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► **To cite this version:**

Aneel Tanwani, Christophe Prieur, Sophie Tarbouriech. Input-to-State Stabilization in  $H^1$ -Norm for Boundary Controlled Linear Hyperbolic PDEs with Application to Quantized Control. CDC 2016 - 55th IEEE Conference on Decision and Control, Dec 2016, Las Vegas, NV, United States. pp.3112-3117, 10.1109/CDC.2016.7798735 . hal-01366133

**HAL Id: hal-01366133**

**<https://hal.science/hal-01366133v1>**

Submitted on 18 Jan 2017

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# Input-to-State Stabilization in $\mathcal{H}^1$ -Norm for Boundary Controlled Linear Hyperbolic PDEs with Application to Quantized Control

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**Abstract**—We consider a system of linear hyperbolic PDEs where the state at one of the boundary points is controlled using the measurements of another boundary point. For this system class, the problem of designing dynamic controllers for input-to-state stabilization in  $\mathcal{H}^1$ -norm with respect to measurement errors is considered. The analysis is based on constructing a Lyapunov function for the closed-loop system, which leads to controller synthesis and the conditions on system dynamics required for stability. As an application of this stability notion, the problem of quantized control for hyperbolic PDEs is considered where the measurements sent to the controller are communicated using a quantizer of finite length. The presence of quantizer yields practical stability only, and the ultimate bounds on the norm of the state trajectory are also derived.

## I. INTRODUCTION

Over the past decade, an enormous amount of interest has grown in designing control algorithms for distributed systems because of their utility in modeling complex physical phenomenon and as a tool for analysis. Hyperbolic partial differential equations (PDEs) represent a class of such infinite dimensional systems which not only model physical laws (such as shallow water equations), but also present a tool for modeling communication delays in control systems. Several results on Lyapunov stability of hyperbolic systems, and feedback control design have been published in the recent past, see the book [1].

*Problem description:* The problem of interest for us is to address feedback control for the class of linear hyperbolic PDEs described by the equation

$$\frac{\partial X}{\partial t}(z, t) + \Lambda \frac{\partial X}{\partial z}(z, t) = 0 \quad (1a)$$

where  $z \in [0, 1]$ , and  $t \in [0, \infty)$ . The matrix  $\Lambda$  is assumed to be diagonal and positive definite. We call  $X : [0, 1] \times \mathbb{R}_+ \rightarrow \mathbb{R}^n$  the state trajectory, and the initial condition is defined as

$$X(z, 0) = X^0(z), \quad z \in (0, 1) \quad (1b)$$

for some function  $X^0 : [0, 1] \rightarrow \mathbb{R}^n$ . The value of the state  $X$  is controlled at the boundary  $z = 0$  through some input  $u : \mathbb{R}_+ \rightarrow \mathbb{R}^m$  so that

$$X(0, t) = HX(1, t) + Bu(t) \quad (2)$$

where  $H \in \mathbb{R}^{n \times n}$  and  $B \in \mathbb{R}^{n \times m}$  are constant matrices. We consider the case when only the measurement of the state  $X$

at the boundary point  $z = 1$  is available for each  $t \geq 0$ . We thus denote the output of the system by

$$y(t) = X(1, t) + d(t) \quad (3)$$

where  $d \in \mathcal{L}^\infty([0, \infty), \mathbb{R}^n)$  is seen as the perturbation in the measurement of the state trajectory at the boundary point.

We are interested in designing a control law  $u$  as a function of the output measurement  $y$ , which stabilizes the system in some appropriate sense. In case there are no perturbations, that is,  $d \equiv 0$ , one typically chooses  $u(t) = Ky(t)$  such that the closed-loop boundary condition

$$X(0, t) = (H + BK)X(1, t) \quad (4)$$

satisfies a certain dissipative condition. This control law yields asymptotic stability of the system with respect to  $\mathcal{H}^2$ -norm [5], or  $\mathcal{C}^1$ -norm [4], depending on the dissipativity criterion imposed on  $H + BK$ . In the presence of perturbations  $d \neq 0$ , one has to modify the stability criteria as the asymptotic stability of the origin can no longer be established. We borrow the notion of *input-to-state stability* (ISS) from the control-theoretic literature on finite-dimensional systems. The basic idea is to regard the perturbation  $d$  as an exogenous input to the closed-loop system and obtain an upper bound on the norm of the state trajectory in terms of a certain norm of the perturbation  $d$ , modulo some decaying term due to the initial condition. The notion further encapsulates the behavior that as  $d(t) \rightarrow 0$  with  $t \rightarrow \infty$ , the corresponding norm of the state trajectory also converges to the origin.

