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EXACT BAYESIAN RESTORATION IN NON-GAUSSIAN MARKOV-SWITCHING TREES

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ABSTRACT. Multiresolution signal and image analysis and multiscale algorithms are of interest in many fields. In particular, efficient Bayesian restoration algorithms have been proposed for some tree-structured Markovian models. In this paper we show that Bayesian filtering and prediction can be performed exactly, with complexity linear in time index, in a particular class of Triplet Markov Trees.

1 INTRODUCTION

Multiresolution signal and image analysis and multiscale algorithms are of interest in many fields (Daubechies *et al.* (1992), Krim *et al.* (1999) and Starck and Bijaoui (1998)). In particular, efficient restoration algorithms in statistical models defined on Hidden Markov trees (HMT) have been developed (see e.g. Chou *et al.* (1994), Laferté *et al.* (2000) or Willsky (2002)).

Let us first briefly recall the definition of a Markov Tree (MT). Let \mathcal{S} be a finite set of indices and let us consider a tree with nodes indexed by \mathcal{S} . Let us consider a partition $\mathcal{S} = \{\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_N\}$, where \mathcal{S}_n are the generations of the tree : \mathcal{S}_1 is the root node r , \mathcal{S}_2 is the set of its children, and so on. Each node s except the root node r has exactly one parent s^- , the set of the children of s is denoted by s^+ , the set of all descendants of s by s^{++} and the set of all ancestors of s by s^{--} . We also denote by $a(s)$ the set of all ancestors of s and s itself (i.e. $a(s) = \{s^{--}, s\}$). Without loss of generality we consider here the case of dyadic trees: each node $s \notin \mathcal{S}_N$ has exactly two children s_1 and s_2 (i.e. $s^+ = \{s_1, s_2\}$) (see Fig. 1). Each node s is associated with a random variable $x(s)$. Also we introduce the notation $x_{\mathcal{S}} = \{x(s), s \in \mathcal{S}\}$. The tree is a Markov one if

$$p(x_{\mathcal{S}}) = p(x_r) \prod_{s \in \mathcal{S} \setminus \mathcal{S}_1} p(x_s | x_{s^-}). \quad (1)$$

Let now $x_{\mathcal{S}} = \{x(s), s \in \mathcal{S}\}$ and $y_{\mathcal{S}} = \{y(s), s \in \mathcal{S}\}$ be two sets of variables defined on the same set \mathcal{S} . Variables $x(s)$ (resp. $y(s)$) are hidden (resp. observed). $(x_{\mathcal{S}}, y_{\mathcal{S}})$ is an HMT if their joint distribution satisfies:

$$p(x_{\mathcal{S}}, y_{\mathcal{S}}) = p(x_r) \prod_{s \in \mathcal{S} \setminus \mathcal{S}_1} p(x_s | x_{s^-}) \prod_{s \in \mathcal{S}} p(y_s | x_{\mathcal{S}}), \quad (2)$$

i.e. x is an MT and $p(y_{\mathcal{S}} | x_{\mathcal{S}}) = \prod_{s \in \mathcal{S}} p(y_s | x_{\mathcal{S}})$. HMT have been extended to Pairwise Markov Trees (PMT) (see Pieczynski (2002) and Desbouvries *et al.* (2006)) defined by:

$$p(z_{\mathcal{S}}) = p(z_r) \prod_{s \in \mathcal{S} \setminus \mathcal{S}_1} p(z_s | z_{s^-}),$$

2 EXACT FILTERING ON SWITCHING-MARKOV TREES

Let $x = \{x_s\}_{s \in \mathcal{S}}$, $y = \{y_s\}_{s \in \mathcal{S}}$ and $r = \{r_s\}_{s \in \mathcal{S}}$ be sets of random variables indexed by \mathcal{S} . Each x_s (resp. y_s) takes its values in \mathbb{R}^q (resp. \mathbb{R}^m) and r_s takes its values in $\Omega = \{\omega_1, \dots, \omega_t\}$.

