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## Ramanujan-like series for $\frac{1}{\pi}$ involving harmonic numbers

John M. Campbell

**Abstract** We introduce new classes of Ramanujan-like series for  $\frac{1}{\pi}$ , by devising methods for evaluating harmonic sums involving squared central binomial coefficients, such as the Ramanujan-type series

$$\sum_{n=1}^{\infty} \frac{\binom{2n}{n}^2 (H_n^2 + H_n^{(2)})}{16^n (2n-1)} = \frac{4\pi}{3} - \frac{32 \ln^2(2) - 32 \ln(2) + 16}{\pi}$$

introduced in this article. While the main technique used in this article is based on the evaluation of a parameter derivative of a beta-type integral, we also show how new integration results involving complete elliptic integrals may be used to evaluate Ramanujan-like series for  $\frac{1}{\pi}$  containing harmonic numbers.

**Keywords** Ramanujan-like series · Harmonic number · Pi formula · Complete elliptic integral

**Mathematics Subject Classification (2000)** Primary 33C75 · 33C20 · Secondary 65B10

### 1 Introduction

The evaluation of summations containing binomial coefficients and harmonic numbers is an interesting topic, and a large amount of mathematical literature is devoted to this area. In this article, we consider the general problem of evaluating infinite series involving harmonic numbers and squared central binomial coefficients.

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Many harmonic sums involving binomial coefficients may be evaluated by applying differential operators to known hypergeometric identities. In particular, the beautiful Ramanujan-like formula

$$\sum_{n=1}^{\infty} \frac{\binom{2n}{n}^2 H_n}{16^n (2n-1)^2} = \frac{12 - 16 \ln(2)}{\pi}$$

which was noted in 2016 by Hongwei Chen may be proven using this method [2]. The author of [2] strongly encourages further exploration of properties of series involving powers of binomial coefficients and harmonic numbers.

Although certain types harmonic summations may be evaluated by applying differential operators to specific hypergeometric identities, it is not in general obvious as to how to evaluate a given a harmonic sum involving powers of binomial coefficients. Inspired in part by [2], we introduce in our present article a very general method for evaluating series with a summand involving a factor of the form  $\binom{2n}{n}^2 H_n$ . Using this method, we prove a variety of new Ramanujan-like formulas for  $\frac{1}{\pi}$ .

The main method introduced in this article for evaluating harmonic sums containing expressions of the form  $\binom{2n}{n}^2$  is based on a parameter derivative of a beta-type integral. This technique is very useful and general, because this method may be used to evaluate series of the form

$$\sum_{n=1}^{\infty} g(n) \binom{2n}{n}^2 H_n$$

for a given function  $g$  on  $\mathbb{N}$ , whereas techniques given by differentiating specific classes of hypergeometric identities will only produce specific results. We also show how a generating function recently considered by Boyadzhiev in [1] may be used to evaluate certain harmonic sums with squared central binomial coefficients, and we also describe a general method for evaluating harmonic sums of the form

$$\sum_{n=1}^{\infty} g(n) \binom{2n}{n}^2 (H_n^2 + H_n^{(2)}).$$

The elegant Ramanujan-type formula

$$\sum_{n=0}^{\infty} (-1)^n \frac{\left(\frac{1}{2}\right)_n^3}{(1)_n^3} (-2 + 3(4n+1)H_n) = -\frac{12 \ln(2)}{\pi}$$

is proven in [3], and several other Ramanujan-like formulas related to harmonic numbers are proven in [3]. Apart from [2] and [3], there does not seem to be very much mathematical literature concerning harmonic sums for  $\frac{1}{\pi}$ . It is thus natural to use a systematic approach towards the construction of new series of this form.

The formulas for  $\frac{1}{\pi}$  introduced in this article can also be proven using some new formulas for definite integrals involving complete elliptic integrals, as we

discuss in Section 4. We are interested in the general problem of evaluating integrals involving complete elliptic integrals in part because there are many applications related to elliptic functions and elliptic integrals in various fields connected to physics and engineering. As discussed in [4], many physical problems involve multiple integrals involving elliptic integrals such that once one of the integrations is completed, the resultant integrand contains an elliptic integral.

## 2 Harmonic sums involving squared central binomial coefficients

In this section, we describe some different techniques for evaluating Ramanujan-like series involving harmonic numbers and expressions of the form  $\binom{2n}{n}^2$ , and in Section 3 we apply our main technique, which is described in Section 2.1, by introducing new harmonic series for  $\frac{1}{\pi}$ .