In this paper, the problem of interest is to obtain such a bound on  $\mathcal{H}^1$ -norm of the state  $X(\cdot, t)$  in terms of the  $\mathcal{L}^\infty$ -norm of  $d_{[0, t]}$ .

*Motivation:* The motivation for aforementioned theoretical problem comes from the application in quantized control. When the measurement  $X(1, t)$  can not be passed precisely to the controller, but has to be encoded using finitely many symbols, one can see  $d$  in (3) as the error between the actual value and the quantized value of the signal  $X(1, t)$ . The quantizers are typically designed to operate over a compact set in the state space. Within this operating region the quantization error remains constant, and hence one expects the state trajectory to converge to a ball around the origin parameterized by the size of quantization error. Hence, to obtain this practical stability, the controller must ensure that the state trajectory remains within the compact set for which the quantizer is designed.

To implement this methodology in the context of PDEs under consideration, the problem is to find a controller which ensures that

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- the maximum norm of  $X(1, \cdot)$  remains bounded within the operating region of the quantizer, and
- input-to-state stability with respect to quantization error is achieved.

*Theoretical Challenges:* If we choose to control the  $\mathcal{L}^2$ -norm of the state trajectory only, the problem is that it doesn't yield any bounds on  $\|X(1, \cdot)\|_\infty$ . One way to obtain the bound on the norm of  $X(1, t)$  is to use the trace theorem [8, Section 5.5, Theorem 1], see also Lemma 1 in Section V. This result gives us a bound on  $X(1, t)$  in terms of  $\|X(\cdot, t)\|_{\mathcal{L}^2} + \|\partial_z X(\cdot, t)\|_{\mathcal{L}^2}$ , or  $\mathcal{H}^1$ -norm of  $X$ . Lyapunov analysis based on  $\mathcal{H}^1$ -norm in the presence of control input becomes complicated.

In applications like quantized control, the perturbations in the output measurements are discontinuous functions of time. In such cases, the use of static controllers of the form  $u(t) = Ky(t)$ , will typically not result in trajectories  $X$  which are differential with respect to spatial variable.

*Objectives:* Our goal in this paper is to design controllers which establish

- existence of sufficiently smooth solutions in the sense that  $X_z$  is well-defined,
- ISS in  $\mathcal{H}^1$ -norm, and
- practical convergence using quantized measurements.

*Our approach:* To achieve the aforementioned objectives, we propose to use a dynamic controller instead of a static one. It has the advantage that the resulting solutions in the closed-loop have desired regularity, that is, existence of  $X_z$  is guaranteed. The addition of dynamic controller introduces a coupling of ODEs and PDEs in the closed-loop, which makes the stability analysis more challenging. We use Lyapunov function based analysis to synthesize the controller and guarantee ISS with respect to the perturbation  $d$ . The results are then used to study the application of quantized control. We establish practical stability of the system, and derive ultimate bounds on the state trajectory in terms of the quantization error. Thus, the closed-loop to be analyzed is depicted in Figure 1.

*Contribution in the Literature:* One finds the Lyapunov stability criteria with  $\mathcal{L}^2$ -norm and dissipative boundary conditions in [2]. Lyapunov stability in  $\mathcal{H}^2$ -norm for nonlinear systems is treated in [5]. Thus, the construction of Lyapunov

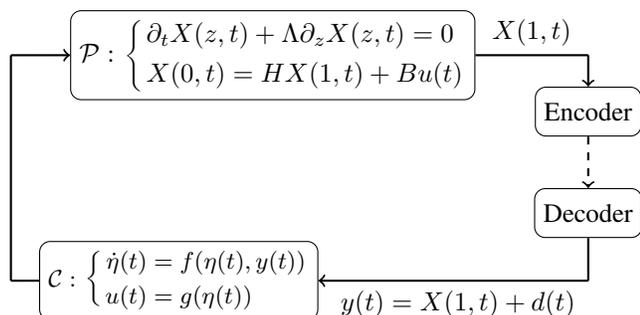


Fig. 1: Communication architecture used for control of hyperbolic system in the presence of quantization.

functions in  $\mathcal{H}^2$ -norm for the hyperbolic PDEs with static control laws can be found in the literature. Because our controller adds dynamics to the closed-loop, the basic idea behind the construction of Lyapunov function for the closed-loop system is to use the ISS property of the hyperbolic PDE and the controller dynamics.

