } \Omega^\varepsilon,\ D^\varepsilon \cdot n^\varepsilon = w^\varepsilon \text{ on } \Gamma^\varepsilon_{eN},\ \varphi^\varepsilon = \varphi^\varepsilon_0 \text{ on } \Gamma^\varepsilon_{eD}, \\ (\sigma^\varepsilon, D^\varepsilon) = M^\varepsilon(x)(e(u), \nabla\varphi) \text{ in } \Omega^\varepsilon, \end{cases} \tag{1}$$

where respectively u^ε, φ^ε, σ^ε, $e(u^\varepsilon)$ and D^ε denotes the displacement and electric potential fields, the stress tensor, the tensor of small strains and the electric displacement, the operator M^ε is such that :

$$\sigma^\varepsilon = a^\varepsilon e(u^\varepsilon) - b^\varepsilon \nabla\varphi^\varepsilon \quad , \quad D^\varepsilon = {b^\varepsilon}^T e(u^\varepsilon) + c^\varepsilon \nabla\varphi^\varepsilon, \tag{2}$$

${b^\varepsilon}^T$ is the transpose of b^ε, a^ε and c^ε are symmetric and positive. Because of the piezoelectric coupling, M^ε is not symmetric. But, under realistic assumptions of boundness of a^ε, b^ε, c^ε and of uniform ellipticity of a^ε and c^ε, and with smooth enough electromechanical loading, the problem admits a unique solution. The very question is to study its behavior when $\varepsilon \to 0$. We will show that, depending on the type of boundary conditions, two limit behaviors, indexed by p with value 1 or 2, can be obtained. Classically [2], we come down to a fixed open set $\Omega = \omega \times (-1, 1)$ through the mapping π^ε :

$$x = (x_1, x_2, x_3) \in \bar{\Omega} \mapsto \pi^\varepsilon x = (x_1, x_2, \varepsilon x_3) \in \bar{\Omega}^\varepsilon, \tag{3}$$

we drop the index ε for the image by $(\pi^\varepsilon)^{-1}$ of the previous geometric sets and let $\Gamma_\pm = \omega \times \{\pm 1\}$, $\Gamma_{\text{lat}} = \partial\omega \times (-1, 1)$. We assume that the electromechanical coefficients satisfy :

$$\exists\, M \in L^\infty(\Omega)^{12} : M^\varepsilon(\pi^\varepsilon x) = M(x),\ \textit{a.e. } x \in \Omega\ ;\ \exists\, \eta_0 > 0 : M(x)h \cdot h \geqslant \eta_0 |h|^2,\ \forall\, h \in \mathbb{R}^{12},\ \textit{a.e. } x \in \Omega. \tag{4}$$

The magnitude of the external loading is chosen as follows :

$$\begin{cases} f^\varepsilon_\alpha(\pi^\varepsilon x) = \varepsilon\, f_\alpha(x),\ f^\varepsilon_3(\pi^\varepsilon x) = \varepsilon^2 f_3(x),\ F^\varepsilon(\pi^\varepsilon x) = \varepsilon^{2-p} F(x),\ \forall\, x \in \Omega, \\ g^\varepsilon_\alpha(\pi^\varepsilon x) = \varepsilon^2\, g_\alpha(x),\ g^\varepsilon_3(\pi^\varepsilon x) = \varepsilon^3\, g_3(x),\ \forall\, x \in \Gamma_{mN} \cap \Gamma_\pm, \\ g^\varepsilon_\alpha(\pi^\varepsilon x) = \varepsilon\, g_\alpha(x),\ g^\varepsilon_3(\pi^\varepsilon x) = \varepsilon^2\, g_3(x),\ \forall\, x \in \Gamma_{mN} \cap \Gamma_{lat}, \\ w^\varepsilon(\pi^\varepsilon x) = \varepsilon^{3-p}\, w(x),\ \forall\, x \in \Gamma_{eN} \cap \Gamma_\pm,\ w^\varepsilon(\pi^\varepsilon x) = \varepsilon^{2-p}\, w(x),\ \forall\, x \in \Gamma_{eN} \cap \Gamma_{lat}, \\ \varphi^\varepsilon_0(\pi^\varepsilon x) = \varepsilon^p\, \varphi_0(x),\ \forall\, x \in \Gamma_{eD}, \end{cases} \tag{5}$$

