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Olivier Allix*, Philippe Vidal

Department of Computational Structural Mechanics, LMT-Cachan, ENS de Cachan/CNRS/Université Paris VI, 61 Avenue du Pree´sident Wilson, 94235 Cachan Cedex, France

Abstract

In this paper, a multi-solution strategy aimed at greatly decreasing the cost of structural identification problems is proposed. It is based on the large time increment method and on the possibility offered by this approach to obtain a family of admissible displacement and stress fields in order to solve a nonlinear time-dependent problem. This technique relies on a time and space decomposition of the solution. It has been extended to solve at a low cost, the inverse problem of identification, for various techniques. In particular, it has been adapted to the minimization of the gradient of the gap between test and simulation. In this first attempt, it is applied to the simple case of bending of an elastic–viscoplastic beam. The issue of the numerical efficiency of this type of strategy is studied.

Keywords: Structural identification; Multi-solution approach; LATIN method

1. Introduction

Today, more and more knowledge of the behavior of structures in service is required. The loadings are complex, multi-axial, nonproportional and, therefore, very difficult to reproduce with homogeneous experiments on test samples. Furthermore, the most critical phenomena, such as failure, happen mostly near the edges, where the loads are applied. These zones are characterized by high gradients and three-dimensional (3D) loads. Typical problems where such a situation occurs include, for example, the delamination of laminated composites [1], the identification of joints and the determination of parameters of constitutive models involving the phenomenon of localization [2].

Thus, in these cases, regardless of the identification strategy adopted, the identification of the behavior is indirect and requires numerous and expensive computations in order to establish satisfactory comparisons

* Corresponding author. Tel.: +33-1-4740-2241; fax: +33-1-4740-2785.
E-mail address: ladeveze@lmt.ens-cachan.fr (O. Allix).

Nomenclature

B	positive operator
E_0	Young modulus
H^-	search direction operator
H^+	search direction operator
H	sensitivity matrix
K_e	fourth order tensor of elastic stiffness moduli
k	structural parameters
k_v, n_v	material constants
l	length of the beam
p	cumulative plastic strain
φ^*	pseudo-potential of dissipation
ψ	free energy
R	isotropic hardening
R_0	initial yield stress
R_C	covariance matrix
sgn	sign
σ^D	deviatoric part of the stress tensor
$\mathbf{S_0}$	space of regular displacements satisfying homogeneous boundary conditions, and of stresses σ self-equilibrated
δ_{kj}	Kronecker's delta
div	divergence
$E()$	average operator
$\langle x \rangle_+$	positive part of x

between experiments and calculations. In this work, a method of calculation aimed at greatly decreasing the cost of these analyses is proposed and studied on a simple case. It uses the fact that a large number of the calculations to be carried out are identical in many respects. To take advantage of this, we try to use the possibilities offered by the "Large Time INcrement method" (LATIN method) proposed by Ladevèze and coworkers [3–6] and, particularly, to reuse a family of admissible displacements, stress fields of a given problem to solve other similar problems much less expensively. The inverse stage of the identification problem involves the resolution of problems different from those encountered in the resolution of the direct problem: therefore, we must also develop methodologies which allow us to solve these problems within the framework of the LATIN method. It is to be noted that these problems have not been solved so far in previous studies related to the LATIN method. They depend on the strategy of identification used. Sample situations were studied.

Common strategies to solve identification problems use algorithms based on the evaluation of the gradient with respect to parameters chosen for reasons of efficiency [7,8]. The evaluation of the gradient can be performed by direct differentiation [8–11]. An interesting alternative is the adjoint state method which was developed for nonlinear mechanical applications, for example in [12–14]. Another technique already used is the estimation of the gradient by interpolation [15,17]. All these methods are based on classical incremental methods of nonlinear calculations. Algorithms based on the evaluation of the cost function (genetic, evolution) are also used in [18–21]. The main disadvantage of these methods is their computational cost because of the large number of function evaluations which must be carried out. To handle the problems

linked to measurement noise, stochastic methods have been elaborated [18–20]. Another technique proposed in this connection is that of the Kalman filters [15,16,22,23].

To take advantage of the LATIN method, we studied different methods which are representative of the difficulties encountered in structural identification, namely, the direct differentiation method, the adjoint state method, the interpolation method and the Kalman filters method.

In this first work, we considered a simple example: a cantilever beam which exhibits a viscoplastic behavior. A white measurement noise (which, here, is simulated) is introduced. In the first part, we present the LATIN method as it is used. Then, we discuss the application of the LATIN method to the various identification strategies. Finally, a comparison of the computational costs of these methods is presented and discussed.

2. The LATIN method

First of all, we recall the principle of the LATIN method. Then, we focus on its multi-solution feature, which seems to us to be a particularly interesting aspect in the context of identification.

2.1. Formulation of the direct problem

The identification is performed in the context of nonlinear evolution problems for small isothermal deformations under quasi-static conditions. The structure studied is defined in a domain Ω bounded by $\partial\Omega$. On a portion $\partial_1\Omega$ of the boundary $\partial\Omega$, the displacement $\underline{U}_d$ is prescribed. On the complementary part $\partial_2\Omega$, a surface force density $\underline{F}_d$ is applied. The structure is also subjected to a volume force density $\underline{f}_d$.

For the purpose of the application of the version of the LATIN method used, the problem to be solved in the context of a viscoplastic model is to find $(\sigma, \underline{U}, \dot{\varepsilon}^p, \dot{p}, R)$ defined on $[0,T]\times\Omega$ belonging to the intersection of the two following spaces A_d and Γ where

- A_d is composed of the static admissibility, the kinematic admissibility, the initial conditions, the compatibility equation and the linear part of the constitutive relations,
- Γ is composed of the nonlinear part of the constitutive relations.

More precisely, A_d is defined by

- kinematic constraints:

$$\underline{U}(t,M) = \underline{U}_d \quad \forall t \in [0,T], \ \forall M \in \partial_1\Omega \tag{1}$$

- initial conditions:

$$\underline{U}\,|_{t=0} = \underline{U}_0 \quad \forall M \in \Omega \tag{2}$$

- equilibrium equations:

$$-\int_\Omega \sigma : \varepsilon(\underline{U}^*)\,d\Omega + \int_\Omega \underline{f}_d \cdot \underline{U}^*\,d\Omega + \int_{\partial_2\Omega} \underline{F}_d \cdot \underline{U}^*\,dS = 0 \quad \forall \underline{U}^*/\underline{U}^* = 0 \text{ on } \partial_1\Omega, \ \forall t \in [0,T] \tag{3}$$

with $\varepsilon = \varepsilon(\underline{U})$.

- linear part of the constitutive relation [24], i.e. the state laws: the model is defined by its free energy

$$\rho\psi = \tfrac{1}{2}K_e^{-1}\sigma : \sigma + \tfrac{1}{2}\beta p^2 \tag{4}$$

which the derivatives yield to the state equations

$$\begin{aligned} R &= \beta \cdot p, \\ \sigma &= K_e(\varepsilon - \varepsilon^p), \end{aligned} \tag{5}$$

K_e is the fourth order tensor of elastic stiffness moduli.

And, Γ is defined by

- nonlinear part of the constitutive relation, i.e., the evolution laws: the model is also defined by its pseudo-potential of dissipation which is given by

$$\varphi^*(\sigma, R) = \frac{k_v}{n_v + 1} \langle z \rangle_+^{n_v+1} \quad \text{with } z = \|\sigma^D\| - R - R_0 \tag{6}$$

k_v and n_v are material constants, R the isotropic hardening, R_0 the initial yield stress, σ^D the deviatoric part of the stress tensor. The symbol $\langle x \rangle_+$ indicates the positive part of x, so that $\langle x \rangle_+ = \frac{1}{2}(x + |x|)$. From this pseudo-potential, the evolution laws are derived:

$$\frac{d}{dt}\begin{bmatrix} \varepsilon^p \\ -p \end{bmatrix} = B\left(\begin{bmatrix} \sigma \\ R \end{bmatrix}\right) = k_v \langle z \rangle_+^{n_v} \begin{bmatrix} \dfrac{\sigma^D}{\|\sigma^D\|} \\ -1 \end{bmatrix} \quad \text{with } \varepsilon^p = p = 0 \text{ at } t = 0 \tag{7}$$

where p (the variable associated with R) is the cumulative plastic strain.

2.1.1. Remarks on the formulation of the state equations

When, as is often the case, the relation between R and p is not linear, we use a normal formulation of the constitutive laws in the sense defined in [24]. In this way, we introduce a new set of internal variables $\tilde{R}$ and $\tilde{p}$ obtained from R and p using the following relation:

$$\tilde{R} = \beta \cdot \tilde{p} \quad \text{and} \quad \tilde{p} = \int_0^p \left(\frac{\partial R}{\partial p}\frac{1}{\beta}\right)^{1/2} dp \quad \text{for } R = R_0 + \beta \cdot p^\alpha, \quad \text{we obtain : } \tilde{p} = \frac{2\sqrt{\alpha}}{\alpha + 1} \cdot p^{(\alpha+1)/2} \tag{8}$$

so that the relation between $\tilde{R}$ and $\tilde{p}$ is linear whereas that between R and p is not. It should be noted that—even though here, for the sake of simplicity, we restrict ourselves to a plasticity flow rule with isotropic hardening—all the following applies to any constitutive law described by internal variables. In the following, we will call variables $\tilde{R}$ and $\tilde{p}$ respectively R and p.

For the constitutive laws studied (which are stable in Drucker's sense [26]), the solution to our problem is unique and corresponds to the intersection of A_d and Γ.

2.2. Principles of the method of resolution

The resolution of the direct problem is carried out using the LATIN method, whose principles have been given by Ladevèze [25]. This method departs from incremental methods in that it allows one to study the whole loading in only one increment [24]. It takes the form of a global iterative method over the whole history of loading considered. At iteration n, two estimates are produced successively:

- An estimate of the solution $s_n = (\dot{\varepsilon}_n^p, \dot{p}_n, \sigma_n, R_n)$ over the whole history of the loading which verifies the equations of A_d. It is worth noting that the conditions of belonging to this space are global but linear.

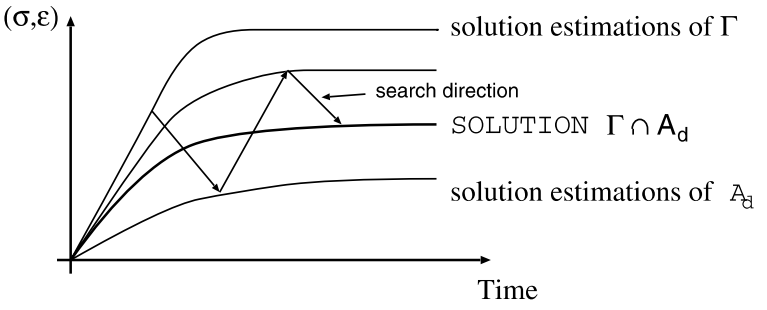

Fig. 1. Iterative process of the LATIN method.

- An estimate of the solution $\hat{s} = (\dot{\hat{\varepsilon}}^{\mathrm{p}}, \dot{\hat{p}}, \hat{\sigma}, \hat{R})$ over the whole history of the loading which satisfies the equations of Γ. The conditions of belonging to this space are nonlinear but local.

The iterative process is illustrated in Fig. 1.

On the previous Fig. 1, each curve schematizes the evolution of a field $(\sigma, \varepsilon, \ldots)$ on the whole time interval $[0, T]$ either satisfying the evolution laws Γ, or the equilibrium and the state equations A_{d}. This figure aims at showing that the iteration process is a global process on the whole time interval, but not a step-by-step iterative procedure.

Therefore, each LATIN iteration is composed of a local stage and a global stage, i.e.

- the local stage: given a known element $s_n = (\dot{\varepsilon}^{\mathrm{p}}_n, \dot{p}_n, \sigma_n, R_n)$ of A_{d}, an element $\hat{s}$ belonging to Γ is obtained by following a given search direction. It is defined by

$$\left\{ \begin{array}{c} \dot{\hat{\varepsilon}}^{\mathrm{p}} - \dot{\varepsilon}^{\mathrm{p}}_n \\ -(\dot{\hat{p}} - \dot{p}_n) \end{array} \right\} + H^{+} \left\{ \begin{array}{c} \hat{\sigma} - \sigma_n \\ \hat{R} - R_n \end{array} \right\} = 0 \tag{9}$$

where the operator H^{+} is the search direction. In general, it is calculated from the previous global stage. Different choices of search directions have already been treated for the resolution of the direct problem. H^{+} can be the elastic operator, or an operator linked to the state laws. In the context of identification, we can use, for example, the search direction which consists of keeping σ and R, which are some values obtained at the global stage. The problem at the local stage then becomes explicit. It is expressed as

Find $\hat{s} = (\dot{\hat{\varepsilon}}^{\mathrm{p}}, \dot{\hat{p}}, \hat{\sigma}, \hat{R})$ such that

$$\left\{ \begin{array}{l} \text{search direction} : H^{+^{-1}} = 0, \ \text{i.e.} \left\{ \begin{array}{l} \hat{\sigma} = \sigma_n \\ \hat{R} = R_n \end{array} \right. \\ \text{evolution law} : \dfrac{\mathrm{d}}{\mathrm{d}t} \left[\begin{array}{c} \hat{\varepsilon}^{\mathrm{p}} \\ -\hat{p} \end{array} \right] = B\left(\left[\begin{array}{c} \hat{\sigma} \\ \hat{R} \end{array} \right] \right) = k_{\mathrm{v}} \langle \hat{z} \rangle_{+}^{n_{\mathrm{v}}} \left[\begin{array}{c} \dfrac{\hat{\sigma}^{\mathrm{D}}}{\|\hat{\sigma}^{\mathrm{D}}\|} \\ -1 \end{array} \right]. \end{array} \right. \tag{10}$$

The second member being known, it is sufficient to integrate the first equation over the complete structure (at each Gauss point in the finite element method) and over the whole time history. The θ-method is used.