In the literature, one finds various instances where the ISS related tools are used for stability analysis of interconnected systems. In the paper [10], an integral ISS Lyapunov function is computed for a networks described by a finite-dimensional nonlinear function. Small gain theorem is crucial when interconnecting ISS systems as exploited in [9], [7].

For infinite dimensional systems, the problem of ISS has attracted attention recently but most of the existing works treat the problem with respect to uncertainties in the dynamics. See, for example [15], where a class of linear and bilinear systems is studied. See also [6] where a linearization principle is applied for a class of infinite-dimensional systems in a Banach space. When focusing on parabolic partial differential equations, some works to compute ISS Lyapunov functions have also appeared, such as [14], [13]. For time-varying hyperbolic PDEs, construction of ISS Lyapunov functions has also been addressed in [17].

However, the results on ISS with respect to measurement errors have not yet appeared in the literature; The only exception being the recent work reported in [11], which derives ISS bounds for 1-D parabolic systems in the presence of boundary disturbances but without the use of Lyapunov-based techniques. Such questions have remained unaddressed for hyperbolic PDEs, which is the topic of this paper. Furthermore, the paper also includes a design element in the sense that the controller that achieves this ISS property is also synthesized. On the other hand, the study of quantized control has only been studied in finite-dimensional systems so far [12], [16], and this paper extends this problem setting to the case of PDEs.

The remainder of this paper is organized as follows: Section II contains the notion of ISS adopted in this paper, and formalizes the problem statement. The design of the proposed controller appears in Section III, and the stability analysis of the closed-loop is carried out in Section IV. The application to the case of quantized control is treated in Section V. An illustrative example with simulations is given in Section VI.

## II. PRELIMINARIES

In this section, we recall solution concepts for hyperbolic PDEs, and the stability notions that would be used later in the paper. Before addressing the class of solutions, we first introduce the notation related.

For a function  $X : [0, 1] \rightarrow \mathbb{R}^n$ , we denote its gradient by  $\partial X$ , or  $\partial_z X$ . The space  $\mathcal{W}^{k,p}([0, 1], \mathbb{R}^n)$  comprises of functions for which the  $k$ -th derivative, denoted  $\partial^k X$ , exists and  $\partial^k X \in \mathcal{L}^p([0, 1], \mathbb{R}^n)$ . We use the shorthand  $\mathcal{H}^1$  for the space  $\mathcal{W}^{1,2}$ . The space  $\mathcal{H}^1$  is naturally equipped with the  $\mathcal{H}^1$ -norm defined as:

$$\|X\|_{\mathcal{H}^1([0,1],\mathbb{R}^n)}^2 := \|X\|_{\mathcal{L}^2([0,1],\mathbb{R}^n)}^2 + \|\partial X\|_{\mathcal{L}^2([0,1],\mathbb{R}^n)}^2.$$

For system class (1), we choose to work with the class of inputs  $u$  that are absolutely continuous functions of time, so that their derivative is defined Lebesgue a.e. For such inputs, we seek a solution  $X \in \mathcal{C}^0([0, T], (\mathcal{H}^1([0, 1], \mathbb{R}^n)))$  where  $\mathcal{C}^0$  denotes the space of continuous functions equipped with supremum norm.

To give conditions for stability, we consider the set  $\mathcal{D}_+^n$  of diagonal positive definite matrices of order  $n \times n$ . For a matrix  $M \in \mathbb{R}^{n \times n}$ , we define  $\rho_p(M)$  as

$$\rho_p(M) := \inf\{\|DMD^{-1}\|_p : D \in \mathcal{D}_+^n\} \quad (5)$$

where  $\|\widetilde{M}\|_p := \max\{\|\widetilde{M}x\|_p : \|x\|_p = 1\}$  denotes the induced norm of the matrix  $\widetilde{M}$ .

### A. Stability Notions

In the literature on stability analysis of hyperbolic PDEs, we find several notions of stability depending on the norm with which the solution space is equipped. The criterion we generalize in this work is based on  $\mathcal{L}^2$ -stability of the state trajectory  $X(\cdot, t)$ .

**Definition 1:** *The system (1) is called exponentially stable with respect to  $\mathcal{L}^2$ -norm if there exist two positive values  $c$  and  $a$  such that for all  $t \geq 0$ , and for all  $X^0 \in \mathcal{L}^2([0, 1], \mathbb{R}^n)$ ,*

$$\|X(\cdot, t)\|_{\mathcal{L}^2([0, 1], \mathbb{R}^n)} \leq ce^{-at} \|X^0\|_{\mathcal{L}^2([0, 1], \mathbb{R}^n)}.$$

However, when there are perturbations in the output measurement, so that  $y(t) = X(1, t) + d(t)$ , then we are interested in the following notion of stability.

