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UNIQUE ERGODICITY OF ASYNCHRONOUS ROTATIONS, AND APPLICATION

FRANÇOIS MAUCOURANT

ABSTRACT. The main result of this paper is an analogue for a continuous family of tori of Kronecker-Weyl's unique ergodicity of irrational rotations. We show that the notion corresponding in this setup to irrationality, namely asynchronicity, is satisfied in some homogeneous dynamical systems. This is used to prove the ergodicity of natural lifts of invariant measures.

RÉSUMÉ. Nous étudions sur une famille continue de tores les rotations dites asynchrones, analogues aux rotations irrationnelles sur les tores classiques. Le résultat principal est l'unique ergodicité de ces rotations sur un monoïde adapté. Nous prouvons que la condition d'asynchronicité est vérifiée dans une famille d'exemples issue de la dynamique homogène, ce qui nous permet de déduire l'ergodicité de relevés de certaines transformations dans des fibrés en tores.

1. INTRODUCTION

1.1. Motivations. The original motivation of this study is the inquiry of ergodic properties of torus extension of homogeneous dynamical systems. Such dynamics have drawn some attention recently - see for example [6], [8], [10] for unipotent actions, and [9] for diagonal actions.

As an informal example, consider a diagonal element $a \in \mathbf{SL}(d, \mathbf{R}) = G_0$ ($d \geq 2$) with positive diagonal entries, acting by left multiplication on the homogeneous space $\mathbf{SL}(d, \mathbf{R})/\mathbf{SL}(d, \mathbf{Z}) = G_0/\Gamma_0$, and let μ be a probability invariant by a and ergodic on G_0/Γ_0 . The main interesting cases for our purpose occur when the measure μ is not algebraic. This dynamical system is a factor of the action of (a, α) , where $\alpha \in \mathbf{R}^d$ is arbitrary, by left multiplication on $(\mathbf{SL}(d, \mathbf{R}) \times \mathbf{R}^d)/(\mathbf{SL}(d, \mathbf{Z}) \times \mathbf{Z}^d) = G/\Gamma$. This latter space is a torus bundle above G_0/Γ_0 . Amongst the possible (a, α) -invariant measures that projects onto μ , there is a particular one, denoted by λ , which decomposes into the Haar measure of

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tori on each fiber. It is natural to ask about its ergodicity with respect to (a, α) .

Following the classical Hopf argument (see [4]), one is naturally led to inquire about ergodic properties of the strong stable foliation of (a, α) . It turns out that this foliation contains the orbits of another action, namely the multiplication by (e, β) on G/Γ , where $\beta \in \mathbf{R}^d$ is an eigenvector for a associated to an eigenvalue < 1 . This action is a unipotent action, but since it is "vertical" (in the sense trivial in the factor G_0/Γ_0), Ratner's theory yield in this case no more information than Kronecker-Weyl's uniform distribution on the torus.

To visualize this action of (e, β) on each fiber, one may think of it as a rotation by a fixed vector β on a varying torus depending on the base-point. Here, we will prefer to think of it as the rotation by a varying vector $f_\beta(x)$ depending on the base-point x , on a fixed torus \mathbf{T}^d . One may hope in this situation that $f_\beta(x) \in \mathbf{T}^d$ is irrational for almost every x . It turns out that under appropriate assumptions, the rotations defined by f_β satisfy a stronger property, namely *asynchronicity*.

As we will see shortly, such rotations on torus bundle above a measured space like $(G_0/\Gamma_0, \mu)$ enjoy strong ergodic properties, enabling us in this setting to prove a unique ergodicity result. In some sense, this can be considered as a weak analogue of Furstenberg's unique ergodicity of horocyclic flow. Finally, we will return to the question of the ergodicity of λ with respect to (a, α) , and related mixing properties.

1.2. Asynchronous rotations. Let $(I, \mathcal{B}(I), \mathcal{L})$ be a standard probability space without atoms, and let K be a torus $\mathbf{T}^d = (\mathbf{R}/\mathbf{Z})^d$, for some integer $d \geq 1$. We think of a measurable map $f : I \rightarrow K$ as the data, for each $x \in I$, of a rotation adding the angle $f(x)$ in a torus indexed by x . Despite what the above motivational example might suggest, in this abstract setting, the case $d = 1$ of rotations on a family of circles above a probability space is already interesting, and contains most of the difficulties.

The set of such measurable maps $\{f : I \rightarrow K\}$, where we identify two maps if they coincide \mathcal{L} -almost everywhere, is naturally an abelian group under pointwise addition of functions. We denote by $(K^I, +)$ this group, by a slight abuse of notation. We would like to study the translation by f in K^I , but as it lacks a nice topology, we consider

a compactification of the group $(K^I, +)$, which will be a monoid, as follows.

Let $\mathcal{M}_{\mathcal{L}}$ be the space of probability measures on $I \times K$ which project to \mathcal{L} on the first factor. To an element $f \in K^I$, we can associate the probability measure \mathcal{D}_f on $I \times K$, supported by its graph, which is the pushforward of \mathcal{L} by the map $x \in I \mapsto (x, f(x)) \in I \times K$. This defines an embedding of K^I into $\mathcal{M}_{\mathcal{L}}$. It is not hard to see that the group law $+$ on K^I correspond to a fiber-wise convolution product $*$ on $\mathcal{M}_{\mathcal{L}}$, which turns $(\mathcal{M}_{\mathcal{L}}, *)$ into an abelian monoid, with neutral \mathcal{D}_0 , where $0 : I \rightarrow K$ is the zero map.

The space $\mathcal{M}_{\mathcal{L}}$ is equipped naturally with a weak-* topology, for which it is a compact metric space. A tricky fact is that the convolution product $(\mu, \nu) \mapsto \mu * \nu$ is *not* continuous of the two variables, but is of each variable separately. A more detailed description of these objects, and explanations of the implied claims, are given in Section 2.

There is a particular element in $\mathcal{M}_{\mathcal{L}}$, the measure $\lambda = \mathcal{L} \otimes Haar_K$. It satisfies the relation: $\forall \mu \in \mathcal{M}_{\mathcal{L}}, \mu * \lambda = \lambda$.

We are interested in studying the dynamics of translation \mathcal{D}_f* on the monoid $\mathcal{M}_{\mathcal{L}}$. Unsurprisingly, we now need a kind of irrationality condition.

Definition. *The angle map $f : I \rightarrow K$ is said to be asynchronous if the image measure $f_*\mathcal{L}$ gives zero mass to any translate of any proper closed subgroup of K . Equivalently, for any non-trivial character $\chi \in \hat{K}$, $(\chi \circ f)_*\mathcal{L}$ has no atoms.*

Intuitively, for $d = 1$, this means that one looks at an action by rotation on a family of circles indexed by $x \in I$, by angles $f(x)$, which are different from one another if picked randomly following the probability \mathcal{L} . For $d \geq 1$, it means that for almost every couple (x, y) , $f(x)$ and $f(y)$ do not belong to the same coset modulo any closed, strict subgroup of K .

Theorem 1. *The following are equivalent.*

- (1) *The angle map $f : I \rightarrow K$ is asynchronous.*
- (2) *The closure $\overline{\{\mathcal{D}_{nf}\}_{n \in \mathbf{Z}}}$ contains λ .*
- (3) *The convolution action of \mathcal{D}_f on $\mathcal{M}_{\mathcal{L}}$ is uniquely ergodic.*

If these are true, the only invariant probability measure is the Dirac measure δ_λ .

The fact that the invariant measure is a Dirac measure implies (see Proposition 3.7) that there exists a subset of the integers $E \subset \mathbf{Z}$, of natural density 1, such that for any $\mu \in \mathcal{M}_{\mathcal{L}}$,

$$\lim_{n \rightarrow \pm\infty, n \in E} \mathcal{D}_{nf} * \mu = \lambda.$$

The question whether λ is an attracting point of the dynamic, that is if

$$\lim_{n \rightarrow \pm\infty} \mathcal{D}_{nf} * \mu = \lambda,$$

or if this fails along some subsequence of zero density, is more delicate, and its answer depends on f .

If $I = [0, 1]$ equipped with Lebesgue measure, $d = 1$, and f is a C^2 map with non-vanishing derivative, then λ is an attracting point (Proposition 3.1), and there is no exceptional subsequence.

However, for an angle map f which is only measurable, the convolution action of \mathcal{D}_f might behave more like an intermittent map with the neutral fixed point λ . An example of this phenomenon is the following. Again, let $I = [0, 1]$ endowed with the Lebesgue measure \mathcal{L} . Let ν be the (probability) Hausdorff measure of dimension $\log 2 / \log 3$ on the usual Cantor set C , viewed as a subset of $K = \mathbf{T}^1$ by identifying 0 and 1. Define $f : [0, 1] \rightarrow \mathbf{T}^1$, by

$$f(x) = \inf\{t \geq 0 : \nu(0, t) \geq x\} \bmod 1.$$

Then $f_*\mathcal{L} = \nu$, and $f(I) \subset C$. Alternatively, f can be defined as the reciprocal, outside of dyadic rationals, of the usual devil's staircase, modulo 1. Since ν does not have any atom, f is an asynchronous angle map. We claim that the sequence $(\mathcal{D}_{3^k f})_{k \geq 1}$ does not intersect a fixed neighborhood of λ . Indeed, since C is invariant by the multiplication $\times 3$ on the circle, the graph of $3^k f$ is contained in $I \times C$, so the measure $\mathcal{D}_{3^k f}$ is supported on $I \times C$, a proper compact subset of $I \times K$. This forbids $\mathcal{D}_{3^k f}$ to be close to λ . Still, by Theorem 1, subsequences like $(\mathcal{D}_{3^k f})_{k \geq 0}$ are scarce, as the points $(\mathcal{D}_{nf})_{n \in \mathbf{Z}}$ tend to λ for a subset of \mathbf{Z} of density one.

1.3. Main example, and a related ergodicity result. As hinted in the motivational paragraph, asynchronous rotations occur naturally

in the context of homogeneous dynamics on torus bundle.

More precisely, let \mathbf{G}_0 be a connected, semisimple algebraic linear group defined over \mathbf{Q} , $G_0 = \mathbf{G}_0(\mathbf{R})$ the group of its \mathbf{R} -points and $\Gamma_0 = \mathbf{G}_0(\mathbf{Z})$ its integer points. By the Borel - Harish-Chandra Theorem, Γ_0 is a lattice in G_0 . We will consider invariant measures on G_0/Γ_0 under some elements $a \in G_0$, under the following assumptions.

Definition. *An element $a \in G_0 - \{e\}$ is said triangularizable with positive eigenvalues if for every finite dimensional representation α of \mathbf{G}_0 defined over \mathbf{Q} , $\alpha(a)$ has only real, positive eigenvalues.*

It is the case, for example, when G_0 is the real split form of \mathbf{G}_0 , meaning the real rank equals the complex rank, and if a is the exponential of a non-zero element of a Cartan subalgebra. It also happens when a is unipotent, but this case is less interesting for our purpose, since by Ratner's Theory, a -invariant ergodic measures are algebraic. This hypothesis implicitly rules out the case where G_0 is the real compact form of \mathbf{G}_0 , as it cannot contain such element a .