- the global stage: given an element $\hat{s} = (\dot{\hat{\varepsilon}}^{\mathrm{p}}, \dot{\hat{p}}, \hat{\sigma}, \hat{R})$ in Γ, a new element s_{n+1} belonging to A_{d} is sought in a given search direction. It is defined by

$$\left\{ \begin{array}{c} \dot{\varepsilon}^{\mathrm{p}}_{n+1} - \hat{\dot{\varepsilon}}^{\mathrm{p}} \\ -(\dot{p}_{n+1} - \hat{\dot{p}}) \end{array} \right\} = H^{-} \left\{ \begin{array}{c} \sigma^{\mathrm{D}}_{n+1} - \hat{\sigma}^{\mathrm{D}} \\ R_{n+1} - \hat{R} \end{array} \right\}. \tag{11}$$

The operator H^{-} is the search direction. A so-called tangent search direction is chosen. For the 1D case, it is expressed as

$$H^{-} = n_{\mathrm{v}} \cdot k_{\mathrm{v}} \langle \hat{z} \rangle_{+}^{n_{\mathrm{v}}-1} \cdot \begin{pmatrix} 1 & -\dfrac{\hat{\sigma}}{|\hat{\sigma}|} \\ -\dfrac{\hat{\sigma}}{|\hat{\sigma}|} & 1 \end{pmatrix}.$$

It depends only on the quantities determined at the previous local stage.

As a finite element scheme is used, the equilibrium equation will be satisfied only at the finite element sense and the constitutive relation at the Gauss points.

2.3. Time/space representation of an element of A_d

For complex problems with a large number of degrees of freedom, the cost of the resolution corresponds mainly to the determination of an element of A_{d} because of the global feature of the equilibrium equations. To decrease the computational time of this stage, it has been proposed and implemented in [27] to represent the corrections at the global stage in the form of a simple loading. This technique is efficient not only in the case of material nonlinearity (elasto-viscoplasticity [27], damage [1]) but also geometric nonlinearity [3]. Thus, the corrections sought are expressed as follows:

$$\begin{aligned} \Delta\varepsilon_n(M,t) &= \varepsilon_{n+1}(M,t) - \varepsilon_n(M,t) = \sum_{i=1}^{m} \alpha_i(t) \cdot A_i(M), \\ \Delta\sigma_n(M,t) &= \sigma_{n+1}(M,t) - \sigma_n(M,t) = \sum_{i=1}^{m} \beta_i(t) \cdot B_i(M), \\ \Delta R_n(M,t) &= R_{n+1}(M,t) - R_n(M,t) = \sum_{i=1}^{m} \gamma_i(t) \cdot C_i(M) \end{aligned} \tag{12}$$

where $\alpha_i(t)$, $\beta_i(t)$, $\gamma_i(t)$ are scalar functions defined on $[0,T]$ with $\alpha_i(0) = \beta_i(0) = \gamma_i(0) = 0$; $A_i(M)$, $B_i(M)$, $C_i(M)$ are fields which must satisfy the condition that Δs_n belongs to $\mathbf{S_0}$ (homogeneous boundary conditions, and σ self-equilibrated).

The pair of time and space functions are built the one after another, usually one or two pairs are sufficient to lead to a correct approximation of the solution of the considered iteration. Thus, the problem at the global stage is solved only approximately, but this approximate technique supplies an element of A_{d}, which is the only requirement of the method at this stage. It should be noted that, in the following, the subscript of the time and space functions designates the global stage at which these functions are constructed.

In order to build the pair of time–space functions, we use an iterative technique of the fixed-point type:

(1) Given the time function, the best space function is built. It corresponds to the minimization of a residue which has the form:

$$\|K(M,t)\,\Delta v(M,t) - F(M,t)\|$$

where K depends on H^- and K_e. This leads to a linear global problem of the same size as an elastic problem. It is described in Appendix A. The initialization can be performed, for example, with the loading.

(2) Using the space function calculated in (1), the best associated time functions are calculated. This involves the resolution of a first-order differential equation. The determination of these functions is given in Section 3.2 (the resulting function is used again in Stage (1) if necessary).

A complete description of the method can be found in [27]. It should be noted that the resolution cost comes essentially from Stage (1).

2.4. Stopping criterion for the direct problem

The quality of the admissible solution to the direct problem is evaluated through the distance between the approximation of A_d at Stage n and the approximation of Γ. If this distance is zero, these two approximations are identical and correspond to the unique solution to the problem. We refer to [28] for more details on the residual which is used.

2.5. Initialization

In this study, we initialize the process by the elastic solution which gives an element of A_d. We search $(\sigma, \underline{U}, \dot{\varepsilon}^p, \dot{p}, R)$ defined on $[0,T] \times \Omega$ such that

$$\begin{aligned}
&\underline{U}(t,M) = \underline{U}_d, \quad \forall t \in [0,T] \ \forall M \in \partial_1\Omega,\\
&\underline{U}\,|_{t=0} = \underline{U}_0 \quad \forall M \in \Omega - \int_\Omega \sigma : \varepsilon(\underline{U}^*)\,d\Omega + \int_\Omega \underline{f}_d \cdot \underline{U}^*\,d\Omega + \int_{\partial_2\Omega} \underline{F}_d \cdot \underline{U}^*\,dS = 0,\\
&\quad \forall \underline{U}^*/\underline{U}^* = 0 \text{ on } \partial_1\Omega, \ \forall t \in [0,T]\\
&R = \varepsilon^p = p = 0,\\
&\sigma = K_e : \varepsilon = K_e : \varepsilon_e.
\end{aligned} \tag{13}$$

3. LATIN method and multi-solution strategy

3.1. Use of the time–space representation in the context of identification

To reduce significantly the cost of these analyses once a first analysis has been carried out for a set of parameters chosen a priori, we take advantage of the possibility offered by the LATIN method to reuse the basis of space functions built during this first resolution. The principle is the following:

Let σ^{ini} be the stress solution corresponding to the initial set. This stress was represented at the end of the iterative process described in the previous section as the product of $N+1$ time functions and $N+1$ space functions designated here by $(B_0^{ini}(M), B_1^{ini}(M), \ldots, B_N^{ini}(M))$. For a problem corresponding to a new set of parameters and for the first global stages, the basis of functions $(B_1^{ini}(M), \ldots, B_N^{ini}(M))$, which corresponds to the homogeneous boundary conditions, is used to determine iteratively the best time functions associated with the previous set of space functions. After a certain number of iterations, if the solution error remains too large, new pairs of time–space functions are generated as described in the previous section. This process is repeated until convergence.

3.2. Calculation of the time functions

At each global stage, we seek to correct N time functions. The process for a global stage will be described. In a following chapter, various means of initializing these time functions for the new set of parameters will be studied. The problem is written as a system of first-order differential equations. At the global stage, the kinematic admissibility (i.e. $\Delta\dot{\varepsilon}_n$ KA at zero) can be expressed as

$$\forall t \in [0,T] \;\; \forall \sigma^* \in \mathbf{S_0}, \;\; \int_\Omega \Delta\dot{\varepsilon}_n : \sigma^* \, \mathrm{d}\Omega = 0 \tag{14}$$

where $\mathbf{S_0}$ is defined in Section 2.3.

Given the search direction, the state laws and Eq. (14), the problem leading to the corrections of the so-called static variables can be written in the form:

Find $\Delta\sigma_n \in \mathbf{S_0}$ and ΔR_n such that

$$\begin{aligned}
&\int_0^T \int_\Omega \left(K_{\mathrm{e}}^{-1} \Delta\dot{\sigma}_n : \sigma^* + \beta^{-1} \Delta\dot{R}_n \cdot R^* + H^- \begin{bmatrix} \Delta\sigma_n^{\mathrm{D}} \\ \Delta R_n \end{bmatrix} \cdot \begin{bmatrix} \sigma^* \\ R^* \end{bmatrix} \right) \mathrm{d}\Omega \, \mathrm{d}t \\
&= \int_0^T \int_\Omega \quad - \Delta\dot{\hat{\varepsilon}}_n^{\mathrm{p}} : \sigma^* + \Delta\dot{\hat{p}}_n \cdot R^* + H^- \begin{bmatrix} \Delta\hat{\sigma}_n^{\mathrm{D}} \\ \Delta\hat{R}_n \end{bmatrix} \begin{bmatrix} \sigma^* \\ R^* \end{bmatrix} \Bigg) \mathrm{d}\Omega \, \mathrm{d}t \quad \forall \sigma^* \in \mathbf{S_0}, \;\; \forall R^*
\end{aligned} \tag{15}$$

where $\Delta\hat{Q}_n = \hat{Q} - Q_n$, and Q represents $\dot{\varepsilon}^{\mathrm{p}}$, $\dot{p}$, σ^{D} and R. The details of this equation are given in Appendix A.

By writing σ^* and R^* in the same form as the corrections (Eq. (12)) in (15), we obtain the following problem:

Hypothesis: $(B_1^{\mathrm{ini}}(M), \ldots, B_N^{\mathrm{ini}}(M))$, $(C_1^{\mathrm{ini}}(M), \ldots, C_N^{\mathrm{ini}}(M))$ are known.

Find $G(t) = \{\beta_1^{\mathrm{C}}(t), \ldots, \beta_N^{\mathrm{C}}(t), \gamma_1^{\mathrm{C}}(t), \ldots, \gamma_N^{\mathrm{C}}(t)\}^{\mathrm{T}}$ such that

$$N\dot{G}(t) + MG(t) = B(t) \quad \forall t \in [0,T] \text{ and } G(t=0) = 0. \tag{16}$$

By using the θ-method, we end up with the system

$$LG(t) = F(t-1). \tag{17}$$

More details are given in Appendix A, especially the resolution of (16) and the determination of the time functions $\{\alpha_1^{\mathrm{C}}(t), \ldots, \alpha_N^{\mathrm{C}}(t)\}$ which comes from the minimization of a residue.

Thus, by solving this linear system we obtain the correction $\{\beta_1^{\mathrm{C}}(t), \ldots, \beta_N^{\mathrm{C}}(t), \gamma_1^{\mathrm{C}}(t), \ldots, \gamma_N^{\mathrm{C}}(t)\}^{\mathrm{T}}$ which must be applied to the time functions for the new set of parameters. At the end of iteration n, an improved solution compared to the previous iteration is obtained. For example, for stresses, we get

$$\begin{aligned}
&\sigma_n(M,t) = \sigma_{n-1}(M,t) + \Delta\sigma_n^{\mathrm{C}}(M,t), \\
&\sigma_n(M,t) = \beta_0^{\mathrm{ini}}(t) \cdot B_0^{\mathrm{ini}}(M) + \sum_{k=1}^{N} \beta_k^{n-1}(t) \cdot B_k^{\mathrm{ini}}(M) + \underbrace{\sum_{k=1}^{N} \beta_k^{\mathrm{C}_n}(t) \cdot B_k^{\mathrm{ini}}(M)}_{\text{correction}}.
\end{aligned} \tag{18}$$

σ_n can be written as

$$\sigma_n(M,t) = \beta_0^{\mathrm{ini}}(t) \cdot B_0^{\mathrm{ini}}(M) + \sum_{k=1}^{N} \underbrace{\left(\beta_k^{n-1}(t) + \beta_k^{\mathrm{C}_n}(t)\right)}_{\beta_k^n(t)} \cdot B_k^{\mathrm{ini}}(M) \tag{19}$$

where $\beta_k^{n-1}(t)$ is the kth time function achieved at iteration $n-1$, $\beta_k^n(t)$ the kth time function obtained at iteration n, $\beta_k^{C_n}(t)$ the correction to the kth time function $\beta_k^{n-1}(t)$ at iteration n.

Finally, according to successive corrections made at every global stage, σ_n can also be written

$$\sigma_n(M,t) = \beta_0^{\text{ini}}(t) \cdot B_0^{\text{ini}}(M) + \sum_{k=1}^{N} \left[\beta_k^0(t) + \sum_{i=1}^{n} \beta_k^{C_i}(t) \right] \cdot B_k^{\text{ini}}(M) \tag{20}$$

where $\beta_k^0(t)$ is the kth time function which initializes the computational method for a new set of parameters, and $\beta_k^{C_i}(t)$ is the correction to the kth time function $\beta_k^{i-1}(t)$ at iteration i.

Remark. This method is very close to the so-called preliminary stage technique proposed and developed by Ladevèze and Bussy (see [29,30]) in order to improve the convergence rate of the LATIN method. The same ideas are used here in a context where they are particularly interesting.