### 2.1 First strategy

Our first strategy for constructing new Ramanujan-type series for  $\frac{1}{\pi}$  is based on the evaluation of integrals of the form  $\int_0^1 \frac{x^n \ln(1-x^2)}{\sqrt{1-x^2}} dx$ . Integrals of this form may be evaluated in a natural way in terms of harmonic numbers, as indicated in Lemma 1 below. We remark that a somewhat similar strategy based on the evaluation of integrals involving an expression of the form  $\frac{x \ln^{2m-1} x}{1-x}$  was recently used to evaluate certain classes of series involving harmonic numbers in [5].

**Lemma 1** *Let  $f: \mathbb{N}_0 \rightarrow \mathbb{C}$  be such that the series*

$$\sum_{n=0}^{\infty} (-1)^n \binom{\frac{1}{2}}{n} f(n) \frac{x^{2n} \ln(1-x^2)}{\sqrt{1-x^2}}$$

*is integrable on  $[0, 1]$ , and write  $g(n) = \frac{f(n)}{16^n \binom{2n-1}{n}}$ . Then  $\sum_{n=0}^{\infty} g(n) \binom{2n}{n}^2 H_n$  is equal to  $\frac{2}{\pi}$  times the sum of*

$$\int_0^1 \left( \sum_{n=0}^{\infty} (-1)^n \binom{\frac{1}{2}}{n} f(n) \frac{x^{2n} \ln(1-x^2)}{\sqrt{1-x^2}} \right) dx$$

*and  $\pi \ln(2) \sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2 f(n)}{16^n (1-2n)}$ .*

*Proof* This follows immediately from the identity

$$\int_0^1 \frac{x^{2n} \ln(1-x^2)}{\sqrt{1-x^2}} dx = -\frac{\pi 2^{-2n} n \Gamma(2n) (H_n + 2 \ln(2))}{\Gamma(n+1)^2},$$

which holds for  $n$  such that  $\Re(n) > -\frac{3}{2}$ . □

The above lemma is especially useful because it is not in general otherwise obvious as to how to evaluate a series of the form  $\sum_{n=0}^{\infty} g(n) \binom{2n}{n}^2 H_n$ .

Since one of the key tools used in this paper is based on the evaluation of integrals of the form  $\int_0^1 \frac{x^n \ln(1-x^2)}{\sqrt{1-x^2}} dx$ , it is worthwhile to note that this integral is a derivative of the beta function with respect to a parameter. In particular, letting

$$B(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt,$$

we have that

$$\frac{1}{2} \frac{\partial}{\partial y} B\left(\frac{n+1}{2}, y+1\right) = \int_0^1 u^n (1-u^2)^y \log(1-u^2) du,$$

so that

$$\int_0^1 \frac{u^n \log(1-u^2)}{\sqrt{1-u^2}} du = \frac{\sqrt{\pi} \Gamma\left(\frac{1+n}{2}\right) \psi^{(0)}\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{1}{2} + \frac{1+n}{2}\right)} - \frac{\sqrt{\pi} \Gamma\left(\frac{1+n}{2}\right) \psi^{(0)}\left(\frac{1}{2} + \frac{1+n}{2}\right)}{2\Gamma\left(\frac{1}{2} + \frac{1+n}{2}\right)}$$

for  $n \in \mathbb{N}_0$ .

## 2.2 Boyadzhiev's generating function

Boyadzhiev recently evaluated the generating function for the sequence

$$\left( \binom{2n}{n} H_n : n \in \mathbb{N}_0 \right)$$

and similar sequences in [1], proving that

$$\sum_{n=0}^{\infty} \binom{2n}{n} H_n x^n = \frac{2}{\sqrt{1-4x}} \ln\left(\frac{1+\sqrt{1-4x}}{2\sqrt{1-4x}}\right)$$

for  $|x| < \frac{1}{4}$ .

Boyadzhiev's generating function may be used to construct certain types of Ramanujan-like series for  $\frac{1}{\pi}$  involving squared central binomial coefficients and harmonic numbers. But there are certain series of this form which cannot feasibly be proven directly using Boyadzhiev's generating function or similar generating functions, and which can be proven easily using Lemma 1, as we later discuss.