Definition. *Let μ be a probability measure on G_0/Γ_0 , invariant and ergodic under the action of a . Such a measure is said to be non-concentrated if for every $H \subset G_0$ closed algebraic, strict subgroup containing a , and every $x \in G_0/\Gamma_0$ such that $Hx\Gamma_0$ is closed, then $\mu(Hx\Gamma_0) < 1$.*

We now consider a fiber bundle over the probability space $(G_0/\Gamma_0, \mu)$, whose fibers are tori.

Let $\rho : \mathbf{G}_0 \rightarrow \mathbf{GL}(V)$ a representation defined over \mathbf{Q} on a finite-dimensional space $V = \mathbf{R}^d$ endowed with the \mathbf{Z} -structure \mathbf{Z}^d . We will always assume that $d \geq 2$, and that ρ is irreducible over \mathbf{Q} . The semidirect product $G = G_0 \ltimes_{\rho} V$ is endowed with the group law

$$\forall (g, v) \in G, \forall (h, w) \in G, (g, v)(h, w) = (gh, v + \rho(g)w).$$

Up to replacing Γ_0 with a subgroup of finite index in a way such that $\rho(\Gamma) \subset \mathbf{GL}(d, \mathbf{Z})$, the set $\Gamma = \Gamma_0 \ltimes_{\rho} \mathbf{Z}^d$ is a subgroup of G , and the map

$$\pi : G/\Gamma \rightarrow G_0/\Gamma_0,$$

is a torus bundle. Indeed, the lattice of $\{e\} \times \mathbf{R}^d$ (for the action of multiplication on the left) stabilizing a point $(x, v)\Gamma$ is precisely $\{e\} \times \rho(x)\mathbf{Z}^d$, thus the fiber of π over $x\Gamma_0$ is the torus $\mathbf{R}^d/\rho(x)\mathbf{Z}^d$. It will be convenient to have measurable coordinates where this fiber bundle is a direct

product.

Let $I \subset G_0$ be a measurable fundamental domain for the action of Γ_0 , and put \mathcal{L} the restriction to I of the Γ_0 -invariant lift of μ . As previously, we denote by K the d -dimensional torus $\mathbf{T}^d = \mathbf{R}^d/\mathbf{Z}^d$. The map

$$\begin{aligned} I \times K &\rightarrow G/\Gamma, \\ (x, \bar{v})_{I \times K} &\mapsto (x, \rho(x)v)\Gamma, \end{aligned}$$

is a measurable bijection, that we will use as an identification between $I \times K$ and G/Γ , the subscript $I \times K$ indicating the coordinates we are using. Likewise, we will identify G_0/Γ_0 with I and μ with \mathcal{L} .

For $\beta \in V$, the action of (e, β) by multiplication on G/Γ on the left, can be read in the $I \times K$ coordinates as the map

$$(x, \bar{v})_{I \times K} \mapsto (x, \bar{v} + \rho(x)^{-1}\beta)_{I \times K},$$

i.e. it is a rotation by an angle map $f_\beta : I \rightarrow K$, with

$$f_\beta(x) = \rho(x)^{-1}\beta \bmod \mathbf{Z}^d.$$

We prove:

Theorem 2. *Assume that $a \in G_0$ is triangularizable with positive eigenvalues, that μ is an a -invariant, non-concentrated, ergodic probability on G_0/Γ_0 , that ρ is irreducible over \mathbf{Q} , and $\dim(V) > 1$. Assume also that $\beta \in V - \{0\}$ is an eigenvector for $\rho(a)$. Then the angle map*

$$\begin{aligned} f_\beta : I &\rightarrow K \\ x &\mapsto \rho(x)^{-1}\beta \bmod \mathbf{Z}^d, \end{aligned}$$

is asynchronous.

Using the identification of G/Γ with $I \times K$, we still denote by λ the measure on G/Γ such that $\pi_*\lambda = \mu$, whose disintegration along each fiber of μ are the Haar measures on each tori.

Direct application of Theorem 1 gives:

Corollary 1.1. *We assume the same hypotheses as Theorem 2. Let \mathcal{M}_μ be the set of probabilities on G/Γ projecting onto μ . The action on \mathcal{M}_μ induced by the left multiplication by (e, β) on G/Γ is uniquely ergodic, with invariant measure δ_λ .*

Now choose any $\alpha \in V$. The action of left multiplication by (a, α) on G/Γ admits the action of a on G_0/Γ_0 as a factor. A natural question is if the measure λ , which is invariant, is ergodic with respect to this

action.

If β is any eigenvector of $\rho(a)$, multiplication by (e, β) is in some sense moving in some part of the stable, unstable or neutral direction (depending on the eigenvalue) of the action of (a, α) . Theorem 2 and Hopf's argument allows us to prove the following ergodicity result:

Theorem 3. *Assume that $a \in G_0$ is triangularizable with positive eigenvalues, that μ is an a -invariant, non-concentrated, ergodic probability on G_0/Γ_0 , and that ρ is irreducible over \mathbf{Q} , of dimension > 1 . Choose $\alpha \in V$, then the action by left multiplication by (a, α) on G/Γ is ergodic with respect to the invariant measure λ .*

If we assume moreover that $\rho(a)$ is not unipotent, then the action of (a, α) on $(G/\Gamma, \lambda)$ is weakly mixing if and only if the action of a on $(G_0/\Gamma_0, \mu)$ is weakly mixing, and the same property holds for strong mixing.

1.4. Plan of the paper. In Section 2, we collect some facts about the topology of $\mathcal{M}_{\mathcal{L}}$. It turns out that its restriction to the set graph measures \mathcal{G} can be given by a natural distance $d_{\mathcal{G}}$, which cannot be extended to $\mathcal{M}_{\mathcal{L}}$.

In Section 3, after dealing with the toy example where f is a C^2 map, we prove Theorem 1. The central tool is a criterion for maps which are the "opposite" of asynchronous (Proposition 3.2): a map $f : K \rightarrow I$ has essentially countable image if and only if the closure of the subgroup generated by \mathcal{D}_f is contained in \mathcal{G} .

In Section 4, we prove that in the algebraic setting, the smallest subgroup of G_0 containing the elements $\gamma \in \Gamma_0$ which allows (by multiplication on the right) to return to a arbitrary set of positive measure in the fundamental domain after applying some power of a on the left, is G_0 itself. This result (Theorem 4), which is the main ingredient of the proof of Theorem 2, relies crucially on the non-concentration of μ .

In Section 5, we prove the asynchronicity of the rotation obtained by the construction in homogeneous dynamics (Theorem 2).

In Section 6, we prove the ergodicity of the extension of the action of a on G_0/Γ_0 (Theorem 3). This mainly relies on the following easy observation: if an angle map $f : I \rightarrow K$ is asynchronous, then for almost every x , the action of translation by $f(x)$ on the torus K is (uniquely) ergodic.

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2. THE SPACE $\mathcal{M}_{\mathcal{L}}$

2.1. Topology of $\mathcal{M}_{\mathcal{L}}$. We recall that $(I, \mathcal{B}(I))$ be a standard measurable space means that I can be endowed with a complete, separable distance d_I , such that $\mathcal{B}(I)$ is the σ -algebra of its Borel sets. The facts about standard probability spaces we will use are summarized in [1, Chapter 1.1]. Choosing such a distance on I defines a topology on the space of probability measures on $I \times K$, and hence on $\mathcal{M}_{\mathcal{L}}$.

Although the weak-* topology of the space of measures on $I \times K$ depends in a strong way on the choice of topology on I , it turns out that:

Lemma 2.1. *The topology induced on $\mathcal{M}_{\mathcal{L}}$ does not depends on the choice of topology on I .*

Proof. Let I_1, I_2 two complete, separable metric space endowed with probabilities \mathcal{L}_i , with a map $\varphi : I_1 \rightarrow I_2$ an isomorphism such that $\varphi_*\mathcal{L}_1 = \mathcal{L}_2$. The topologies induced on measures on $I_i \times K$ are generated by the open sets:

$$\mathcal{U}^i(F, \delta, \mu) = \left\{ \nu : \left| \int_{I_i \times K} F d\mu - \int_{I_i \times K} F d\nu \right| < \delta \right\},$$

where $F : I_i \times K \rightarrow \mathbf{C}$ is continuous with compact support for the relevant topology on $I_i \times K$.

Denote by $\tilde{\varphi}$ the map $I_1 \times K \rightarrow I_2 \times K$, $\tilde{\varphi}(x, y) = (\varphi(x), y)$. To show that $\tilde{\varphi}_* : \mathcal{M}_{\mathcal{L}_1} \rightarrow \mathcal{M}_{\mathcal{L}_2}$ is a homeomorphism, it is sufficient by symmetry to show its continuity.

We fix a neighborhood $\mathcal{U}^2(F, \epsilon, \tilde{\varphi}_*\mu)$, and wish to show that its preimage contains some neighborhood of the initial point $\mathcal{U}^1(G, \delta, \tilde{\varphi}_*\mu)$, for some G, δ .

The map φ from I_1 to I_2 is measurable. By Lusin's Theorem, for every $\delta > 0$, there is a compact set $J \subset I_1$, such that $\mathcal{L}_1(J) > 1 - \delta$, on which φ is continuous. Let $F : I_2 \times K \rightarrow \mathbf{C}$ be continuous, by Tietze-Urysohn's Theorem, there exists a continuous function $G_\delta : I_1 \times K \rightarrow \mathbf{C}$ which extends the continuous map $F \circ \tilde{\varphi} : J \times K \rightarrow \mathbf{C}$. Moreover, since F is bounded, G_δ can be chosen such that $\|G_\delta\|_\infty = \|F\|_\infty$. If $\nu \in \mathcal{M}_{\mathcal{L}_1}$,

$$\left| \int_{I_2 \times K} F d\tilde{\varphi}_*\mu - \int_{I_2 \times K} F d\tilde{\varphi}_*\nu \right| \leq 2\delta\|F\|_\infty + \left| \int_{J \times K} G_\delta d\mu - \int_{J \times K} G_\delta d\nu \right|,$$

because $\mu((I_2 - \varphi J) \times K) = \mathcal{L}_2(I_2 - \varphi J) = \mathcal{L}_1(I_1 - J) < \delta$ and the same holds for ν . If we choose $\delta > 0$ such that $2\delta\|F\|_\infty < \epsilon/2$, we have:

$$\left| \int_{I_2 \times K} F d\tilde{\varphi}_* \mu - \int_{I_2 \times K} F d\tilde{\varphi}_* \nu \right| \leq \epsilon/2 + \left| \int_{I_1 \times K} G_\delta d\mu - \int_{I_1 \times K} G_\delta d\nu \right|,$$

and therefore, provided that $\delta < \epsilon/2$,

$$\mathcal{M}_{\mathcal{L}_1} \cap \mathcal{U}^1(G_\delta, \delta, \mu) \subset \tilde{\varphi}_*^{-1}(\mathcal{U}^2(F, \epsilon, \tilde{\varphi}_* \mu) \cap \mathcal{M}_{\mathcal{L}_2}).$$

As any neighborhood of $\varphi_* \mu$ contains finite intersections of sets of the form $\mathcal{U}^2(F, \epsilon, \varphi_* \mu)$, this implies that $\tilde{\varphi}_* : \mathcal{M}_{\mathcal{L}_1} \rightarrow \mathcal{M}_{\mathcal{L}_2}$ is continuous, as required. \square

A corollary of this discussion is that we can assume for example that $I = [0, 1]$ and \mathcal{L} is the Lebesgue measure on this interval, endowed with its usual topology. Since in this case, the set of probability measures on $I \times K$ is a compact, separable metric space, it follows that $\mathcal{M}_{\mathcal{L}}$ is also compact, separable and metric.