here (f, F, g, w) is an element of $L^2(\Omega)^3 \times L^2(\Omega) \times L^2(\Gamma_{mN})^3 \times L^2(\Gamma_{eN})$ independent of ε and we assume that φ_0 admits an extension to Ω, still noted φ_0, in $H^1(\Omega) = \{\psi \in L^2(\Omega);\ \nabla\psi \in L^2(\Omega)^3\}$. In the sequel, for every domain

G of $\mathbb{R}^n$ we will note $H^1_\Gamma(G)$ the subset of the Sobolev space $H^1(G)$ whose elements vanish on $\Gamma \subset \partial G$. With the electromechanical state $s_p^\varepsilon = (u_p^\varepsilon, \varphi_p^\varepsilon)$ defined on Ω^ε is associated the scaled state $s_p(\varepsilon) = (u_p(\varepsilon), \varphi_p(\varepsilon))$ defined on Ω by :

$$\forall\, x^\varepsilon = \pi^\varepsilon x \in \Omega^\varepsilon, \quad (u_p^\varepsilon)_\alpha\,(x^\varepsilon) = \varepsilon (u_p)_\alpha\,(x), \quad (u_p^\varepsilon)_3\,(x^\varepsilon) = (u_p)_3\,(x), \quad \varphi_p^\varepsilon\,(x^\varepsilon) = \varepsilon^p \varphi_p\,(x), \tag{6}$$

$s_p(\varepsilon)$ is then the solution of the following mathematical problem $S(\varepsilon,\Omega)_p$ equivalent to the genuine physical problem :

$$\boxed{S(\varepsilon,\Omega)_p : \text{Find } s_p(\varepsilon) \in (0,\varphi_0) + S_\varepsilon;\ m_p(\varepsilon)(s_p(\varepsilon), r) = L(r),\ \forall\, r \in S_\varepsilon = \{r = (v,\psi) \in H^1_{\Gamma_{mD}}(\Omega)^3 \times H^1_{\Gamma_{eD}}(\Omega)\}},$$

with :

$$\begin{cases} m_p(\varepsilon)(s,r) = \int_\Omega M(x) k_p(\varepsilon,s)\cdot k_p(\varepsilon,r)\,dx,\ k_p(\varepsilon,r) = k_p(\varepsilon,(v,\psi)) = (e(\varepsilon,v), (\nabla_{(p)}(\varepsilon,\psi))), \\ e(\varepsilon,v)_{\alpha\beta} = e(v)_{\alpha\beta}, e(\varepsilon,v)_{\alpha 3} = \varepsilon^{-1} e(v)_{\alpha 3}, e(\varepsilon,v)_{33} = \varepsilon^{-2} e(v)_{33},\ 2e(v)_{ij} = \partial_i v_j + \partial_j v_i, \\ \nabla_{(p)}(\varepsilon,\psi)_\alpha = \varepsilon^{p-1}\partial_\alpha\psi, \nabla_{(p)}(\varepsilon,\psi)_3 = \varepsilon^{p-2}\partial_3\psi, \\ L(r) = L(v,\psi) = \int_\Omega f\cdot v\,dx + \int_\Omega F\psi\,dx + \int_{\Gamma_{mN}} g\cdot v\,ds + \int_{\Gamma_{eN}} w\psi\,ds. \end{cases} \tag{7}$$