**Definition 2:** *The system (1) is called input-to-state stable with respect to disturbance  $d$  if there exists a class  $\mathcal{KL}$  function  $\beta$  and a class  $\mathcal{K}$  function  $\gamma$  such that, for all  $t \geq 0$ ,*

$$\|X(\cdot, t)\|_{\mathcal{H}^1([0, 1], \mathbb{R}^n)} \leq \beta(\|X^0\|_{\mathcal{H}^1([0, 1], \mathbb{R}^n)}, t) + \gamma(\|d_{[0, t]}\|_\infty), \quad (6)$$

where  $d_{[0, t]}$  denotes the restriction of the function  $d$  on the interval  $[0, t]$ .

### B. Problem Statement

In [2], the criterion for exponential stability in  $\mathcal{L}^2$ -norm, or  $\mathcal{H}^2$ -norm is to choose feedback control  $u = KX(1, t)$  such that  $\rho_2(H + BK) < 1$ . The goal is to build on this stability condition, and develop a methodology for designing (dynamic) controllers that result in input-to-state stabilization of the system.

More precisely, we consider the problem of designing a dynamic controller with ODEs, which has the form

$$\dot{\eta}(t) = f(\eta(t), y(t)) \quad (7a)$$

$$u(t) = g(\eta(t)) \quad (7b)$$

where  $f, g$  are sufficiently regular functions, which need to be chosen appropriately.

In this article, we ask ourselves whether there exists a controller from the class of controllers described by (7) which achieves the following objectives:

- ISS in  $\mathcal{H}^1$ -norm
- Boundedness of  $X(1, \cdot)$ .

The reason for emphasizing the use of dynamic controllers is that we are looking for a way to bound  $|X(1, t)|$ , which in our knowledge is only possible if a bound on the  $\mathcal{H}^1$ -norm of  $X(\cdot, t)$  is obtained. Existence of solutions  $X$  in the space  $\mathcal{H}^1([0, 1], \mathbb{R}^n)$  requires us to use inputs which are at least absolutely continuous. If we allow perturbations  $d$  to be discontinuous, static controllers would not yield smooth enough solutions. The dynamic controller is therefore added to smoothen the discontinuity effect of the perturbations.

## III. CONTROL DESIGN AND EXISTENCE OF SOLUTIONS

As a solution to the problem formulated in the previous section, we now specify the controller dynamics, and give comments on the existence of solutions for the closed-loop system. The stability of the closed-loop system is then analyzed in next section using Lyapunov-based methods where the stability conditions are also stated precisely.

### A. Controller Design

The controller that we choose for our purposes is described by the following equations:

$$\begin{aligned} \dot{\eta}(t) &= -\alpha(\eta(t) - y(t)) \\ &= -\alpha\eta(t) + \alpha X(1, t) + \alpha d(t) \end{aligned} \quad (8a)$$

$$\eta(0) = \eta^0 \in \mathbb{R}^n \quad (8b)$$

$$u(t) = K\eta(t), \quad (8c)$$

where  $\eta^0$  is the initial condition for the controller dynamics, and where the conditions on the constant  $\alpha > 0$ , and the matrix  $K \in \mathbb{R}^{m \times n}$  will be stated in the next section (see Theorem 1).

For the system in the closed-loop, the dynamics of the state trajectory  $X$  are given by

$$\frac{\partial X}{\partial t}(z, t) + \Lambda \frac{\partial X}{\partial z}(z, t) = 0 \quad (9a)$$

$$X(z, 0) = X^0(z), \quad \forall z \in [0, 1], \quad (9b)$$

$$X(0, t) = HX(1, t) + BK\eta(t). \quad (9c)$$

For what follows, we are also interested in analyzing the dynamics of  $\partial_z X =: X_z$ , which are derived as follows:

$$\frac{\partial X_z}{\partial t}(z, t) + \Lambda \frac{\partial X_z}{\partial z}(z, t) = 0. \quad (10)$$

To obtain the boundary condition, from (9c), we have

$$X_t(0, t) = HX_t(1, t) + BK\dot{\eta}(t)$$

Substituting  $X_t(z, t) = -\Lambda X_z(z, t)$  for each  $z \in [0, 1]$ , we get

$$X_z(0, t) = \Lambda^{-1} H \Lambda X_z(1, t) - \Lambda^{-1} BK\dot{\eta}(t). \quad (11)$$

### B. Well-posedness of the Closed-Loop

We give some remarks on how to establish the existence of solutions in the closed-loop. One can view system (1) being driven by the exogenous signal  $\eta$  which is absolutely continuous in time. Under compatibility conditions on the initial condition  $(X^0, \eta^0)$ , one can generalize the result of [3, Theorem 2.4] to show that there exists a solution in  $X \in \mathcal{C}^0([0, T], \mathcal{H}^1([0, 1], \mathbb{R}^n))$ .