2.2. Graphs and measures. For a measurable map $g : I \rightarrow K$, we define the *graph measure* \mathcal{D}_g , of g as the direct image of \mathcal{L} by the map $x \in I \mapsto (x, g(x)) \in I \times K$. Two measurable maps $I \rightarrow K$ define the same graph measure if and only if they are equal \mathcal{L} -almost everywhere.

Let \mathcal{G} be the set of graph measures, this is a subset of $\mathcal{M}_{\mathcal{L}}$, equipped with the induced topology. For f, g measurable maps from I to K , define the distance $d_{\mathcal{G}}(\mathcal{D}_f, \mathcal{D}_g)$ of their graphs as the infimum of numbers $\epsilon > 0$ such that there exists a compact subset $E \subset I$, with $\mathcal{L}(E) > 1 - \epsilon$, such that for any $x \in E$, $d(f(x), g(x)) < \epsilon$. Clearly, this defines a distance on \mathcal{G} . We will denote by $B_{\mathcal{G}}(\mathcal{D}_f, \epsilon) \subset \mathcal{G}$ the ball of radius ϵ around \mathcal{D}_f for this distance.

Lemma 2.2. *The topology induced on \mathcal{G} by the topology of $\mathcal{M}_{\mathcal{L}}$ is equal to the distance topology given by $d_{\mathcal{G}}$.*

Proof. We first prove that the distance topology is finer than the weak-* topology, restricted to \mathcal{G} . Let \mathcal{D}_f be a graph measure, a basis of neighborhoods for the topology induced by $\mathcal{M}_{\mathcal{L}}$ is given by finite intersection of the form

$$\mathcal{N} = \mathcal{G} \cap_j \mathcal{U}(F_j, \delta, \mathcal{D}_f),$$

where like in the proof of Lemma 2.1, the neighborhoods \mathcal{U} are defined by

$$\mathcal{U}(F, \delta, \mu) = \left\{ \nu : \left| \int_{I \times K} F d\mu - \int_{I \times K} F d\nu \right| < \delta \right\}.$$

Given such a finite family of continuous function with compact support F_j and $\delta > 0$, since the functions F_j are uniformly continuous, there exists $\epsilon > 0$ such that $|F_j(x, y) - F_j(x, y')| \leq \frac{\delta}{3}$ for any $(y, y') \in K^2$ such that $d(y, y') \leq \epsilon$, and any j . We also assume that

$$\epsilon \leq \min_j \frac{\delta}{3\|F_j\|_\infty}.$$

Let \mathcal{D}_g be in the ball of radius ϵ around \mathcal{D}_f , we want to check that it lies in \mathcal{N} . By definition of the distance, there exists a set $E \subset I$ of measure $\mathcal{L}(E) > 1 - \epsilon$, such that for any $x \in E$, $d(f(x), g(x)) < \epsilon$. Therefore,

$$\begin{aligned} & \left| \int_{I \times K} F_j d\mathcal{D}_f - \int_{I \times K} F_j d\mathcal{D}_g \right| \\ & \leq 2\epsilon\|F_j\|_\infty + \left| \int_E F_j(x, f(x)) d\mathcal{L}(x) - \int_E F_j(x, g(x)) d\mathcal{L}(x) \right|, \end{aligned}$$

so

$$\left| \int_{I \times K} F_j d\mathcal{D}_f - \int_{I \times K} F_j d\mathcal{D}_g \right| \leq \frac{2\delta}{3}.$$

This implies that $\mathcal{D}_g \in \mathcal{N}$, so $B_{\mathcal{G}}(\mathcal{D}_f, \epsilon) \subset \mathcal{N}$ for sufficiently small ϵ , as required.

Conversely, consider a ball $B_{\mathcal{G}}(\mathcal{D}_f, \epsilon)$ around some \mathcal{D}_f of radius ϵ . By Lusin's Theorem, there exists a compact set $E \subset I$, of measure $> 1 - \epsilon/4$, such that f is continuous on E . Define $F : E \times K \rightarrow \mathbf{C}$ by

$$F(x, y) = \inf \left(1, \frac{d(y, f(x))}{\epsilon} \right),$$

this is a continuous function with values in $[0, 1]$. By the Tietze-Urysohn Theorem, F can be extended to a continuous function on $I \times K$, with values in $[0, 1]$. We claim that

$$\mathcal{U}(F, \epsilon/4, \mathcal{D}_f) \cap \mathcal{G} \subset B_{\mathcal{G}}(\mathcal{D}_f, \epsilon).$$

To prove this, let $g \in K^I$ such that $\mathcal{D}_g \in \mathcal{U}(F, \epsilon/4, \mathcal{D}_f)$. Then

$$\int_{E \times K} F d\mathcal{D}_g \leq \int_{I \times K} F d\mathcal{D}_g + \epsilon/4 \leq \int_{I \times K} F d\mathcal{D}_f + \epsilon/4 \leq \epsilon/2.$$

By Markov's Inequality, $F(x, g(x)) = 1$ on a subset of E of measure at most $\epsilon/2$, which implies that $d(g(x), f(x)) < \epsilon$ on a subset of measure at least $1 - \epsilon$. This concludes the proof. \square

2.3. Disintegration along \mathcal{L} . Any $\mu \in \mathcal{M}_{\mathcal{L}}$ can be disintegrated as a family of measures $(\mu^x)_{x \in I}$, such that for any continuous test-function with compact support $F : I \times K \rightarrow \mathbf{C}$,

$$\mu(F) = \int_I \left(\int_K F(x, y) d\mu^x(y) \right) d\mathcal{L}(x).$$

Moreover, the map $x \mapsto \mu^x$ is measurable, and uniquely defined modulo zero sets. See e.g. [7, Th. 5.8].

For $\mu \in \mathcal{M}_{\mathcal{L}}$, $\chi \in \hat{K}$ a character, $x \in I$, recall that the Fourier-Stieltjes coefficient of μ^x is defined by

$$\hat{\mu}^x(\chi) = \int_K \chi(y) d\mu^x(y),$$

and this formula makes sense for \mathcal{L} -almost every x .

2.4. Convolution product, and Fourier-Stieltjes coefficients. For μ_1, μ_2 two measures in $\mathcal{M}_{\mathcal{L}}$, we define the fiberwise convolution product of μ_1, μ_2 by

$$\mu_1 * \mu_2(F) = \int_I \left(\int_{K^2} F(x, y + z) d\mu_1^x(y) d\mu_2^x(z) \right) d\mathcal{L}(x),$$

where $F : I \times K \rightarrow \mathbf{C}$ a continuous test-function with compact support. Equivalently, $(\mu * \nu)^x$ is the usual convolution product of μ^x and ν^x .

The following Lemma, whose proof is left to the reader, summarizes elementary properties of this fiberwise convolution product.

Lemma 2.3. *The following holds.*

(1)

$$\forall (f, g) \in (K^I)^2, \mathcal{D}_{f+g} = \mathcal{D}_f * \mathcal{D}_g.$$

(2) \mathcal{D}_0 is the neutral element of the commutative monoid $(\mathcal{M}_{\mathcal{L}}, *)$, where $0 : I \rightarrow K$ is the map almost everywhere zero.

(3) For $(\mu_1, \mu_2) \in \mathcal{M}_{\mathcal{L}}$ and $\chi \in \hat{K}$ a character, the fiberwise Fourier-Stieltjes coefficients satisfies for almost every $x \in I$,

$$\widehat{(\mu_1 * \mu_2)^x}(\chi) = \hat{\mu}_1^x(\chi) \hat{\mu}_2^x(\chi).$$

(4) The set of invertible elements for $*$ is \mathcal{G} , the set of graph measures.

(5) A measure $\mu \in \mathcal{M}_{\mathcal{L}}$ is in \mathcal{G} if and only if for all $\chi \in \hat{K}$, $|\hat{\mu}^x(\chi)| = 1$ for almost every x .

- (6) If $d = 1$, a measure $\mu \in \mathcal{M}_{\mathcal{L}}$ is in \mathcal{G} if and only if the first Fourier-Stieltjes coefficient has modulus 1 for almost every x , that is

$$\left| \int_{\mathbf{T}^1} e^{2i\pi y} d\mu^x(y) \right| = 1.$$

Remark that, if $f : [0, 1] \rightarrow \mathbf{T}^1$, $f(x) = x \bmod 1$, then one can check by hand (or see e.g. Proposition 3.1) that \mathcal{D}_{nf} tends to λ as $n \rightarrow \pm\infty$, but

$$\mathcal{D}_{nf} * \mathcal{D}_{-nf} = \mathcal{D}_0 \neq \lambda * \lambda,$$

so the fiberwise convolution product is not continuous. However, we have:

Lemma 2.4. *For any $\nu \in \mathcal{M}_{\mathcal{L}}$, the convolution map*

$$*\nu : \mathcal{M}_{\mathcal{L}} \rightarrow \mathcal{M}_{\mathcal{L}},$$

$$\mu \mapsto \mu * \nu,$$

is continuous.