Indexes α, β run from 1 to 2, whereas indexes i, j run from 1 to 3. We make the assumption that the extension of φ_0 does not depend on x_3 when $p = 1$, while $w = 0$ on $\Gamma_{eN} \cap \Gamma_{\text{lat}}$ or $\Gamma_{eN} \cap \Gamma_{\text{lat}} = \emptyset$ and the closure of the projection of Γ_{eD} on ω coincides with $\bar{\omega}$ when $p = 2$. Let $H^1_{\partial_3} = \{\psi \in L^2(\Omega);\ \partial_3\psi \in L^2(\Omega)\}$ and

$$\begin{cases} V_p = \{v \in H^1_{\Gamma_{mD}}(\Omega)^3;\ e_{i3}(v) = 0\},\ X_1 = H^1_{\Gamma_{mD}}(\Omega)^3 \times H^1(\Omega),\ X_2 = H^1_{\Gamma_{mD}}(\Omega)^3 \times H^1_{\partial_3}, \\ \Phi_1 = \{\psi \in H^1_{\Gamma_{eD}}(\Omega);\ \partial_3\psi = 0\},\ \Phi_2 = \{\psi \in H^1_{\partial_3};\ \psi = 0 \text{ on } \Gamma_{eD} \cap \Gamma^\pm\},\ S_p = V_p \times \Phi_p, \\ \widetilde{m}_p(s,r) = \int_\Omega \widetilde{M}_p\, k_p(s)\cdot k_p(r)\,dx,\ k_1(s) = (e_{\alpha\beta}(u), \partial_\alpha\varphi),\ k_2(s) = (e_{\alpha\beta}(u), \partial_3\varphi), \end{cases} \tag{8}$$

where $\widetilde{M}_p$ is the condensation of M with respect to the components of $(e(v), \nabla\psi)$ used in $k_p(v,\psi)$. We have the convergence result ([3]) :

Theorem 1 *When ε goes to 0, the family $s_p(\varepsilon)$ of the unique solution of $S(\varepsilon,\Omega)_p$ converges in X_p to the unique solution s_p of* $\boxed{S(\Omega)_p : \textit{Find } s_p \in (0,\varphi_0) + S_p \textit{ such that } \widetilde{m}_p(s_p, r) = L(r),\ \forall\, r \in S_p}$.

In order to obtain a physically meaningful result, $s_p(\varepsilon)$ has to be descalled.

CONCLUSION

We emphasize on the fact that *sensors and actuators enter respectively in the cases $p = 1$ and $p = 2$*. As in the case of purely elastic plates, the limit state fields involve simplified kinematics : the displacements are of Kirchhoff-Love type and, in the case $p = 1$, the electric potential does not depend on x_3. From the very definition of S_1 and of a classical characterization of Kirchhoff-Love displacements (see [2]), $S(\Omega)_1$ is actually a bi-dimensional problem set in ω. Moreover, if $\int_{-1}^{1} x_3 \widetilde{M}_1(x_1,x_2,x_3)\,dx_3 = 0$ (which is implied by x_3-even electro-mechanic coefficients) appears a decoupling between the flexural and membrane displacements in the sense that they are solutions of two independent variational equations, where the one corresponding to the flexion does not involve the electric potential. Similar results were obtained in [4] through formal asymptotic expansions and in [5] through rigorous but somewhat different mathematical arguments as ours. As observed in [6] in the particular case when a^ε accounts for homogeneous isotropic elasticity, if $\widetilde{M}_2$ and F do not depend on x_3, then the limit electric potential φ_2 is a second degree polynomial in x_3.

Considering the influence of crystalline symmetries, notice that interesting properties of $\widetilde{M}_p$ can be derived ([3]). In particular, when $p = 1$, there is an electromechanical decoupling for the classes 2, 222, $2mm$, 4, $\bar{4}$, 422, $4mm$, $\bar{4}2m$, 6, 622, $6mm$ and 23 ; when $p = 2$, this decoupling occurs with the classes m, 32, 422, $\bar{6}$, 622 and $\bar{6}m2$; so we claim that actuators should not be designed with these kinds of material.

It can be shown that in the dynamic case, under the realistic quasi-electrostatic approximation, the limit behavior depends further more on the relative magnitudes of the density and of the thickness of the plate. In view of applications, it should be of interest to study the controllability of the plate. Our work should be an introductory to these studies...

References

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