4. Methods of resolution of the inverse problem

4.1. Strategy of identification

The principles of the LATIN method described above present a high degree of flexibility (uncoupled equations, approximation of the complete structural response over the whole considered history of loading,...). Thus, several strategies of identification can be considered in order to reduce computational costs. The simplest one to implement is the identification at the local stage without taking the previous stages into account. In fact, the only equations which must be satisfied are the evolution laws. The gradient is, therefore, easy to calculate. Besides, the cost of this operation is relatively small compared to the global stage. This strategy was evaluated in [31]. This study led to its rejection and emphasized the necessity to take the admissibility conditions into account in the calculation of the gradient. Therefore, in this paper, the identification will be carried out according to these remarks. The gradient will be calculated at the global stage taking the previous stages into account.

4.2. Deterministic approach

The identification problem is treated as an optimization problem which can be formulated as:

Find the structural parameters k minimizing a distance function between calculation and experiments $\|q(k) - q^{\exp}\|^2$, where $q(k)$ satisfies the direct problem. $q(k)$ can represent, for instance, the strains, the stresses, in the framework of field measurements, or the displacements,...

In order to solve this problem numerically, and for simplicity's sake, we will assume that the discrete values for the time integration and for the experimental data coincide for the calculated values and for the observations.

The strategy of optimization considered is a B.F.G.S. algorithm [32] which is based on the evaluation of the gradient. Therefore, we need to calculate the gradient of the distance computation/experiments, and particularly the term $\mathrm{d}q(k)/\mathrm{d}k$ which is not easy to calculate. The evaluation of this quantity is described in Section 5.

In the B.F.G.S. algorithm, after gradient calculation, several direct calculations are made in order to determine the descent step (line search). These operations can take advantage of the multi-solution aspect of the LATIN method. The basis of space functions is constructed for the initial set of parameters. This basis is then used for all subsequent calculations. The initialization of the associated time functions can be

performed either by using the first calculation, or by an intermediate calculation which can be, for example, the calculation of the gradient. In some cases, the basis of space functions must be enriched by additional functions. Thus, the B.F.G.S. algorithm is well suited to the multi-solution feature of the LATIN method.

4.3. Adaptation of the adjoint state method to the LATIN method

Another deterministic method is available to minimize a cost function under constraints by calculating the gradient of this function with respect to parameters, i.e. the adjoint state method.

We recall that the inverse problem is formulated as the minimization of the distance computation/experiments, designated by D, with the constraint that it satisfies the direct problem. The problem is then written as:

Find k^* such that

$$D(k^*) = D(k^*, \sigma(k^*)) = \min D(k, \sigma) \begin{cases} k, \sigma \text{ satisfying } \sigma(k) \text{ is} \\ \text{solution of the direct problem.} \end{cases} \tag{21}$$

The minimization of a functional under constraints is equivalent to the determination of the saddle point of a Lagrangian, where all variables are assumed to be independent.

The formulation of the adjoint problem is obtained by considering the stationarity conditions on the adjoint variables and integrating these equations by part in space and in time. Thus, we obtain the adjoint problem, which is retrogressive in time because of the final conditions which appear during the integration by part. For more details, see the description of the distance D, the Lagrangian chosen and the resulting problem in Appendix B. This problem is solved like the direct problem using the principles of the LATIN method. On the one hand, the linear and local equations can be divided between the two spaces A_d and Γ (see Section 2.2). On the other hand, the specific representation of the solution poses no particular problem: one must simply express the problem in a retrogressive way using the θ-method. In fact, the differential equation (16) is still available. The main difference is that the initial condition is replaced by a final one. Then, the resulting equation (17) is written in a retrogressive way. Thus, it is possible to benefit from the multi-solution feature.

This method is formulated again, as in the previous paragraph, as the minimization of a cost function. Moreover, it enables the evaluation of the gradient with respect to parameters, as we will see in Section 5.2. This minimization can be carried out using the B.F.G.S. algorithm.

4.4. Stochastic approach: adaptation of the extended Kalman filter to the LATIN method

The stochastic method of identification used is Kalman's filtering technique. This method enables one to take into account the measurement noise by giving it an optimal weight. Furthermore, it presents some interesting characteristics. On the one hand, the filtered estimation at time t is accompanied by its covariance matrix, which gives a measure of its uncertainty. On the other hand, this stochastic method makes use of all a priori available information concerning the variable and the model errors. Finally, it is not necessary to have more measurable quantities than unknowns. Appendix C gives more details on this technique.

In this algorithm, it is necessary to estimate the quantities $h(\hat{x}_t)$, which contain all the nonlinearities of the problem, and the sensitivity matrix $H(\hat{x}_t)$. Here, $\hat{x}_t$ is the estimation of the model parameter over an interval $[t-1, t]$. The first quantity is obtained by direct calculation using the LATIN method with a particular representation of the solution. For this purpose, a first calculation is carried out over the whole space and the whole time interval for the initial set of parameters. Thus, the basis of the space functions which belong to $\mathbf{S_0}$ is constructed. Then, in order to calculate $h(\hat{x}_t)$ on $[0, t]$, we can either initialize the time functions on $[0, t]$ using the first calculation (initial basis) or reuse the previous calculation on $[0, t-1]$ and complete the

initialization at the time t using the first calculation. Since the basis functions belong to $\mathbf{S_0}$, so does the initialization. This is done in order to get a richer initialization. Then, the calculation of $h(\hat{x}_t)$ is carried out as before by performing local stages and preliminary stages successively until convergence. As far as the calculation of $H(\hat{x}_t)$ is concerned, the process will be described in detail in Section 5.1. In fact, it follows the same procedure as the direct calculation. We will also see an easier way to evaluate this sensitivity matrix.

5. Calculation of the gradient with respect to parameters

The methods described above revolve around on the gradient with respect to parameters. It has an essential role. Thus, we try to estimate the influence of a modification of the parameters on the calculated solution. However, this is not easily done in the case of viscoplastic behavior. In the following paragraphs, several methods of evaluation of the gradient (direct differentiation, adjoint state, interpolation) will be described within the scope of the LATIN method. These have already been used for nonlinear incremental calculations. This is one of the key points of our approach which must be studied in detail.

5.1. The direct differentiation method

We saw in Section 4 that the solution to the identification problem is based on the calculation of the gradient with respect to parameters. This presentation of the direct differentiation is tailored to our problem, where the representation of the solution is particular. Indeed, in the context of the LATIN method, the solutions are expressed as sum of products of time functions and space functions. For example, the stress can be written

$$\sigma_{n+1}(M,t) = \sigma_0(M,t) + \sum_{k=1}^{n} \beta_k(t) \cdot B_k(M) \quad \text{with } m = 1 \tag{22}$$

and in the case where the initial stress is defined by a single couple:

$$\sigma_0(M,t) = \beta_0(t) \cdot B_0(M). \tag{23}$$

The gradient with respect to parameters is obtained directly:

$$\frac{\mathrm{d}\sigma_{n+1}(M,t)}{\mathrm{d}k} = \frac{\mathrm{d}\sigma_0(M,t)}{\mathrm{d}k} + \sum_{k=1}^{n} \left[\frac{\mathrm{d}\beta_k(t)}{\mathrm{d}k} \cdot B_k(M) + \beta_k(t) \cdot \frac{\mathrm{d}B_k(M)}{\mathrm{d}k} \right]. \tag{24}$$

The other static and kinematic quantities are expressed in the same way.

The expression (24) shows that the gradient of the time and space functions with respect to the parameters must be calculated. Thus, we need to study the influence of a modification of the parameters on these two types of functions. To obtain these quantities, we will again consider the problems at the global stage.

5.1.1. Calculation of the gradient of the static variable corrections

We saw that Problem no. 2 (calculation of the time functions) is expressed as a system of first-order differential equations (16). The calculation of the gradient follows the same procedure as the determination of the time functions:

- $\mathrm{d}B(M)/\mathrm{d}k$ and $\mathrm{d}C(M)/\mathrm{d}k$ are known.
- The calculation of the gradient is given by

$$\frac{\mathrm{d}G(t)}{\mathrm{d}k} = L^{-1} \cdot \left(\frac{\mathrm{d}F(t-1)}{\mathrm{d}k} - \frac{\mathrm{d}L}{\mathrm{d}k} \cdot G(t) \right). \tag{25}$$

We have here a recurrence formula on the time steps and on the LATIN iterations (with respect to the previous local and global stages). It is worth noting that L was already inverted in the direct problem.

In Problem no. 1 (calculation of the space functions), the gradient of the time functions is considered to be known. By considering Eq. (15) and writing σ^* and R^* in the same form as the corrections, the spacial problem can be written as $K(M)\,\Delta v(M) = F(M)$ (dualized problem). From $\Delta v(M)$, we deduce $B_i(M)$ and $C_i(M)$. So, it is important to calculate $\mathrm{d}\Delta v(M)/\mathrm{d}k$ so as to have $\mathrm{d}B_i(M)/\mathrm{d}k$ and $\mathrm{d}C_i(M)/\mathrm{d}k$. The expression of the gradient of Δv with respect to parameters is derived from the dualized problem (coming from (15)) as

$$\frac{\mathrm{d}\Delta v}{\mathrm{d}k} = K^{-1} \left(\frac{\mathrm{d}F}{\mathrm{d}k} - \frac{\mathrm{d}K}{\mathrm{d}k} \cdot \Delta v \right). \tag{26}$$

Then, the gradient of the space functions with respect to the parameters can be derived from this expression. It depends on the quantities from the previous local and global stages. One should note again that the matrix K was already inverted in the direct problem.

5.1.2. Calculation of the gradient of the strain corrections

The two subproblems in the calculation of strain corrections (cf. Appendix A) being identical to those of the previous section, the gradient will be estimated in the same way.

5.1.3. Calculation of the gradient of the quantities in the local stage

It is worth noting that the expression of the gradient of the quantities in the previous local stage is present in Eqs. (25) and (26), therefore, these quantities must be calculated.

The analytical calculation of the gradient of the computation/experiments distance is performed according to the method proposed by Mahnken and Stein [8], which is adjusted to the equations governing the local stage. It is based on the numerical scheme used, and leads to a recursive formula.

At the local stage, for a 1D problem, the behavior follows the equations:

$$\begin{cases} \dot{\hat{p}} = k_{\mathrm{v}} \langle |\hat{\sigma}| - \hat{R} - R_0 \rangle_+^{n_{\mathrm{v}}} \\ \dot{\hat{\varepsilon}}^{\mathrm{p}} = \dot{\hat{p}} \operatorname{sgn}(\hat{\sigma}) \end{cases} \tag{27}$$

where $\operatorname{sgn}(\hat{\sigma})$ stands for sign of $\hat{\sigma}$. With the scheme considered, the problem is written in terms of

- $G_t(k, Y_t, Y_{t-1}, \hat{\sigma}_t, \hat{\sigma}_{t-1}, \hat{R}_t, \hat{R}_{t-1}) = \left\{ \begin{matrix} g_{1,t} \\ g_{2,t} \end{matrix} \right\} = 0,$

$$\begin{aligned} &\text{where } Y_t = [\hat{\varepsilon}_t^{\mathrm{p}}, \hat{p}_t] \\ &\text{and } \begin{cases} g_{1,t} = \hat{\varepsilon}_t^{\mathrm{p}} - \hat{\varepsilon}_{t-1}^{\mathrm{p}} - \Delta\hat{\varepsilon}_t^{\mathrm{p}} \\ g_{2,t} = \hat{p}_t - \hat{p}_{t-1} - \Delta\hat{p}_t \end{cases} \text{ (numerical scheme)} \\ &\quad \text{with } \begin{cases} \Delta\hat{p}_t = k_{\mathrm{v}} \cdot \Delta t \langle |\hat{\sigma}| - \hat{R} - R_0 \rangle_+^{n_{\mathrm{v}}}|_\tau \\ \Delta\hat{\varepsilon}_t^{\mathrm{p}} = \Delta\hat{p}_t \operatorname{sgn}(\hat{\sigma}) \end{cases} \\ &\quad \text{and } \quad q|_\tau = (1-\theta) \cdot q(t-1) + \theta \cdot q(t), \quad \theta \in [0,1], \end{aligned} \tag{28}$$

- the search direction.

For the calculation of the derivative, we consider that

$$G_t(k, Y_t(k), Y_{t-1}(k), \hat{\sigma}_t(k), \hat{\sigma}_{t-1}(k), \hat{R}_t(k), \hat{R}_{t-1}(k)) = 0. \tag{29}$$

G_t depends on the parameters k implicitly. Hence, by differentiating, we obtain

$$\frac{\mathrm{d}Y_t}{\mathrm{d}k} = -\left[\frac{\partial G_t}{\partial Y_t}\right]^{-1} \cdot \left\{ \frac{\partial G_t}{\partial k} + \frac{\partial G_t}{\partial Y_{t-1}} \frac{\mathrm{d}Y_{t-1}}{\mathrm{d}k} + \frac{\partial G_t}{\partial \hat{\sigma}_t} \frac{\mathrm{d}\hat{\sigma}_t}{\mathrm{d}k} + \frac{\partial G_t}{\partial \hat{\sigma}_{t-1}} \frac{\mathrm{d}\hat{\sigma}_{t-1}}{\mathrm{d}k} + \frac{\partial G_t}{\partial \hat{R}_t} \frac{\mathrm{d}\hat{R}_t}{\mathrm{d}k} + \frac{\partial G_t}{\partial \hat{R}_{t-1}} \frac{\mathrm{d}\hat{R}_{t-1}}{\mathrm{d}k} \right\}. \tag{30}$$

This expression can be viewed as a recursive formula between $\mathrm{d}Y_t/\mathrm{d}k$ and $\mathrm{d}Y_{t-1}/\mathrm{d}k$.