Proof. It is sufficient to check that the preimage by $*\nu$ of any set of the form $\mathcal{U}(F, \epsilon, \mu * \nu)$, for any F, ϵ, μ , contains a set of the form $\mathcal{U}(G, \delta, \mu)$ for some G and some $\delta > 0$. Let $\mathcal{U}(F, \epsilon, \mu * \nu)$ be such a neighborhood of $\mu * \nu$, and let δ such that $\delta(4\|F\|_{\infty} + 1) < 1$. As the map $x \mapsto \nu^x$ is measurable, again by Lusin's Theorem, it is continuous on a set E of measure $1 - \delta$. Define

$$H(x, y) = \int_K F(x, y + z) d\nu^x(z).$$

It follows from the continuity of F and the continuity of $x \mapsto \nu^x$ that this is a continuous map on $E \times K$, and moreover bounded by $\|F\|_{\infty}$. Thus it can be extended to a bounded continuous map, say G , on $I \times K$, still bounded by $\|F\|_{\infty}$. Notice that for any $\eta \in \mathcal{M}_{\mathcal{L}}$,

$$\int_{E \times K} F d(\eta * \nu) = \int_E \int_K \left(\int_K F(x, y + z) d\nu^x(z) \right) d\eta^x(y) d\mathcal{L}(x) = \int_{E \times K} G d\eta.$$

Let $\eta \in \mathcal{U}(G, \delta, \mu)$, then

$$\begin{aligned}
 \left| \int_{I \times K} Fd(\eta * \nu) - \int_{I \times K} Fd(\mu * \nu) \right| &\leq 2\delta \|F\|_\infty \\
 &+ \left| \int_{E \times K} Fd(\eta * \nu) - \int_{E \times K} Fd(\mu * \nu) \right| \\
 &\leq 2\delta \|F\|_\infty + \left| \int_{E \times K} Gd\eta - \int_{E \times K} Gd\mu \right| \\
 &\leq 4\delta \|F\|_\infty + \delta.
 \end{aligned}$$

By the choice of δ , this implies that $*\nu(\mathcal{U}(G, \delta, \mu)) \subset \mathcal{U}(F, \epsilon, \mu * \nu)$, as announced. \square

The Fourier-Stieltjes coefficients are also not continuous, as can be seen using the previous example of map f such that \mathcal{D}_{nf} tends to λ . However, they can only decrease in the limit:

Lemma 2.5. *For any character $\chi \in \hat{K}$, the map from $\mathcal{M}_{\mathcal{L}}$ to \mathbf{R}^+ ,*

$$H_\chi : \mu \mapsto \int_I |\hat{\mu}^x(\chi)| d\mathcal{L}(x),$$

is lower semi-continuous

Proof. It will be convenient to assume in this proof that I is the unit interval and \mathcal{L} the Lebesgue measure. Let $\epsilon > 0$, χ a character, and $\mu_i \in \mathcal{M}_{\mathcal{L}}$. We wish to prove that for ν sufficiently close to μ , $H_\chi(\nu) \geq H_\chi(\mu) - 4\epsilon$.

Let $J \subset I$ be a compact continuity subset for $x \mapsto \mu^x$, of measure $> 1 - \epsilon$, obtained by Lusin's Theorem. Find a finite partition of J into intervals, $J = \cup_i J_i$, such that for all $(x, y) \in J_i$,

$$\left| \int_K \chi(k) d\mu^x(k) - \int_K \chi(k) d\mu^y(k) \right| < \epsilon.$$

We then have

$$\left| \int_{J_i} \left| \int_K \chi(k) d\mu^x(k) \right| dx - \int_{J_i} \int_K \chi(k) d\mu^x(k) dx \right| < \epsilon \mathcal{L}(J_i),$$

Since the boundary of the interval is of zero Lebesgue measure, then $\nu|_{J_i \times K}$ is close to $\mu|_{J_i \times K}$ whenever ν is close to μ .

Let ν be close enough to μ such that

$$\left| \int_{J_i \times K} \chi d\mu - \int_{J_i \times K} \chi d\nu \right| < \epsilon \mathcal{L}(J_i).$$

We have

$$\int_{J_i} \left| \int_K \chi(k) d\nu^x(k) \right| dx \geq \left| \int_{J_i \times K} \chi d\nu \right|.$$

Therefore,

$$\begin{aligned} \int_{J_i} \left| \int_K \chi(k) d\nu^x(k) \right| dx &\geq \left| \int_{J_i \times K} \chi d\mu \right| - \epsilon \mathcal{L}(J_i). \\ \int_{J_i} \left| \int_K \chi(k) d\nu^x(k) \right| dx &\geq \int_{J_i} \left| \int_K \chi d\mu^x \right| dx - 2\epsilon \mathcal{L}(J_i). \end{aligned}$$

Adding over the intervals J_i , we obtain

$$\int_J \left| \int_K \chi(k) d\nu^x(k) \right| dx \geq \int_J \left| \int_K \chi d\mu^x \right| dx - 2\epsilon,$$

So

$$H_\chi(\nu) \geq H_\chi(\mu) - 4\epsilon.$$

□

2.5. Restriction to a set of positive measure. If $E \subset I$ is a set of positive measure, and $\chi \in \hat{K} - \{1\}$ a nontrivial character, it will be useful to consider the normalized restricted measure $\mathcal{L}'(F) = \mathcal{L}(F \cap E) / \mathcal{L}(E)$, and the space $\mathcal{M}_{\mathcal{L}'}$ of probability measures on $I \times \mathbf{T}^1$ which projects to \mathcal{L}' . There is a natural map

$$\begin{aligned} \pi : \mathcal{M}_{\mathcal{L}} &\rightarrow \mathcal{M}_{\mathcal{L}'}, \\ \pi(\eta) &= \frac{1}{\mathcal{L}(E)} (id, \frac{1}{2i\pi} \log \chi)_* \eta|_{E \times K}, \end{aligned}$$

where $(id, \frac{1}{2i\pi} \log \chi) : I \times K \rightarrow I \times \mathbf{T}^1$.

Lemma 2.6. *The natural map $\pi : \mathcal{M}_{\mathcal{L}} \rightarrow \mathcal{M}_{\mathcal{L}'}$, satisfies the following.*

- (1) *It is surjective and continuous,*
- (2) *It is a monoid morphism for the fiberwise convolution products,*
- (3) *It sends a graph measure \mathcal{D}_f on the graph measure $\mathcal{D}_{(2i\pi)^{-1} \log \chi \circ f}$, and*

$$\hat{\mu}^x(\chi) = \int_{\mathbf{T}^1} e^{2i\pi y} d(\pi(\mu))^x(y).$$

Proof. These properties are easy to see in the case where $I = [0, \mathcal{L}(E)] \cup [1, 2 - \mathcal{L}(E)]$ endowed with the Lebesgue measure, and $E = [0, \mathcal{L}(E)]$. The general case can be deduced from this model case by Borel isomorphism, since the topology of $\mathcal{M}_{\mathcal{L}}$ does not depend on the distance on I . □

3. ASYNCHRONOUS MAPS

3.1. A simple example. As stated in the introduction, if f has enough regularity properties, it turns out that λ is the limit point of the dynamic of $*\mathcal{D}_f$ on $\mathcal{M}_{\mathcal{L}}$. This result will not be used in the sequel.

Proposition 3.1. *Assume $f : [0, 1] \rightarrow \mathbf{R}/\mathbf{Z}$ is a C^2 -map, such that f' does not vanish. Then for all $\mu \in \mathcal{M}_{\mathcal{L}}$, $\mathcal{D}_{kf} * \mu \rightarrow \lambda$ when $k \rightarrow \pm\infty$.*

Proof. By continuity of $*\mu$, it is sufficient to check that \mathcal{D}_{kf} tends to λ as k tends to infinity. To do so, compute the Fourier-Stieltjes coefficients

$$\hat{\mathcal{D}}_{kf}(n, m) = \int_{[0,1]} e^{2i\pi(nx+mkf(x))} dx.$$

If $m = 0$, this coefficient is 1 or 0, depending on whether $n = 0$ or not. If $m \neq 0$, we can write

$$\hat{\mathcal{D}}_{kf}(n, m) = \int_{[0,1]} \frac{e^{2i\pi nx}}{2i\pi mk f'(x)} \frac{\partial}{\partial x} (e^{2i\pi mk f(x)}) dx,$$

and integration by parts immediately shows that $\hat{\mathcal{D}}_{kf}(n, m) = O(\frac{1}{k})$ when $k \rightarrow \pm\infty$ with n, m fixed. \square

3.2. Strategy of the proof of Theorem 1. Here we explain the difficult part, which is 1) \Rightarrow 2). The implication 3) \Rightarrow 1) will be dealt with in Section 3.8, and the implication 2) \Rightarrow 3) is proved below. We consider the closure $Z = \overline{\{\mathcal{D}_{nf}\}_{n \in \mathbf{Z}}}$, and wish to prove that $\lambda \in Z$. The first step is to prove the following criterion.

Proposition 3.2. *Let $f : I \rightarrow K$ be a measurable map. The following are equivalent:*

- (1) $f_*\mathcal{L}$ is purely atomic.
- (2) The closure $\overline{\{\mathcal{D}_{nf}\}_{n \in \mathbf{Z}}}$ is contained in the set of graph measures \mathcal{G} .

*If these properties are true, $\overline{\{\mathcal{D}_{nf}\}_{n \in \mathbf{Z}}}$ is then a compact subgroup of $(\mathcal{G}, *)$.*

Therefore, if f is asynchronous, the closure Z always contains measures which are not graphs - and this is still true for the restriction of f to any set of positive measure. The second step is to show that this fact is sufficient to deduce that $\lambda \in \overline{\{\mathcal{D}_{nf}\}_{n \in \mathbf{Z}}}$. This is done in section 3.7 by considering the minimum of the integral of the absolute value of Fourier-Stieltjes coefficients of elements of $\overline{\{\mathcal{D}_{nf}\}_{n \in \mathbf{Z}}}$, knowing, thanks to the first step and Lemma 2.3 (5), that it cannot be 1.

3.3. Proof of 2) \Rightarrow 3). Let us check that $\lambda \in \overline{\{\mathcal{D}_{nf}\}_{n \in \mathbf{Z}}}$ implies that $*\mathcal{D}_f$ is uniquely ergodic. Let $(n_i)_{i \geq 1}$ be a sequence such that $\mathcal{D}_{n_i f}$ converges weakly to λ when $i \rightarrow +\infty$. Let $\mu \in \mathcal{M}_{\mathcal{L}}$, then by one-sided continuity of convolution,

$$\mu * \mathcal{D}_{n_i f} \xrightarrow{i \rightarrow +\infty} \mu * \lambda = \lambda.$$

Let m be any invariant measure on $\mathcal{M}_{\mathcal{L}}$, $F : \mathcal{M}_{\mathcal{L}} \rightarrow \mathbf{R}$ be a continuous function. Then

$$\int_{\mathcal{M}_{\mathcal{L}}} F(\mu) dm(\mu) = \int_{\mathcal{M}_{\mathcal{L}}} F(\mu * \mathcal{D}_{n_i f}) dm(\mu),$$

by invariance of m . The Lebesgue dominated convergence Theorem implies

$$\int_{\mathcal{M}_{\mathcal{L}}} F(\mu) dm(\mu) \rightarrow \int_{\mathcal{M}_{\mathcal{L}}} F(\lambda) dm(\mu) = F(\lambda),$$

which means that m is the Dirac measure at λ , as required.

3.4. Preliminary Lemmas.

3.4.1. Local density. The first technical Lemma states in a quantitative way that rotations in K spend a positive proportion of time away from zero, uniformly in the choice of the angle away from zero. For a subset $E \subset \mathbf{Z}$, we denote for $M \geq 1$,

$$\mathbf{P}_M(E) = \frac{1}{2M+1} |E \cap [-M, M]|,$$

the measure of E with respect to the uniform probability on $[-M, M]$. For $\mathbf{z} \in K$, put

$$\mathcal{E}_{\mathbf{z}} = \{n \in \mathbf{Z} : d(n\mathbf{z}, 0) \geq 1/5\},$$

where d is the supremum distance on $K = \mathbf{T}^d$.

