Bifurcation of rotating patches from Kirchhoff vortices

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TAOUFIK HMIDI AND JOAN MATEU

Abstract. In this paper we prove the existence of countable branches of rotating patches bifurcating from the ellipses at some implicit angular velocities.

1. Introduction

In this paper we deal with the vortex motion for incompressible Euler equations in two-dimensional space. The formulations velocity-vorticity is given by the nonlinear transport equation

$$\begin{cases} \partial_t \omega + v \cdot \nabla \omega = 0, \\ v = \nabla^{\perp} \Delta^{-1} \omega, \\ \omega_{|t=0} = \omega_0, \end{cases} \tag{1}$$

where ω denotes the vorticity of the velocity field $v = (v^1, v^2)$ and it is given by $\omega = \partial_1 v^2 - \partial_2 v^1$. The second equation in (1) is nothing but the Biot-Savart law which can be written with a singular operator as follows: by identifying $v = (v_1, v_2)$ with $v_1 + iv_2$, we write

$$v(t,z) = \frac{i}{2\pi} \int_{\mathbb{C}} \frac{\omega(t,\xi)}{\overline{z} - \overline{\xi}} \, dA(\xi), \quad z \in \mathbb{C}, \tag{2}$$

with dA being the planar Lebesgue measure. Global existence of classical solutions is a consequence of the transport structure of the vorticity equation, for more details about this subject we refer to [1, 7]. For less regular initial data Yudovich proved in [31] that the system (1) admits a unique global solution in the weak sense when the initial vorticity ω_0 lies in $L^1 \cap L^\infty$. This allows to deal rigorously with the vortex patches which are the characteristic function of bounded domains. Therefore, it follows that when $\omega_0 = \chi_{D_0}$ with D_0 a bounded domain then the solution of (1) preserves this structure and $\omega(t) = \chi_{D_t}$, with $D_t = \psi(t, D_0)$ being the image of D_0 by the flow. In the special case where D_0 is the open unit disc the vorticity is radial and thus we get a steady flow. Another remarkable exact solution was discovered by Kirchhoff [20] who proved that an ellipse D_0 performs a steady rotation about its center. More precisely, if the center is assumed to be the origin then $D_t = e^{it\Omega} D_0$, where the angular velocity Ω is determined by the semi-axes a and b of the ellipse through the formula $\Omega = ab/(a+b)^2$. These ellipses are often referred in the literature as Kirchhoff vortices. For a proof, see for instance [1, p.304] and [21, p.232].

The existence of general class of rotating patches, called also V-states, was discovered numerically by Deem and Zabusky [9]. Later on, Burbea gave an analytical proof and showed the existence of the m-fold symmetric V-states for each integer $m \geq 2$ and in this countable family the case $m = 2$ corresponds to the known Kirchhoff's ellipses. Burbea's approach consists in using some complex analysis tools combined with the bifurcation theory. Notice that in this framework, the rotating patches appear as a countable collection of curves bifurcating from Rankine vortices (trivial solution) at the discrete angular velocities set $\left\{\frac{m-1}{2m}, m \geq 2\right\}$. It is extremely interesting to look at the pictures of the limiting V-states done in [30], which are the end points of each branch. The boundary develops corners at right angles. Recently, the authors studied in [15] the boundary regularity of the V-states close to the disc and proved that the boundaries are in fact of class C^∞

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and convex. More recently, Castro, Córdoba and Gómez-Serrano proved in [5] the analyticity of the V-states close to the disc. Notice that the existence and the regularity of the V-states for more singular nonlinear transport equations arising in geophysical flows as the surface quasi-geostrophic equations has been studied recently in [4, 5, 13].
Another connected subject which has been investigated very recently in a series of paper [12, 14, 17, 16] is the existence of doubly connected V-states (patches with one hole).
The main goal of this paper is to study the second bifurcation of rotating patches from Kirchhoff ellipses corresponding to $m = 2$. This subject was first examined by Kamm in [19], who gave numerical evidence of the existence of some branches bifurcating from the ellipses, see also [27]. We mention that the first bifurcation occurs at the aspect ratio 3 corresponding to the transition regime from stability to instability. In the paper [23] of Luzzatto-Fegiz and Williamson one can find more details about the diagram for the first bifurcations and some illustrations of the limiting V-states. Another central problem which has been studied since the work of Love [22] is the linear and nonlinear stability of the ellipses. For instance, we mention the following papers [18, 28]. As to the linear stability of the m-folds symmetric V-states, it was conducted by Burbea and Landau in [3]. However the nonlinear stability of these structures in a small neighborhood of Rankine vortices was done by Wan in [29]. For further numerical discussions, see also [6, 11, 25].
In the current paper we intend to give an analytical proof of the bifurcation from the ellipses. Our result reads as follows.

Theorem 1. *Consider the family of the ellipses $\mathcal{E} : Q \in (0,1) \mapsto \mathcal{E}_Q$ given by the parametrization*

$$\mathcal{E}_Q = \Big\{ w + \frac{Q}{w}, w \in \mathbb{T} \Big\}.$$

Let

$$\mathcal{S} \triangleq \Big\{ Q \in (0,1), \exists\ m \geq 3,\ 1 + Q^m - \frac{1-Q^2}{2} m = 0 \Big\}.$$

Then for each $Q \in \mathcal{S}$ there exists a nontrivial curve of rotating vortex patches bifurcating from the curve $\mathcal{E}$ at the ellipse $\mathcal{E}_Q$. Moreover the boundary of these V-states are $C^{1+\alpha}$, $\forall \alpha \in (0,1)$.

Before giving some details about the proof, we shall give first some remarks.

Remark. In a very recent paper [5], Castro, Córdoba and Gómez-Serrano proved the analyticity of the V-states close to the ellipses. Our approach seems to be easier but not so deep and cannot lead to the analyticity of the V-states. Notice that what one could expect from iterating our method is to get the regularity $C^{n+\alpha}$ for each $n \in \mathbb{N}$ but the proof does not guarantee a uniform existence interval with respect to the parameter n.

Remark. In contrast to the bifurcation from the disc where we get a collection of m folds, the V-states of Theorem 1 are in general one or two-folds. For m even we can show from the proof that the V-states are symmetric with respect to the origin.

Now we shall sketch the proof of Theorem 1 which is mainly based upon the bifurcation theory via Crandall-Rabinowitz theorem. We shall look for a parametrization of the boundary ∂D of the rotating patches as a small perturbation of a given ellipse. This parametrization takes the form $\Phi : \mathbb{T} \to \partial D$, with $\mathbb{T}$ is the unit circle and

$$\Phi(w) = w + Q\overline{w} + \sum_{n \geq 2} a_n w^n, \quad Q \in (0,1), \quad a_n \in \mathbb{R}.$$

Observe that when all the coefficients a_n vanish then this parametrization corresponds to an ellipse, where $Q = \frac{a-b}{a+b}$, with a and b being the major axis and the minor axis, respectively. As we shall see in the next section, the function Φ satisfies the nonlinear equation

$$G(\Omega, \Phi(w)) \triangleq \mathrm{Im}\left\{ \left(2\Omega\, \overline{\Phi(w)} + \fint_{\mathbb{T}} \frac{\overline{\Phi(\xi)} - \overline{\Phi(w)}}{\Phi(\xi) - \Phi(w)}\, \Phi'(\xi) d\xi \right) w\, \Phi'(w) \right\} = 0, \quad \forall w \in \mathbb{T}.$$

Setting $\alpha_Q(w) = w + Q\overline{w}$ then we retrieve the Kirchhoff solutions, meaning that,

$$G(\frac{1-Q^2}{4}, \alpha_Q(w)) = 0, \quad \forall w \in \mathbb{T}.$$

Now we introduce the function

$$F(Q, f(w)) = \mathrm{Im}\Big\{G\big(\tfrac{1-Q^2}{4}, \alpha_Q(w) + f(w)\big)\Big\}.$$

Then by this transformation the ellipses lead to a family of trivial solutions: $F(Q,0) = 0, \forall Q \in (0,1)$. Therefore it is legitimate at this stage to look for non trivial solutions by using the bifurcation techniques in the spirit of Burbea's work [2]. As we shall see, the computations of the linearized operator $\mathcal{L}_Q \triangleq \partial_f F(Q,0)$ are a little bit more involved that the radial case but they can still be done in an explicit way. We shall see from this part that the dispersion set $\mathcal{S}$ introduced in Theorem 1 corresponds in fact to the values of Q such that the kernel of the operator $\mathcal{L}_Q$ is one -dimensional. We shall also check that all the assumptions of Crandall-Rabinowitz theorem are satisfied and therefore the proof of the main result will follow immediately.

Notation. We need to fix some notation that will be frequently used along this paper. We denote by C any positive constant that may change from line to line. We denote by $\mathbb{D}$ the unit disc and its boundary, the unit circle, is denoted by $\mathbb{T}$. Let $f : \mathbb{T} \to \mathbb{C}$ be a continuous function, we define its mean value by,

$$\fint_{\mathbb{T}} f(\tau)d\tau \triangleq \frac{1}{2i\pi}\int_{\mathbb{T}} f(\tau)d\tau,$$

where $d\tau$ stands for the complex integration.