#### IV. STABILITY RESULT

The first main contribution of the paper is to present conditions on the controller dynamics (8) which results in ISS of system (1) with respect to the measurement disturbances  $d$ . To state the result, we introduce some notation. For scalars  $\mu > 0$  and  $0 < \nu < 1$ , let  $\rho := e^{-\mu} - \nu$ ; let  $F := BK$ , and  $Q := F^\top D^2 F$  for  $D \in \mathcal{D}_+^n$ ; and finally, let  $G := H^\top D^2 F$ . We denote by  $\Omega$  the matrix

$$\begin{bmatrix} \rho\beta_1 D^2 & -\beta_1(G+Q) & 0 \\ -\beta_1(G+Q) & 2\alpha\beta_3 - (\beta_1 + \alpha^2\beta_2)Q & \beta_3 I + \alpha\beta_2 G \\ 0 & \beta_3 I + \alpha\beta_2 G & (\rho D^2 + Q + 2G)\beta_2 \end{bmatrix}$$

in which  $\alpha, \beta_1, \beta_2, \beta_3$  are some positive constants.

**Theorem 1:** *Assume that there exist scalars  $\mu, \nu > 0$ , a matrix  $D \in \mathcal{D}_+^n$ , the gain matrix  $K$ , and the positive constants  $\alpha, \beta_1, \beta_2, \beta_3$  in the definition of  $\Omega$  such that*

$$\|D(H+BK)D^{-1}\|_2 \leq \nu < 1, \quad (12a)$$

$$\Omega > \delta I \quad (12b)$$

for some scalar  $\delta > 0$ . Then, the closed-loop system is ISS in  $\mathcal{H}^1$ -norm with respect to measurement disturbances  $d$ .

**Remark 1:** In the statement of Theorem 1, condition (12a) requires  $\rho_2(H+BK) < 1$ , which also appears in the more general context of nonlinear systems [5] when analyzing stability with respect to  $\mathcal{H}^2$ -norm. However, the condition (12b) is introduced in our work to compensate for the lack of proportional gain in the feedback law. It definitely restricts the class of systems that can be treated with our approach and relaxing this condition or obtaining different criteria is a topic of further investigation.  $\circ$

**Remark 2:** At this moment, we do not have a precise characterization of the parameters of system (1) for which (12) admits a solution. As a particular instance, assume that (12a) holds with  $K = 0$ . In that case, the matrix  $\Omega$  simplifies greatly as  $Q = G = 0$ . Using the Schur complement, one can immediately find the constants  $\alpha, \beta_1, \beta_2, \beta_3$  that result in  $\Omega$  being positive definite, and hence satisfying (12b). By applying the continuity argument for solutions of matrix inequalities with respect to parameter variations, the solution to (12b) will also hold for  $K \neq 0$ , but sufficiently small.  $\circ$

Due to space constraints, the detailed proof of Theorem 1 is not provided here, but in the remainder of the section we sketch an outline describing the main steps of the proof.

*Proof sketch:* The primary idea is to introduce a Lyapunov function and analyze its derivative with respect to time. As a candidate, we choose  $V : \mathcal{H}^1([0, 1], \mathbb{R}^n) \times \mathbb{R}^n \rightarrow \mathbb{R}_+$  given by

$$V := V_1 + V_2 + V_3 \quad (13)$$

where  $V_1 : \mathcal{H}^1([0, 1], \mathbb{R}^n) \rightarrow \mathbb{R}_+$  is defined as,

$$V_1(X) := \int_0^1 X^\top(z) P_1 X(z) e^{-\mu z} dz,$$

in which  $P_1 := \beta_1 D^2 \Lambda^{-1}$  is a diagonal positive definite matrix, with  $\beta_1$  and  $D$  satisfying the hypotheses of Theorem 1.