Lemma 3.3. *Let $c_0 = \frac{1}{4} - \frac{\sin^2(\pi/5)}{2} > 0$. For all $\eta \in (0, 1/2)$, there exist an integer M such that for all $\mathbf{z} \in K$ with $d(\mathbf{z}, 0) > \eta$,*

$$\mathbf{P}_M(\mathcal{E}_{\mathbf{z}}) \geq c_0.$$

Proof. For $\mathbf{z} = (z_1, \dots, z_d) \in K$, define

$$h(\mathbf{z}) = \sup_{j \in \{1, \dots, d\}} \sin^2(z_j \pi).$$

We have

$$1_{d(\mathbf{z}, 0) > 1/5} \geq h(\mathbf{z}) - \sin^2(\pi/5).$$

Therefore, for any $M \geq 1$,

$$\mathbf{P}_M(\mathcal{E}_z) \geq \frac{1}{2M+1} \sum_{n=-M}^M (h(nz) - \sin^2(\pi/5)),$$

Since a sum of suprema is greater than the supremum of the sums,

$$\mathbf{P}_M(\mathcal{E}_z) \geq -\sin^2(\pi/5) + \frac{1}{2M+1} \sup_j \left(\sum_{n=-M}^M \sin^2(z_j \pi) \right),$$

$$\mathbf{P}_M(\mathcal{E}_z) \geq -\sin^2(\pi/5) + \frac{1}{(2M+1)} \sup_j \left(\sum_{n=-M}^M \frac{1}{2} - \frac{1}{4} (e^{2i\pi n z_j} + e^{-2i\pi n z_j}) \right),$$

$$\mathbf{P}_M(\mathcal{E}_z) \geq 2c_0 - \frac{1}{4(2M+1)} \inf_j \left(\frac{e^{-2i\pi M z_j} - e^{2i\pi(M+1)z_j}}{1 - e^{2i\pi z_j}} + \frac{e^{2i\pi M z_j} - e^{-2i\pi(M+1)z_j}}{1 - e^{-2i\pi z_j}} \right),$$

$$\mathbf{P}_M(\mathcal{E}_z) \geq 2c_0 - \frac{1}{2M+1} \inf_j \frac{1}{|1 - e^{2i\pi z_j}|}.$$

Since $d(\mathbf{z}, 0) > \eta$, any

$$M \geq \frac{1}{2c_0 |1 - e^{2i\pi \eta}|}$$

satisfies the property. \square

3.4.2. *Criterion for locally constant maps.* We will need the following criterion.

Lemma 3.4. *Let $f : I \rightarrow K$ be a measurable map. Assume that for any $\epsilon > 0$, there exists an integer $H_\epsilon > 0$, such that for all $\eta > 0$, there exists a set $E_{\epsilon, \eta} \subset I$ of measure $\mathcal{L}(E_{\epsilon, \eta}) > 1 - \epsilon$ such that $f(E_{\epsilon, \eta})$ is contained in at most H_ϵ balls of radius η . Then $f_* \mathcal{L}$ is a purely atomic measure.*

Proof. Let $\epsilon > 0$, for any $k > 0$, $f(E_{\epsilon, 2^{-k}})$ is contained in H_ϵ balls of radius 2^{-k} (up to adding some unnecessary balls). Denote $x_k^1, \dots, x_k^{H_\epsilon}$ the center of these balls. As K^{H_ϵ} is compact, there exists a converging subsequence, that is $x_{\psi(k)}^j \rightarrow x_\infty^j$ for any $j \in \{1, \dots, H_\epsilon\}$. Define

$$F_\epsilon = \bigcap_{N \geq 1} \bigcup_{k \geq N} E_{\epsilon, 2^{-\psi(k)}}.$$

We have $\mathcal{L}(F_\epsilon) \geq 1 - \epsilon$, and clearly,

$$f(F_\epsilon) \subset \bigcap_{N \geq 1} \bigcup_{k \geq N} f(E_{\epsilon, 2^{-\psi(k)}}) \subset \bigcap_{N \geq 1} \bigcup_{k \geq N} \bigcup_j B(x_{\psi(k)}^j, 2^{-\psi(k)}),$$

and therefore,

$$f(F_\epsilon) \subset \{x_\infty^1, \dots, x_\infty^{H_\epsilon}\}.$$

Therefore, if $f_*\mathcal{L} = \nu_1 + \nu_2$ is the decomposition of this measure in a purely atomic part ν_1 and a measure ν_2 without atoms, then

$$\nu_1(K) \geq f_*\mathcal{L}(\{x_\infty^1, \dots, x_\infty^{H_\epsilon}\}) \geq 1 - \epsilon,$$

This concludes the proof. \square

3.5. Proof of Proposition 3.2, 1) \Rightarrow 2). Assume that $f_*\mathcal{L}$ is purely atomic. Let $(\alpha_k)_{k \geq 0} \in K$ be the atoms. We assume that the sequence is infinite; otherwise, the following have to be adapted accordingly in a straightforward manner. Let $E_k = f^{-1}(\alpha_k)$. The family of sets $(E_k)_{k \geq 0}$ partition I modulo a zero-set. Consider the map:

$$\begin{aligned} T : \quad K^{\mathbf{N}} &\rightarrow \mathcal{M}_{\mathcal{L}}, \\ (\beta_k)_{k \geq 0} &\rightarrow \mathcal{D}_g, \text{ with } g(x) = \beta_k \text{ for any } x \in E_k. \end{aligned}$$

It is not hard to see that this is a continuous group morphism for the product topology on $K^{\mathbf{N}}$. The image of T is therefore a compact subgroup of \mathcal{G} containing \mathcal{D}_f , and therefore its iterates and the closure of the iterates. This concludes the easy part of Proposition 3.2.

3.6. Proof of Proposition 3.2, 2) \Rightarrow 1). Assume that $Z = \overline{\{\mathcal{D}_{nf}\}_{n \in \mathbf{Z}}}$ is contained in \mathcal{G} . We will show that f satisfies the assumptions of Lemma 3.4. Thus, let $\epsilon > 0$, we have first to choose an integer H_ϵ .

3.6.1. Choice of H_ϵ . Since Z is a closed subset of the compact set $\mathcal{M}_{\mathcal{L}}$, it is compact. Since by Lemma 2.2, the topology on \mathcal{G} is given by the distance $d_{\mathcal{G}}$, this implies that the compact set Z is contained in the union of finitely many balls of the form $B_{\mathcal{G}}(\mathcal{D}_{g_k}, \epsilon/2)$, for some finite family of $\mathcal{D}_{g_k} \in Z$. Since $\{\mathcal{D}_{nf}\}_{n \in \mathbf{Z}}$ is a dense subset of its closure, there exists integers n_k such that $\mathcal{D}_{n_k f} \in B_{\mathcal{G}}(\mathcal{D}_{g_k}, \epsilon/2)$. So there exists an integer $N_\epsilon = \sup_k |n_k|$, such that

$$(1) \quad \overline{\{\mathcal{D}_{nf}\}_{n \in \mathbf{Z}}} \subset \bigcup_{|m| \leq N_\epsilon} B_{\mathcal{G}}(\mathcal{D}_{mf}, \epsilon).$$

By Lusin's Theorem, there exists a compact subset $F \subset I$, of measure greater than $1 - \epsilon$, such that f is continuous on F . It is in fact uniformly continuous because F is compact. So there exists $\delta > 0$ such that:

$$(2) \quad \forall (x, x') \in F^2, d_I(x, x') < \delta \Rightarrow d_K(f(x), f(x')) < \frac{1}{6N_\epsilon}.$$

Here and in the rest of the proof, we assumed that $I = [0, 1]$ equipped with the Lebesgue measure \mathcal{L} and the usual distance d_I . We now choose

$$(3) \quad H_\epsilon = \frac{3}{\delta}.$$

3.6.2. *Construction of $E_{\epsilon,\eta}$.* Having defined H_ϵ , we now consider a parameter $\eta > 0$. By Lemma 3.3, there exist $M > 0$ (depending on η) such that

$$(4) \quad \forall \mathbf{z} \in K \text{ s.t. } d_K(\mathbf{z}, 0) > \eta, \mathbf{P}_M(\mathcal{E}_{\mathbf{z}}) \geq c_0.$$

By (1), for all $n \in \mathbf{Z}$, there exists an integer m_n with $|m_n| \leq N_\epsilon$ such that $d_G(\mathcal{D}_{nf}, \mathcal{D}_{m_n f}) \leq \epsilon$. By definition of the distance d_G , there exists a set E_n of measure greater than $1 - \epsilon$ such that

$$(5) \quad \forall x \in E_n, d_K(nf(x), m_n f(x)) < \epsilon.$$

One difficulty is that the set of 'bad' points E_n^c depends on n . Since

$$\int_I \frac{1}{2M+1} \sum_{n=-M}^M 1_{E_n^c}(x) d\mathcal{L}(x) \leq \epsilon,$$

if we define $G(x) = \{n \in \mathbf{Z} : x \in E_n\}$ the set of 'good' times n for x , then

$$\int_I \mathbf{P}_M(G(x)^c) d\mathcal{L}(x) \leq \epsilon.$$

By Markov's inequality,

$$\mathcal{L} \left\{ x : \mathbf{P}_M(G(x)^c) \geq \frac{c_0}{4} \right\} \leq \frac{4\epsilon}{c_0}.$$

This implies that

$$\mathcal{L} \left\{ x \in F : \mathbf{P}_M(G(x)) \geq 1 - \frac{c_0}{4} \right\} \geq 1 - \left(\frac{4}{c_0} + 1 \right) \epsilon.$$

Therefore, there exists a compact subset $E_{\epsilon,\eta}$ of F of measure greater than $1 - \left(\frac{4}{c_0} + 2 \right) \epsilon$ such that

$$(6) \quad \forall x \in E_{\epsilon,\eta}, \mathbf{P}_M(G(x)) \geq 1 - \frac{c_0}{4}.$$

3.6.3. *Assumptions of Lemma 3.4.* We are now in position to check the assumptions of Lemma 3.4. By compactness of $E_{\epsilon,\eta}$, it is contained in the union of finitely many balls in I centered on points of $E_{\epsilon,\eta}$, of radius $\delta/3$. By the Vitali covering Lemma (which applies, since we assumed that $I = [0, 1]$), there exists x_1, \dots, x_p points of $E_{\epsilon,\eta}$ such that the balls $B(x_i, \delta)$ cover $E_{\epsilon,\eta}$, and the balls $B(x_i, \delta/3)$ are disjoint. Therefore, the number p of balls satisfies $p \leq H_\epsilon$.

