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# iISS formation tracking control of autonomous vehicles 

Mohamed Maghenem Antonio Loría Elena Panteley


#### Abstract

We present a cascades-based controller for the problem of formation-tracking control in a group of mobile robots. We consider general models composed of a kinematics equation and a generic dynamics equation at the force level. For each robot, a local controller ensures a follow-the-leader task. Then, only one robot possesses the information of the reference trajectory. We establish uniform global asymptotic stability in closed-loop. Our analysis relies on the construction of an original strict Lyapunov function for the position tracking error dynamics and an inductive argument based on cascades-systems theory.


## I. Introduction

For a group of autonomous vehicles, the formation-tracking control problem roughly consists in making them form a spatial configuration and move along a reference path while keeping the pattern [1]. It naturally stems from the well-studied leader-follower tracking control problem, in which one robot follows the trajectory described by a virtual leader -see e.g., [2].
For the purpose of control design, autonomous vehicles are often modelled as unicycle systems, having two Cartesian coordinates to determine translation and one for orientation, this is the so-called kinematic model. More complete models include an additional forces-balance equation, expressed, e.g., in Lagrangian form [3] or in Hamiltonian one [4]. Yet, even at the kinematic level alone, tracking control imposes certain difficulties that stem from the nonholonomy of the robot; see, e.g., [5] where a general framework for consensus of nonholonomic systems in chain form is presented.

When the full model is considered, a common approach used in the literature is backstepping control -see e.g., [6], [7], [8] to mention only a few. Notably, the controllers in the last two references use partialstate or output feedback. In [7] an observer is designed based on a transformed model that is linear in the unmeasured velocities, In [8] a similar trend is followed along with the virtual structure and path-tracking approaches. See also [9], in which the output-feedback controller employs observer-generated estimates of relative (Cartessian) positions between the leader and the follower. Under parametric uncertainty, an adaptive state-feedback controller that guarantees the convergence of tracking errors is proposed in [10].

Other problems in formation control focus on achieving or maintaining a configuration while following a trajectory. In [11] necessary and sufficient conditions for the solubility of this problem, under distributed control, are given. Under the assumption that robot is modelled as a point-mass (second-order integrators), time-varying configurations are considered in [12].

In this note we address the problem of formation tracking control in a leader-follower topological configuration. This means that each robot follows one leader and communicates its coordinates to one or several followers that stand not necessarily physically close. The reference trajectory is generated by a virtual robot, which may be known by only one, or by several robots. Our controllers are based on a recursive repetition of a time-varying nonlinear tracking controller for the kinematics model, based
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on [2]. In this case, we establish via Lyapunov's direct method that the origin is uniformly globally asymptotically stable. For the complete model, we show that any controller ensuring the stabilization at the force level with sufficiently fast rate of convergence, leads to uniform global asymptotic stability.

Our main contributions lie in the construction of a strict Lyapunov function for the kinematics model and the proofs of uniform global asymptotic stability. Even though Lyapunov's direct method is also used, e.g., in [8], our main condition is that either the forward or the angular reference velocity is persistently exciting. This is a weaker condition than what is considered in the literature, including the references cited previously. For instance, neither velocity is required to be strictly positive and either one can equal to zero while the other is time-varying non-negative. On the other hand, although our controllers use state feedback (in contrast to [9], [7], [8]) they are mathematically very simple, hence easy to implement.

Our main results are stated in Section II; firstly for leader-follower tracking control and then, for formation tracking of large groups. The constructive stability proofs are provided in Section III, before concluding with some remarks in Section IV.