Let X and Y be two normed spaces. We denote by $\mathcal{L}(X,Y)$ the space of all continuous linear maps $T : X \to Y$ endowed with its usual strong topology. We shall denote by $N(T)$ and $R(T)$ the kernel and the range of T, respectively. Finally, if F is a subspace of Y, then Y/F denotes the quotient space.

2. Formulation of the problem

Following [16, 17] one can see that the boundary of any smooth V-states χ_{D_t}, with $D_t = e^{it\Omega}D$ is subject to the equation

$$\mathrm{Re}\left\{\left(2\Omega\,\overline{z} + \frac{1}{2\pi i}\int_{\partial D_0}\frac{\overline{\zeta}-\overline{z}}{\zeta - z}d\zeta\right)z'\right\} = 0, \quad \forall z \in \partial D. \tag{3}$$

Recall that a curve γ of the complex plane $\mathbb{C}$ is said a regular Jordan curve if it admits a parametrization $\Phi : \mathbb{T} \to \gamma$ which is simple and of class C^1 such that $\Phi'(w) \neq 0, \quad \forall w \in \mathbb{T}$. Note that in this case the curve γ encloses a simply connected domain. Now to solve the equation (3) we shall restrict ourselves to domains whose boundaries are parametrized by a regular Jordan curve $\Phi : \mathbb{T} \to \mathbb{C}$. A tangent vector to the boundary at the point $\Phi(w)$ is determined by $z' = iw\Phi'(w)$ and therefore (3) becomes

$$\mathrm{Im}\left\{\left(2\Omega\,\overline{\Phi(w)} + \fint_{\mathbb{T}}\frac{\overline{\Phi(\xi)}-\overline{\Phi(w)}}{\Phi(\xi)-\Phi(w)}\Phi'(\xi)d\xi\right)w\,\Phi'(w)\right\} = 0, \quad \forall w \in \mathbb{T}. \tag{4}$$

We shall define the object G by

$$G(\Omega, \Phi(w)) \triangleq \left(2\Omega\,\overline{\Phi(w)} + \fint_{\mathbb{T}}\frac{\overline{\Phi(\xi)}-\overline{\Phi(w)}}{\Phi(\xi)-\Phi(w)}\Phi'(\xi)d\xi\right)w\,\Phi'(w). \tag{5}$$

It is easily seen that the equation (4) is invariant by rotation and dilation. Moreover, one can deduce from this formulation Kirchhoff's result which states that an ellipse of the semi-axes a and

b rotates with the angular velocity $\Omega = \frac{ab}{(a+b)^2}$. Indeed, note that in this case the ellipse may be parametrized by the conformal parametrization,

$$\Phi(w) = \frac{a+b}{2}\Big(w + \frac{Q}{w}\Big), \quad Q = \frac{a-b}{a+b}.$$

In the sequel we shall use the notation

$$\alpha_Q(w) \triangleq w + \frac{Q}{w}, \quad w \in \mathbb{T}.$$

By straightforward computations we get

$$\overline{\alpha_Q(w)}\, w\, \alpha_Q'(w) = 1 - Q^2 + Q(w^2 - \overline{w}^2).$$

Using residue theorem and taking $r > 1$ we get

$$\begin{aligned} \fint_{\mathbb{T}} \frac{\overline{\alpha_Q(\xi)} - \overline{\alpha_Q(w)}}{\alpha_Q(\xi) - \alpha_Q(w)}\, \alpha_Q'(\xi) d\xi &= Q \fint_{r\mathbb{T}} \frac{\xi}{\alpha_Q(\xi) - \alpha_Q(w)} \alpha_Q'(\xi) d\xi - \overline{\alpha_Q(w)} \\ &= Q\alpha_Q(w) - \alpha_Q(\overline{w}) \\ &= \frac{Q^2 - 1}{w}. \end{aligned}$$

It follows that

$$\big(\frac{2}{a+b}\big)^2 G(\Omega, \alpha_Q(w)) = 2\Omega\, Q(w^2 - \overline{w}^2) + (Q^2 - 1)\Big(1 - 2\Omega - Q\overline{w}^2\Big).$$

Thus

$$\big(\frac{2}{a+b}\big)^2 \mathrm{Im}\{G(\Omega, \alpha_Q(w))\} = Q\big(4\Omega + Q^2 - 1\big)\mathrm{Im}(w^2)$$

and consequently (4) is satisfied provided that

$$\Omega = \frac{1 - Q^2}{4} = \frac{ab}{(a+b)^2}.$$

This can be written in the form

$$\mathrm{Im}\, \Big\{G\big(\tfrac{1-Q^2}{4}, \alpha_Q(w)\big)\Big\} = 0, \quad \forall\, w \in \mathbb{T}. \tag{6}$$

Now we shall introduce the function

$$F(Q, f(w)) = \mathrm{Im}\Big\{G\big(\tfrac{1-Q^2}{4}, \alpha_Q(w) + f(w)\big)\Big\}. \tag{7}$$

From the preceding discussion we readily get

$$F(Q, 0) = 0, \quad \forall Q \in (0, 1). \tag{8}$$

To prove Theorem 1 we need to show the existence of nontrivial solutions of the equation defining the V-states :

$$F(Q, f(w)) = 0, \quad \forall\, w \in \mathbb{T}.$$

It will be done using the bifurcation theory through Crandall-Rabinowitz theorem [8]. For the completeness of the paper we recall this basic theorem and it will be sometimes referred to as C-R theorem.

Theorem 2. *Let X, Y be two Banach spaces, V a neighborhood of 0 in X and let $F : \mathbb{R} \times V \to Y$ with the following properties:*

1. *$F(\lambda, 0) = 0$ for any $\lambda \in \mathbb{R}$.*
2. *F is C^1 and $F_{\lambda x}$ exists and are continuous.*
3. *$N(\mathcal{L}_0)$ and $Y/R(\mathcal{L}_0)$ are one-dimensional.*
4. *Transversality assumption: $F_{\lambda x}(0,0)x_0 \notin R(\mathcal{L}_0)$, where*

$$N(\mathcal{L}_0) = span\{x_0\}, \quad \mathcal{L}_0 \triangleq \partial_x F(0,0).$$

If Z is any complement of $N(\mathcal{L}_0)$ in X, then there is a neighborhood U of $(0,0)$ in $\mathbb{R}\times X$, an interval $(-a,a)$ and continuous functions $\varphi:(-a,a)\to\mathbb{R}$, $\psi:(-a,a)\to Z$ such that $\varphi(0)=0$, $\psi(0)=0$ and

$$F^{-1}(0)\cap U=\Big\{\big(\varphi(\varepsilon),\varepsilon x_0+\xi\psi(\varepsilon)\big)\,;\,|\varepsilon|<a\Big\}\cup\Big\{(\lambda,0)\,;\,(\lambda,0)\in U\Big\}.$$

Now we shall give a precise statement of Theorem 1. For this purpose we should fix the spaces X and Y used in C-R theorem. They are given by,

$$X=\Big\{h\in C^{1+\alpha}(\mathbb{T}),h(w)=\sum_{n\geq 2}a_n w^n,\quad a_n\in\mathbb{R}\Big\} \tag{9}$$

and

$$Y=\Big\{g\in C^{\alpha}(\mathbb{T}),\ g(w)=\sum_{n\geq 1}g_n\,e_n,g_n\in\mathbb{R},\ w\in\mathbb{T}\Big\},\quad e_n(w)\triangleq \mathrm{Im}(w^n). \tag{10}$$

Theorem 3. *Consider the family of ellipses $\mathcal{E}:Q\in(0,1)\mapsto\mathcal{E}_Q$ given by the parametrization*

$$\mathcal{E}_Q=\Big\{w+\frac{Q}{w},w\in\mathbb{T}\Big\}.$$

Let

$$\mathcal{S}\triangleq\Big\{Q\in(0,1),\ \exists\ m\geq 3,\ 1+Q^m-\frac{1-Q^2}{2}m=0\Big\}.$$

Then for each $Q=Q_m\in\mathcal{S}$ there exists a nontrivial curve of rotating vortex patches bifurcating from the curve $\mathcal{E}$ at the ellipse $\mathcal{E}_Q$. Moreover the boundary of these V-states are $C^{1+\alpha}$.
More precisely, let Z_m be any complement of the vector $v_m=\frac{w^{m+1}}{1-Qw^2}$ in the space X. Then there exist $a>0$ and continuous functions $Q:(-a,a)\to\mathbf{R}$, $\psi:(-a,a)\to Z_m$ satisfying $Q(0)=Q_m$, $\psi(0)=0$, such that the bifurcating curve at this point is described by,

$$F(Q(\varepsilon),\varepsilon\frac{w^{m+1}}{1-Qw^2}+\varepsilon\psi(\varepsilon))=0.$$

In particular the boundary of the V-states rotating is described by

$$\gamma_\varepsilon:\mathbb{T}\to\mathbb{C},\quad \gamma_\varepsilon(w)=w+\frac{Q(\varepsilon)}{w}+\varepsilon\frac{w^{m+1}}{1-Qw^2}+\varepsilon\psi(\varepsilon).$$

The proof consists in checking all the assumptions of Theorem 2. This will be done in details in the next sections.