We consider the following particular TMT model(see Fig.2):

$$(R_S, Y_S) \text{ is a Markov Tree;} \quad (3)$$

$$X_s = F_{s^-}(R_{s^-}, Y_{s^-})X_{s^-} + U_{s^-}; \quad (4)$$

where $\{U_s\}_{s \in \mathcal{S}}$ are independent zero-mean random vectors, such that for each $s \in \mathcal{S}$, U_s is independent from (R_S, Y_S) and from X_r . Note that in (4) vectors U_s are not necessarily Gaussian.

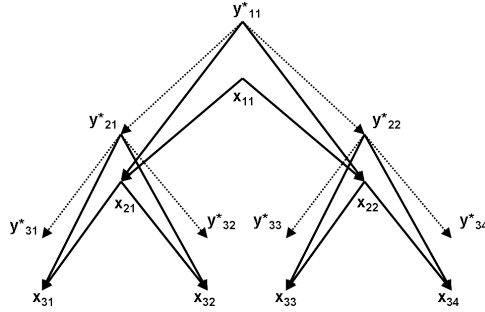


Figure 2. Switch-Markov Tree, with $y^* = (r, y)$

In this section we aim at computing $E[X_s | Y_{a(s)} = y_{a(s)}]$ and $Cov(X_s | Y_{a(s)} = y_{a(s)})$ for any $s \in \mathcal{S}$.

Notation

For each $p \in \mathcal{S}$ and $s \in \mathcal{S}$ let us first set:

$$M_p(r_p, y_{a(s)}) = \int_{\mathbb{R}^q} x_p p(x_p, r_p | y_{a(s)}) dx_p \quad (5)$$

If the covariance matrix Σ_p of U_p exists for all p , let us set:

$$V_p(r_p, y_{a(s)}) = \int_{\mathbb{R}^q} x_p x_p^T p(x_p, r_p | y_{a(s)}) dx_p \quad (6)$$

Of course, $E[X_s | Y_{a(s)} = y_{a(s)}]$ and $Cov(X_s | Y_{a(s)} = y_{a(s)})$ can be computed from $M_s(r_s, y_{a(s)})$ and $V_s(r_s, y_{a(s)})$ as:

$$E[X_s | Y_{a(s)} = y_{a(s)}] = \sum_{r_s} M_s(r_s, y_{a(s)}) \text{ and}$$

$$\text{Cov}(X_s|Y_{a(s)} = y_{a(s)}) = \sum_{r_s} V_s(r_s, y_{a(s)}) - \left(\sum_{r_s} M_s(r_s, y_{a(s)}) \right) \left(\sum_{r_s} M_s(r_s, y_{a(s)}) \right)^T.$$

In the following we thus focus on the computation of $M_s(r_s, y_{a(s)})$ and $V_s(r_s, y_{a(s)})$.

Proposition

Let (X_S, R_S, Y_S) satisfy (3)-(4), with given transition $p(r_s, y_s | r_{s-}, y_{s-})$. Then $M_s(r_s, y_{a(s)})$ can be recursively computed with linear complexity in time by the following way:

$$M_s(r_s, y_{a(s)}) = \frac{1}{p(y_s | y_{a(s-)})} \sum_{r_{s-}} p(r_s, y_s | r_{s-}, y_{s-}) F_{s-}(r_{s-}, y_{s-}) M_{s-}(r_{s-}, y_{a(s-)}) \quad (7)$$

with

$$p(y_s | y_{a(s-)}) = \frac{p(y_{a(s)})}{p(y_{a(s-)})} = \frac{\sum_{r_s} p(r_s, y_{a(s)})}{\sum_{r_{s-}} p(r_{s-}, y_{a(s-)})}$$

and

$$p(r_s, y_{a(s)}) = \sum_{r_{s-}} p(r_{s-}, y_{a(s-)}) p(r_s, y_s | r_{s-}, y_{s-})$$

Furthermore if the covariance matrix Σ_s of U_s exists for all $s \in \mathcal{S}$ it is possible to compute $V_s(r_s, y_{a(s)})$ as:

$$V_s(r_s, y_{a(s)}) = \frac{1}{p(y_s | y_{a(s-)})} \sum_{r_{s-}} p(r_s, y_s | r_{s-}, y_{s-}) [F_{s-}(r_{s-}, y_{s-}) V_{s-}(r_{s-}, y_{a(s-)}) F_{s-}(r_{s-}, y_{s-})^T + \Sigma_{s-}] \quad (8)$$