- For the vertical search direction, the quantities $\mathrm{d}\hat{\sigma}_t/\mathrm{d}k$, $\mathrm{d}\hat{\sigma}_{t-1}/\mathrm{d}k$, $\mathrm{d}\hat{R}_t/\mathrm{d}k$, $\mathrm{d}\hat{R}_{t-1}/\mathrm{d}k$ are known from the previous global stage.
- $\partial G_t/\partial Y_t = Id$ and $\partial G_t/\partial Y_{t-1} = (\partial G_t/\partial Y_t) - 2Id$ (chosen scheme).
- The other derivatives with respect to parameters, or the quantities $\hat{\sigma}$ or $\hat{R}$ are calculated using the evolution laws.

The calculation of the gradient of G_t with respect to two parameters is given for the vertical search direction:

$$\frac{\partial G_t}{\partial \beta} = k_{\mathrm{v}} \cdot n_{\mathrm{v}} \Delta t \cdot p_n|_\tau \langle |\sigma_n| - \beta \cdot p_n - R_0 \rangle_+^{n_{\mathrm{v}}-1}|_\tau \operatorname{sgn}(\sigma_n), \tag{31}$$

$$\frac{\partial G_t}{\partial R_0} = k_{\mathrm{v}} \cdot n_{\mathrm{v}} \cdot \Delta t \langle |\sigma_n| - \beta \cdot p_n - R_0 \rangle_+^{n_{\mathrm{v}}-1}|_\tau \operatorname{sgn}(\sigma_n). \tag{32}$$

Thus, a time dependence occurs in the evaluation of the gradient.

Remark. It is possible to calculate the gradient for other search directions. This differs from the previous case in the calculation of the Jacobian. Indeed, the quantity $\Delta\hat{\varepsilon}^{\mathrm{p}}$ depends on the quantities $\hat{\varepsilon}^{\mathrm{p}}$ and/or $\hat{p}$, depending on the case.

Thus, the calculation of the gradient with respect to parameters introduced in this paper is specific to the LATIN method. Indeed, the gradient depends (implicitly) on all the iterations in the method. Furthermore, it is based on the process of determining the time and space functions which are used to represent the solution. In fact, during each iteration, it is sufficient to determine the gradient of the correction which must be applied to the static and kinematic quantities. Thus, this method takes into account the admissibility conditions as well as the behavior. For the part related to the evolution laws, a method proposed by Mahnken and Stein [8] was modified to suit our context.

In order to estimate the influence of the modification of parameters on the solution during the multi-solution process, we evaluate the gradient in a way similar to that of the direct calculation. A first calculation with the initial set of parameters k_0 is performed. Using the method described above, we obtain a basis of space gradient functions $(\mathrm{d}B_0/\mathrm{d}k|_{k_0}, \mathrm{d}B_1/\mathrm{d}k|_{k_0}, \ldots, \mathrm{d}B_N/\mathrm{d}k|_{k_0})$ and the gradient of the associated time functions. Then, for the new sets of parameters, we initialize the basis with $(\mathrm{d}B_0/\mathrm{d}k|_{k_0}, \mathrm{d}B_1/\mathrm{d}k|_{k_0}, \ldots, \mathrm{d}B_N/\mathrm{d}k|_{k_0})$. Once the space functions are known, the gradients of the associated time functions for this new set are determined by solving the two subproblems in time. Hence, since this is inherent to the LATIN calculation, the gradients of the static and kinematic quantities are estimated by improving only the gradient of the time functions (preliminary stage). Sometimes, the basis of space functions may not be sufficient to represent the solution for a new set of parameters. If so, it is necessary to complete it with new functions. The calculation of the gradient follows the same procedure again. Thus, it should be noted that for these calculations the gradient of the space functions in the basis is always calculated at k_0, regardless of the new

sets of parameters in the subsequent calculations. All this reasoning is also valid for the strains. So, we obtain an approximation of the gradient.

The calculation of the gradient is expressed in terms of a sum of products of space functions and time functions. Thus, the multi-solution feature is retained in the sensitivity study, which is interesting because the calculation of the gradient concerns only the time functions, hence a gain in comparison with the classical incremental methods.

5.2. The adjoint state method

The solutions to the adjoint problem formulated in Section 4.3 can be expressed as a sum of products of time functions and space functions. The calculation of the gradient depends on these adjoint quantities. It can be written as

$$\frac{\mathrm{d}D}{\mathrm{d}k} = \int_0^T \int_\Omega \left[\frac{\mathrm{d}K_{\mathrm{e}}^{-1}}{\mathrm{d}k}\dot{\boldsymbol{\sigma}} : \sigma^* + \frac{\partial^2 \varphi^*}{\partial A_{\varepsilon^{\mathrm{p}}}\,\partial k} \cdot A_{\varepsilon^{\mathrm{p}}}^* + \frac{\partial^2 \varphi^*}{\partial R\,\partial k} R^* - \frac{\partial^2 \dot{\psi}}{\partial(-p)\,\partial k}(-p^*) \right] \mathrm{d}\Omega\,\mathrm{d}t \tag{33}$$

if $(\underline{U}, \sigma, \varepsilon^{\mathrm{p}}, -p, R)$ and $(\underline{U}^*, \sigma^*, \varepsilon^{\mathrm{p}*}, -p^*, R^*)$ are the solutions to the direct and adjoint problems respectively. In fact, we differentiate the Lagrangian (cf. (B.3)) with respect to parameters, where all the variables are supposed independent.

In this equation, $A_{\varepsilon^{\mathrm{p}}}$ is associated with σ. In our case, we also have $\partial^2\psi/\partial\varepsilon^{\mathrm{p}}\,\partial k = 0$.

Thus, the calculation of the gradient poses no particular problem as long as the two problems (direct and adjoint) are solved. It is just a matter of evaluating an integral.

In this approach, we assume that the pseudo-potential φ^* and the free energy ψ are differentiable twice. This limits the choice to a constitutive law of the viscoplastic type.

In the minimization algorithm, the basis of space functions generated by the first adjoint calculation is reused in every gradient calculation. Therefore, it is sufficient to calculate the associated time functions in the local and preliminary stages of the adjoint problem alternatively until convergence. The identification problem is based on two different calculations:

- The direct problem utilizing the basis of space functions from the first calculation.
- The calculation of the gradient utilizing the so-called adjoint basis of the space functions from the first adjoint calculation.

Since a number of direct calculations and gradients are necessary, the full benefit of the LATIN method's multi-solution feature can be derived in solving nearly identical direct and adjoint problems.

However, it is worth noting that this method is completely dependent on the cost function which is used. Another one can be used (with displacements, ...), but the adjoint problem must be reformulated.

5.3. The interpolation method

Another means of determining the gradient with respect to parameters is the interpolation method. This has already been used in [15,17] along with a classical incremental method of resolution for the direct problem, in the framework of the Kalman filters. Here, it is applied to the LATIN method. It is a simple method. It allows the number of computations to be reduced, especially in the context of the LATIN method. Our aim is to interpolate the gradient of the strains with respect to parameters. This interpolation technique can be applied to all the methods.

For a case involving m parameters, 3^m values of strains are necessary for a quadratic function. To simplify the notations, we consider only two parameters. Thus, the quantities are evaluated as follows (Fig. 2):

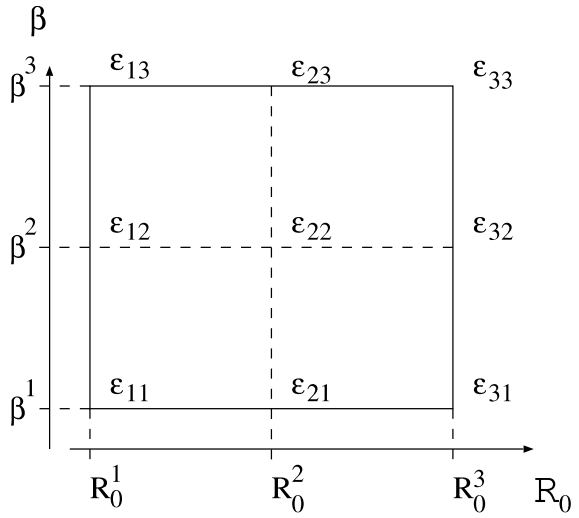

Fig. 2. Meshing with R_0 and β.

$$\tilde{\varepsilon}(R_0,\beta)=\sum_{i=1}^{3}\sum_{j=1}^{3}N_i^{R_0}N_j^{\beta}\varepsilon_{ij},\ \text{where } N_i=\frac{\prod_{\substack{j=1\\ j\neq i}}^{3}(k-k^j)}{\prod_{\substack{j=1\\ j\neq i}}^{3}(k^i-k^j)}\ \text{with } k=R_0 \text{ or } \beta.$$

Since the strains are expressed as products of time functions and space functions, the interpolation for every calculation is performed only on the time functions. Indeed, the space functions remain identical for all cases. The expression of these time functions is

$$\alpha_k(t)=\sum_{i=1}^{3}\sum_{j=1}^{3}N_i^{R_0}N_j^{\beta}\cdot\alpha_k^{ij}(t). \tag{34}$$

It should be noted that this interpolation concerns the strains field.

This method is very well suited to the multi-solution technique. Indeed, as soon as the basis of space functions for the first calculation is constructed (ε_{11} for example), it can be utilized in the solution of the other problems (for ε_{ij} with $i\neq 1$ and $j\neq 1$) because they are nearly identical. Therefore, the calculation is faster than with a classical method. The numerical results are given in Section 6.4.2.

6. Numerical results

6.1. Example of a cantilever beam

Every method described previously was applied numerically on the example of a cantilever beam.

The beam is subjected to a prescribed displacement at one end. The problem is illustrated below (Fig. 3):

The beam characteristics are

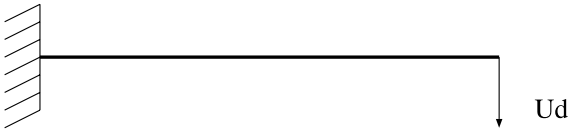

Fig. 3. Cantilever beam.

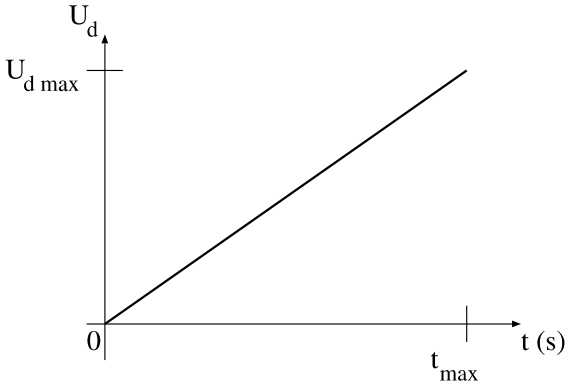

Fig. 4. Evolution of the prescribed displacement with time.

$E_0 = 292914$ MPa, $n_v = 2$
$R_0 = 0.7$ MPa, $k_v = 2.25$
$\beta = 2$, $u_{d\,max} = 3.33e^{-2}$ m
$l = 1$ m, $t_{max} = 0.4$ s

The parameters to identify are initialized as $R_0 = 0.3$ MPa and $\beta = 1$.

The constraint loading at the beam end is a monotonic with respect to time. It is represented in Fig. 4.

For the experimental data which are simulated, the strain field is used.

6.2. Results of the identification

In the examples considered, the experimental results were simulated numerically for $R_0 = 0.7$ MPa and $\beta = 2$. In addition, a measurement noise with a magnitude of 10% and a Gaussian distribution was inserted. It is only applied to the measurements. The deterministic cases in the absence of measurement noise had been calculated earlier in order to verify the convergence of the method. Some examples can be found in [31].

We note that the construction of the basis for the initial values $R_0 = 0.3$ MPa and $\beta = 1$ requires six space functions for R and two functions for the stresses. The beam is discretized into 10 elements.