To construct a Ramanujan-type series for  $\frac{1}{\pi}$  using Boyadzhiev's generating function one may use the following integral identity:

$$\frac{\binom{2n}{n}}{4^n} = \frac{2}{\pi} \int_0^{\infty} \frac{dx}{(x^2+1)^{n+1}}.$$

To illustrate this idea, as well as the technique described in Section 2.1, we offer two corresponding proofs of the following new Ramanujan-like formula.

**Theorem 1**  $\sum_{n=1}^{\infty} \frac{\binom{2n}{n}^2 H_n}{16^n (2n-1)} = \frac{8 \ln(2) - 4}{\pi}$ .

*Proof #1:* This follows immediately from Lemma 1 by letting  $f(n) = 1$ .  $\square$

*Proof #2:* By Boyadzhiev's generating function, we have that

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n} H_n x^{2n-2}}{4^n} = \frac{2 \ln \left( \frac{\sqrt{1-x^2}+1}{2\sqrt{1-x^2}} \right)}{x^2 \sqrt{1-x^2}}.$$

Integrating both sides of this equality, we have that  $\sum_{n=0}^{\infty} \frac{\binom{2n}{n} H_n y^{2n-1}}{4^n (2n-1)}$  is equal to:

$$\frac{2 \left( -\sqrt{1-y^2} - \sqrt{1-y^2} \ln \left( \frac{1}{\sqrt{1-y^2}} + 1 \right) + \sqrt{1-y^2} \ln(2) + 1 \right)}{y}.$$

Therefore,  $\sum_{n=0}^{\infty} \frac{\binom{2n}{n} H_n 2 \left( \frac{1}{x^2+1} \right)^{n+1}}{4^n \pi (2n-1)}$  is equal to

$$\frac{4 \left( \sqrt{x^2+1} - \sqrt{x^2} \ln \left( \frac{1}{\sqrt{\frac{x^2}{x^2+1}} + 1} \right) + \sqrt{x^2} (\ln(2) - 1) \right)}{\pi (x^2+1)^{3/2}}.$$

Mathematica is able to evaluate the integral

$$\int_0^{\infty} \frac{4 \left( \sqrt{x^2+1} - \sqrt{x^2} \ln \left( \frac{1}{\sqrt{\frac{x^2}{x^2+1}} + 1} \right) + \sqrt{x^2} (\ln(2) - 1) \right)}{\pi (x^2+1)^{3/2}} dx$$

as  $\frac{8 \ln(2) - 4}{\pi}$ , thus proving the above evaluation of  $\sum_{n=1}^{\infty} \frac{\binom{2n}{n}^2 H_n}{16^n (2n-1)}$ .  $\square$

In Section 4, we offer another elegant proof of Theorem 1, through the use of integrals involving complete elliptic integrals.

Our main strategy introduced in Section 2.1 for evaluating Ramanujan-type series for  $\frac{1}{\pi}$  involving harmonic numbers is much more general and powerful compared to the alternative technique outlined in Section 2.2. For example, it seems that it would not be feasible to use Boyadzhiev's generating function to prove the following result due to Hongwei Chen, although Lemma 1 may be used to prove the following theorem.

**Theorem 2**  $\sum_{n=1}^{\infty} \frac{\binom{2n}{n}^2 H_n}{16^n (2n-1)^2} = \frac{12-16 \ln(2)}{\pi}$  [2].

*Proof #1:* This follows immediately from Lemma 1 by letting  $f(n) = \frac{1}{2n-1}$ .  $\square$

By Boyadzhiev's generating function, we have that  $\sum_{n=0}^{\infty} \frac{\binom{2n}{n} H_n y^{2n-2}}{4^n (2n-1)}$  is equal to

$$\frac{2 \left( -\sqrt{1-y^2} - \sqrt{1-y^2} \ln \left( \frac{1}{\sqrt{1-y^2}} + 1 \right) + \sqrt{1-y^2} \ln(2) + 1 \right)}{y^2}.$$

Integrating the above expression yields a very complicated expression involving the polylogarithm function. Mathematica is not able to evaluate the corresponding integral needed to evaluate the series  $\sum_{n=1}^{\infty} \frac{\binom{2n}{n}^2 H_n}{16^n (2n-1)^2}$  using the identity  $\frac{\binom{2n}{n}}{4^n} = \frac{2}{\pi} \int_0^{\infty} \frac{dx}{(x^2+1)^{n+1}}$ . This illustrates how the strategy introduced in Section 2.1 may be used to construct new Ramanujan-like series for  $\frac{1}{\pi}$  which cannot be evaluated directly following the technique given in Section 2.2. We later offer an alternative proof of Theorem 2 using definite integrals involving complete elliptic integrals. We also offer an alternative proof of the following theorem using integrals containing complete elliptic integrals.