We wish to prove that

$$f(E_{\epsilon,\eta}) \subset \cup_{i=1}^p B_K(f(x_i), \eta).$$

Let $x \in E_{\epsilon, \eta}$, there exists $i \in \{1, \dots, p\}$ such that $d_I(x, x_i) < \delta$. It is sufficient to show that $d_K(f(x), f(x_i)) < \eta$, so we proceed by contradiction and assume from now on that $d_K(f(x), f(x_i)) > \eta$.

Let $\mathbf{z} = f(x) - f(x_i)$. We first choose a suitable n . As $\mathbf{P}_M(\mathcal{E}_{\mathbf{z}}) \geq c_0$ by Equation (4), $\mathbf{P}_M(G(x)) \geq 1 - \frac{c_0}{4}$, and $\mathbf{P}_M(G(x_i)) \geq 1 - \frac{c_0}{4}$ by Equation (6), there exists an integer $n \in G(x) \cap G(x_i) \cap \mathcal{E}_{\mathbf{z}}$, for which we have simultaneously:

$$\begin{aligned} d_K(n(f(x) - f(x_i)), 0) &> \frac{1}{5}, \\ d_K(nf(x), m_n f(x)) &< \epsilon, \\ d_K(nf(x_i), m_n f(x_i)) &< \epsilon. \end{aligned}$$

This implies that

$$(7) \quad d_K(m_n \mathbf{z}, 0) > \frac{1}{5} - 2\epsilon.$$

But by Equation (2), that we also have

$$d_K(f(x), f(x_i)) < \frac{1}{6N_\epsilon},$$

and since $|m_n| \leq N_\epsilon$, we have

$$d_K(m_n \mathbf{z}, 0) < \frac{1}{6},$$

and this contradicts the Inequality (7) provided that $2\epsilon < \frac{1}{5} - \frac{1}{6}$, which we can freely assume.

So all the assumption of Lemma 3.4 are satisfied (up to a harmless multiplication of ϵ by the fixed number $\frac{4}{c_0} + 2$), this proves that $f_*\mathcal{L}$ is purely atomic and concludes the proof of Proposition 3.2.

3.7. Fourier-Stieltjes Transforms. We now wish to prove that, under the assumption that f is asynchronous, that $\lambda \in Z = \overline{\{\mathcal{D}_{nf}\}_{n \in \mathbf{Z}}}$.

Lemma 3.5. *The set Z is closed under the convolution product.*

Proof. Let $\mu, \nu \in Z$. There exists sequences n_i, m_i so that $\mathcal{D}_{n_i f}$ and $\mathcal{D}_{m_i f}$ tends to μ, ν respectively when $i, j \rightarrow +\infty$. Thus, by fixing j and letting $i \rightarrow \infty$

$$\mathcal{D}_{n_i f} * \mathcal{D}_{m_j f} \rightarrow \mu * \mathcal{D}_{m_j f},$$

by continuity of $*\mathcal{D}_{m_j f}$. So $\mu * \mathcal{D}_{m_j f}$ belongs to Z . Now we let j go to infinity and use continuity of $*\mu$, to conclude that $\mu * \nu \in Z$. \square

To prove that $\lambda \in Z$, we first choose a sequence $(a_\chi)_{\chi \in \hat{K} - \{1\}}$ indexed by the set $\hat{K} - \{1\}$ of nontrivial characters, such that $a_\chi > 0$ and $\sum_\chi a_\chi = 1$. Consider the following function on Z

$$H(\mu) = \sum_{\chi \in \hat{K} - \{1\}} a_\chi H_\chi(\mu),$$

where H_χ was defined in Lemma 2.5. As H_χ is bounded by 1, the sum is convergent, and $H(\mu) = 0$ if and only if $\mu = \lambda$. The function H is lower-semicontinuous by Lemma 2.5, and admits therefore a minimum at some point $\nu \in Z$. We wish to prove that $H(\nu) = 0$, which implies that $\nu = \lambda$ is in Z .

Assume by contradiction that this minimum is positive. This implies that there exists a nontrivial character χ such that the set

$$E = \{x : \hat{\nu}^x(\chi) \neq 0\},$$

is of positive measure.

Lemma 3.6. *Let $\eta \in Z$. Then for almost all $x \in E$, we have $|\hat{\eta}^x(\chi)| = 1$.*

Proof. Since Z is closed under convolution, and ν is a minimum, we have

$$H(\eta * \nu) \geq H(\nu),$$

so by Lemma 2.3 (3), we have

$$\sum_\chi a_\chi \int_I |\hat{\nu}^x(\chi)| (|\hat{\eta}^x(\chi)| - 1) \geq 0.$$

As the Fourier-Stieltjes coefficients are bounded by 1, this implies that for almost every x , either $\hat{\nu}^x(\chi) = 0$, or $|\hat{\eta}^x(\chi)| = 1$. \square

Denote by \mathcal{L}' the restriction to E of the measure \mathcal{L} , renormalized to be a probability. The set I endowed with the probability \mathcal{L}' is a standard probability space without atoms, and the map $(2i\pi)^{-1} \log \chi \circ f$ is an asynchronous rotation $(I, \mathcal{L}') \rightarrow \mathbf{T}^1$. Denote by $\mathcal{M}_{\mathcal{L}'}$ the set of measures on $I \times \mathbf{T}^1$ which projects to \mathcal{L}' on the first coordinate. Let $Z' \subset \mathcal{M}_{\mathcal{L}'}$ be the closure of the convolutions of $\mathcal{D}_{(2i\pi)^{-1} \log \chi \circ f}$. By Proposition 3.2, there exist a point $\mu \in Z'$ which is not a graph measure. By Lemma 2.3 (6), there is a subset F of E of positive measure such that for all $x \in F$,

$$\left| \int_{\mathbf{T}^1} e^{2i\pi y} d\mu^x(y) \right| < 1.$$

The projection map $\pi : \mathcal{M}_{\mathcal{L}} \rightarrow \mathcal{M}_{\mathcal{L}'}$ defined in Section 2.5, whose properties are summarized in Lemma 2.6, satisfies $\pi(Z) = Z'$.

This means that there exists a measure $\eta \in Z$ such that $\pi(\eta) = \mu$, so in particular for almost every $x \in E$, $((2i\pi)^{-1} \log \chi)_* \eta^x = \mu^x$. For almost every $x \in F$, the associated Fourier-Stieltjes coefficient must be of absolute value < 1 (by construction of μ) and equal to 1 by Lemma 3.6. Contradiction.

This proves that $H(\nu) = 0$, so $\nu = \lambda$ and $\lambda \in Z$ as required, finishing the proof of 1) \Rightarrow 2) in Theorem 1.

3.8. Proof of Theorem 1, 3) \Rightarrow 1). Let us prove the contrapositive of 3) \Rightarrow 1). We assume that f is not asynchronous, and aim to produce an invariant measure, different from δ_λ . There exists χ a non-trivial character for which $(\chi \circ f)_* \mathcal{L}$ has at least one atom. Therefore, there exists a subset $E \subset I$ of positive measure on which the normalized restriction \mathcal{L}' of \mathcal{L} satisfies that $((2i\pi)^{-1} \log \chi \circ f)_* \mathcal{L}'$ is purely atomic. Again, we consider the restriction to the subset E as in Section 2.5. By Proposition 3.2, the normalized restriction to E of $\mathcal{D}_{(2i\pi)^{-1} \log \chi \circ f}$ lies in a compact subgroup G of $\mathcal{M}_{\mathcal{L}'}$, the set of measures on $I \times \mathbf{T}^1$ that projects to \mathcal{L}' .

We have a continuous surjection $\pi : \mathcal{M}_{\mathcal{L}} \rightarrow \mathcal{M}_{\mathcal{L}'}$. The preimage of G is a compact subset, invariant by convolution by \mathcal{D}_f , not containing λ , so it carries an invariant measure different from δ_λ .

3.9. Sets of natural density one.

Proposition 3.7. *Assume $f : I \rightarrow K$ is asynchronous. Then there exists a set $E \subset \mathbf{Z}$ of full natural density such that for all $\mu \in \mathcal{M}_{\mathcal{L}}$,*

$$\lim_{n \rightarrow \pm\infty, n \in E} \mathcal{D}_{nf} * \mu = \lambda.$$

Proof. We consider the measure on $\mathcal{M}_{\mathcal{L}}$,

$$\nu_N = \frac{1}{2N+1} \sum_{|k| \leq N} \delta_{\mathcal{D}_{kf}}.$$

As any weak limit of ν_N is \mathcal{D}_f -invariant and $\mathcal{M}_{\mathcal{L}}$ is compact, by unique ergodicity of $(\mathcal{M}_{\mathcal{L}}, \mathcal{D}_f)$, ν_N converges to δ_λ when N goes to $+\infty$. This implies that for any neighborhood \mathcal{U} of λ , the proportion of $\{\mathcal{D}_{kf}\}_{|k| \leq N}$

outside \mathcal{U} goes to zero as $N \rightarrow +\infty$.

Let $(\mathcal{U}_m)_{m \geq 1}$ be a decreasing basis of neighborhood of λ , and

$$E_m = \{k \in \mathbf{Z} : \mathcal{D}_{kf} \in \mathcal{U}_m\}.$$

Let N_m be an integer such that for all $N \geq N_m$,

$$\mathbf{P}_N(E_m) \geq 1 - \frac{1}{m},$$

where \mathbf{P}_N is the uniform probability on $[-N, N]$. We can modify the sequence $(N_m)_{m \geq 0}$ to be strictly increasing, and choose $N_1 = -1$. Let E be the subset

$$E = \bigcup_{m \geq 1} E_m \cap \{k \in \mathbf{Z} : N_m < |k| \leq N_{m+1}\}.$$

Notice that since the sets E_m are decreasing with m , if $n \leq N_{m+1}$,

$$E \cap [-n, n] \subset E_m.$$

Thus, for n such that $N_m < n \leq N_{m+1}$, we have

$$\mathbf{P}_n(E) \geq 1 - \frac{1}{m}.$$

This proves that E is a set of natural density one. By construction, we have

$$\lim_{n \rightarrow \pm\infty, n \in E} \mathcal{D}_{nf} = \lambda.$$

By continuity of the convolution with μ , the latter limit holds for the sequence $\mathcal{D}_{nf} * \mu$ with the same set E . \square

4. ON THE SMALLEST ALGEBRAIC GROUP CONTAINING RETURN ELEMENTS

The following Theorem, which will be a crucial ingredient of the proof of Theorem 2, might be of independent interest.

Theorem 4. *Let G_0 be the group of real points of an algebraic group \mathbf{G}_0 defined over \mathbf{Q} , without nontrivial \mathbf{Q} -characters, $\Gamma_0 = \mathbf{G}_0(\mathbf{Z})$ be its integer points, $a \in G_0$ be triangularizable with positive eigenvalues, and μ an a -invariant measure on G_0/Γ_0 . We assume that the measure μ is ergodic and non-concentrated. Let $I \subset G_0$ be a fundamental domain for Γ_0 , and denote by \mathcal{L} the lift to I of μ . Let $E \subset I$ be a subset of positive \mathcal{L} -measure. Define*

$$P_E = \{\gamma \in \Gamma_0 : \mathcal{L}(\cup_{k \in \mathbf{Z}} a^k E \gamma \cap E) > 0\},$$

the set of elements of Γ_0 associated to return times in E . Then the smallest algebraic subgroup of G_0 containing P_E is G_0 .