## II. Problem formulation and its solution

## A. Tracking control

Let us consider the force-controlled model of a mobile robot:

$$
\begin{align*}
& \left\{\begin{aligned}
\dot{x} & =v \cos \theta \\
\dot{y} & =v \sin \theta \\
\dot{\theta} & =\omega
\end{aligned}\right.  \tag{1}\\
& \left\{\begin{aligned}
\dot{v} & =f_{1}(t, v, \omega, z)+u_{1} \\
\dot{\omega} & =f_{2}(t, v, \omega, z)+u_{2}
\end{aligned}\right. \tag{2}
\end{align*}
$$

where $v$ and $\omega$ denote the forward and angular velocities respectively, the first two elements of $z:=$ $\left[\begin{array}{lll}x & y & \theta\end{array}\right]^{\top}$ correspond to the Cartesian coordinates of a point on the robot with respect to a fixed reference frame, and $\theta$ denotes the robot's orientation with respect to the same frame. The two control inputs are the torques $u_{1}, u_{2}$.
Equations (1) correspond to the kinematic model while the last two correspond to the force-balance equations which may be expressed, e.g., in Lagrangian form -see [3], [7]. The control strategy in this paper consists in decoupling the stabilisation task at both levels. For (1) we establish robustness results hence, our main statements are valid for any controller that guarantees the stabilisation of (2) with "fast" convergence (for instance, but not only, local exponential).

The tracking-control problem consists in making the robot to follow a fictitious reference vehicle modelled by

$$
\begin{aligned}
\dot{x}_{r} & =v_{r} \cos \theta_{r} \\
\dot{y}_{r} & =v_{r} \sin \theta_{r} \\
\dot{\theta}_{r} & =\omega_{r},
\end{aligned}
$$

and which moves about with reference velocities $v_{r}(t)$ and $\omega_{r}(t)$. More precisely, it is desired to steer the differences between the Cartesian coordinates to some values $d_{x}, d_{y}$, and to zero the orientation angles and the velocities of the two robots, that is, the quantities

$$
p_{\theta}=\theta_{r}-\theta, \quad p_{x}=x_{r}-x-d_{x}, \quad p_{y}=y_{r}-y-d_{y} .
$$

The distances $d_{x}, d_{y}$ define the position of the robot with respect to the (virtual) leader. In general, these may be functions that depend on time and the state or may be assumed to be constant, depending on the desired path to be followed. In this paper, we consider these distances to be defined as piece-wise constant functions $-c f$. [13].

Then, as it is customary, we transform the error coordinates $\left[p_{\theta}, p_{x}, p_{y}\right.$ ] of the leader robot from the global coordinate frame to local coordinates fixed on the robot, that is, we define

$$
\left[\begin{array}{l}
e_{\theta}  \tag{3}\\
e_{x} \\
e_{y}
\end{array}\right]:=\left[\begin{array}{ccc}
1 & 0 & 0 \\
0 & \cos \theta & \sin \theta \\
0 & -\sin \theta & \cos \theta
\end{array}\right]\left[\begin{array}{l}
p_{\theta} \\
p_{x} \\
p_{y}
\end{array}\right]
$$

In these new coordinates, the error dynamics between the virtual reference vehicle and the follower becomes

$$
\begin{align*}
& \dot{e}_{\theta}=\omega_{r}(t)-\omega  \tag{4a}\\
& \dot{e}_{x}=\omega e_{y}-v+v_{r}(t) \cos \left(e_{\theta}\right)  \tag{4b}\\
& \dot{e}_{y}=-\omega e_{x}+v_{r}(t) \sin \left(e_{\theta}\right) \tag{4c}
\end{align*}
$$

which is to be completed with Eqs (2). Hence, the control problem reduces to steering the trajectories of (4) to zero via the inputs $u_{1}$ and $u_{2}$ in (2). A natural method consists in designing virtual control laws at the kinematic level, that is, $w^{*}$ and $v^{*}$, and control inputs $u_{1}$ and $u_{2}$, depending on the latter, such that the origin $(e, \tilde{v}, \tilde{w})=(0,0,0)$ with

$$
\tilde{v}:=v-v^{*}, \quad \tilde{\omega}:=\omega-\omega^{*}, \quad e=\left[\begin{array}{lll}
e_{\theta} & e_{x} & e_{y} \tag{5}
\end{array}\right]^{\top}
$$