3. Regularity of the functional

This section is devoted to the study of the regularity assumptions stated in C-R theorem. We shall study the nonlinear functional F defining the V-states already seen in (7). It is given through the functional G as follows,

$$G(\Omega,\Phi(w))=\left(2\Omega\,\overline{\Phi(w)}+\fint_{\mathbb{T}}\frac{\overline{\Phi(\xi)}-\overline{\Phi(w)}}{\Phi(\xi)-\Phi(w)}\,\Phi'(\xi)d\xi\right)w\,\Phi'(w)$$

and

$$F(Q,f(w))=\mathrm{Im}\Big\{G\big(\tfrac{1-Q^2}{4},\alpha_Q(w)+f(w)\big)\Big\}.$$

For $r\in(0,1)$ we denote by B_r the open ball of X (this space was introduced in (9)) with center 0 and radius r,

$$B_r=\Big\{f\in X,\quad \|f\|_{C^{1+\alpha}}<r\Big\}.$$

We shall make use at several stages of the following lemma, for more details see [24, p. 419].

Lemma 1. *Let T be a singular operator defined by*

$$T\phi(w) = \int_{\mathbb{T}} K(\xi, w)\phi(\xi)d\xi.$$

Assume that the kernel of the operator T satisfies

(1) *K is measurable on $\mathbb{T}\times\mathbb{T}$ and*

$$|K(\xi, w)| \leq C_0, \quad \forall\, \xi, w \in \mathbb{T}.$$

(2) *For each $\xi \in \mathbb{T}$, $w \mapsto K(\xi, w)$ is differentiable in $\mathbb{T}\backslash\{\xi\}$ and*

$$|\partial_w K(\xi, w)| \leq \frac{C_0}{|w-\xi|}, \quad \forall\, w \in \mathbb{T}\backslash\{\xi\}.$$

Then for every $0 < \alpha < 1$

$$\|T\phi\|_\alpha \leq C_0\|\phi\|_{L^\infty}.$$

The main result of this section reads as follows.

Proposition 1. *Let $\varepsilon \in (0,1)$ and $r_\varepsilon = \frac{1-\varepsilon}{2}$, then the following holds true.*

(1) *The function $F : (0,\varepsilon)\times B_{r_\varepsilon} \to Y$ is of class C^1.*
(2) *The partial derivative $\partial_Q\partial_f F : (0,\varepsilon)\times B_{r_\varepsilon} \to Y$ is continuous.*

Proof. (**1**) To get this result it suffices to prove that $\partial_Q F, \partial_f F :]0,\varepsilon[\times B_{r_\varepsilon} \to Y$ exist and are continuous. We shall first compute $\partial_f F(Q, f)$. This will be done by showing first the existence of the Gâteaux derivative and second its continuity in the strong topology. Before dealing with this problem we should first show that the functional F is well-defined. For this purpose it suffices to show that the functional G sends X into $C^\alpha(\mathbb{T})$ and the Fourier coefficients of $G((1-Q^2)/4, \alpha_Q + f))$ are real when f belongs to X. As to the second claim we follow the Arxiv version of the paper [15] and for the sake of simplicity we shall skip the details and sketch just the basic ideas of the proof. First, we write

$$\begin{aligned} G((1-Q^2)/4, \alpha_Q(w) + f(w)) &= \frac{1-Q^2}{2}\big[1 + Qw^2 + w\overline{f(w)}\big]\big[1 - Q\overline{w}^2 + f'(w)\big] \\ &+ w\Phi_f'(w)\fint_{\mathbb{T}} \frac{\overline{\Phi_f(\xi)} - \overline{\Phi_f(w)}}{\Phi_f(\xi) - \Phi_f(w)}\Phi_f'(\xi)d\xi \\ &\triangleq G_1(Q, f(w)) + w\Phi_f'(w)\, G_2(Q, f) \end{aligned} \tag{11}$$

with the notation $\Phi_f = \alpha_Q + f$. It is clear that G_1 is polynomial in the variable Q and bilinear on f and f'. Therefore using the algebra structure of $C^\alpha(\mathbb{T})$ one gets

$$\|G_1(Q, f))\|_\alpha \leq C\|\Phi_f\|_\alpha\|\Phi_f\|_{1+\alpha}.$$

This implies in particular that $G_1 : (0,1)\times X \to Y$ is of class C^∞. Now we shall focus on the second part G_2. Fix $Q \in (0,\varepsilon)$ and put $r_\varepsilon = \frac{1-\varepsilon}{2}$, then for $f \in B_{r_\varepsilon}$ we get

$$\frac{1-\varepsilon}{2}|w-\xi| \leq \big|\alpha_Q(w) + f(w) - \alpha_Q(\xi) - f(\xi)\big|, \quad \forall\, \xi, w \in \mathbb{T}. \tag{12}$$

Indeed,

$$\begin{aligned} \big|\alpha_Q(w) - \alpha_Q(\xi)\big| &= |w-\xi|\,\Big|1 - \frac{Q}{w\xi}\Big| \\ &\geq (1-Q)|\xi - w| \\ &\geq (1-\varepsilon)|\xi - w|. \end{aligned}$$

We combine this with the mean-value theorem applied to f which is holomorphic inside the unit disc

$$\begin{aligned} |f(w)-f(\xi)| &\leq \|f'\|_{L^\infty}|w-\xi| \\ &\leq \frac{1-\varepsilon}{2}|w-\xi|. \end{aligned}$$

Now using Lemma 1 we get that $G_2(Q,f)\in C^\alpha(\mathbb{T})$. This concludes the fact that F is well-defined. Next, we shall prove that for $f\in X$ with $\|f\|_{1+\alpha}<r_\varepsilon$ the Gâteaux derivative $\partial_f G_2$ exists and is continuous. Straightforward computations show that this derivative is given by: for $h\in X$,

$$\partial_f G_2(Q,f)h(w)=\sum_{j=1}^{3} I_j(Q,f)h(w), \tag{13}$$

with

$$\begin{aligned} I_1(Q,f)h(w) &= \fint_{\mathbb{T}} \frac{\Phi_f(\overline{\xi})-\Phi_f(\overline{w})}{\Phi_f(\xi)-\Phi_f(w)}h'(\xi)d\xi \\ &= \fint_{\mathbb{T}} K_1(\xi,w)h'(\xi)d\xi, \end{aligned}$$

$$\begin{aligned} I_2(Q,f)h(w) &= \fint_{\mathbb{T}} \frac{h(\overline{\xi})-h(\overline{w})}{\Phi_f(\xi)-\Phi_f(w)}\Phi_f'(\xi)d\xi \\ &= \fint_{\mathbb{T}} K_2(\xi,w)\Phi_f'(\xi)d\xi \end{aligned}$$

and

$$\begin{aligned} I_3(Q,f)h(w) &= -\fint_{\mathbb{T}} \frac{\big(\Phi_f(\overline{\xi})-\Phi_f(\overline{w})\big)\big(h(\xi)-h(w)\big)}{\big(\Phi_f(\xi)-\Phi_f(w)\big)^2}\Phi_f'(\xi)d\xi \\ &= \fint_{\mathbb{T}} K_3(\xi,w)\Phi_f'(\xi)d\xi. \end{aligned}$$

Notice that we have used the fact that the Fourier coefficients of Φ_f are real and therefore $\overline{\Phi_f(w)}=\Phi_f(\overline{w})$. It is easy to check according to (12) that

$$|K_1(\xi,w)|=1, \quad |\partial_w K(\xi,w)|\leq C_0|w-\xi|^{-1}$$

and thus we deduce from Lemma 1 that

$$\|I_1(Q,f)h\|_\alpha \lesssim \|h'\|_{L^\infty}\lesssim \|h\|_{1+\alpha}.$$

For the second term I_2 we have the following estimates for the kernel

$$\begin{aligned} |K_2(\xi,w)| &= \frac{|h(\overline{\xi})-h(\overline{w})|}{|\Phi_f(\xi)-\Phi_f(w)|} \\ &\leq \frac{2}{1-\varepsilon}\|h'\|_{L^\infty} \\ |\partial_w K_2(\xi,w)| &\leq C_0\|h'\|_{L^\infty}|w-\xi|^{-1}. \end{aligned}$$

Once again from Lemma 1 one gets,

$$\|I_2(Q,f)h\|_\alpha \lesssim \|h'\|_{L^\infty}\|\Phi_f'\|_{L^\infty}\lesssim \|h\|_{1+\alpha}.$$

The last term can be estimated similarly to the previous one and we get

$$\|I_3(Q,f)h\|_\alpha \lesssim \|h'\|_{L^\infty}\|\Phi_f'\|_{L^\infty}\lesssim \|h\|_{1+\alpha}.$$