6.2.1. Deterministic approach

6.2.1.1. Direct differentiation method. The gradient was calculated by the direct differentiation method. The identification of the two parameters R_0 and β led to the values $R_0 = 0.68$ MPa, and $\beta = 2.06$. The computation/experiment distance then reached its residual value due to the measurement noise. It took four

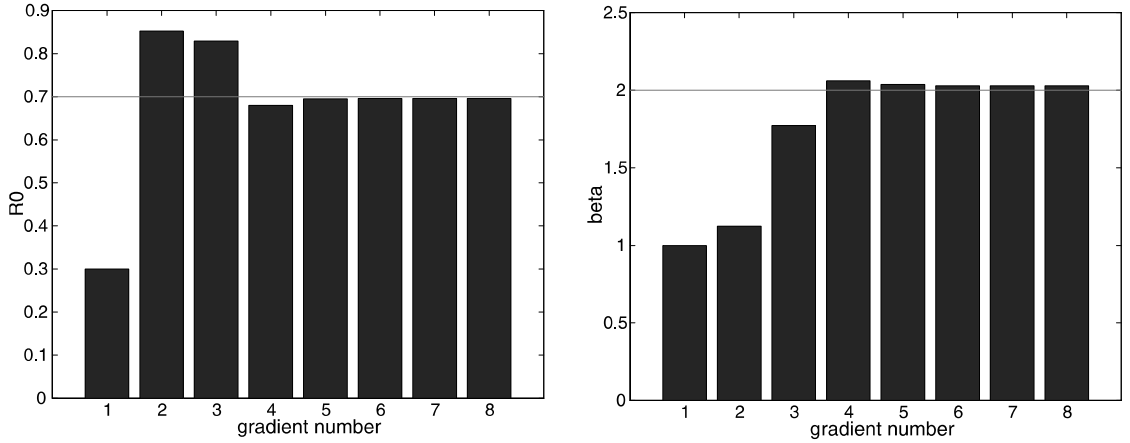

Fig. 5. Evolution of the parameters versus time.

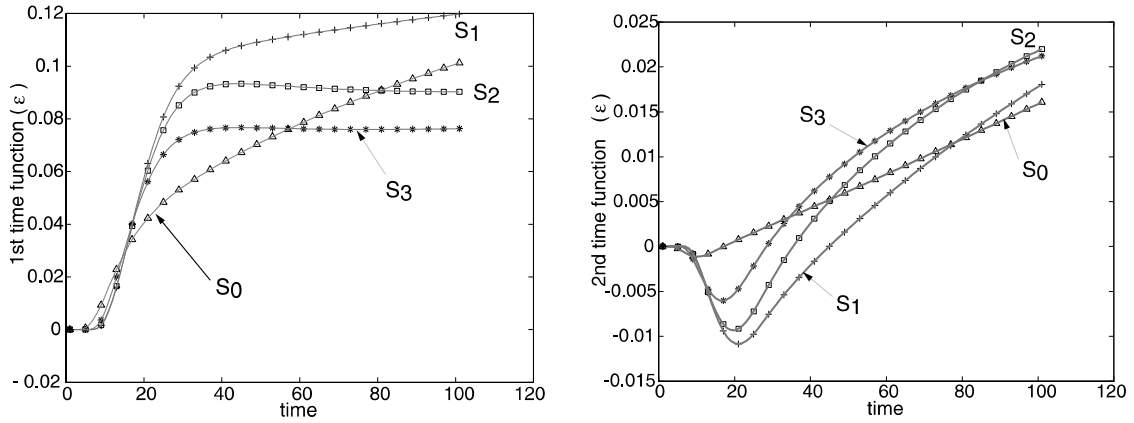

Fig. 6. Evolution of time functions 1 and 2 associated with the strain, for different sets of parameters of the gradient calculation.

gradient calculations and 11 direct calculations to obtain this minimum. Fig. 5 shows the evolution of these parameters. Figs. 6–9 show the evolution of the time functions related to the stresses and strains for the new sets of parameters S_k (see Fig. 10) against the minimization steps associated with the gradients with respect to the parameters. We recall that in the optimization algorithm chosen (see Fig. 10) the values of the parameters change when a gradient calculation is performed: therefore, it takes several direct calculations associated with S_{ki} to determine the descent step.

6.2.1.2. Adjoint state method. The final parameters obtained in the context of the adjoint state method were $R_0 = 0.69$ MPa and $\beta = 2.02$, which is satisfactory considering the measurement noise. Again, the residual error of the cost function was reached. The comparison of these results with direct differentiation for the same cost function shows that there are some identical solutions for the final values of the parameters as well as for their evolution during the minimization.

6.2.1.3. Interpolation method. The results obtained are identical to those of the direct differentiation method. The final values of the parameters are $R_0 = 0.68$ MPa and $\beta = 2.07$. Again, the residual error due to the

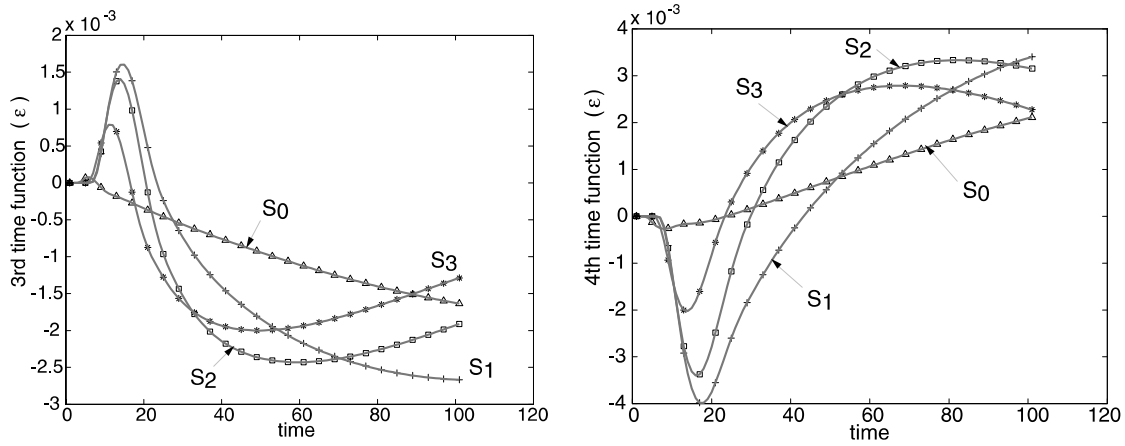

Fig. 7. Evolution of time functions 3 and 4 associated with the strain, for different sets of parameters of the gradient calculation.

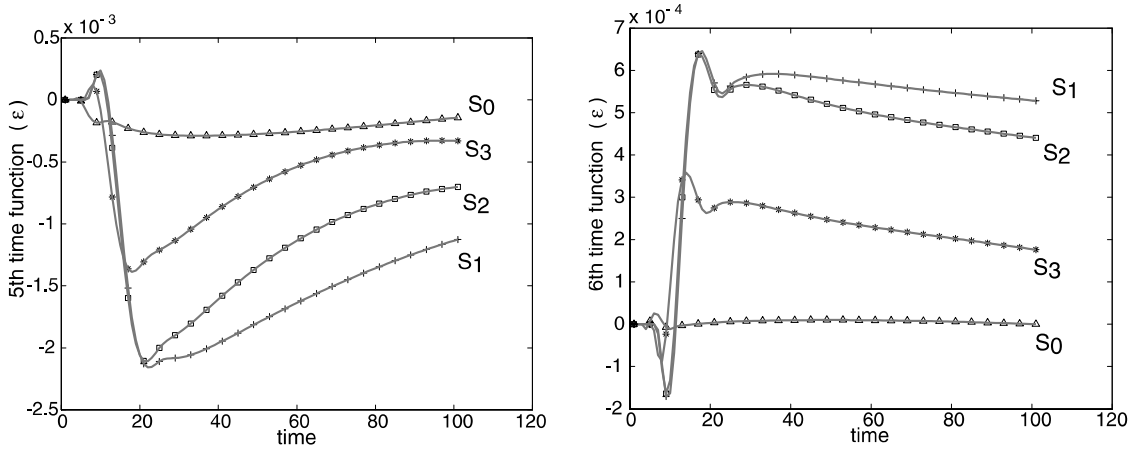

Fig. 8. Evolution of time functions 5 and 6 associated with the strain, for different sets of parameters of the gradient calculation.

measurement noise was reached. The total number of direct calculations and gradient calculations are still 11 and 4 respectively.

In conclusion, these three methods of gradient calculation led to similar values of the gradient, but at very different costs.

6.2.2. The Kalman filter

6.2.2.1. Direct differentiation method. The following curves show the stabilization of the parameters with time. The final parameters are $R_0 = 0.69$ MPa and $\beta = 1.96$ (Fig. 11).

The comparison between the identified and the experimental constitutive curves at the final time step and at two points of the structure shows good agreement (Fig. 12).

6.2.2.2. Interpolation method. Once the mesh has been built, the Kalman filter allows one to identify the parameters by using the interpolation of the time functions. Their evolution is given in Fig. 13. These curves

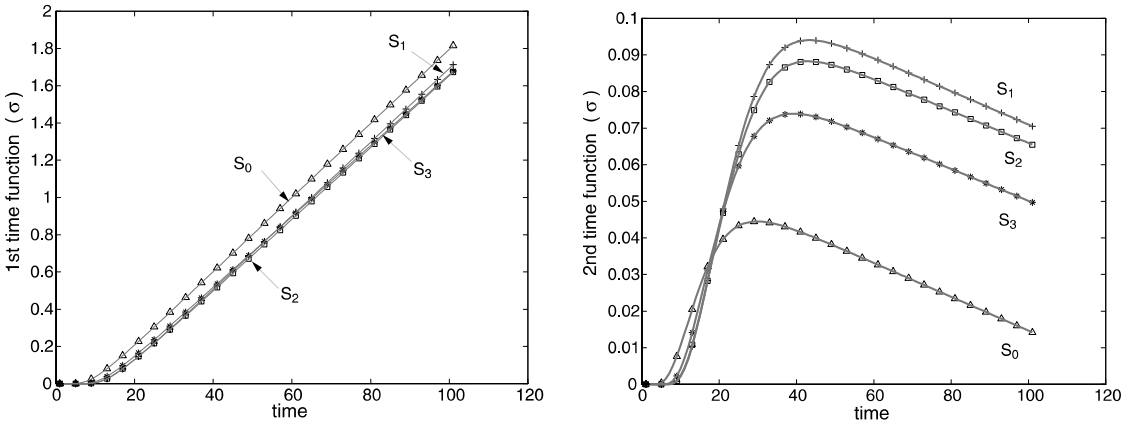

Fig. 9. Evolution of time functions 1 and 2 associated with the stress, for different sets of parameters of the gradient calculation.

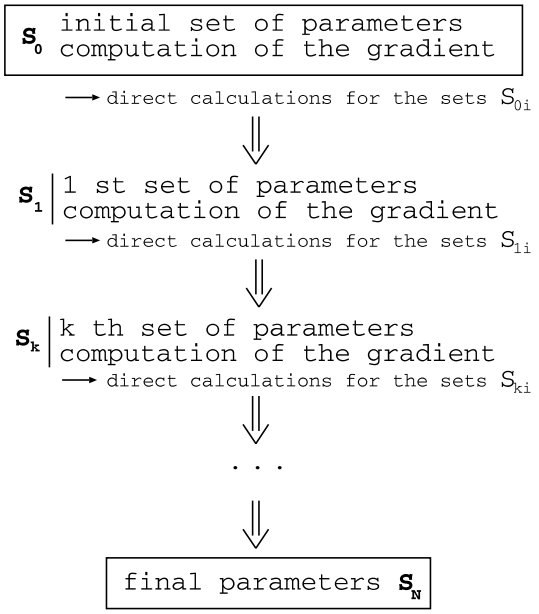

Fig. 10. Optimization algorithm.

show the stabilization of the parameters with time and their convergence towards the expected value despite the approximation on the calculated quantities. The resulting final values are $R_0 = 0.70$ MPa and $\beta = 1.93$, which is acceptable considering the measurement noise.

6.3. Comparison of the values of the three types of gradient

The similar behavior of the results obtained previously with the three methods of gradient evaluation can be explained by comparing the values of the gradients with respect to the two identified parameters (see

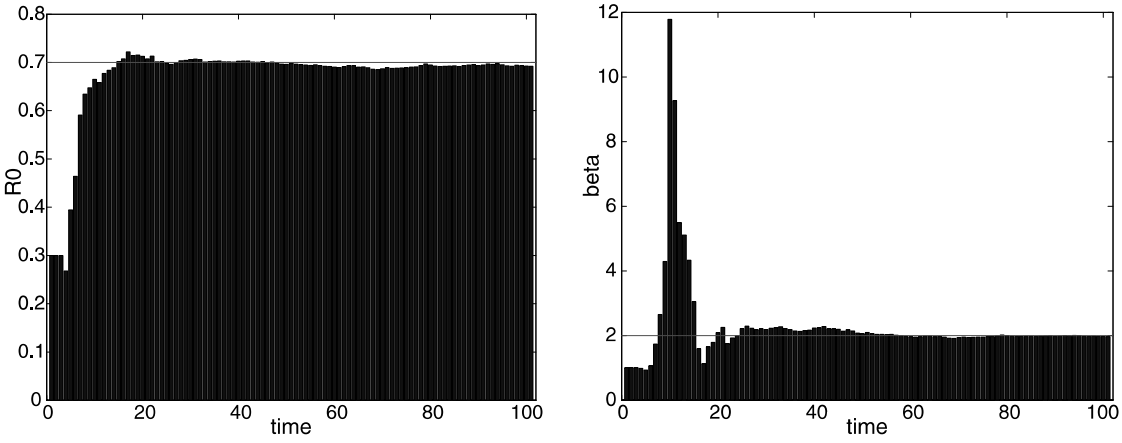

Fig. 11. Evolution of the parameters with time.