**Theorem 3**  $\sum_{n=1}^{\infty} \frac{\binom{2n}{n}^2 H_n}{16^n (2n-3)} = \frac{120 \ln(2) - 68}{27\pi}$ .

*Proof #1:* This follows immediately from Lemma 1 by letting  $f(n)$  be equal to  $\frac{2n-1}{2n-3}$ .  $\square$

### 2.3 Series involving an expression of the form $H_n^2 + H_n^{(2)}$

In Section 5, we show how the evaluation of integrals of the form

$$\int_0^1 \frac{x^n \ln^2(1-x^2)}{\sqrt{1-x^2}} dx$$

may be used to prove new formulas for Ramanujan-like series, such as the elegant formula

$$\sum_{n=1}^{\infty} \frac{\binom{2n}{n}^2 (H_n^2 + H_n^{(2)})}{16^n (2n-1)} = \frac{4\pi}{3} + \frac{16(2 \ln(2) - 2 \ln^2(2) - 1)}{\pi}$$

introduced in this article.

### 3 New Ramanujan-type series for $\frac{1}{\pi}$ involving harmonic numbers

We begin by presenting new Ramanujan-like series for  $\frac{1}{\pi}$  with a summand of the form  $\frac{\binom{2n}{n}^2 H_n}{16^n p_n}$ , letting  $p_n$  denote a polynomial with integer coefficients. Lemma 1 may be used to evaluate summations of this form in the case whereby  $p_n$  is a polynomial with two different factors, but this case can be reduced to simpler cases by decomposing the appropriate fraction into simpler fractions. So, we omit consideration of summands of the form  $\frac{\binom{2n}{n}^2 H_n}{16^n p_n}$  in the case whereby  $p_n$  has two different factors.

The Ramanujan-type formulas given in this section are new in the sense that state-of-the-art computer algebra systems are not able to directly evaluate the series given in this section. For example, Mathematica 11 is only able to evaluate the series  $\sum_{n=1}^{\infty} \frac{\binom{2n}{n}^2 H_n}{16^n (2n-1)}$  as

$${}_2F_1^{(0,0,1,0)}\left(-\frac{1}{2}, \frac{1}{2}, 1, 1\right).$$

The Ramanujan-like formulas given in this section are also new in the sense that these results have apparently not appeared in any mathematical literature concerning summations involving harmonic numbers and binomial coefficients, with the exception of the formula which is given in Theorem 2. The following new Ramanujan-like formulas are easily verified using Lemma 1.

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{\binom{2n}{n}^2 H_n}{16^n (n+2)} &= \frac{16}{9} + \frac{16 - 80 \ln(2)}{9\pi} \\ \sum_{n=1}^{\infty} \frac{\binom{2n}{n}^2 H_n}{16^n (n+3)} &= \frac{256}{225} + \frac{416 - 1424 \ln(2)}{225\pi} \\ \sum_{n=1}^{\infty} \frac{\binom{2n}{n}^2 H_n}{16^n (n+4)} &= \frac{1024}{1225} + \frac{6416 - 18288 \log(2)}{3675\pi} \\ \sum_{n=1}^{\infty} \frac{\binom{2n}{n}^2 H_n}{16^n (2n-1)} &= \frac{8 \ln(2) - 4}{\pi} \\ \sum_{n=1}^{\infty} \frac{\binom{2n}{n}^2 H_n}{16^n (2n-3)} &= \frac{120 \ln(2) - 68}{27\pi} \\ \sum_{n=1}^{\infty} \frac{\binom{2n}{n}^2 H_n}{16^n (2n-5)} &= \frac{-6508 + 10680 \log(2)}{3375\pi} \\ \sum_{n=1}^{\infty} \frac{\binom{2n}{n}^2 H_n}{16^n (2n-1)^2} &= \frac{12 - 16 \ln(2)}{\pi} \\ \sum_{n=1}^{\infty} \frac{\binom{2n}{n}^2 H_n}{16^n (2n-3)^2} &= \frac{164 - 176 \ln(2)}{27\pi} \end{aligned}$$

$$\sum_{n=1}^{\infty} \frac{\binom{2n}{n}^2 H_n}{16^n (2n-5)^2} = \frac{70724 - 67760 \log(2)}{16875\pi}.$$