Putting together the preceding estimates we get

$$\|\partial_f G_2(Q,f)h\|_\alpha \le C\|h\|_{1+\alpha}.$$

This shows the existence of Gâteaux derivative and now we intend to prove the continuity of the map $f \mapsto \partial_f G_2(Q,f)$ from X to $\mathcal{L}(X,Y)$. This is a consequence of the following estimate that we shall prove now: for $f, g \in B_{r_\varepsilon}$, one has

$$\|\partial_f G_2(Q,f)h - \partial_f G_2(Q,g)h\|_\alpha \le C\|f-g\|_{1+\alpha}\|h\|_{1+\alpha}. \tag{14}$$

First we write

$$\begin{aligned} I_1(Q,f)h(w) - I_1(Q,g)h(w) &= \fint_{\mathbb{T}} \frac{\big(\Phi_f(\overline{\xi}) - \Phi_f(\overline{w})\big)\big((g-f)(\xi) - (g-f)(w)\big)}{(\Phi_f(\xi)-\Phi_f(w))(\Phi_g(\xi)-\Phi_g(w))} h'(\xi)d\xi \\ &+ \fint_{\mathbb{T}} \frac{(f-g)(\overline{\xi}) - (f-g)(\overline{w})}{\Phi_g(\xi)-\Phi_g(w)} h'(\xi)d\xi \\ &\triangleq \fint_{\mathbb{T}} K_4(\xi,w)h'(\xi)d\xi. \end{aligned}$$

By the mean value theorem we may check that

$$|K_4(\xi,w)| \le C\|f'-g'\|_{L^\infty}$$

and

$$|\partial_w K_4(\xi,w)| \le C\|f'-g'\|_{L^\infty}\|\xi-w|^{-1}.$$

Thus we obtain by using Lemma 1

$$\begin{aligned} \|I_1(Q,f)h - I_1(Q,g)h\|_\alpha &\le C_0\|f'-g'\|_{L^\infty}\|h'\|_{L^\infty} \\ &\le C\|f-g\|_{1+\alpha}\|h\|_{1+\alpha}. \end{aligned}$$

To estimate $I_2(Q,f)h - I_2(Q,g)h$ we shall use the identity

$$\frac{\Phi'_f(\xi)}{\Phi_f(\xi)-\Phi_f(w)} - \frac{\Phi'_g(\xi)}{\Phi_g(\xi)-\Phi_g(w)} = \frac{f'(\xi)-g'(\xi)}{\Phi_f(\xi)-\Phi_f(w)} - \Phi'_g(\xi)\frac{(f-g)(\xi)-(f-g)(w)\big)}{(\Phi_f(\xi)-\Phi_f(w))(\Phi_g(\xi)-\Phi_g(w))}$$

and thus

$$\begin{aligned} I_2(Q,f)h(w) - I_2(Q,g)h(w) &= \fint_{\mathbb{T}} \frac{h(\overline{\xi})-h(\overline{w})}{\Phi_f(\xi)-\Phi_f(w)}(f'(\xi)-g'(\xi))d\xi \\ &- \fint_{\mathbb{T}} \frac{\big(h(\overline{\xi})-h(\overline{w})\big)\big((f-g)(\xi)-(f-g)(w)\big)}{(\Phi_f(\xi)-\Phi_f(w))(\Phi_g(\xi)-\Phi_g(w))}\Phi'_g(\xi)d\xi \\ &\triangleq \fint_{\mathbb{T}} K_5(\xi,w)(f'(\xi)-g'(\xi))d\xi - \fint_{\mathbb{T}} K_6(\xi,w)\Phi'_g(\xi)d\xi. \end{aligned}$$

The kernels can be estimated as follows

$$|K_5(\xi,w)| \le C\|h'\|_{L^\infty}, |\partial_w K_5(\xi,w)| \le C\|h'\|_{L^\infty}|w-\xi|^{-1}$$

and

$$|K_6(\xi,w)| \le C_0\|h'\|_{L^\infty}\|f'-g'\|_{L^\infty}, |\partial_w K_6(\xi,w)| \le C_0\|h'\|_{L^\infty}\|f'-g'\|_{L^\infty}|w-\xi|^{-1}.$$

So Lemma 1 implies that

$$\begin{aligned} \|I_2(Q,f)h - I_2(Q,g)h\|_\alpha &\le C_0\|f'-g'\|_{L^\infty}\|h'\|_{L^\infty} \\ &\le C_0\|f-g\|_{1+\alpha}\|h\|_{1+\alpha}. \end{aligned}$$

It remains to check the continuity of I_3. We write

$$\begin{aligned} I_3(Q,f)h(w) &- I_3(Q,g)h(w) = -\fint_{\mathbb{T}} \frac{\big(\Phi_f(\overline{\xi})-\Phi_f(\overline{w})\big)(h(\xi)-h(w))}{\big(\Phi_f(\xi)-\Phi_f(w)\big)^2}(f-g)'(\xi)d\xi \\ &- \fint_{\mathbb{T}} \frac{\big[(f-g)(\overline{\xi})-(f-g)(\overline{w})\big](h(\xi)-h(w))}{\big(\Phi_f(\xi)-\Phi_f(w)\big)^2}\Phi_g'(\xi)d\xi \\ &+ \fint_{\mathbb{T}} K_7(\xi,w)\Phi_g'(\xi)d\xi \\ &\triangleq \sum_{j=1}^{3} I_3^j(f,g)(h)(w) \end{aligned}$$

with

$$\begin{aligned} K_7(\xi,w) &= \frac{\big(\overline{\Phi_g}(\xi)-\overline{\Phi_g}(w)\big)\Big(\{\Phi_g(\xi)-\Phi_g(w)\}+\{\Phi_f(\xi)-\Phi_f(w)\}\Big)}{\big(\Phi_f(\xi)-\Phi_f(w)\big)^2\big(\Phi_g(\xi)-\Phi_g(w)\big)^2} \\ &\times \big(h(\xi)-h(w)\big)\big[(f-g)(\xi)-(f-g)w)\big]. \end{aligned}$$

The first term can be written in the form

$$I_3^1(f,g)(w) = \fint_{\mathbb{T}} K_8(\xi,w)(f-g)'(\xi)d\xi$$

where the kernel K_8 satisfies

$$|K_8(\xi,w)| \le C\|h'\|_{L^\infty} \quad \text{and} \quad |\partial_w K_8(\xi,w)| \le C\|h'\|_{L^\infty}|\xi-w|^{-1}.$$

This yields in view of Lemma 1

$$\begin{aligned} \|I_3^1(f,g)\|_\alpha &\le C\|h'\|_{L^\infty}\|f'-g'\|_{L^\infty} \\ &\le C\|h\|_{1+\alpha}\|f-g\|_{1+\alpha}. \end{aligned}$$

The second term can written under the form

$$I_3^1(f,g)(w) = \fint_{\mathbb{T}} K_9(\xi,w)\Phi_g'(\xi)d\xi$$

and the kernel K_9 satisfies

$$|K_9(\xi,w)| \le C_0\|f'-g'\|_{L^\infty}\|h'\|_{L^\infty} \quad \text{and} \quad |\partial_w K_8(\xi,w)| \le C_0\|f'-g'\|_{L^\infty}\|h'\|_{L^\infty}\frac{1}{|\xi-w|}$$

which yields in view of Lemma 1

$$\begin{aligned} \|I_3^2(f,g)\|_\alpha &\le C\|h'\|_{L^\infty}\|f'-g'\|_{L^\infty}\|\Phi_g'\|_{L^\infty} \\ &\le C\|h\|_{1+\alpha}\|f-g\|_{1+\alpha}. \end{aligned}$$

For the third term we can check that the kernel K_7 satisfies

$$|K_7(\xi,w)| \le C\|f'-g'\|_{L^\infty}\|h'\|_{L^\infty} \quad \text{and} \quad |\partial_w K_7(\xi,w)| \le C\|f'-g'\|_{L^\infty}\|h'\|_{L^\infty}\frac{1}{|\xi-w|}$$

which gives according to Lemma 1

$$\begin{aligned} \|I_3^3(f,g)\|_\alpha &\le C\|h'\|_{L^\infty}\|f'-g'\|_{L^\infty}\|\Phi_g'\|_{L^\infty} \\ &\le C\|h\|_{1+\alpha}\|f-g\|_{1+\alpha}. \end{aligned}$$

Putting together the preceding estimates we get

$$\|I_3(Q,f)h - I_3(Q,g)h\|_\alpha \le C\|h\|_{1+\alpha}\|f-g\|_{1+\alpha}.$$

This achieves the proof of (14) and therefore the Gâteaux derivative is Lipschitz and thus it is continuous on the variable f. Therefore we conclude at this stage that the Fréchet derivative exists and coincides with Gâteaux derivative. See [10] for more information.
We shall now study the regularity of G_2 with respect to Q. This reduces to studying the regularity of $Q \mapsto G_2(Q,f)$ given by

$$G_2(Q,f) = \fint_{\mathbb{T}} \frac{\{\alpha_Q(\bar{\xi})) - \alpha_Q(\overline{w})\} + f(\bar{\xi}) - f(\bar{w})}{\{\alpha_Q(\xi)) - \alpha_Q(w)\} + f(\xi) - f(w)} \big(\alpha_Q'(\xi) + f'(\xi)\big) d\xi.$$