Similarly,  $V_2 : \mathcal{H}^1([0, 1], \mathbb{R}^n) \rightarrow \mathbb{R}_+$  is given by

$$V_2(X) := \int_0^1 \partial X^\top(z) P_2 \partial X(z) e^{-\mu z} dz,$$

where  $P_2 := \beta_2 D \Lambda$  is a diagonal positive definite matrix with  $\beta_2$  satisfying the conditions in Theorem 1. Finally,  $V_3 : \mathcal{H}^1([0, 1], \mathbb{R}^n) \times \mathbb{R}^n \rightarrow \mathbb{R}_+$  is given by

$$V_3(X, \eta) = (\eta - X(1))^\top P_3 (\eta - X(1)),$$

where  $P_3 := \beta_3 I_{n \times n}$  is a symmetric positive definite matrix with  $\beta_3 > 0$ .

It is evident that there exist constants  $c_1, c_2 > 0$  such that, for all  $X \in \mathcal{H}^1([0, 1], \mathbb{R}^n)$ , and  $\eta \in \mathbb{R}^n$ ,

$$c_1(\|X\|_{\mathcal{H}^1}^2 + |\eta - X(1)|^2) \leq V(X, \eta) \leq c_2(\|X\|_{\mathcal{H}^1}^2 + |\eta - X(1)|^2).$$

The next step is to compute the derivative of  $V_i$ ,  $i = 1, 2, 3$ , and invoke the conditions listed in (12) to obtain an upper bound on the time-derivative of  $V$  along the solutions to (8)-(9) as follows:

$$\dot{V}(X(t), \eta(t)) \leq -\sigma V(X(t), \eta(t)) + \chi |d(t)|^2 \quad (14)$$

for some constant  $\sigma, \chi > 0$ . It readily follows that

$$V(X(t), \eta(t)) \leq e^{-\sigma t} V(X(0), \eta(0)) + \gamma (\|d_{[0,t]}\|_\infty),$$

for a suitable function  $\gamma$  of class  $\mathcal{K}$ , which is the desired estimate for ISS property.

#### V. QUANTIZED CONTROL

We are interested in studying stabilization of the system (1) when the output  $X(1, \cdot)$  is quantized using a set of finite alphabets, and cannot be transmitted to the control precisely. To define a quantizer, we first specify a set of finite alphabets  $\mathcal{Q} := \{q_1, q_2, \dots, q_N\}$ . A quantizer with sensitivity  $\Delta_q > 0$ , and range  $M_q > 0$ , is then a function  $q : \mathbb{R}^n \rightarrow \mathcal{Q}$  having the property that

$$|q(x) - x| \leq \Delta_q \quad \text{if} \quad |x| \leq M_q \quad (15)$$

and

$$|q(x)| \geq M_q - \Delta_q \quad \text{if} \quad |x| > M_q. \quad (16)$$

In other words, within the space  $\mathbb{R}^n$ , where the measurements of  $X(1, \cdot)$  take values, we take a ball of radius  $M_q$ , and partition it into  $N$  regions. Each of these regions is identified with a symbol  $q_i$  from the set  $\mathcal{Q}$ . If  $|X(1, t)| \leq M_q$ , the controller receives a valid symbol and knows the variable  $X(1, t)$ , modulo the error due to sensitivity of the quantizer. When the measurements are out of the range of the quantizer, then the quantizer just sends an out of bounds flag and no upper bound on the error between  $X(1, t)$  and its quantized value can be obtained in that case. For this paper, we limit ourselves to the case of static quantizers, that is, the parameters of the quantizer are assumed to be fixed which introduces a bounded measurement error determined by the

sensitivity of the quantizer. The controller (8) takes the form

$$\dot{\eta}(t) = -\alpha \eta(t) + \alpha q(X(1, t)) \quad (17a)$$

$$u(t) = K\eta(t). \quad (17b)$$

The ratio between the range and the sensitivity of the quantizer  $M_q/\Delta_q$  determines the rate at which the information is communicated by the quantizer on average. The basic idea of the quantized control in finite-dimensional systems is to show that the state of the system converges to a certain ball around the origin if this rate is sufficiently large (to dominate the most unstable mode) [16]. In the same spirit, we derive a lower bound on the ratio  $M_q/\Delta_q$  which is required to achieve practical stability in the presence of quantization errors.

To state this result, we need the following lemma which relates  $|X(1, t)|$  with the value of the Lyapunov function  $V$  considered in the previous section.