We remark that analogues of Lemma 1 may be used to evaluate series involving expressions of the form  $H_{2n}$ . For example, an analogue of Lemma 1 may be used to prove that

$$\sum_{n=1}^{\infty} \frac{2^{-6n} \binom{2n}{n} \binom{4n}{2n} H_{2n}}{2n-1}$$

is equal to the following expression involving the dilogarithm:

$$\frac{1}{4\pi} (2\text{Li}_2(17 - 12\sqrt{2}) - 4\sqrt{2} + 20\sqrt{2} \ln(2) + \ln(19601 - 13860\sqrt{2}) + \cos^{-1}(3)^2 + \cosh^{-1}(17)^2 + 2 \ln(3 - 2\sqrt{2}) \sinh^{-1}(1)) - \frac{\pi}{12}.$$

We remark that the Ramanujan-like series introduced in this section may be rewritten as double series using identities such as the following series identity:

$$\sum_{n=1}^{\infty} \frac{1}{n(m+n)} = \frac{\psi^{(0)}(m+1) + \gamma}{m}.$$

For example, it is easily seen that the formula

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2 H_n}{16^n (2n-3)} = \frac{120 \ln(2) - 68}{27\pi}$$

may be rewritten as the following elegant double hypergeometric series formula.

$$\frac{96 \ln(2) - 88}{9\pi} = \sum_{m,n \geq 0} \frac{\binom{2m}{m}^2}{16^m (2m-1)(n+1)(m+n+2)}.$$

#### 4 Definite integrals involving complete elliptic integrals

It is convenient for our purposes to define the complete elliptic integral of the first kind so that  $\mathbf{K}(k) = \frac{1}{2}\pi {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; k\right)$ . Similarly we define the complete elliptic integral of the second kind so that  $\mathbf{E}(k) = \frac{1}{2}\pi {}_2F_1\left(\frac{1}{2}, -\frac{1}{2}; 1; k\right)$ . The series for  $\frac{1}{\pi}$  given in the previous section are closely related to properties concerning complete elliptic integrals. To illustrate this idea, we begin by showing how the known integral formula

$$\int_0^1 \ln(1-x) \mathbf{K}(x) dx = 8 \ln(2) - 8$$

may be used to construct yet another proof of Theorem 1.

*Proof #3 of Theorem 1:* Using integration by parts with respect to the integral

$$\int_0^1 \ln(1-x) \mathbf{K}(x) dx = 8 \ln(2) - 8,$$

by letting  $u = \ln(1 - x)$  and  $v' = \mathbf{K}(x)$ , with

$$v(x) = 2x\mathbf{K}(x) - 2\mathbf{K}(x) + 2\mathbf{E}(x) - 2,$$

we have that  $8 \log(2) - 8$  is equal to

$$\int_0^1 \frac{-2 + 2\mathbf{E}(x) - 2\mathbf{K}(x) + 2x\mathbf{K}(x)}{1 - x} dx.$$

Therefore,

$$8 \log(2) - 8 = -2 \int_0^1 \frac{1 - \mathbf{E}(x)}{1 - x} dx + \int_0^1 \frac{-2\mathbf{K}(x) + 2x\mathbf{K}(x)}{1 - x} dx.$$

So, we have that

$$2 - 4 \ln(2) = \int_0^1 \frac{1 - \mathbf{E}(x)}{1 - x} dx \quad (1)$$

So, since

$$1 - \mathbf{E}(x) = \frac{1}{2}\pi \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(-\frac{1}{2}\right)_n (1 - x^n)}{(1)_n n!},$$

we have that

$$\frac{1 - \mathbf{E}(x)}{1 - x} = \frac{1}{2}\pi \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(-\frac{1}{2}\right)_n \left(\frac{1-x^n}{1-x}\right)}{(1)_n n!}.$$

By interating both sides of this equality, we find that

$$2 - 4 \log(2) = \frac{1}{2}\pi \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(-\frac{1}{2}\right)_n H_n}{(1)_n n!},$$

as desired.  $\square$

The equality given in (1) turns out to be unexpectedly useful, since (1) may be used to construct an alternative proof of Theorem 2, as shown below. The formula in (1) can also be used to construct another proof of Theorem 3. We remark that state-of-the-art computer algebra systems are not able to evaluate the definite integral given in (1), and that this integral does not seem to appear in any mathematical literature concerning definite integrals.