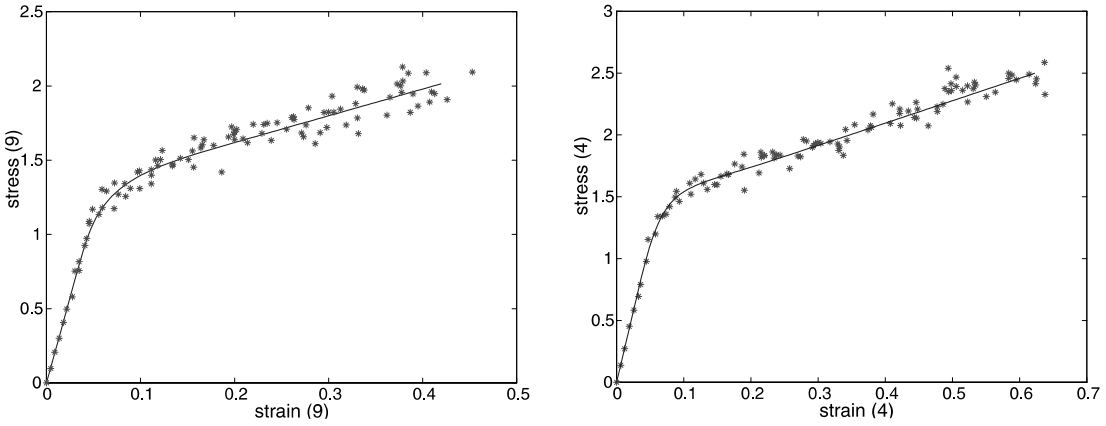

Fig. 12. Comparison between the identified and experimental strain/stress curves at $x = 0.179$ and 0.42 m.

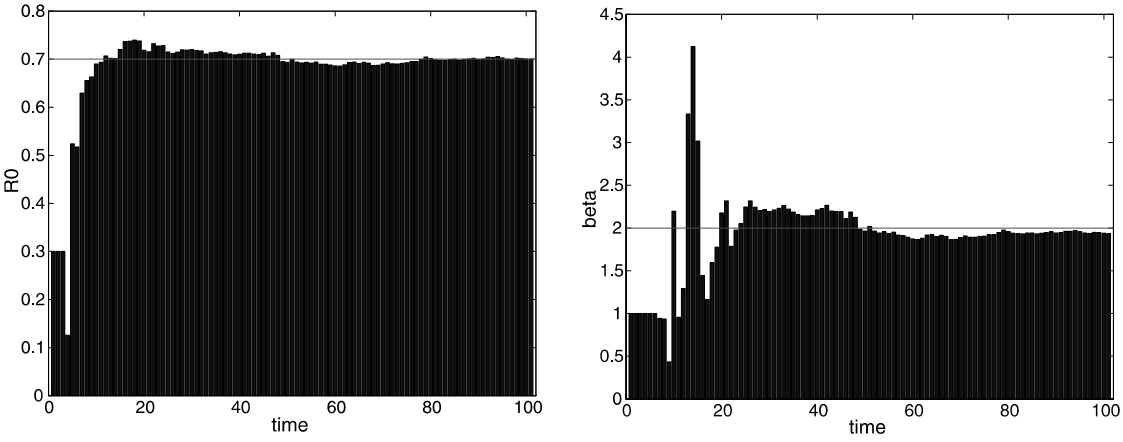

Fig. 13. Evolution of the parameters with time.

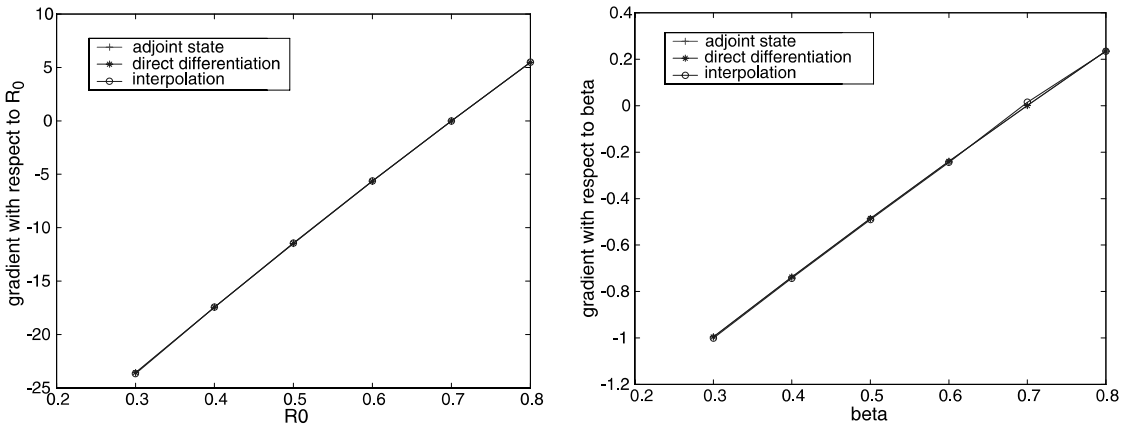

Fig. 14. Comparison of the gradient values with respect to R_0 and β for the adjoint state, direct differentiation, and interpolation approaches.

Fig. 14). These figures emphasize the nearly identical numerical results obtained with the three techniques. In these examples, the values of the gradient resulting from the adjoint state can be considered as our reference. So, the approximation which consists in using only the effect of the parameters k variation on time functions, and not on space basis functions, is quite good.

6.4. Comparison of computation costs

6.4.1. Comparison in terms of number of iterations

The deterministic and the stochastic approaches are compared, with the gradient being estimated by the direct differentiation method. Table 1 gives the total number of preliminary stages for the direct calculation without taking the gradient into account. It is important to indicate that no space function was added to the initial basis during the multi-solution. Thus, only preliminary stages were performed.

This table allows us to compare only different direct calculations. The results demonstrate the advantage of the time/space functions of the LATIN method. These reduce the number of iterations necessary to reach the convergence of the direct calculation because the initialization of every calculation is better. Moreover, it is recommended to initialize the calculation associated with the parameters S_{ki} with the previous one, i.e. the one associated with S_{ki-1} (see Fig. 10) rather than the one associated with the parameters S_k. This is demonstrated by the results obtained when the calculation associated with S_k (designated by ref.

Table 1
Comparison of the number of iterations for the deterministic approach and the Kalman filter

Deterministic approach		
With time/space functions	Ref. initialization: 1st computation	59 Preliminary stages
With time/space functions	Ref. initialization: gradient	34 Preliminary stages
Without time/space functions		78 Iterations
Kalman filter		
With time/space functions	Ref. initialization: 1st computation	250 Preliminary stages
With time/space functions	Ref. initialization: gradient	110 Preliminary stages
Without time/space functions		200 Iterations

initialization in Table 1) is used as the initialization for the direct calculation. The gain on the examples considered is a factor of about 2 compared to the approach with no time/space functions.

We also notice that the Kalman filtering technique is expensive in that it requires between three and four times as many preliminary stages.

6.4.2. Comparison in terms of number of operations

In this section, we estimate the computational cost in terms of number of operations, in order to compare the different methods. In this context, the cost of numerical operations of the finite element analysis with the LATIN method was evaluated (see [33]). We also utilized the fact that the LATIN method has the same behavior in 2D as in 3D. Thus, the results for the 1D examples were extended to the 2D and 3D cases in terms of number of iterations. The discretization is made such that the bandwidth should be minimized.

The time/space functions of the LATIN method present some advantages concerning the computational cost. These are most noticeable on 3D calculations (also in 2D), where the gains are quite significant. They become especially high when the number of degrees of freedom increases. We notice this if we compare the cost of a global stage with no time/space functions, of the same with the time/space functions and of the preliminary stage, in the case of 1D and 3D beams (Figs. 15 and 16). Therefore, the method is particularly attractive for structural identification when structures and loadings can be complex. Moreover, multiple nearly identical problems are solved due to the multi-solution feature. The gain would be even greater because the cost of the preliminary stage is less than that of a global stage. For instance, for a 3D case with 35,000 d.o.f.s, the gain is about 50.

As far as the computational method for the gradient is concerned, the interpolation method is easy to implement and can be applied to all examples. The cost of constructing a mesh is relatively small because of the time/space functions of the LATIN method. The number of iterations is reduced: we counted 20 preliminary stages (Fig. 17), once the basis was generated. At the same time, the cost of each stage is less, which accounts for the total gain (see Figs. 18 and 19). The gain becomes very significant as the number of d.o.f.s increases. Furthermore, these first calculations require very little storage because they involve the time functions for every set of parameters and a single basis of space functions. This is much less than what

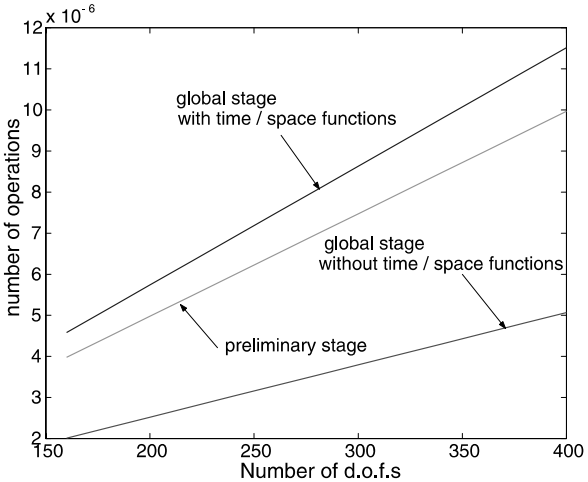

Fig. 15. Cost comparison of three different stages: global stage with time/space functions, global stage without time/space functions, preliminary stage, for a 1D beam.

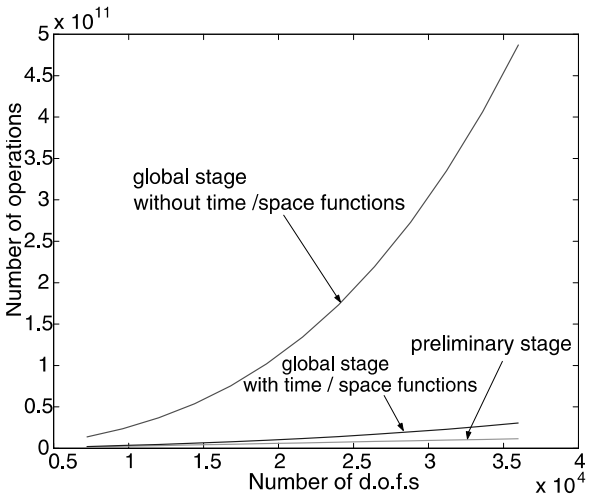

Fig. 16. Cost comparison of three different stages: global stage with time/space functions, global stage without time/space functions, preliminary stage, for a 3D beam.

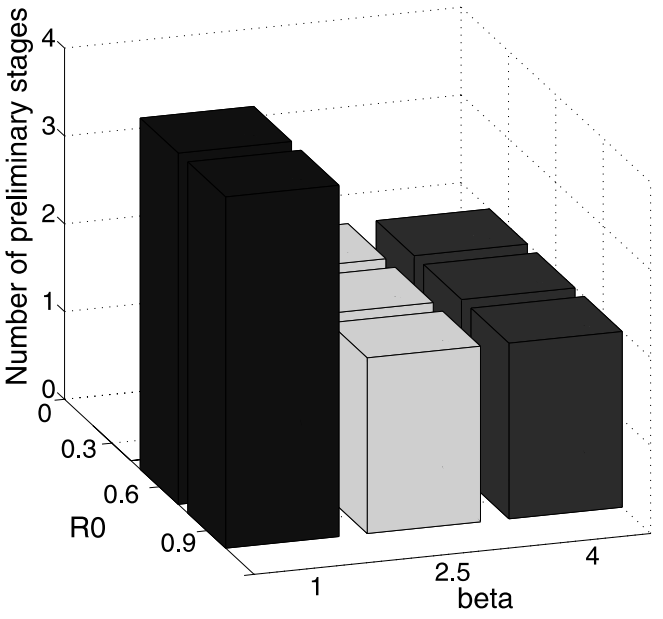

Fig. 17. Number of preliminary stages per direct calculation for the mesh.

is needed for the storage of stresses over the whole time and space interval for every set of parameters. Therefore, the LATIN method presents a double advantage compared to classical methods.

The limit of the interpolation method lies in the a priori unknown values of the parameters. Thus, the mesh must be very coarse at first to ensure that it contains the final value. Then, numerous meshes must be built, at the risk of increasing the computational cost greatly. Furthermore, for functions with high

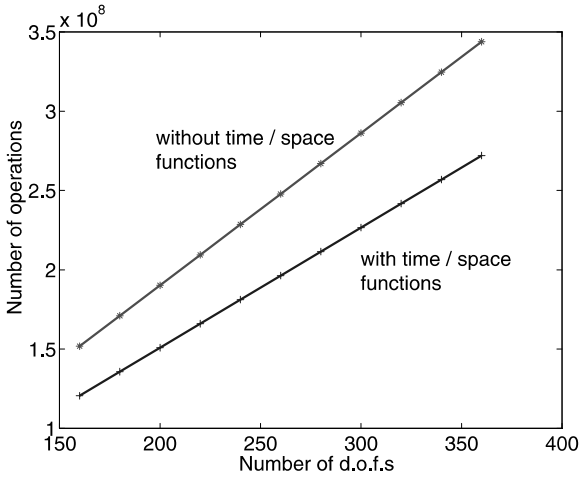

Fig. 18. Cost of the 1D mesh.