Easy computations yields

$$\begin{aligned}
\partial_Q G_2(Q,f) &= \fint_{\mathbb{T}} \frac{\xi - w}{\Phi_f(\xi) - \Phi_f(w)} \big(\alpha_Q'(\xi) + f'(\xi)\big) d\xi \\
&- \fint_{\mathbb{T}} \frac{\Phi_f(\bar{\xi}) - \Phi_f(\overline{w})}{\Phi_f(\xi) - \Phi_f(w)} \frac{1}{\xi^2} d\xi \\
&- \fint_{\mathbb{T}} \frac{\big(\Phi_f(\bar{\xi}) - \Phi_f(\overline{w})\big)(\overline{\xi - w})}{\Big(\Phi_f(\xi) - \Phi_f(w)\Big)^2} \big(\alpha_Q'(\xi) + f'(\xi)\big) d\xi.
\end{aligned}$$

As before, using Lemma 1 we get for $(Q,f) \in (0,\varepsilon) \times B_{r_\varepsilon}$

$$\|\partial_Q G_2(Q,f)\|_\alpha \leq C_\varepsilon.$$

Reproducing the same analysis we get for any $k \in \mathbb{N}$

$$\|\partial_Q^k G_2(Q,f)\|_\alpha \leq C_{k,\varepsilon}.$$

Similarly we obtain that $\partial_Q G_2 : (0,\varepsilon) \times B_{r_\varepsilon} \to C^\alpha(\mathbb{T})$ exists and is continuous. Using that $\partial_f G_2$ is continuous we deduce that $G_2 : (0,\varepsilon) \times B_{r_\varepsilon} \to C^\alpha(\mathbb{T})$ is C^1 and it follows that $F : (0,\varepsilon) \times B_{r_\varepsilon} \to Y$ is also C^1. By induction one can show that $G_2 : (0,\varepsilon) \times B_{r_\varepsilon} \to C^\alpha(\mathbb{T})$ is in fact C^∞.

(**2**) We shall check that $\partial_Q \partial_f G_2 : (0,\varepsilon) \times B_{r_\varepsilon} \to C^\alpha(\mathbb{T})$ is continuous. According to (13) we obtain

$$\partial_Q \partial_f G_2(Q,f)h(w) = \sum_{j=1}^{3} \partial_Q I_j(Q,f)h(w).$$

• Estimate of $\partial_Q I_1(Q,f)h$. From its expression we write

$$\partial_Q I_1(Q,f)h(w) = \fint_{\mathbb{T}} \partial_Q K_1(\xi,w) h'(\xi) d\xi.$$

Straightforward computations yield

$$\begin{aligned}
\partial_Q I_1(Q,f)h(w) &= \fint_{\mathbb{T}} \frac{\xi - w}{\Phi_f(\xi) - \Phi_f(w)} h'(\xi) d\xi \\
&- \fint_{\mathbb{T}} \frac{\big(\Phi_f(\bar{\xi}) - \Phi_f(\overline{w})\big)(\overline{\xi - w})}{\Big(\Phi_f(\xi) - \Phi_f(w)\Big)^2} h'(\xi) d\xi.
\end{aligned}$$

Using Lemma 1 we get

$$\begin{aligned}
\|\partial_Q I_1(Q,f)h\|_\alpha &\leq C\|h'\|_{L^\infty} \\
&\leq C\|h\|_{1+\alpha}.
\end{aligned}$$

• Estimate of $\partial_Q I_2(Q,f)h$. One may write

$$\begin{aligned}\partial_Q I_2(Q,f)h(w) &= -⨍_{\mathbb{T}} \frac{h(\bar{\xi})-h(\overline{w})}{\Phi_f(\xi)-\Phi_f(w)}\frac{1}{\xi^2}d\xi \\ &- ⨍_{\mathbb{T}} \frac{(\bar{\xi}-\bar{w})\big(h(\bar{\xi})-h(\overline{w})\big)}{\Big(\Phi_f(\xi)-\Phi_f(w)\Big)^2}\Phi_f'(\xi)d\xi.\end{aligned}$$

Using again Lemma 1 we find

$$\begin{aligned}\|\partial_Q I_2(Q,f)h\|_\alpha &\leq C\|h'\|_{L^\infty} \\ &\leq C\|h\|_{1+\alpha}.\end{aligned}$$

• Estimate of $\partial_Q I_3(Q,f)h$. We have

$$\begin{aligned}\partial_Q I_3(Q,f)h(w) &= ⨍_{\mathbb{T}} \frac{\big(\Phi_f(\bar{\xi})-\Phi_f(\overline{w})\big)(h(\xi)-h(w))}{\big(\Phi_f(\xi)-\Phi_f(w)\big)^2}\frac{1}{\xi^2}d\xi \\ &- ⨍_{\mathbb{T}} \frac{\big(\xi-w\big)(h(\xi)-h(w))}{\big(\Phi_f(\xi)-\Phi_f(w)\big)^2}\Phi_f'(\xi)d\xi \\ &+ 2⨍_{\mathbb{T}} \frac{\big(\bar{\xi}-\overline{w}\big)\big(\Phi_f(\bar{\xi})-\Phi_f(\overline{w})\big)(h(\xi)-h(w))}{\big(\Phi_f(\xi)-\Phi_f(w)\big)^3}\Phi_f'(\xi)d\xi \\ &= ⨍_{\mathbb{T}} K_9(\xi,w)\frac{1}{\xi^2}d\xi + ⨍_{\mathbb{T}} K_{10}(\xi,w)\Phi_f'(\xi)d\xi.\end{aligned}$$

We can check that

$$|K_9(\xi,w)|+|K_{10}(\xi,w)|\leq C\|h'\|_{L^\infty} \quad \text{and} \quad |\partial_w K_9(\xi,w)|+|\partial_w K_{10}(\xi,w)|\leq C\|h'\|_{L^\infty}\frac{1}{|w-\xi|}$$

and therefore we get by Lemma 1,

$$\|\partial_Q I_3(Q,f)h\|_\alpha \leq C\|h\|_{1+\alpha}.$$

Finally we obtain

$$\|\partial_Q I_j(Q,f)h\|_\alpha \leq C\|h\|_{1+\alpha}.$$

Reproducing the same analysis we get for any $k\in\mathbb{N}$

$$\|\partial_Q^k I_j(Q,f)h\|_\alpha \leq C(k)\|h\|_{1+\alpha}$$

and consequently

$$\|\partial_Q^k \partial_f G_2(Q,f)h\|_\alpha \leq C\|h\|_{1+\alpha}. \tag{15}$$

On the other hand the same analysis used for proving (14) shows that

$$\forall Q\in(0,\varepsilon), f,g\in B_{r_\varepsilon}, \quad \|\partial_Q I_j(Q,f)h-\partial_Q I_j(Q,g)h\|_\alpha \leq C\|h\|_{1+\alpha}\|f-g\|_{1+\alpha}$$

and thus

$$\big\|\partial_Q\partial_f G_2(Q,f)h-\partial_Q\partial_f G_2(Q,g)h\big\|_\alpha \leq C\|h\|_{1+\alpha}\|f-g\|_{1+\alpha}. \tag{16}$$

Combining (15) for the case $k=1$ with (16) we conclude that $\partial_Q\partial_f G_2:(0,\varepsilon)\times B_{r_\varepsilon}\to C^\alpha(\mathbb{T})$ is continuous. This achieves the proof of the proposition. □

4. Study of the linearized equation

The main goal of this section is to study some spectral properties of the linearized operator of the functional $f \in X \mapsto F(Q,f)$ in a neighborhood of zero. This operator is defined by

$$\mathcal{L}_Q h(w) \triangleq \frac{d}{dt} F(Q, t\,h(w))_{|t=0}, \quad h \in X,$$

where F was defined in (7) and the space X in (9).
Now we introduce the following set

$$\mathcal{S} \triangleq \Big\{ Q \in (0,1),\, \exists\, m \geq 3, 1 + Q^m - \frac{1-Q^2}{2} m = 0 \Big\}.$$

For given $m \geq 3$ the function $f_m : Q \in (0,1) \mapsto 1 + Q^m - \frac{1-Q^2}{2} m$ is strictly increasing and satisfies

$$f_m(0) = 1 - \frac{m}{2} < 0, \quad f_m(1) = 2.$$

Consequently, there is only one $Q_m \in (0,1)$ with $f_m(Q_m) = 0$. This allows to construct a function $m \mapsto Q_m$. As the map $n \mapsto f_n(Q)$ is strictly decreasing then one can readily prove that the sequence $m \mapsto Q_m$ is strictly increasing. Moreover, it is not difficult to prove the asymptotic behavior

$$Q_m \approx 1 - \frac{\alpha}{m}, \quad m \to \infty$$

where α is the unique solution of

$$1 + e^{-\alpha} - \alpha = 0.$$

Now we shall establish the following properties for $\mathcal{L}_Q$ which yields immediately Theorem 1 and Theorem 3 according to Crandall-Rabinowitz theorem.