*Proof #2 of Theorem 2:* Rewriting the equality  $\int_0^1 \mathbf{K}(x) dx = 2$  as

$$\int_0^1 \left( -\frac{\mathbf{K}(x)}{2(-1+x)} + \frac{x\mathbf{K}(x)}{2(-1+x)} \right) dx = 1,$$

from (1), we have that

$$\int_0^1 \left( \frac{-1 + \mathbf{E}(x)}{-1+x} - \frac{\mathbf{K}(x)}{2(-1+x)} + \frac{x\mathbf{K}(x)}{2(-1+x)} \right) dx = 3 - 4 \ln(2).$$

Rewrite this equality as follows:

$$\int_0^1 \pi \frac{\left( \frac{2(x-1)\mathbf{K}(x)}{\pi} + \frac{4\mathbf{E}(x)}{\pi} \right) - 4}{\pi(x-1)} dx = \frac{12 - 16 \ln(2)}{\pi}.$$

This equality may be rewritten as follows:

$$\int_0^1 \frac{-4 + \pi {}_2F_1\left(-\frac{1}{2}, -\frac{1}{2}; 1; x\right)}{\pi(-1+x)} dx = \frac{12 - 16 \ln(2)}{\pi}.$$

But since

$$\sum_{n=0}^{\infty} \frac{16^{-n} (1-x^n) \binom{2n}{n}^2}{(-1+2n)^2(1-x)} = \frac{\pi {}_2F_1\left(-\frac{1}{2}, -\frac{1}{2}; 1; x\right) - 4}{\pi(x-1)},$$

by integrating both sides of the above equality appropriately we thus have that  $\sum_{n=1}^{\infty} \frac{\binom{2n}{n}^2 H_n}{16^n (2n-1)^2} = \frac{12-16 \ln(2)}{\pi}$  as desired.  $\square$

*Proof #2 of Theorem 3:* Begin by evaluating  $\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2 \left(\frac{1-x^n}{1-x}\right)}{16^n (2n-3)}$  as

$$\frac{10 - 3\pi {}_3F_2\left(-\frac{3}{2}, \frac{1}{2}, \frac{1}{2}; -\frac{1}{2}, 1; x\right)}{9\pi(-1+x)},$$

which is equal to

$$\frac{10 - 3\pi \left( \frac{4(1-x)\mathbf{K}(x)}{3\pi} + \frac{2(4x+1)\mathbf{E}(x)}{3\pi} \right)}{9\pi(x-1)}.$$

So, the infinite series given in Theorem 3 is equal to the following integral:

$$\int_0^1 \frac{10 - 3\pi \left( \frac{2(1+4x)\mathbf{E}(x)}{3\pi} + \frac{4(1-x)\mathbf{K}(x)}{3\pi} \right)}{9\pi(-1+x)} dx.$$

Therefore, the series given in Theorem 3 is equal to:

$$\frac{8}{9\pi} + \int_0^1 \left( \frac{10}{9\pi(-1+x)} - \frac{2\mathbf{E}(x)}{9\pi(-1+x)} - \frac{8x\mathbf{E}(x)}{9\pi(-1+x)} \right) dx,$$

which is equal to:

$$\frac{8}{9\pi} - \frac{2}{9\pi} \int_0^1 \frac{1 - \mathbf{E}(x)}{1-x} dx - \frac{8}{9\pi} \int_0^1 \frac{1 - x\mathbf{E}(x)}{1-x} dx.$$

We can find a closed-form evaluation of the above expression using (1), since  $\frac{1-x\mathbf{E}(x)}{1-x} = \mathbf{E}(x) + \frac{1-\mathbf{E}(x)}{1-x}$ .  $\square$

Similar approaches may be used to construct alternative proofs for the other formulas for  $\frac{1}{\pi}$  given in Section 3. We remark that new formulas for integrals containing complete elliptic integrals, such as the formula

$$\int_0^1 \ln^2(1-x) d\mathbf{E}(x) = 8 - \frac{2\pi^2}{3} + 16(\ln(2) - 1) \ln(2),$$

may be proven by rewriting the harmonic numbers in the series given in Section 3 and Section 5 using integrals such as  $\int_0^1 x^m \ln(1-x) dx$ . Many integrals of this form can also be proven using integration by parts and by applying known results concerning complete elliptic integrals, so we omit a full investigation of these integrals.