**Lemma 1:** *There exists a constant  $C > 0$  such that*

$$|X(1, t)|^2 \leq C V(X(\cdot, t), \eta(t)), \quad \forall t \geq 0, \quad (18)$$

for the Lyapunov function  $V$  defined in (13).

The proof of Lemma 1 is omitted but we emphasize that the value of  $C$  in (18) can be computed directly in terms of closed-loop system data.

**Theorem 2:** *Assume that the conditions of Theorem 1 hold, and the initial condition  $X^0$  and  $\eta(0)$  satisfy*

$$C V(X^0, \eta(0)) \leq M_q^2 \quad (19)$$

where the constant  $C$  is obtained from (18). With the constants  $\sigma, \chi$  appearing in (14), if the quantizer is designed such that

$$\frac{M_q^2}{\Delta_q^2} > \frac{C\chi}{\sigma}, \quad (20)$$

then the following items hold:

- The output  $X(1, t)$  remains within the range of the quantizer for all  $t \geq 0$ , that is,

$$|X(1, t)| \leq M_q, \quad \forall t \geq 0.$$

- The state of the system remains ultimately bounded in  $\mathcal{H}^1$ -norm, that is, there exists  $T$  such that for all  $t \geq T$

$$V(X(\cdot, t), \eta(t)) \leq \gamma_q(\Delta_q)$$

where  $\gamma_q(s) = \mu \frac{\chi}{\sigma} s^2 (1 + \varepsilon)$ , for some sufficiently small  $\varepsilon > 0$ , is a class  $\mathcal{K}$  function.

*Proof of Theorem 2.* In the light of condition (20), fix  $\varepsilon > 0$  such that

$$\frac{\chi}{\sigma} \Delta_q^2 (1 + \varepsilon) \leq \frac{M_q^2}{C}.$$

When the controller uses quantized measurements of  $X(1, t)$ , the derivative of the Lyapunov function in (14) satisfies

$$\dot{V}(X(t), \eta(t)) \leq -\sigma V(X(t), \eta(t)) + \chi |q(X(1, t)) - X(1, t)|^2.$$

Thus, for the chosen  $\varepsilon > 0$ , if

$$\frac{\chi}{\sigma} \Delta_q^2 (1 + \varepsilon) \leq V(X(t), \eta(t)) \leq \frac{M_q^2}{C}$$

then, using (18),  $|X(1, t)| \leq M_q$  implying that  $|q(X(1, t)) - X(1, t)| \leq \Delta_q$ , and hence

$$\dot{V}(X(t), \eta(t)) \leq -\varepsilon \frac{\chi}{\sigma} \Delta_q^2.$$

From the constraints imposed on the initial condition of the system, it readily follows from the above inequality that

$$|X(1, t)| \leq C V(X(t), \eta(t)) \leq M_q, \quad \forall t \geq 0,$$

and hence the quantization error is always upper bounded by  $\Delta_q$ . The uniform decrease in the value of  $V$  also guarantees that

$$V(X(t), \eta(t)) \leq \frac{\chi}{\sigma} \Delta_q^2 (1 + \varepsilon).$$

for sufficiently large  $t$ .  $\square$

## VI. EXAMPLE

For the illustration of our results, the simulations for a  $2 \times 2$  hyperbolic system are reported in this section. The system we consider is of the form (1) with

$$\Lambda := \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

and the boundary condition is described by

$$H = \begin{bmatrix} 1.25 & 0 \\ 0 & 1.25 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Selecting the matrix  $K = \begin{bmatrix} 0 & 0.5 \\ -0.25 & -0.5 \end{bmatrix}$ , it could be checked the boundary damping condition  $\rho_2(H + BK) < 1$  of [5] is satisfied, and thus the exponential stability of (1a) with the boundary condition (4) follows. Select the following initial condition which satisfies the first-order compatibility condition for the existence of solutions in  $\mathcal{H}^1([0, 1], \mathbb{R}^n)$ :

$$X_1(z, 0) = \cos(4\pi z) - 1, \quad X_2(z, 0) = \cos(2\pi z) - 1,$$

for  $z \in [0, 1]$ .

Now to implement the controller with measurement disturbances and illustrate Theorem 2, let us consider the quantizer given by  $q(x) = \lfloor \ell x \rfloor / \ell$  with the parameter  $\ell$ . The error due to quantization in this case is  $\Delta_q = 1/\ell$ , and for the sake of simplicity we take the range to be sufficiently large.