To prove this, let H be the smallest algebraic subgroup of G_0 containing P_E . Our aim is to show that $H = G_0$. This will be done in the following sequence of Lemmata.

4.1. Closure of $H\Gamma_0$.

Lemma 4.1. *The set $H\Gamma_0$ is closed*

Proof. Notice that H is defined over \mathbf{Q} , since P_E consists of integer points. We claim that the non-trivial \mathbf{Q} -characters of H are of order 2. Indeed, if c is such a character defined over \mathbf{Q} , the image by c of the subgroup generated by $P_E \cap H^0 \subset \Gamma_0 = \mathbf{G}(\mathbf{Z})$ consists of rational with bounded denominators, and is a multiplicative subgroup, so $c(P_E) \subset \{-1, +1\}$. Therefore, P_E is contained in $\text{Ker}(c^2)$, an algebraic group defined over \mathbf{Q} . By definition of H , $H \subset \text{Ker}(c^2)$, so $H = \text{Ker}(c^2)$ as required. In particular, $H/(H \cap \Gamma_0)$ is of finite volume, by the Theorem of Borel and Harish-Chandra [2, Corollaire 13.2]. By [2, Proposition 8.1], this also implies that $H^0\Gamma_0$ is a closed subset of G_0/Γ_0 , where H^0 is the connected component of the identity of H , in the Zariski topology (a subgroup of finite index). This implies that $H\Gamma_0$ is closed. \square

4.2. Reduction step.

Lemma 4.2. *To prove Theorem 4, we can (and will) assume that for all $k \in \mathbf{Z}$ and $\gamma \in \Gamma_0$ such that $a^k E \gamma \cap E \neq \emptyset$, then $\gamma \in P_E$.*

Proof. Consider the subset

$$F = E - \bigcup_{(k,\gamma) \in \mathbf{Z} \times \Gamma_0 \text{ s.t. } \mathcal{L}(a^k E \gamma \cap E) = 0} a^k E \gamma.$$

Clearly, F is a subset of E of the same measure, and $P_F = P_E$. So it is sufficient to prove the statement of Theorem 4 for F instead of E , and F satisfies the above property. \square

4.3. Invariance of $xH\Gamma_0$.

Lemma 4.3. *For \mathcal{L} -almost every $x \in E$, $a \in xHx^{-1}$.*

Proof. By a Theorem of Chevalley [3, Thm 5.1], there exists a finite dimensional representation α of \mathbf{G} such that H is the stabilizer of a line D , that is $H = \{g \in G_0 : \alpha(g)D = D\}$. By Poincaré recurrence Theorem, for \mathcal{L} -almost every $x \in E$, there exists a sequence $n_k \rightarrow +\infty$ and $\gamma_k \in \Gamma_0$ such that $a^{n_k} x \gamma_k \rightarrow x$ and $a^{n_k} x \gamma_k \in E$. Fix such an $x \in E$.

By Lemma 4.2, we know that $\gamma_k \in P_E \subset H$. It follows that $\alpha(\gamma_k)D = D$, so since $a^{n_k}x\gamma_k \rightarrow x$, we have

$$(8) \quad \lim_{k \rightarrow +\infty} \alpha(a)^{n_k} \alpha(x)D = \alpha(x)D.$$

By assumption, a is triangularizable with positive eigenvalues, so $\alpha(a)$ has only positive, real eigenvalues. We claim that (8) implies that $\alpha(x)D$ is contained in one of its eigenspaces.

Let

$$\alpha(a) = \delta + \eta,$$

be the Jordan-Chevalley decomposition of $\alpha(a)$, that is: δ and η commutes, η is nilpotent, δ diagonalizable (with positive, real eigenvalues). If p is the nilpotent index of η , for k such that $n_k > p$,

$$\alpha(a)^{n_k} = \sum_{i=0}^{p-1} \binom{n_k}{i} \delta^{n_k-i} \eta^i.$$

Let $v \in \alpha(x)D - \{0\}$, and $v = \sum_{\theta} v_{\theta}$ be its decomposition along the eigenspaces of δ corresponding to the eigenvalues $\{\theta\}$ of δ . Then

$$\alpha(a)^{n_k}(v) = \sum_{\theta} \left(\sum_{i=0}^{p-1} \binom{n_k}{i} \theta^{n_k-i} \eta^i(v_{\theta}) \right).$$

As a function of n_k , this is a combination of polynomials and powers of eigenvalues. If θ_0 is the highest eigenvalue θ for which $v_{\theta} \neq 0$, and i_0 is the largest i for which $\eta^i(v_{\theta_0}) \neq 0$, then we have the asymptotic as $k \rightarrow +\infty$,

$$\alpha(a)^{n_k}(v) \sim \binom{n_k}{i_0} \theta_0^{n_k-i_0} \eta^{i_0}(v_{\theta_0}).$$

However, we know that projectively, $\alpha(a)^{n_k} \alpha(x)D \rightarrow \alpha(x)D$, so v is colinear to $\eta^{i_0}(v_{\theta_0})$. As η preserves the eigenspace of δ associated to θ_0 ,

$$\alpha(a) \eta^{i_0}(v_{\theta_0}) = \theta_0 \eta^{i_0}(v_{\theta_0}) + \eta^{i_0+1}(v_{\theta_0}) = \theta_0 \eta^{i_0}(v_{\theta_0}),$$

because by definition of i_0 , $\eta^{i_0+1}(v_{\theta_0}) = 0$. This shows that $\eta^{i_0}(v_{\theta_0})$ is an eigenvector, and so is v .

We have proved that $\alpha(x)D$ is contained in an eigenspace of $\alpha(a)$. So D is stabilized by $\alpha(x^{-1}ax)$, meaning that $a \in xHx^{-1}$, as required. \square

4.4. Conclusion of the proof of Theorem 4.

Lemma 4.4. *We have $H = G_0$.*

Proof. By ergodicity of a with respect to μ , for μ -almost every $x\Gamma_0$, $a^{\mathbf{Z}}x\Gamma_0$ is dense in the support of μ . By the previous Lemma, we have also for almost every $x \in E$, $a \in xHx^{-1}$, so

$$a^{\mathbf{Z}}x\Gamma_0 \subset xH\Gamma_0.$$

Consider a typical $x \in E$ satisfying both of these properties. By Lemma 4.1, $xH\Gamma_0$ is a closed set. By density of the a -orbit of x in the support of μ , this implies that $\text{supp}(\mu) \subset xH\Gamma_0$, so

$$\mu(xH\Gamma_0) = \mu((xHx^{-1})x\Gamma_0) = 1.$$

By assumption, μ is non-concentrated, so $H = G_0$ necessarily. \square

5. PROOF OF THEOREM 2

The proof is by contradiction. We assume that f_β is not asynchronous. By translate of a \mathbf{Q} -subspace of $V = \mathbf{R}^d$, we mean a set $T \subset V$ of the form $T = v + W$, where $v \in V$, and $W \subset V$ a subspace of V defined over \mathbf{Q} ; in particular, there is no rationality assumption on v , and T itself does not have to be defined over \mathbf{Q} . The \mathbf{Q} -subspace W is called the direction of T .

If non-empty, the intersection of two translates of \mathbf{Q} -subspaces is again a translate of \mathbf{Q} -subspace. This property allows us to define, for a set $E \subset I$ of positive \mathcal{L} -measure, the set T_E which is the smallest translate of \mathbf{Q} -subspace containing $\{\rho(x)^{-1}\beta\}_{x \in E}$. We denote by W_E its direction.

Lemma 5.1. *There exist $E \subset I$ of positive measure such that $W_E \neq V$.*

Proof. Since f_β is not asynchronous, so for some non-trivial character $\chi \in \hat{K}$, $\chi \circ f_\beta$ is constant on a set F of positive \mathcal{L} -measure. There exists $\mathbf{n} \in \mathbf{Z}^d - \{0\}$, such that

$$\chi(\bar{v}) = e^{2i\pi \langle \mathbf{n}, v \rangle}.$$

So the set $\{\langle \mathbf{n}, \rho(x)^{-1}\beta \rangle\}_{x \in F}$ is contained in a countable set of the form $c + \mathbf{Z}$, for some $c \in \mathbf{R}$. This implies that at least one of the sets

$$F_m = \{x \in F : \langle \mathbf{n}, \rho(x)^{-1}\beta \rangle = c + m\},$$

for $m \in \mathbf{Z}$, has positive \mathcal{L} -measure. By construction, for such a $m \in \mathbf{Z}$,

$$T_{F_m} \subset \{v \in V : \langle \mathbf{n}, v \rangle = c + m\},$$

the right-hand side set being the translate of a proper \mathbf{Q} -subspace, $E = F_m$ satisfies the Lemma. \square

We now fix $E \subset I$ a set of positive measure, such that W_E is of minimal possible dimension (it exists). By Lemma 5.1, $W_E \neq V$. Like in Theorem 4, we define

$$P_E = \{\gamma \in \Gamma_0 : \mathcal{L}(\cup_{k \in \mathbf{Z}} a^k E \gamma \cap E) > 0\}.$$

Lemma 5.2. *For all $\gamma \in P_E$, $\rho(\gamma)W_E = W_E$.*

Proof. Recall that β is an eigenvector for $\rho(a)$, denote by κ the corresponding eigenvalue. By definition of P_E , there exists $k \in \mathbf{Z}$ with $\mathcal{L}(a^k E \gamma \cap E) > 0$. Let $F = a^k E \gamma \cap E$, then for all $x \in F$, there exist $y \in E$ such that $x = a^k y \gamma$. We have:

$$\rho(x)^{-1}\beta = \rho(\gamma^{-1}y^{-1}a^{-k})\beta = \kappa^{-k}\rho(\gamma)^{-1}\rho(y)^{-1}\beta \in \kappa^{-k}\rho(\gamma)^{-1}T_E.$$

Note that since $\rho(\gamma)^{-1}$ is a matrix with integer coefficients, $\kappa^{-k}\rho(\gamma)^{-1}T_E$ is also the translate of a \mathbf{Q} -subspace, containing $\{\rho(x)^{-1}\beta\}_{x \in F}$. By definition of T_F , this means that

$$T_F \subset \kappa^{-k}\rho(\gamma)^{-1}T_E.$$

Since $F \subset E$, $T_F \subset T_E$, and because E was chosen such that T_E is of minimal possible dimension, we have $T_E = T_F$, so from the rank-nullity Theorem,

$$\kappa^{-k}\rho(\gamma)^{-1}T_E = T_E.$$

This implies equality for the directions, $\rho(\gamma)^{-1}W_E = W_E$, and multiplication by $\rho(\gamma)$ concludes the proof. \square

Lemma 5.3. *We have $W_E = \{0\}$, that is, the map $x \mapsto \rho(x)^{-1}\beta$ is constant on E .*

Proof. The subgroup $Stab_{G_0}(W_E) = \{g \in G_0 : \rho(g)W_E = W_E\}$ is an algebraic subgroup containing P_E , by Lemma 5.2. By Theorem 4, this group is G_0 . Since ρ is irreducible over \mathbf{Q} and W_E is defined over \mathbf{Q} , $W_E = \{0\}$, or $W_E = V$. But the latter cannot happen, because of the choice of E . \square

From now on, we fix some $x_0 \in E$. By the previous Lemma, T_E is the point $\rho(x_0)^{-1}\beta$.