## 5 Summations with squared harmonic numbers

Given the variety of integration results and Ramanujan-type series related to Lemma 1, it is natural to consider generalizations of the strategy introduced in Section 2.1. We begin by considering the integral

$$\int_0^1 \frac{x^n \ln^2(1-x^2)}{\sqrt{1-x^2}} dx,$$

which is equal to the following expression for  $\Re(n) > -5$ .

$$\frac{1}{4n\Gamma\left(\frac{n}{2}+1\right)\Gamma\left(\frac{n}{2}\right)} \left( \sqrt{\pi}\Gamma\left(\frac{n+1}{2}\right) \left( 4\Gamma\left(\frac{n}{2}+1\right) \left( \psi^{(0)}\left(\frac{n}{2}+1\right)^2 - \psi^{(1)}\left(\frac{n}{2}+1\right) \right) + n\Gamma\left(\frac{n}{2}\right) \left( 4(\gamma + \ln(4))\psi^{(0)}\left(\frac{n}{2}+1\right) + \pi^2 + 2\gamma^2 + 8\ln^2(2) + 8\gamma\ln(2) \right) \right) \right).$$

**Theorem 4**  $\sum_{n=1}^{\infty} \frac{\binom{2n}{n}^2 (H_n^2 + H_n^{(2)})}{16^n (2n-1)} = \frac{4\pi}{3} + \frac{16(2\ln(2) - 2\ln^2(2) - 1)}{\pi}.$

*Proof* Since

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n} \binom{\frac{1}{2}}{n} \ln^2(1-x^2)}{\sqrt{1-x^2}} = \ln^2(1-x^2),$$

we have that

$$\sum_{n=0}^{\infty} (-1)^n \binom{\frac{1}{2}}{n} \left( \int_0^1 \frac{x^{2n} \ln^2(1-x^2)}{\sqrt{1-x^2}} dx \right) = 8 - \frac{\pi^2}{3} + 4\ln^2(2) - 8\ln(2).$$

Expanding the above summand using the above evaluation of the expression  $\int_0^1 \frac{x^n \ln^2(1-x^2)}{\sqrt{1-x^2}} dx$  together with Theorem 1 may be used to evaluate  $\sum_{n=1}^{\infty} \frac{\binom{2n}{n}^2 (H_n^2 + H_n^{(2)})}{16^n (2n-1)}$ .

The following propositions may be proven by analogy with our proof of Theorem 4.

$$\textbf{Proposition 1} \quad \sum_{n=1}^{\infty} \frac{\binom{2n}{n}^2 (H_n^2 + H_n^{(2)})}{16^n (2n-1)^2} = \frac{64 + 64 \ln^2(2) - 96 \ln(2)}{\pi} - \frac{8\pi}{3}.$$

$$\textbf{Proposition 2} \quad \sum_{n=1}^{\infty} \frac{\binom{2n}{n}^2 (H_n^2 + H_n^{(2)})}{16^n (2n-3)} = \frac{20\pi}{27} + \frac{16(-53 - 90 \ln^2(2) + 102 \ln(2))}{81\pi}.$$

$$\textbf{Proposition 3} \quad \sum_{n=1}^{\infty} \frac{2^{-4n} \binom{2n}{n}^2 (H_n^2 + H_n^{(2)})}{(2n-3)^2} = \frac{32(268 + 9 \ln(2)(22 \ln(2) - 41))}{243\pi} - \frac{88\pi}{81}.$$

It is often useful to translate summations such as the series given in the above propositions using integral identities such as  $\int_0^1 n x^{n-1} \ln^2(1-x) dx = (H_n)^2 + H_n^{(2)}$ . For example, we have that

$$\int_0^1 \mathbf{K}(x) \ln^2(1-x) dx = 48 - \frac{4\pi^2}{3} + 32(\ln(2) - 2) \ln(2) \quad (2)$$

from Proposition 1, together with the aforementioned integral identity. We remark that such integration results often may be used to prove elegant formulas for double series involving harmonic numbers. For example, it is easily seen that

$$\sum_{m,n \geq 0} \frac{16^{-m} \binom{2m}{m}^2 H_n}{(n+1)(m+n+2)} = \frac{48 + 32(\ln(2) - 2) \ln(2)}{\pi} - \frac{4\pi}{3}$$

using (2) together with the identity  $\sum_{n=0}^{\infty} \frac{x^{n+1} H_n}{n+1} = \frac{1}{2} \ln^2(1-x)$ .