Proposition 2. *The following assertions hold true.*

(1) *Let $h(w) = \sum_{n\geq 2} a_n w^n \in X$, then*

$$\mathcal{L}_Q h = \sum_{n \geq 1} g_{n+1} e_n; \quad e_n(w) = Im(w^n),$$

with

$$\begin{aligned} g_2 &= -\frac{1}{2}(1+Q)^2 a_2, \\ g_3 &= -2Q^2 a_3, \\ g_{n+1} &= \Big(\frac{1-Q^2}{2} n - 1 - Q^n\Big)\big(a_{n+1} - Q a_{n-1}\big), \quad \forall n \geq 3. \end{aligned}$$

(2) *The kernel of $\mathcal{L}_Q$ is nontrivial if and only if $Q = Q_m \in \mathcal{S}$ and it is a one-dimensional vector space generated by*

$$v_m(w) = \frac{w^{m+1}}{1 - Qw^2}.$$

(3) *The range of $\mathcal{L}_Q$ is of co-dimension one in Y and it is given by*

$$R(\mathcal{L}_Q) = \Big\{ g \in C^\alpha(\mathbb{T}), g = \sum_{\substack{n\geq 1 \\ n \neq m}} g_{n+1} e_n, \quad g_n \in \mathbb{R} \Big\}.$$

(4) *Transversality assumption: for any $Q = Q_m \in \mathcal{S}$,*

$$\partial_Q \mathcal{L}_Q v_m \notin R(\mathcal{L}_Q).$$

Proof. (**1**) − (**2**) We shall first slightly transform the expression of G seen in (5). We may write

$$G(\Omega,\Phi(w))=(2\Omega-1)\overline{\Phi(w)}w\Phi'(w)+w\Phi'(w)\fint_{\mathbb{T}}\frac{\overline{\Phi(\xi)}\Phi'(\xi)}{\Phi(\xi)-\Phi(w)}d\xi.$$

Notice that the last integral and all the singular integrals that will appear later in this proof are understood as the limit from the interior in the following sense: For $f:\mathbb{T}\to\mathbb{C}$ a continuous function and $w\in\mathbb{T}$, we denote by

$$\fint_{\mathbb{T}}\frac{f(\xi)}{\Phi(\xi)-\Phi(w)}d\xi=\lim_{\substack{z\in\mathbb{D}^\star\\ z\mapsto w}}\fint_{\mathbb{T}}\frac{f(\xi)}{\Phi(\xi)-\Phi(z)}d\xi,$$

with $\mathbb{D}^\star$ being the interior of $\mathbb{D}$. This definition is justified by the fact that the points in the unit disc which are located close to the boundary $\mathbb{T}$ are sent by the map Φ inside the domain enclosed by the Jordan curve $\Phi(\mathbb{T})$, since Φ is as small perturbation of the outside conformal map of the ellipse α_Q. Recall also that $\Omega=\frac{1-Q^2}{4}$. Now we shall compute

$$Lh(w)\triangleq\frac{d}{dt}G(\Omega,\alpha_Q(w)+th(w))_{|t=0}.$$

The relation with $\mathcal{L}_Q$ is given by $\mathcal{L}_Q h=\operatorname{Im}Lh$. For simplicity set $\alpha:=\alpha_Q$ then performing straightforward calculations one can check that

$$-Lh(w)=I_1(w)-\sum_{j=2}^{5}I_j(w),$$

with

$$I_1(w)=\frac{1+Q^2}{2}\Big((1+Qw^2)h'(w)+(w-Q\overline{w})h(\overline{w})\Big),$$

$$I_2(w)=wh'(w)\fint_{\mathbb{T}}\frac{\alpha(\bar\xi)}{\alpha(\xi)-\alpha(w)}\alpha'(\xi)d\xi,$$

$$I_3(w)=w\alpha'(w)\fint_{\mathbb{T}}\frac{\alpha'(\xi)}{\alpha(\xi)-\alpha(w)}h(\bar\xi)d\xi,$$

$$I_4(w)=w\alpha'(w)\fint_{\mathbb{T}}\frac{\alpha(\bar\xi)}{\alpha(\xi)-\alpha(w)}h'(\xi)d\xi$$

and

$$I_5(w)=-w\alpha'(w)\fint_{\mathbb{T}}\alpha(\bar\xi)\frac{h(\xi)-h(w)}{\big(\alpha(\xi)-\alpha(w)\big)^2}\alpha'(\xi)d\xi.$$

• *Computation of* I_2. Let $z\in\mathbb{D}$ being in a very small tubular neighborhood of $\mathbb{T}$ (such that $Q/z\in\mathbb{D}$) then by residue theorem we get

$$\begin{aligned}
\fint_{\mathbb{T}}\frac{\alpha(\bar\xi)}{\alpha(\xi)-\alpha(z)}\alpha'(\xi)d\xi &= \fint_{\mathbb{T}}\frac{(1+Q\xi^2)(\xi^2-Q)}{\xi^2(\xi-z)(\xi-\frac{Q}{z})}d\xi\\
&= \fint_{\mathbb{T}}\frac{\xi^2-Q}{\xi^2(\xi-z)(\xi-\frac{Q}{z})}d\xi+Q\fint_{\mathbb{T}}\frac{\xi^2}{(\xi-z)(\xi-\frac{Q}{z})}d\xi\\
&= Q\fint_{\mathbb{T}}\frac{\xi^2}{(\xi-z)(\xi-\frac{Q}{z})}d\xi\\
&= Q\Big(\frac{z^2}{z-\frac{Q}{z}}+\frac{\frac{Q^2}{z^2}}{\frac{Q}{z}-z}\Big)\\
&= Q\big(z+\frac{Q}{z}\big).
\end{aligned}$$

Therefore letting z go to w we find

$$I_2(w) = Qw\big(w + \frac{Q}{w}\big)h'(w). \tag{17}$$

• *Computation of* I_3. From the explicit formula of α we get

$$\fint_{\mathbb{T}} \frac{\alpha'(\xi)}{\alpha(\xi)-\alpha(z)}\overline{h}(\xi)d\xi = \fint_{\mathbb{T}} \frac{\xi^2 - Q}{\xi(\xi - z)\big(\xi - \frac{Q}{z}\big)}\overline{h}(\xi)d\xi.$$

Since $\overline{h}$ is analytic outside the open unit disc and is at least of order two in $\frac{1}{\xi}$ at infinity then by residue theorem (always with $z \in \mathbb{D}$ such that $Q/z \in \mathbb{D}$)

$$\begin{aligned}\fint_{\mathbb{T}} \frac{\alpha'(\xi)}{\alpha(\xi)-\alpha(z)}\overline{h}(\xi)d\xi &= \fint_{\mathbb{T}} \frac{\xi^2 - Q}{\xi(\xi - z)\big(\xi - \frac{Q}{z}\big)}\overline{h}(\xi)d\xi \\ &= 0.\end{aligned}$$

Consequently we find

$$I_3(w) = 0. \tag{18}$$

• *Computation of* I_4. Let $z \in \mathbb{D}$ with $Q/z \in \mathbb{D}$, since h' is holomorphic inside the unit disc then we deduce by residue theorem

$$\begin{aligned}\fint_{\mathbb{T}} \frac{\overline{\alpha}(\xi)}{\alpha(\xi)-\alpha(z)}h'(\xi)d\xi &= \fint_{\mathbb{T}} \frac{1 + Q\xi^2}{(\xi - z)(\xi - \frac{Q}{z})}h'(\xi)d\xi \\ &= \frac{1 + Qz^2}{z - \frac{Q}{z}}h'(z) + \frac{1 + \frac{Q^3}{z^2}}{\frac{Q}{z} - z}h'\big(\frac{Q}{z}\big).\end{aligned}$$

Thus we obtain

$$I_4(w) = (1 + Qw^2)h'(w) - \big(1 + Q^3\overline{w}^2\big)h'\big(Q\overline{w}\big). \tag{19}$$

• *Computation of* I_5. Using residue theorem as in the preceding cases we find

$$\begin{aligned}\fint_{\mathbb{T}} \overline{\alpha}(\xi)\frac{h(\xi) - h(w)}{\big(\alpha(\xi) - \alpha(w)\big)^2}\alpha'(\xi)d\xi &= \fint_{\mathbb{T}} \frac{(1 + Q\xi^2)(\xi^2 - Q)}{\xi(\xi - w)^2\big(\xi - \frac{Q}{w}\big)^2}\big(h(\xi) - h(w)\big)d\xi \\ &= -h(w)\fint_{\mathbb{T}} \frac{(1 + Q\xi^2)(\xi^2 - Q)}{\xi(\xi - w)^2\big(\xi - \frac{Q}{w}\big)^2}d\xi \\ &+ \fint_{\mathbb{T}} \frac{(1 + Q\xi^2)(\xi^2 - Q)}{(\xi - w)^2\big(\xi - \frac{Q}{w}\big)^2}\frac{h(\xi)}{\xi}d\xi \\ &= -Qh(w) + J(w),\end{aligned}$$

with

$$J(w) \triangleq \fint_{\mathbb{T}} \frac{(1 + Q\xi^2)(\xi^2 - Q)}{(\xi - w)^2\big(\xi - \frac{Q}{w}\big)^2}\frac{h(\xi)}{\xi}d\xi.$$