The time-evolution of the solutions for the first and second component of  $X$ , as well as the state of the dynamic controller  $\eta$  are plotted in Figure 2 for  $\ell = 0.05$ . It could be seen that the solution to (2) and (17) converges to a neighborhood of the origin as the time increases. Further numerical simulations show that the size of this neighborhood shrinks as the sensitivity of the quantizer gets smaller, that is, the parameter  $\ell$  of the quantizer is increased, as predicted by Theorem 2. This is readily observed in the simulations of closed-loop trajectories reported in Figure 3 for  $\ell = 5$ .

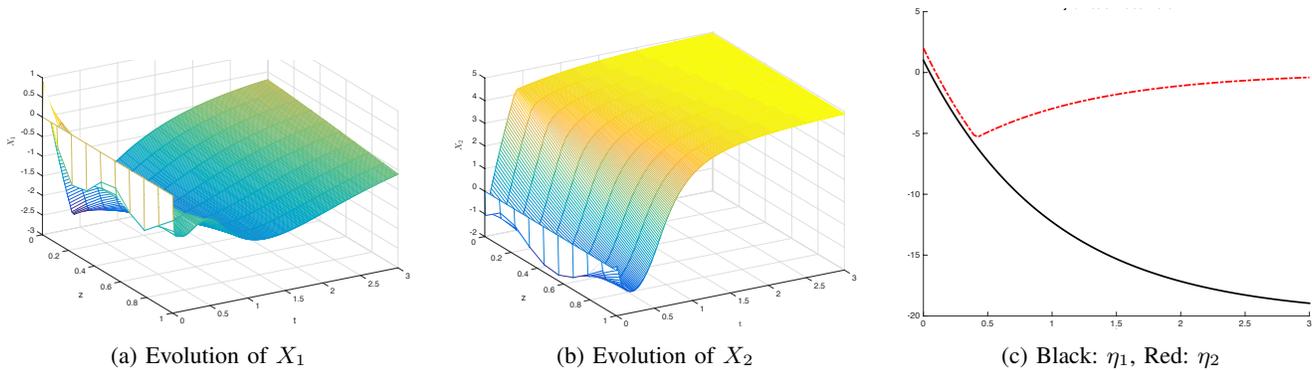


Fig. 2: Evolution of closed-loop trajectories with  $\ell = 0.05$ .

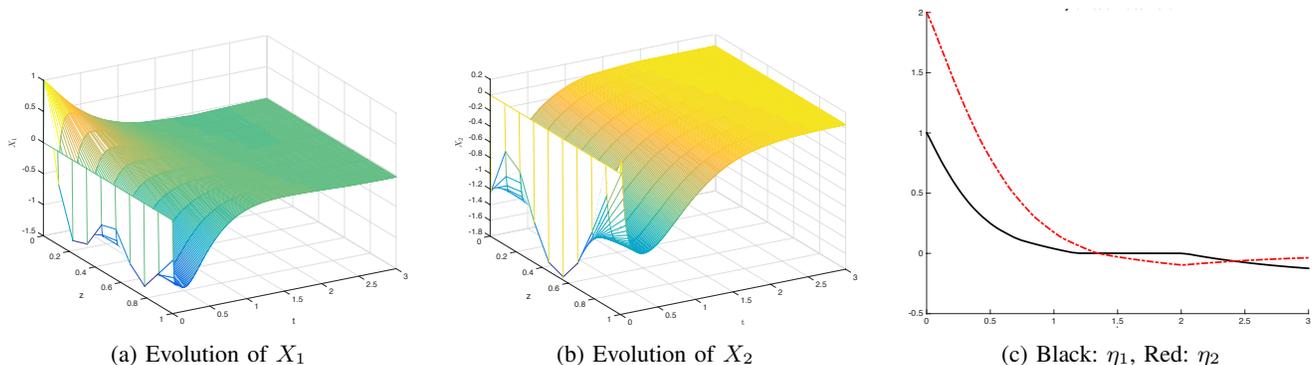


Fig. 3: Evolution of closed-loop trajectories with  $\ell = 5$ .

## VII. CONCLUSIONS

We considered the problem of stabilization of boundary controlled linear hyperbolic PDEs in the presence of measurement errors in the output. A class of dynamic controllers is proposed under certain conditions which allow us to achieve ISS in  $\mathcal{H}^1$ -norm with respect to these disturbances. The results are used for the case when the output measurements are quantized over a finite alphabet set before being passed to the controller. If the initial condition of the system is within the range of the quantizer, the resulting state trajectory is shown to converge to a ball parameterized by the quantization error. Lower bounds on the cardinality of the alphabet set for the quantizer to achieve stability are also given.

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