## 6 Variations

Recall that the main techniques introduced in this article for constructing series for  $\frac{1}{\pi}$  involving harmonic numbers are based upon the evaluation of the following integrals:

$$\int_0^1 \frac{x^n \ln(1-x^2)}{\sqrt{1-x^2}} dx,$$

$$\int_0^1 \frac{x^n \ln^2(1-x^2)}{\sqrt{1-x^2}} dx.$$

Intuitively, our approach towards constructing new series for  $\frac{1}{\pi}$  involved the integration of “variations” of the binomial expansion of  $\sqrt{1-x^2}$  using the above integrals (see Lemma 1 and Section 5), so that the expression  $\frac{1}{\sqrt{1-x^2}}$  would in some sense “cancel” with an expression similar to  $\sqrt{1-x^2}$ , thus yielding a relatively simple logarithmic integral being equal to  $\pi$  times a Ramanujan-like series involving harmonic numbers. It is natural to consider simple variations of this strategy, based on the evaluation definite integrals of the form  $\int_0^1 \frac{x^n e(x)}{\sqrt{1-x^2}} dx$  and similar integrals, letting  $e(x)$  denote an elementary function.

*Example 1* The interesting formula

$$\frac{\pi - \tanh(\pi)}{\sqrt{\pi}} = \sum_{n=0}^{\infty} \frac{\binom{2n}{n} \left( (-i-n)_{-\frac{1}{2}} + (i-n)_{-\frac{1}{2}} \right)}{4^n}$$

is easily seen to hold by evaluating

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^n \binom{-\frac{1}{2}}{n} \sin(\ln(x))}{\sqrt{1-x}} = -\frac{\sin(\ln(x))}{x-1},$$

and integrating both sides of this equation by evaluating  $\int_0^1 \frac{x^n \sin(\ln(x))}{\sqrt{1-x}} dx$ .

*Example 2* It is easily seen that

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n} (H_{-1-i-n} - H_{-1+i-n})}{4^n (2n-1)}$$

is equal to

$$\frac{4 \cosh(\pi) \left( \Gamma(1+i) \Gamma\left(\frac{3}{2}-i\right) - \Gamma(1-i) \Gamma\left(\frac{3}{2}+i\right) \right)}{5\sqrt{\pi}}$$

by evaluating

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^n \binom{\frac{1}{2}}{n} \sin(\ln(x))}{-1+x} = -\frac{\sin(\ln(x))}{\sqrt{1-x}},$$

and integrating both sides of this equality, by evaluating  $\int_0^1 \frac{x^n \sin(\ln(x))}{-1+x} dx$ .

*Example 3* The equality

$$-\frac{4i \coth(\pi)}{5\sqrt{\pi}} = \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{4^n (2n-1)} \left( \frac{1}{(-i-n)_{\frac{1}{2}}} - \frac{1}{(i-n)_{\frac{1}{2}}} \right)$$

is easily seen to hold by evaluating

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n} \binom{\frac{1}{2}}{n} \sin^2(\ln(x))}{\sqrt{1-x^2}} = \sin^2(\ln(x))$$

and integrating both sides of this equation.

We currently leave it as an open problem to further investigate variations of the techniques given in Section 2.

## 7 Conclusion

The evaluation of integrals involving complete elliptic integrals based on the evaluation of harmonic sums given by analogues of Lemma 1 may be an interesting area to explore. Also, we currently leave the natural problem of evaluating Ramanujan-like series involving higher powers of central binomial coefficients and harmonic numbers as an open problem.

Recall that one of the main tools used in our paper is based on the following parameter derivative for a beta-type integral:

$$\frac{1}{2} \frac{\partial}{\partial y} B\left(\frac{n+1}{2}, y+1\right) = \int_0^1 u^n (1-u^2)^y \log(1-u^2) du.$$

It would be interesting to further explore the use of other kinds of parameter derivatives of beta-type integrals in the construction of Ramanujan-like series involving  $\frac{1}{\pi}$  and definite integrals involving complete elliptic integrals.

While there are many new formulas which can be obtained by applying parameter derivatives to hypergeometric identities, the main technique indicated in Section 2.1 can be applied in a very general way to compute series involving products of the form  $\binom{2n}{n}^2 H_n$ , as discussed in Section 1. However, we encourage the exploration of new formulas for  $\frac{1}{\pi}$  which can be obtained from parameter derivatives of hypergeometric identities.

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