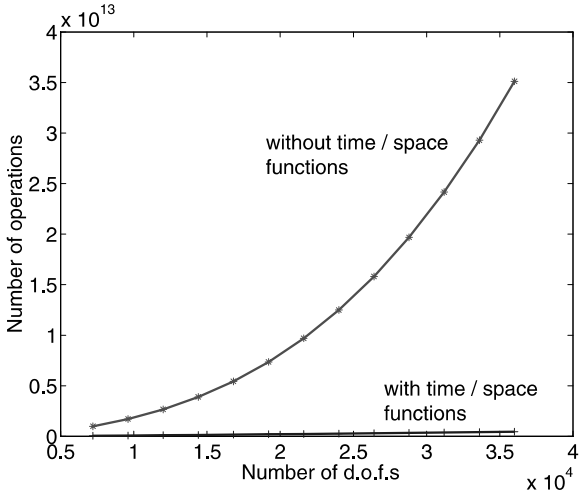

Fig. 19. Cost of the 3D mesh.

gradient, the approximation would be very different from the real value of the gradient. Nevertheless, the method described would still be advantageous compared to a classical incremental method, especially if the number of calculations increases.

The adjoint state method has attractive features. It is less expensive than the direct differentiation method (see Figs. 20 and 21). This would be particularly significant when a large number of parameters need to be identified. In the gradient calculation, the adjoint quantities can be used regardless of the parameter to be identified. On the other hand, if the cost function changes, the whole adjoint problem must be reformulated. Besides, this method seems to be well adapted to the LATIN method, as all of its principles can be applied. Again we see the advantage of using the particular representation of the solution.

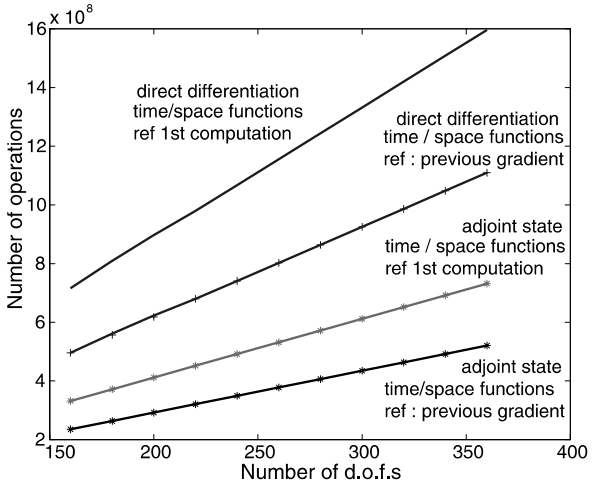

Fig. 20. Computational cost comparison between the direct differentiation and adjoint state approaches, with time/space functions, for a 1D beam.

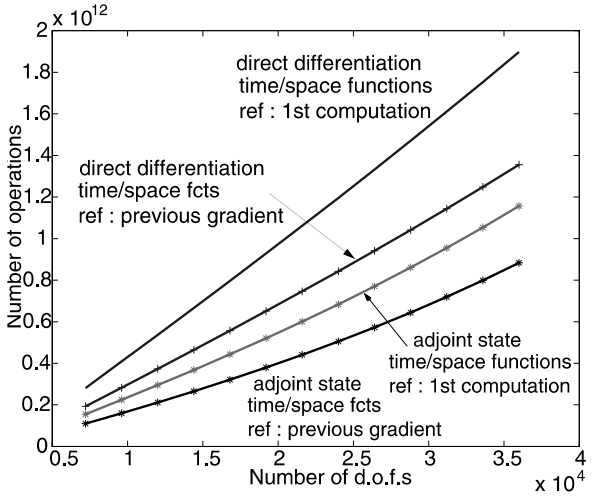

Fig. 21. Computational cost comparison between the direct differentiation and adjoint state approaches, with time/space functions, for a 3D beam.

The Kalman filters present interesting characteristics in the presence of measurement noise. Besides, it is important to mention that they lend themselves easily to the handling of incomplete measurements: for example, it is not necessary to carry out a projection of fields. The method also gives a measure of the uncertainty on the identified parameters through the covariance matrix. In all results, the final covariance was very small, which proves the quality of the identification. But the method seems more expensive than the deterministic approach (Figs. 22 and 23). In particular, it requires that the gradients with respect to the parameters be evaluated at every iteration. We can again emphasize the interest of the time/space functions, which greatly decrease the computational costs.

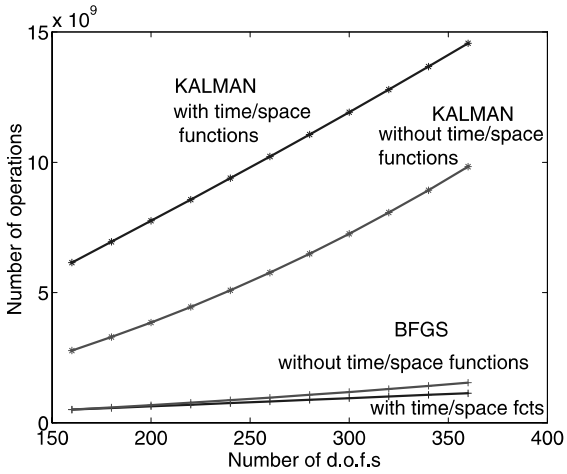

Fig. 22. Computational cost comparison between the stochastic approach and the deterministic approach in 1D (extended Kalman filters) with and without time/space functions.

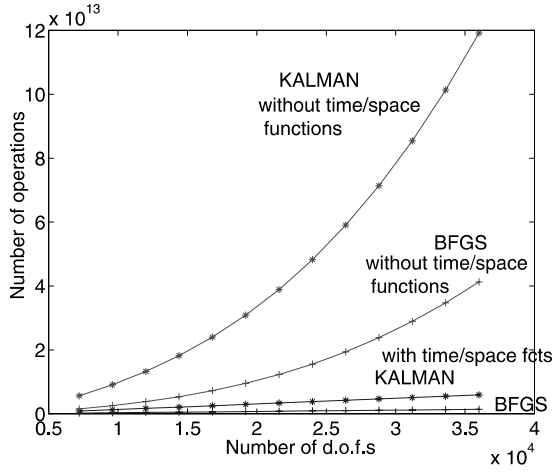

Fig. 23. Computational cost comparison between the stochastic approach and the deterministic approach in 3D (extended Kalman filters) with and without time/space functions.

7. Conclusion

Two approaches, one deterministic and one stochastic, were implemented. They led to the development of sensitivity techniques (direct differentiation, adjoint state, interpolation) within the framework of the LATIN method. These are particularly important for the two types of approach studied.

All the methods gave stable results with respect to measurement noise. They also demonstrated the interest of the particular representation of solutions in order to benefit from the multi-solution feature of the LATIN method. All the results clearly show a significant gain in terms of computational cost. This gain would be particularly important as the number of d.o.f.s and the number of computations to be performed

get large. This gain concerns not only the number of iterations necessary to reach convergence (quality of the initialization), but also the cost of each iteration. Thus, the method is completely adapted to structural identification and will be more efficient than the classical incremental methods.

Concerning the sensitivity methods implemented, the adjoint state method seems to present advantages in terms of cost, especially when many parameters need to be identified.

The interpolation method seems easy to implement and allows one to benefit from the advantages of the LATIN method. But because the parameters are initially unknown it can require numerous computations. Nevertheless, the method described here will be always less expensive than a classical incremental method.

Finally, the method used for the identification in this paper shows promising results, especially for a large number of degrees of freedom.

At present, we are studying more realistic problems where the identification involves necessarily structural computation i.e. the case of delamination.

Appendix A. Calculation of the time and space functions

A.1. Calculation of the time functions

The calculation of the time functions is detailed. The problem is written as a system of first-order differential equations. This system corresponding to Eq. (16) is given below:

$$\begin{aligned} &N_{ki}^{1}\cdot\dot{\beta}_{i}^{\mathrm{C}}(t)+M_{ki}^{11}\cdot\beta_{i}^{\mathrm{C}}(\tau)+M_{ki}^{12}\cdot\gamma_{i}^{\mathrm{C}}(\tau)=B_{k}^{1}(\tau),\\ &N_{ki}^{2}\cdot\dot{\gamma}_{i}^{\mathrm{C}}(t)+M_{ki}^{21}\cdot\gamma_{i}^{\mathrm{C}}(\tau)+M_{ki}^{22}\cdot\beta_{i}^{\mathrm{C}}(\tau)=B_{k}^{2}(\tau) \end{aligned} \tag{A.1}$$

with

$$\begin{aligned} N_{ki}^{1}&=\int_{\Omega}K_{\mathrm{e}}^{-1}C_{i}^{\mathrm{ini}}(M):C_{k}^{\mathrm{ini}}(M)\,\mathrm{d}\Omega,\\ N_{ki}^{2}&=\int_{\Omega}\frac{D_{i}^{\mathrm{ini}}(M):D_{k}^{\mathrm{ini}}(M)}{\beta}\,\mathrm{d}\Omega \end{aligned} \tag{A.2}$$

and

$$\begin{aligned} M_{ki}^{11}&=\int_{\Omega}H_{11}C_{i}^{\mathrm{ini}}(M):C_{k}^{\mathrm{ini}}(M)\,\mathrm{d}\Omega,\\ M_{ki}^{12}&=\int_{\Omega}H_{12}D_{i}^{\mathrm{ini}}(M):C_{k}^{\mathrm{ini}}(M)\,\mathrm{d}\Omega,\\ M_{ki}^{21}&=\int_{\Omega}H_{22}D_{i}^{\mathrm{ini}}(M):D_{k}^{\mathrm{ini}}(M)\,\mathrm{d}\Omega,\\ M_{ki}^{22}&=\int_{\Omega}H_{21}C_{i}^{\mathrm{ini}}(M):D_{k}^{\mathrm{ini}}(M)\,\mathrm{d}\Omega \end{aligned} \tag{A.3}$$

and

$$\begin{aligned} B_{k}^{1}&=\int_{\Omega}\{H_{11}\,\Delta\hat{\sigma}+H_{12}\,\Delta\hat{R}-\Delta\dot{\hat{\varepsilon}}^{\mathrm{p}}\}:C_{k}^{\mathrm{ini}}(M)\,\mathrm{d}\Omega,\\ B_{k}^{2}&=\int_{\Omega}\{H_{21}\,\Delta\hat{\sigma}+H_{22}\,\Delta\hat{R}+\Delta\dot{\hat{p}}\}:D_{k}^{\mathrm{ini}}(M)\,\mathrm{d}\Omega \end{aligned} \tag{A.4}$$

and

$$H^- = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}. \tag{A.5}$$

By using the θ-method, we end up with Eq. (17).

A.2. Calculation of the space functions

Here, we give some details so as to obtain Eq. (15). Then, we describe the process which allows us to calculate the space functions.

First, we write Eq. (11) under the form

$$\begin{Bmatrix} \Delta\dot{\varepsilon}_n^{\mathrm{p}} - \Delta\dot{\hat{\varepsilon}}_n^{\mathrm{p}} \\ -(\Delta\dot{p}_n - \Delta\dot{\hat{p}}_n) \end{Bmatrix} = H^- \begin{Bmatrix} \Delta\sigma_n^{\mathrm{D}} - \Delta\hat{\sigma}_n^{\mathrm{D}} \\ \Delta R_n - \Delta\hat{R}_n \end{Bmatrix} \tag{A.6}$$

where

$$\begin{cases} \Delta Q_n = Q_{n+1} - Q_n, \\ \Delta\hat{Q}_n = \hat{Q} - Q_n \end{cases}$$

and Q represents $\dot{\varepsilon}^{\mathrm{p}}$, $\dot{p}$, σ^{D}, and R.

Then, we use the state laws

$$\begin{cases} \Delta\dot{\varepsilon}_n = K_{\mathrm{e}}^{-1}\,\Delta\dot{\sigma}_n + \Delta\dot{\varepsilon}_n^{\mathrm{p}}, \\ \Delta\dot{p}_n - \dfrac{\Delta\dot{R}_n}{\beta} = 0. \end{cases} \tag{A.7}$$

By substituting $\Delta\dot{\varepsilon}_n^{\mathrm{p}}$ and $\Delta\dot{p}_n$ with the expression given by (A.6), we can formulate the following problem:

Find $\Delta\sigma_n \in \mathbf{S_0}$ and ΔR_n such that

$$\begin{aligned} &\int_0^T \int_\Omega \left(K_{\mathrm{e}}^{-1}\,\Delta\dot{\sigma}_n : \sigma^* + \beta^{-1}\,\Delta\dot{R}_n \cdot R^* + H^- \begin{bmatrix} \Delta\sigma_n^{\mathrm{D}} \\ \Delta R_n \end{bmatrix} \cdot \begin{bmatrix} \sigma^* \\ R^* \end{bmatrix} \right) \mathrm{d}\Omega\,\mathrm{d}t \\ &\quad = \int_0^T \int_\Omega \quad - \Delta\dot{\hat{\varepsilon}}_n^{\mathrm{p}} : \sigma^* + \Delta\dot{\hat{p}}_n \cdot R^* + H^- \begin{bmatrix} \Delta\hat{\sigma}_n^{\mathrm{D}} \\ \Delta\hat{R}_n \end{bmatrix} \cdot \begin{bmatrix} \sigma^* \\ R^* \end{bmatrix} \right) \mathrm{d}\Omega\,\mathrm{d}t, \quad \forall\sigma^* \in \mathbf{S_0}, \ \forall R^* \end{aligned} \tag{A.8}$$

where σ^* and R^* are replaced by the same form as the corrections, i.e.: $\sigma^* = \beta_n^*(t)B_n(M) + \beta_n(t)B_n^*(M)$ and $R^* = \gamma_n^*(t)C_n(M) + \gamma_n(t)C_n^*(M)$. By integrating this equation with regard to time, we obtain two equations. We substitute the equation related to R^* in the first to eliminate $C_n(M)$. So, we deduce a dualized problem under the form: $K(M)\,\Delta v(M) = F(M)$. Then, we can calculate $B_n(M)$ and $C_n(M)$ from the values of $\Delta v(M)$.