is uniformly globally asymptotically stable.
The stabilization problem for (4), that is neglecting the dynamics (2) so that $\omega \equiv \omega^{*}, v \equiv v^{*}$, has been broadly and long studied in the literature. For instance, in [2] the authors proposed the controller

$$
\begin{align*}
v^{*} & :=v_{r}(t) \cos \left(e_{\theta}\right)+k_{x} e_{x}  \tag{6a}\\
\omega^{*} & :=\omega_{r}(t)+k_{\theta} e_{\theta}+v_{r}(t) k_{y} e_{y} \phi\left(e_{\theta}\right) \tag{6b}
\end{align*}
$$

where $\phi$ is the so-called 'sync' function defined by

$$
\begin{equation*}
\phi\left(e_{\theta}\right):=\frac{\sin \left(e_{\theta}\right)}{e_{\theta}} \tag{7}
\end{equation*}
$$

and established (non-uniform) convergence of the tracking errors. Then, for the same controller but under slightly relaxed conditions, uniform global asymptotic stability for the closed-loop system is established in [14]. In this paper, we establish the same property and, in addition, for the first time we provide a strict Lyapunov function. As in [14], our standing assumption is that either the forward or the angular reference velocities are persistently exciting that is, that there exist positive numbers $\mu$ and $T$ such that

$$
\begin{equation*}
\int_{t}^{t+T}\left[\omega_{r}(s)^{2}+v_{r}(s)^{2}\right] d s \geq \mu \quad \forall t \geq 0 \tag{8}
\end{equation*}
$$

The design of the controller (6) is motivated by the resulting structure of the error dynamics for the tracking errors, which is reminiscent of nonlinear adaptive control systems. Indeed, by setting $\omega=\omega^{*}$ and $v=v^{*}$, we obtain

$$
\left[\begin{array}{c}
\dot{e}_{\theta}  \tag{9}\\
\dot{e}_{x} \\
\dot{e}_{y}
\end{array}\right]=\underbrace{\left[\begin{array}{ccc}
-k_{\theta} & 0 & -v_{r}(t) k_{y} \phi\left(e_{\theta}\right) \\
0 & -k_{x} & \omega^{*}(t, e) \\
v_{r}(t) \phi\left(e_{\theta}\right) & -\omega^{*}(t, e) & 0
\end{array}\right]}_{A_{v_{r}}(t, e)}\left[\begin{array}{c}
e_{\theta} \\
e_{x} \\
e_{y}
\end{array}\right]
$$

Then, we obtain the crucial property that the trivial solution for this system is uniformly globally stable (it is uniformly stable and all solutions are uniformly globally bounded). To see this, note that the total derivative of $V_{1}: \mathbb{R}^{3} \rightarrow \mathbb{R}_{\geq 0}$, defined as

$$
\begin{equation*}
V_{1}(e)=\frac{1}{2}\left[e_{x}^{2}+e_{y}^{2}+\frac{1}{k_{y}} e_{\theta}^{2}\right] \tag{10}
\end{equation*}
$$

corresponds to

$$
\begin{equation*}
\dot{V}_{1}(e)=-k_{x} e_{x}^{2}-k_{\theta} e_{\theta}^{2} \leq 0 \tag{11}
\end{equation*}
$$

Furthermore, after [15], it may be concluded that the origin of this system is uniformly globally asymptotically stable provided that the vector $\left[-v_{r}(t) k_{y} \phi\left(e_{\theta}\right) \quad \omega^{*}(t, e)\right]$, subject to $e_{\theta}=0$, is $\delta$ persistently exciting with respect to $e_{y}$. Roughly, this holds provided that this vector is persistently exciting for any $e_{y} \neq 0$; condition which, actually, reduces to (8). Thus, our first statement is the following.