Proof

By using the Bayes formula, the fact that (X_S, Y_S, R_S) is a Markov Tree and the model (3)-(4) we have:

$$\begin{aligned} p(x_s, r_s | y_{a(s)}) &= \sum_{r_{s-}} \int p(x_s, r_s, x_{s-}, r_{s-} | y_{a(s-)}, y_s) dx_{s-} \\ &= \frac{1}{p(y_s | y_{a(s-)})} \sum_{r_{s-}} \int p(x_s, r_s, x_{s-}, r_{s-}, y_s | y_{a(s-)}) dx_{s-} \\ &= \frac{1}{p(y_s | y_{a(s-)})} \sum_{r_{s-}} \int p(x_{s-}, r_{s-} | y_{a(s-)}) p(r_s, y_s | r_{s-}, y_{s-}) p(x_s | x_{s-}, r_{s-}, y_{s-}) dx_{s-} \quad (9) \end{aligned}$$

We next multiply (9) by x_s and integrate with respect to x_s . Since $\{U_s\}$ are independent, zero-mean and independent from (R_S, Y_S) :

$$M_s(r_s, y_{a(s)}) = \frac{1}{p(y_s | y_{a(s-)})} \sum_{r_{s-}} \int p(x_{s-}, r_{s-} | y_{a(s-)}) p(r_s, y_s | r_{s-}, y_{s-}) F_{s-}(r_{s-}, y_{s-}) x_s dx_{s-}$$

Finally:

$$\begin{aligned}
M_s(r_s, y_{a(s)}) &= \frac{1}{p(y_s | y_{a(s^-)})} \sum_{r_{s^-}} p(r_s, y_s | r_{s^-}, y_{s^-}) F_{s^-}(r_{s^-}, y_{s^-}) \int x_{s^-} p(x_{s^-}, r_{s^-} | y_{a(s^-)}) dx_{s^-} \\
&= \frac{1}{p(y_s | y_{a(s^-)})} \sum_{r_{s^-}} p(r_s, y_s | r_{s^-}, y_{s^-}) F_{s^-}(r_{s^-}, y_{s^-}) M_{s^-}(r_{s^-}, y_{a(s^-)})
\end{aligned}$$

which brings us to the end of the proof. (8) is obtained similarly.

3 EXACT PREDICTION IN SWITCHING-MARKOV TREES

We consider the following particular TMT:

$$R_{\mathcal{S}} \text{ is a Markov Tree;} \quad (10)$$

$$(R_{\mathcal{S}}, Y_{\mathcal{S}}) \text{ is a Markov Tree;} \quad (11)$$

$$X_s = F_{s^-}(R_{s^-})X_{s^-} + W_{s^-}; \quad (12)$$

where $\{W_s\}_{s \in \mathcal{S}}$ are independent zero-mean random vectors, such that for each $s \in \mathcal{S}$, W_s is independent from $(R_{\mathcal{S}}, Y_{\mathcal{S}})$ and from X_r . Note that in (12) (as in (4)) vectors W_s are not necessarily Gaussian.

In this section we aim at computing $E[x_p | y_{a(s)}]$ and $Cov(x_p | y_{a(s)})$ for any $s \in \mathcal{S}$ and $p \in s^{++}$. As above we focus on the computation of $M_p(r_p, y_{a(s)})$ and $V_p(r_p, y_{a(s)})$, defined in (5) and (6).

Proposition

Let $(X_{\mathcal{S}}, R_{\mathcal{S}}, Y_{\mathcal{S}})$ satisfy (10)-(12), with given transition $p(r_s, y_s | r_{s^-}, y_{s^-})$. Then $M_p(r_p, y_{a(s)})$ can be recursively computed with linear complexity in time by the following scheme:

- Compute $M_s(r_s, y_{a(s)})$ with the algorithm presented in the last section;
- For each $p \in s^{++}$ compute

$$M_p(r_p, y_{a(s)}) = \sum_{r_{p^-}} p(r_p | r_{p^-}) F_{p^-}(r_{p^-}) M_{p^-}(r_{p^-}, y_{a(s)}). \quad (13)$$