Set

$$F_1(\xi) = \frac{(1 + Q\xi^2)(\xi^2 - Q)}{\big(\xi - \frac{Q}{w}\big)^2}\frac{h(\xi)}{\xi} \triangleq K_1(\xi)\frac{h(\xi)}{\xi},$$

and

$$F_2(\xi) = \frac{(1 + Q\xi^2)(\xi^2 - Q)}{\big(\xi - w\big)^2}\frac{h(\xi)}{\xi} \triangleq K_2(\xi)\frac{h(\xi)}{\xi}.$$

Then we get successively,

$$K_1'(w) = \frac{2Qw^2}{w - \frac{Q}{w}} \qquad K_2'\big(\frac{Q}{w}\big) = \frac{\frac{2Q^3}{w^2}}{\frac{Q}{w} - w}$$

which give in turn

$$F_1'(w) = \frac{1}{w - \frac{Q}{w}}\Big\{(Qw - \frac{1}{w})h(w) + (1 + Qw^2)h'(w)\Big\}$$

$$F_2'(\frac{Q}{w}) = \frac{1}{\frac{Q}{w} - w}\Big\{\big(\frac{Q^2}{w} - \frac{w}{Q}\big)h(\frac{Q}{w}) + \big(1 + \frac{Q^3}{w^2}Q\big)h'\Big(\frac{Q}{w}\Big)\Big\}.$$

Applying once again residue theorem we obtain

$$J(w) = F_1'(w) + F_2'(Q\overline{w}).$$

It follows that

$$(20) I_5(w) = (1 - Q^2)\overline{w}h(w) - (1 + Qw^2)h'(w) + \Big(Q^2\overline{w} - \frac{w}{Q}\Big)h(Q\overline{w}) + \big(1 + Q^3\overline{w}^2\big)h'\big(Q\overline{w}\big).$$

Putting together the identities (17), (18), (19) and (20) one gets

$$\sum_{j=2}^{5} I_j(w) = (1 - Q^2)\overline{w}h(w) + Q(Q + w^2)h'(w) + \Big(Q^2\overline{w} - \frac{w}{Q}\Big)h(Q\overline{w})$$

and consequently,

$$\begin{aligned} -Lh(w) &= \frac{1 + Q^2}{2}\Big\{(1 + Qw^2)h'(w) + (w - Q\overline{w})\overline{h}(w)\Big\} + (Q^2 - 1)\overline{w}h(w) \\ &- Q(w^2 + Q)h'(w) + \big(\frac{w}{Q} - Q^2\overline{w}\big)h\big(Q\overline{w}\big). \end{aligned}$$

This can also be written in the form

$$\begin{aligned} -Lh(w) &= \Big[\frac{1 - Q^2}{2} + Q\frac{Q^2 - 1}{2}w^2\Big]h'(w) + (Q^2 - 1)\overline{w}h(w) \\ &+ \frac{1 + Q^2}{2}\big(w - Q\overline{w}\big)\overline{h}(w) + \big(\frac{w}{Q} - Q^2\overline{w}\big)h\big(Q\overline{w}\big) \\ &\triangleq L_1h(w) + L_2h(w) + L_3h(w) + L_4h(w). \end{aligned}$$

It is easy to check that

$$\begin{aligned} L_1h(w) &= \frac{1 - Q^2}{2}(1 - Qw^2)\sum_{n\geq 2} na_n w^{n-1} \\ &= \frac{1 - Q^2}{2}\Big(2a_2w + 3a_3w^2\Big) + \sum_{n\geq 3} c_n w^n, \end{aligned}$$

with

$$c_n = \frac{1 - Q^2}{2}\Big((n + 1)a_{n+1} - Q(n - 1)a_{n-1}\Big), \quad \forall n \geq 3.$$

The computation of L_2h is obvious,

$$\begin{aligned} L_2h(w) &= (Q^2 - 1)\sum_{n\geq 2} a_n w^{n-1} \\ &= (Q^2 - 1)\Big(a_2w + a_3w^2\Big) + (Q^2 - 1)\sum_{n\geq 3} a_{n+1}w^n. \end{aligned}$$

Therefore we get

$$L_1h(w)+L_2h(w) = \frac{1-Q^2}{2}a_3w^2+\sum_{n\geq 3}d_nw^n, \tag{21}$$

with

$$d_n=\frac{1-Q^2}{2}(n-1)\big(a_{n+1}-Qa_{n-1}\big), \quad \forall n\geq 3. \tag{22}$$

As to the third term, we easily find

$$\begin{aligned} L_3h(w) &= \frac{1+Q^2}{2}\big(w-Q\overline{w}\big)\sum_{n\geq 2}\frac{a_n}{w^n} \\ &= \frac{1+Q^2}{2}\big(\frac{a_2}{w}+\frac{a_3}{w^2}\big)+\frac{1+Q^2}{2}\sum_{n\geq 3}\frac{a_{n+1}-Qa_{n-1}}{w^n}. \end{aligned}$$

Performing the same analysis yields

$$\begin{aligned} L_4h(w) &= \big(\frac{w}{Q}-\frac{Q^2}{w}\big)\sum_{n\geq 2}\frac{a_nQ^n}{w^n} \\ &= \sum_{n\geq 2}\frac{a_nQ^{n-1}}{w^{n-1}}-Q\sum_{n\geq 2}\frac{a_nQ^{n+1}}{w^{n+1}} \\ &= \frac{Qa_2}{w}+\frac{Q^2a_3}{w^2}+\sum_{n\geq 3}\frac{Q^n\big(a_{n+1}-Qa_{n-1}\big)}{w^n}. \end{aligned}$$

Consequently,

$$\begin{aligned} L_3h(w)+L_4h(w) &= \frac{(1+Q)^2}{2}\frac{a_2}{w}+\frac{1+3Q^2}{2}\frac{a_3}{w^2} \\ &+ \sum_{n\geq 3}\frac{\widehat{d_n}}{w^n}, \end{aligned} \tag{23}$$

with

$$\widehat{d_n} = \Big(\frac{1+Q^2}{2}+Q^n\Big)\big(a_{n+1}-Qa_{n-1}\big). \tag{24}$$

According to (21) and (23) we get

$$\begin{aligned} -\mathcal{L}_Qh(w) &= \operatorname{Im}Lh(w) \\ &= -\frac{(1+Q)^2}{2}a_2e_1(w)-2Q^2a_3e_2(w) \\ &+ \sum_{n\geq 3}(d_n-\widehat{d_n})e_n(w). \end{aligned}$$

with the notation $e_n(w)=\operatorname{Im}(w^n)$. It follows that

$$\begin{aligned} -\mathcal{L}_Qh(w) &= -\frac{1}{2}(1+Q)^2a_2\,e_1(w)-2Q^2a_3\,e_2(w) \\ &+ \sum_{n\geq 3}(d_n-\widehat{d_n})\,e_n(w). \end{aligned} \tag{25}$$

Combining (22) and (24) we obtain

$$d_n-\widehat{d_n}=\Big(\frac{1-Q^2}{2}n-1-Q^n\Big)\big(a_{n+1}-Qa_{n-1}\big). \tag{26}$$

Therefore the equation $\mathrm{Im}\,\mathcal{L}h(w)=0$ is equivalent to the linear system

$$\begin{aligned}(1+Q)^2a_2 &= 0\\ Q^2a_3 &= 0\\ \Big(\frac{1-Q^2}{2}n-1-Q^n\Big)\big(a_{n+1}-Qa_{n-1}\big) &= 0,\quad \forall n\geq 3.\end{aligned}$$

Since $Q\in(0,1)$ then necessarily

$$a_2=a_3=0.$$

The last equation is equivalent to

$$\Big(1+Q^n-\frac{1-Q^2}{2}n\Big)\big(a_{n+1}-Qa_{n-1}\big)=0,\quad \forall n\geq 3. \tag{27}$$

Let $m\geq 3$, then we know from the beginning of this section the existence of only one solution $Q=Q_m\in(0,1)$ of the equation

$$1+Q^m-\frac{1-Q^2}{2}m=0.$$

Moreover, the left part of this equality defines a strictly decreasing function in m implying that

$$1+Q^n-\frac{1-Q^2}{2}n\neq 0,\quad \forall n\neq m.$$

Thus (27) is equivalent to

$$a_{n+1}=Qa_{n-1},\forall n\geq 3,\quad n\neq m.$$

Thus the dynamical system (27) combined with the vanishing two first values admits the following solutions

$$\forall n\geq 0, a_{m+1+2n}=Q^n a_{m+1}\quad\text{and}\quad 0\quad\text{otherwise}$$