Lemma 5.4. *For any $\gamma \in P_E$, $\rho(\gamma) \in Stab(\mathbf{R}\rho(x_0)^{-1}\beta)$.*

Proof. Let $\gamma \in P_E$. Thus there exist $k \in \mathbf{Z}$ such that $\mathcal{L}(a^k E \gamma \cap E) > 0$. In the proof of Lemma 5.2, we saw that

$$\kappa^{-k}\rho(\gamma)^{-1}T_E = T_E.$$

But since $T_E = \{\rho(x_0)^{-1}\beta\}$, this means that $\rho(\gamma)^{-1}$ stabilizes the line through $\rho(x_0)^{-1}\beta$. \square

The end of the proof of Theorem 2 is given by the following contradictory Claim.

Lemma 5.5. *The space V is one-dimensional.*

Proof. The group

$$\{g \in G : \rho(g) \in \text{Stab}(\mathbf{R}\rho(x_0)^{-1}\beta)\},$$

is an algebraic group containing P_E . By Theorem 4, it follows that $\rho(x_0)^{-1}\beta$ is a common eigenvector for all elements of $\rho(G_0)$ (and so is β). Were the representation ρ irreducible over \mathbf{R} , this would be sufficient to conclude; however we assumed only \mathbf{Q} -irreducibility, and have no particular rationality assumption on β .

Since G_0 is semisimple and connected, the eigenvalue associated to $\rho(x_0)^{-1}\beta$ is 1 for every $g \in G_0$. Let $V_1(g)$ denote the eigenspace associated to the eigenvalue 1 for the operator $\rho(g)$. Let $Z = \bigcap_{\gamma \in P_E} V_1(\gamma)$. This subspace Z is defined over \mathbf{Q} , because P_E consists of integral points. The set of $g \in G_0$ acting trivially on Z is an algebraic subgroup containing P_E , so again is G_0 . Since $\rho(x_0)^{-1}\beta \in Z$, Z is of positive dimension. By \mathbf{Q} -irreducibility of ρ , $Z = V$ is an irreducible representation where G_0 acts trivially, so is one-dimensional. \square

6. ERGODICITY AND MIXING

We now prove Theorem 3. By assumption, a is triangularizable with positive eigenvalues. We separate the proof in two cases.

Case 1: $\rho(a)$ is unipotent.

Let $\beta \in V - \{0\}$ be an eigenvector for $\rho(a)$, its eigenvalue is 1.

Notice that the actions of (e, β) and (a, α) commute: since $\rho(a)\beta = \beta$, we have

$$(a, \alpha)(e, \beta) = (a, \alpha + \rho(a)\beta) = (a, \rho(e)\alpha + \beta) = (e, \beta)(a, \alpha).$$

Consider the ergodic decomposition of λ with respect to the action of (a, α) : there exists a measure m on the set of ergodic, (a, α) -invariant measures on G/Γ , such that

$$\lambda = \int \nu dm(\nu).$$

If we apply the projection map $\pi : G/\Gamma \rightarrow G_0/\Gamma_0$ to this equality, we obtain

$$\mathcal{L} = \mu = \int (\pi)_* \nu dm(\nu),$$

where $(\pi)_*\nu$ are a -invariant. Since μ is a -ergodic, we have that $(\pi)_*\nu = \mu$, m -almost surely. Therefore, m is supported on \mathcal{M}_μ . But since (e, β) commutes with (a, α) , $(e, \beta)_*m$ is the measure associated to the ergodic decomposition of $(e, \beta)_*\lambda = \lambda$. This implies that m is (e, β) -invariant, and by Corollary 1.1, m is the Dirac measure at λ . This concludes the proof of the ergodicity.

Case 2: $\rho(a)$ is not unipotent. Thus $\rho(a)$ has some of its eigenvalues different from 1. Since G_0 is semisimple, $\det(\rho(a)) = 1$ so there exist at least one eigenvalue $\kappa < 1$. Let $\beta \in V$ be an eigenvector of $\rho(a)$ associated to κ .

Notice that , for $k \geq 0$,

$$(a, \alpha)^k(e, \beta) = (a^k, \rho(a)^k\beta + \sum_{i=0}^{k-1} \rho(a)^i\alpha) = (e, \kappa^k\beta)(a, \alpha)^k,$$

and $(e, \kappa^k\beta) \rightarrow (e, 0)$ when $k \rightarrow +\infty$. This implies that the distance (with respect to a distance $d_{G/\Gamma}$ on G/Γ induced by a right- G -invariant riemannian distance on G) between $(a, \alpha)^k(x, v)\Gamma$ and $(a, \alpha)^k(e, \beta)(x, v)\Gamma$ tends to zero as k tends to $+\infty$. In other words, the strong stable distribution for (a, α) , defined by

$$W^{ss}((x, v)\Gamma) = \{(y, w)\Gamma : \lim_{k \rightarrow +\infty} d_{G/\Gamma}((a, \alpha)^k(x, v)\Gamma, (a, \alpha)^k(y, w)\Gamma) = 0\},$$

is invariant under the action of (e, β) .

We first prove the claims about ergodicity and weak-mixing.

Let $f \in L^2(G/\Gamma, \lambda) - \{0\}$ be an eigenvector for the Koopman operator of (a, α) , that is

$$f((a, \alpha)(x, v)\Gamma) = \omega f((x, v)\Gamma),$$

for λ -almost every $(x, v)\Gamma$, for some $\omega \in \mathbf{C}$ of modulus one. Recall that ergodicity states that any such eigenvector associated to $\omega = 1$ is constant almost everywhere, and weak-mixing that any such eigenvector is constant almost everywhere and moreover $\omega \neq 1$.

To prove ergodicity or weak-mixing, we may (and will) restrict to the case where f is bounded. By the Hopf argument, and more precisely the version proved by Coudène [4], f is W^{ss} -invariant, meaning that there exists a full measure set $\Omega \subset G/\Gamma$ such that for all $(x, v)\Gamma \in \Omega$

Ω , $(y, w)\Gamma \in \Omega$ such that $(y, w)\Gamma \in W^{ss}((x, v)\Gamma)$, $f((x, v)\Gamma) = f((y, w)\Gamma)$.

By Theorem 2, the angle map $x \mapsto \rho(x)^{-1}\beta$ is asynchronous. In particular, for every strict linear rational (closed) subtorus $\mathcal{T} \subset K$, the set of x such that $\rho(x)^{-1}\beta$ does not belong to \mathcal{T} is of full measure. As the set of such subtorus is countable, this implies that for μ -almost every x , the translation on K given by $\rho(x)^{-1}\beta$ is ergodic. Via a linear change of variable, this means that for μ -almost every $x\Gamma_0 \in G_0/\Gamma_0$, the translation by (e, β) on the fiber above $x\Gamma_0$ is ergodic.

As f is (e, β) -invariant, for μ -almost every $x\Gamma_0$, f is almost everywhere constant on the fiber above $x\Gamma_0$, and merely depends on $x\Gamma_0$. Write $F \in L^2(G_0/\Gamma_0, \mu)$ for its almost everywhere value, that is $f = F \circ \pi$, λ -almost everywhere, where $\pi : G/\Gamma \rightarrow G_0/\Gamma_0$ is the fiber bundle. We have then

$$F(ax\Gamma_0) = \omega F(x\Gamma_0).$$

If $\omega = 1$, then F is a -invariant, and by ergodicity of a , F is constant μ -a.e., and so f is constant λ -a.e. . This proves the ergodicity.

Assume now that the action of a is weakly mixing on $(G_0/\Gamma_0, \mu)$. Let $f \in L^2(G/\Gamma, \lambda)$ be like previously an eigenvector for the Koopman operator of (a, α) , $F \in L^2(G_0/\Gamma_0, \mu)$ its almost sure value depending on the fiber. By what we saw before, F is an eigenvector for the Koopman operator of a , so $\omega = 1$ by weak-mixing of a . By ergodicity, F is constant, as was to be proved.

We now assume that the action of a on $(G_0/\Gamma_0, \mu)$ is strongly mixing, and wish to prove that (a, α) is also strongly mixing. Recall that strong mixing of a is equivalent to the fact that for all $F \in L^2(G_0/\Gamma_0, \mu)$, $F \circ a^k$ converges weakly to a constant as $k \rightarrow +\infty$.

Let $f \in L^2(G/\Gamma, \lambda)$. Let $g \in L^2(G/\Gamma, \lambda)$ be any weak limit of $f \circ (a, \alpha)^k$ as $k \rightarrow +\infty$ along a subsequence. By another result of Coudène [5], generalizing a result of Babillot, g is W^{ss} -invariant. By ergodicity of (e, β) on almost every fiber, $g = G \circ \pi$ almost surely, where $G \in L^2(G_0/\Gamma_0, \mu)$. Define

$$F(x\Gamma_0) = \int_{\mathbf{R}^d/\rho(x)\mathbf{Z}^d} f((x, v)\Gamma) d\text{Haar}_{\mathbf{R}^d/\rho(x)\mathbf{Z}^d}(v),$$

the mean value of f on each fiber. Let $H \in L^2(G_0/\Gamma_0, \mu)$ be a test-function. Then

$$\int_{G/\Gamma} (f \circ (a, \alpha)^k)(H \circ \pi) d\lambda = \int_{G_0/\Gamma_0} (F \circ a^k).H d\mu.$$

Taking the limits in the left-hand side and right-hand side respectively along the subsequence, using the strong mixing property for a , gives:

$$\int_{G/\Gamma} (G \circ \pi)(H \circ \pi) d\lambda = \left(\int_{G_0/\Gamma_0} F d\mu \right) \left(\int_{G_0/\Gamma_0} H d\mu \right),$$

in other words,

$$\int_{G_0/\Gamma_0} G H d\mu = \left(\int_{G_0/\Gamma_0} F d\mu \right) \left(\int_{G_0/\Gamma_0} H d\mu \right),$$

which implies that G is μ -almost everywhere the constant $\int_{G_0/\Gamma_0} F d\mu = \int_{G/\Gamma} f d\lambda$. Therefore the only possible weak limit of $f \circ (a, \alpha)^k$ is the above constant. By weak compactness of the ball of radius $\|f\|_2$ in $L^2(G/\Gamma, \lambda)$, this proves that this sequence must converge weakly to $\int_{G/\Gamma} f d\lambda$, as required.

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