Furthermore if the covariance matrix Σ_s of W_s exists for all $s \in \mathcal{S}$ it is possible to compute $V_p(r_p, y_{a(s)})$ as follows:

- Compute $V_s(r_s, y_{a(s)})$ with the algorithm presented in the last section;
- For each $p \in s^{++}$ compute

$$V_p(r_p, y_{a(s)}) = \sum_{r_{p^-}} p(r_p | r_{p^-}) [F_{p^-}(r_{p^-}) V_{p^-}(r_{p^-}, y_{a(s)}) F_{p^-}(r_{p^-})^T + \Sigma_{p^-}]. \quad (14)$$

Proof

We have:

$$\begin{aligned} p(x_p, r_p | y_{a(s)}) &= \sum_{r_{p^-}} \int p(x_p, r_p, x_{p^-}, r_{p^-} | y_{a(s)}) dx_{p^-} \\ &= \sum_{r_{p^-}} \int p(x_{p^-}, r_{p^-} | y_{a(s)}) p(x_p, r_p | x_{p^-}, r_{p^-}, y_{a(s)}) dx_{p^-} \\ &= \sum_{r_{p^-}} \int p(x_{p^-}, r_{p^-} | y_{a(s)}) p(x_p | r_p, x_{p^-}, r_{p^-}, y_{a(s)}) p(r_p | x_{p^-}, r_{p^-}, y_{a(s)}) dx_{p^-} \end{aligned} \quad (15)$$

Then from (10) and (12), $p(x_p | r_p, x_{p^-}, r_{p^-}, y_{a(s)})$ reduces to $p(x_p | x_{p^-}, r_{p^-})$ and $p(r_p | x_{p^-}, r_{p^-}, y_{a(s)})$ reduces to $p(r_p | r_{p^-})$. We next multiply (15) by x_p and integrate with respect to x_p to get (13). (14) is obtained similarly.

REFERENCES

- BARDEL, N., DESBOUVRIES, F. (2009): Exact bayesian prediction in non-Gaussian Markov-switching model, *Proceedings of the XIIIth International Conference on Applied Stochastic Models and Data Analysis (ASMDA)*, June 30 - July 3, Vilnius, Lithuania.
- CHOU, K.C., WILLISKY, A.S., BENVENISTE, A. (1994): Multiscale recursive estimation, data fusion, and regularization, *IEEE Trans. Autom. Control*, vol. 39, pp. 464-478.
- DAUBECHIES, I., MALLAT, S.M., WILLISKY, A.S. (1992): Special issue on wavelet transforms and multiresolution signal analysis (Eds.), *IEEE Trans. Inform. Theory*, vol. 38, pp. 529-860.
- DESBOUVRIES, F., LECOMTE, J., PIECZYNSKI, W. (2006): Kalman Filtering in pairwise Markov trees, *Signal Processing*, vol. 86, Number 5, pp. 1049-1054.
- KRIM, H., WILLINGER, W., JUDITSKI, A., TSE, D. (1999): Special issue on multiscale statistical signal analysis and its applications (Eds.), *IEEE Trans. Inform. Theory*, vol. 45, pp. 825-1062.
- LAFERTE, J.M., PEREZ, P., HEITZ, F. (2000): Discrete Markov image modeling and inference on the quadtree, *IEEE Trans. Image Process.*, vol. 9, pp. 390-404.
- PIECZYNSKI, W. (2002): Arbres de Markov Couple - Pairwise Markov Trees, *C.R.A.S - Mathématiques*, Vol. 335, pp. 79-82.
- PIECZYNSKI, W. (2008): Exact calculation of optimal filter in semi-Markov switching model, *Fourth World Conference of the International Association for Statistical Computing (IASC 2008)*, December 5-8, Yokohama, Japan.
- STARCK, F.M.J.-L., BIAOUI, A. (1998): *Multiscale Image Processing and Data Analysis*. Cambridge, U.K.: Cambridge Univ. Press.
- TUGNAIT, J.K. (1982): Adaptive estimation and identification for discrete systems with Markov jump parameters, *IEEE Transactions on Automatic Control*, vol.25, pp. 1054-1065.
- WILLISKY, A.S. (2002): Multiresolution Markov models for signal and image processing, *Proc. IEEE*, vol. 90, no.8, pp. 1396-1458.