A.3. Calculation of the time functions $\alpha(t)$

We search $\Delta\varepsilon_n$ under the form $\alpha(t)\cdot A(M)$. From the static problem, we have $\Delta\sigma_n$ and ΔR_n. Then, from the search direction, we can calculate $\Delta\dot{\hat{\varepsilon}}(M,t)$ which is not under the form $\alpha(t)\cdot A(M)$. So, we search $\Delta\varepsilon_n(M,t) = \alpha(t)\cdot A(M)$ such that

$$\int_0^T \int_\Omega \mathrm{Tr}[K_{\mathrm{e}}x : x^*]\,\mathrm{d}\Omega\,\mathrm{d}t = 0 \quad \forall x^* \text{ kinematically admissible to zero} \tag{A.9}$$

with $x = \dot{\alpha}(t)\cdot A(M) - \Delta\dot{\hat{\varepsilon}}(M,t)$. The determination of $\alpha(t)$ and $A(M)$ is identical to those of the functions $\beta(t)$, $\gamma(t)$ and $B(M)$, $C(M)$.

Appendix B. Adjoint state method

We will recall the principles of the adjoint state method. In particular, we define the computation/experiments distance and the Lagrangian which are used and we define the resulting problem. The cost functional is chosen as follows:

$$D(k) = \text{distance computation/experiments} = 100 \times \frac{\frac{1}{2}\int_0^T \int_\Omega K_e^{-1}(\sigma - \sigma^{\exp}) : (\sigma - \sigma^{\exp})\,\mathrm{d}\Omega\,\mathrm{d}t}{\frac{1}{2}\int_0^T \int_\Omega K_e^{-1}\sigma^{\exp} : \sigma^{\exp}\,\mathrm{d}\Omega\,\mathrm{d}t}. \tag{B.1}$$

D depends implicitly on k. The minimization under constraint is equivalent to the determination of the saddle point of the following Lagrangian:

$$L(u, \sigma, \varepsilon^p, -p, R, u^*, \sigma^*, \varepsilon^{p^*}, -p^*, R^*, k) = \int_0^T L_t(u, \sigma, \varepsilon^p, -p, R, u^*, \sigma^*, \varepsilon^{p^*}, -p^*, R^*, k)\,\mathrm{d}t. \tag{B.2}$$

The quantities $(*)$ are the adjoint quantities. They play the same role as Lagrange multipliers:

$$\begin{aligned}
&L(u, \sigma, \varepsilon^p, -p, R, u^*, \sigma^*, \varepsilon^{p^*}, -p^*, R^*, k)\\
&\quad = \int_0^T \left\{ 100 \times \frac{\frac{1}{2}\int_0^T \int_\Omega K_e^{-1}(\sigma - \sigma^{\exp}) : (\sigma - \sigma^{\exp})\,\mathrm{d}\Omega\,\mathrm{d}t}{\frac{1}{2}\int_0^T \int_\Omega K_e^{-1}\sigma^{\exp} : \sigma^{\exp}\,\mathrm{d}\Omega\,\mathrm{d}t} + \int_\Omega \operatorname{div}\dot{\sigma}\underline{U^*}\,\mathrm{d}\Omega - \int_{\partial_2\Omega} \dot{\sigma} n \cdot \underline{U^*}\,\mathrm{d}\Gamma \right.\\
&\qquad + \int_{\partial_1\Omega} (\underline{\dot{U}} - \underline{\dot{U}}_d) \cdot \sigma^* n\,\mathrm{d}\Gamma + \int_{\partial\Omega_{\text{clamped}}} \underline{\dot{U}} \cdot \sigma^* n\,\mathrm{d}\Gamma - \int_\Omega (\varepsilon(\underline{\dot{U}}) - K_e^{-1}\dot{\sigma} - \dot{\varepsilon}^p) : \sigma^*\,\mathrm{d}\Omega\\
&\qquad - \int_\Omega \left(\dot{\varepsilon}^p - \frac{\partial\varphi^*}{\partial A_{\varepsilon^p}}\right) \cdot A^*_{\varepsilon^p}\,\mathrm{d}\Omega - \int_\Omega \left(-\dot{p} - \frac{\partial\varphi^*}{\partial R}\right) \cdot R^*\,\mathrm{d}\Omega - \int_\Omega (\dot{A}_{\varepsilon^p} - \dot{\sigma}) : \varepsilon^{p^*}\,\mathrm{d}\Omega\\
&\qquad \left. - \int_\Omega \left(\dot{R} + \frac{\dot{\partial}\psi}{\partial(-p)}\right) \cdot (-p)^*\,\mathrm{d}\Omega \right\} \mathrm{d}t.
\end{aligned} \tag{B.3}$$

The existence and the unicity of the saddle point of L are not guaranteed. Nevertheless, if one exists, the necessary conditions of Lagrangian stationarity allow one to characterize it.

By integrating these equations by part in space and in time, the problem can be written in the form:

- equilibrium equations in Ω:

$$\begin{aligned}
&\operatorname{div}\sigma^* = 0,\\
&\varepsilon(\underline{\dot{U}}^*) = K_e^{-1}\dot{\sigma} + \dot{\varepsilon}^p - \operatorname{norm} e K_e^{-1}(\sigma - \sigma^{\exp}) \quad \text{with } \operatorname{norm} e = \frac{100}{\int_0^T \int_\Omega K_e^{-1}\sigma^{\exp} : \sigma^{\exp}\,\mathrm{d}\Omega\,\mathrm{d}t},\\
&\dot{A}^*_{\varepsilon^p} = \dot{\sigma},\\
&\dot{\varepsilon}^{p^*} = -\frac{\partial^2\varphi^*}{\partial A^2_{\varepsilon^p}} : A^*_{\varepsilon^p} - \frac{\partial^2\varphi^*}{\partial R\,\partial A_{\varepsilon^p}} R^*,\\
&-\dot{p}^* = -\frac{\partial^2\varphi^*}{\partial R^2} R^* - \frac{\partial^2\varphi^*}{\partial A_{\varepsilon^p}\,\partial R} A^*_{\varepsilon^p},\\
&\dot{R}^* = -\frac{\partial^2\psi}{\partial(-p)^2}(-\dot{p})^*;
\end{aligned} \tag{B.4}$$

- boundary conditions:

$$\begin{aligned} &\dot{\sigma}^* n = 0 \quad \text{on } \partial_2\Omega, \\ &\underline{\dot{U}^*} = 0 \quad \text{on } \partial\Omega_{\text{clamped}} \text{ and } \partial_1\Omega; \end{aligned} \tag{B.5}$$

- final conditions at T:

$$\begin{cases} \operatorname{div}\sigma^*(T) = 0 \\ \varepsilon(\underline{U^*})(T) = K_{\text{e}}^{-1}\sigma^*(T) + \varepsilon^{\text{p}^*}(T) \\ A^*_{\varepsilon^{\text{p}}}(T) = \sigma^*(T) \\ -p^*(T) = 0 \\ \varepsilon^{\text{p}^*}(T) = 0 \\ R^*(T) = 0 \end{cases} \quad \text{in } \Omega$$

$$\begin{aligned} &\sigma^* n(T) = 0 \quad \text{on } \partial_2\Omega, \\ &\underline{U^*}(T) = 0 \quad \text{on } \partial\Omega_{\text{clamped}} \text{ and } \partial_1\Omega. \end{aligned} \tag{B.6}$$

For the viscoplastic model (1D):

$$\psi(\sigma, p) = \frac{\beta}{2} \cdot p^2 + \frac{\sigma^2}{2E}. \tag{B.7}$$

Appendix C. The Kalman filter

We call x_t the state vector which contains all the unknown parameters and y_t the observation vector containing the measured quantities (displacements, strains, ...). The subscript t designates a time step. Thus, these vectors can be time-dependent variables. For example, y_t contains measurements at the time steps of the loading process on the time interval $[t_0, t_{\text{end}}]$ and at some positions in the structure. Thus, the Kalman filter is applied to a nonlinear system which is discretized in time and in space. The time interval $[t_0, t_{\text{end}}]$ is partitioned according to $\bigcup_{t=1}^{t=N} [t-1, t]$. During an interval $[t-1, t]$, the estimation of the model parameters at $t-1$, designated by $\hat{x}_{t-1}$, and the measurements z_t at time t are given. The extended Kalman filter updates the estimate $\hat{x}_{t-1}$ using the information gained from measurements z_t to give a new estimate $\hat{x}_t$ on the interval $[t-1, t]$.

We have the state vector as a steady feature of the system:

$$x_{t-1} = x_t \quad \forall t. \tag{C.1}$$

The parameters are independent of time. We also assume that the only uncertainties come from measurements: modeling errors are not considered.

The observation equation is given by

$$y_t = h(x_t) + v_t \quad \text{with } t = 1, \ldots, N. \tag{C.2}$$

The vector h contains all the nonlinearities of the system. The vector v_t describes the measurement noise. It is a white noise, with zero expected value, characterized by its covariance matrix:

$$E(v_t) = 0 \tag{C.3}$$

and

$$E(v_k v_t^{\text{T}}) = \delta_{kj} R_{\text{C}} k \tag{C.4}$$

where E is an average operator, δ_{kj} the Kronecker's delta, and R_{C} the covariance matrix.

The Kalman filter can be applied to the linearization of the response function. In this case, $h(x_t)$ is expanded into a Taylor series up to the first order around the estimate $\hat{x}_{t-1}$:

$$y_t = h(\hat{x}_{t-1}) + \underbrace{\frac{\partial h(\hat{x}_{t-1})}{\partial x^{\mathrm{T}}}}_{H(\hat{x}_{t-1})}(\hat{x}_t - \hat{x}_{t-1}) + v_t \tag{C.5}$$

where T stands for transposed.

H is called the sensitivity matrix. It requires that the response of the system be a continuous, continuously differentiable function.

At time t, the extended Kalman filter improves the previous estimate $\hat{x}_{t-1}$ by taking into account the measurements y_t at time t. The estimation equation becomes

$$\hat{x}_t = \hat{x}_{t-1} + K_t \underbrace{[y_t - h(\hat{x}_{t-1})]}_{\otimes}. \tag{C.6}$$

K_t is the Kalman gain matrix and $\otimes$ is the residual, i.e. the difference between the current response of the system at time t and its estimate based on the filtered vector $\hat{x}_{t-1}$ at time $t-1$.

K_t is defined by

$$K_t = P_{t-1}H_t^{\mathrm{T}}\left(H_t P_{t-1} H_t^{\mathrm{T}} + R_{\mathrm{C}}\right)^{-1}, \tag{C.7}$$

$\hat{x}_t$ is associated with the covariance matrix of the error estimate:

$$P_t = P_{t-1} - P_{t-1}H_t^{\mathrm{T}}\left(H_t P_{t-1} H_t^{\mathrm{T}} + R_{\mathrm{C}}\right)^{-1} H_t P_{t-1}. \tag{C.8}$$

These expressions guarantee an estimator with minimum variance.

To get an estimate at each time t, an iterative procedure is necessary:

1. initialization $\hat{x}_t^0 = \hat{x}_{t-1}$;
2. calculation of $\hat{x}_t^i = \hat{x}_{t-1} + K_t(\hat{x}_t^{i-1})[y_t - h_t(\hat{x}_t^{i-1}) - H(\hat{x}_t^{i-1})(\hat{x}_{t-1} - \hat{x}_t^{i-1})]$;
3. calculation of $K_t(\hat{x}_t^{i-1}) = P_{t-1}H(\hat{x}_t^{i-1})^{\mathrm{T}}[H(\hat{x}_t^{i-1})P_{t-1}H(\hat{x}_t^{i-1})^{\mathrm{T}} + R_{\mathrm{C}}]^{-1}$; (C.9)
4. evaluation of P_t.

The procedure is repeated until $((\hat{x}_t^i - \hat{x}_t^{i-1})/(\hat{x}_t^i)) \preceq \mathrm{TOL}$. Thus, we obtain $\hat{x}_t$. The final estimate is given by $\hat{x}_N$.

It should be noted that the original Kalman filter was defined for certain linear cases, which is why here we speak about the "extended" Kalman filter for the nonlinear case.

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