This means that the associated kernel is one dimensional generated by the eigenfunction

$$v_m(w)=\sum_{n\geq 0}Q^n\,w^{m+1+2n}=\frac{w^{m+1}}{1-Qw^2}.$$

Note that v_m is holomorphic in the open ball $B(0,\frac{1}{Q})$ and since $Q\in(0,1)$ we deduce that $v_m\in X$.
(**3**) Let $Q\in\mathcal{S}$ and m being the frequency such that $Q=Q_m$. We will show that the range $R(\mathcal{L}_{Q_m})$ coincides with the closed subspace

$$\mathcal{Y}\triangleq\Big\{g\in C^\alpha(\mathbb{T});\quad g=\sum_{\substack{n\geq 1\\ n\neq m}}g_{n+1}\,e_n,\quad g_n\in\mathbb{R}\Big\}.$$

From (25) and (26) one sees that the range of $\mathcal{L}_Q$ is contained in $\mathcal{Y}$. Conversely, let $g\in\mathcal{Y}$ we shall look for $h(w)=\sum_{n\geq 2}a_nw^n\in C^{1+\alpha}(\mathbb{T})$ such that $\mathcal{L}_Qh=g$. Once again from (25) this is equivalent to

$$\begin{aligned}-\frac{1}{2}(1+Q)^2a_2 &= g_2,\\ -2Q^2a_3 &= g_3,\\ \Big(\frac{1-Q^2}{2}n-1-Q^n\Big)\big(a_{n+1}-Qa_{n-1}\big) &= g_{n+1},\quad \forall n\geq 3,\quad n\neq m.\end{aligned}$$

This determines uniquely the sequence $(a_n)_{2\leq n\leq m}$ and for $n\geq m+1$ one has the recursive formula

$$a_{n+1}-Qa_{n-1}=\frac{g_{n+1}}{\frac{1-Q^2}{2}n-1-Q^n},\quad \forall n\geq m+1.$$

The only free coefficient is a_{m+1} and therefore the solutions of the above system form one-dimensional affine space. To prove that any pre-image h belongs to $C^{1+\alpha}(\mathbb{T})$ it suffices to show it for the function $H(w)=\sum_{n\geq m+2}a_nw^n$. Set

$$G(w)=\sum_{n\geq m+1}\frac{g_{n+1}}{\frac{1-Q^2}{2}n-1-Q^n}w^n,\quad R_m(w)=\sum_{n\geq m+1}g_{n+1}w^n.$$

Then

$$\begin{aligned}H(w)&=w\sum_{n\geq m+1}a_{n+1}w^n\\&=wQ\sum_{n\geq m+1}a_{n-1}w^n+\sum_{n\geq m+1}\frac{g_{n+1}}{\frac{1-Q^2}{2}n-1-Q^n}w^{n+1}\\&=w^2Q\sum_{n\geq m}a_nw^n+wG(w)\\&=w^2QH(w)+w^2Q(a_mw^m+a_{m+1}w^{m+1})+wG(w).\end{aligned}$$

Therefore

$$H(w)=\frac{1}{1-Qw^2}\Big(w^2Q(a_mw^m+a_{m+1}w^{m+1})+wG(w)\Big).$$

The problem reduces then to check that $G\in C^{1+\alpha}(\mathbb{T})$. We split G into two terms as follows

$$\begin{aligned}G(w)&=\sum_{n\geq m+1}\frac{g_{n+1}}{\frac{1-Q^2}{2}n-1}w^n+\sum_{n\geq m+1}\frac{Q^ng_{n+1}}{(\frac{1-Q^2}{2}n-1)(\frac{1-Q^2}{2}n-1-Q^n)}w^n\\&=G_1+G_2.\end{aligned}$$

Since the sequence (g_n) is bounded then for large n one gets

$$\frac{Q^n|g_{n+1}|}{(\frac{1-Q^2}{2}n-1)(\frac{1-Q^2}{2}n-1-Q^n)}\leq C\frac{Q^n}{(1-Q^2)^2}.$$

This shows that $G_2\in C^k(\mathbb{T})$ for all $k\in\mathbb{N}$. Let us now prove that $G_1\in C^{1+\alpha}(\mathbb{T})$. First from the embedding $C^\alpha(\mathbb{T})\hookrightarrow L^\infty(\mathbb{T})\hookrightarrow L^2(\mathbb{T})$ one obtains

$$\sum_n|g_{n+1}|^2\lesssim\|g\|^2_{C^\alpha}.$$

Therefore by Cauchy-Schwarz

$$\begin{aligned}\|G_1\|_{L^\infty}&\lesssim\sum_{n\geq m+1}\frac{|g_{n+1}|}{n}\\&\lesssim\Big(\sum_{n\geq m+1}|g_{n+1}|^2\Big)^{\frac12}\\&\lesssim\|g\|_{C^\alpha}.\end{aligned}$$

It remains to prove that $G_1' \in C^\alpha(\mathbb{T})$. Differentiating term by term the series we get

$$\begin{aligned} G_1' &= \sum_{n\geq m+1} \frac{n\, g_{n+1}}{\frac{1-Q^2}{2}n-1} w^{n-1} \\ &= \frac{2}{1-Q^2}\sum_{n\geq m+1} g_{n+1} w^{n-1} + \frac{2}{1-Q^2}\sum_{n\geq m+1} \frac{g_{n+1}}{\frac{1-Q^2}{2}n-1} w^{n-1} \\ &= \frac{2}{1-Q^2}\frac{1}{w}R_m(w) + \frac{4}{(1-Q^2)^2}\sum_{n\geq m+1}\frac{g_{n+1}}{n} w^{n-1} \\ &+ \frac{4}{(1-Q^2)^2}\sum_{n\geq m+1}\frac{g_{n+1}}{n(\frac{1-Q^2}{2}n-1)} w^{n-1} \\ &\triangleq G_3(w)+G_4(w)+G_5(w). \end{aligned}$$

The function G_3 is clearly in $C^\alpha(\mathbb{T})$ according to the assumption $g \in C^\alpha$. The function G_4 belongs to $L^\infty(\mathbb{T})$ and $G_4' \in C^\alpha(\mathbb{T})$. Indeed,

$$\begin{aligned} \|G_4\|_{L^\infty} &\lesssim \sum_{n\geq p+1}\frac{|g_{n+1}|}{n} \\ &\lesssim \Big(\sum_{n\geq m+1}|g_{n+1}|^2\Big)^{\frac{1}{2}} \\ &\lesssim \|G\|_{L^\infty}. \end{aligned}$$

Moreover

$$(wG_4)' = \frac{4}{(1-Q^2)^2}\sum_{n\geq m+1} g_{n+1}w^{n-1} = \frac{4}{(1-Q^2)^2}\frac{1}{w}R_m(w).$$

This gives $(wG_4)' \in C^\alpha(\mathbb{T})$ and thus $G_4 \in C^{1+\alpha}(\mathbb{T})$. On the other hand

$$wG_5'(w) = \frac{4}{(1-Q^2)^2}\sum_{n\geq m+1}\frac{g_{n+1}}{\frac{1-Q^2}{2}n-1}w^{n-1}.$$

Arguing as before we see that $wG_5' \in L^\infty(\mathbb{T})$ and belongs also to $C^\alpha(\mathbb{T})$. This shows that $G_1' \in C^\alpha(\mathbb{T})$ which gives that $G_1 \in C^{1+\alpha}(\mathbb{T})$. This shows finally that any pre-image of g belongs to the space X.
(**4**) Let $m \geq 3$ be an integer and $Q = Q_m$ the associated element in the set $\mathcal{S}$. We have seen that the kernel is one-dimensional generated by

$$v_m(w) = \frac{w^{m+1}}{1-Qw^2} = \sum_{n\geq 0} Q^n w^{m+1+2n}.$$

We shall compute $\partial_Q\partial_f F(Q_m,0)v_m$ which coincides with $\big\{\partial_Q\mathcal{L}_Q v_m\big\}_{Q=Q_m}$. The transversality condition that we shall check is

$$\big\{\partial_Q\mathcal{L}_Q v_m\big\}_{Q=Q_m} \notin R(\mathcal{L}_{Q_m}).$$

From the structure of the range of $\mathcal{L}$ this is equivalent to prove that the coefficient of e_m in the decomposition $\big\{\partial_Q\mathcal{L}_Q v_m\big\}_{Q=Q_m}$ is not zero. From (25) and (26) this coefficient is given by

$$\begin{aligned} \{\partial_Q(d_m-\widehat{d}_m)\}_{Q=Q_m} &= -m(Q_m+Q_m^{m-1})(a_{m+1}-Q_m a_{m-1}) + (\frac{1-Q_m^2}{2}m-1-Q_m^m)a_{m-1} \\ &= -m(Q_m+Q_m^{m-1})+0 \\ &\neq 0. \end{aligned}$$

We have used the fact that $a_{m+1} = 1$ and $a_{m-1} = 0$. This achieves the transversality assumption and therefore the proof of Proposition 2 is complete. □

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