Proposition 1 (Kinematic model): For the system (9) assume that (8) holds and there exist $\bar{\omega}_{r}, \overline{\dot{\omega}}_{r}, \bar{\nu}$, $\bar{\nu}>0$ such that

$$
\begin{equation*}
\left|\omega_{r}\right|_{\infty} \leq \bar{\omega}_{r}, \quad\left|\dot{\omega}_{r}\right|_{\infty} \leq \bar{\omega}_{r}, \quad\left|v_{r}\right|_{\infty} \leq \bar{v}_{r}, \quad\left|\dot{v}_{r}\right|_{\infty} \leq \overline{\dot{v}}_{r} . \tag{12}
\end{equation*}
$$

Then, the origin is uniformly globally asymptotically stable and locally exponentially stable, for any positive values of the control gains $k_{x}, k_{y}$, and $k_{\theta}$.

Beyond the statement itself, our (first) contribution lies in the original proof of Proposition 1 (see Section III) which is based on Lyapunov's direct method. Concretely, following the methods of [16], we show that there exists a positive definite radially unbounded function $V: \mathbb{R}_{\geq 0} \times \mathbb{R}^{3} \rightarrow \mathbb{R}_{\geq 0}$ defined as the functional

$$
\begin{equation*}
V(t, e):=P_{[3]}\left(t, V_{1}\right) V_{1}(e)-\omega_{r}(t) e_{x} e_{y}+v_{r}(t) P_{[1]}\left(t, V_{1}\right) e_{\theta} e_{y} \tag{13}
\end{equation*}
$$

where $P_{[k]}: \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is a smooth function such that $P_{[k]}\left(\cdot, V_{1}\right)$ is uniformly bounded and $P_{[k]}(t, \cdot)$ is a polynomial of degree $k$, designed such that the total derivative of $V$ along the trajectories of (9) satisfies

$$
\begin{equation*}
\dot{V}(t, e) \leq-\frac{\mu}{T} V_{1}(e)-k_{x} e_{x}^{2}-k_{\theta} e_{\theta}^{2} \tag{14}
\end{equation*}
$$

The value of having a strict Lyapunov function for (9) may not be overestimated. Notably, this allows to carry on with a robustness analysis vis-a-vis of the dynamics (2). In other words, to solve the tracking control problem for (1), (2) it is only left to design $u_{1}$ and $u_{2}$ such that, given the references $v^{*}$ and $\omega^{*}$, the origin of the closed-loop dynamics

$$
\begin{align*}
\dot{\tilde{v}} & =f_{1 c l}(t, \tilde{v}, \tilde{\omega}, e)  \tag{15a}\\
\dot{\tilde{\omega}} & =f_{2 c l}(t, \tilde{v}, \tilde{\omega}, e) \tag{15b}
\end{align*}
$$

is globally asymptotically stable uniformly in the initial conditions and in $e$. The most obvious choice corresponds to the linearizing feedbacks $u_{1}:=\dot{v}^{*}-f_{1}(t, v, \omega, e)-k_{1} \tilde{v}$ and $u_{2}=\dot{\omega}^{*}-f_{1}(t, v, \omega, e)-k_{2} \tilde{\omega}$. In Section II-C we present another example of an effective controller at force level. However, in general, the design of the control inputs $u_{1}$ and $u_{2}$ depends on the problem setting and is beyond the scope of this technical note.

On the other hand, it is remarkable that the overall error dynamics takes the convenient form

$$
\begin{align*}
\dot{e} & =A_{v_{r}}(t, e) e+B(e) \eta  \tag{16a}\\
\dot{\eta} & =F_{c l}(t, \tilde{v}, \tilde{\omega}, e), \quad F_{c l}:=\left[f_{1 c l} f_{2 c l}\right] \tag{16b}
\end{align*}
$$

where

$$
B(e):=\left[\begin{array}{cc}
0 & -1  \tag{17}\\
-1 & e_{y} \\
0 & -e_{x}
\end{array}\right], \quad \eta:=\left[\begin{array}{c}
\tilde{v} \\
\tilde{\omega}
\end{array}\right] .
$$

Now, for the purpose of analysis, we replace $e$ with $e(t)$ in (16b) so the closed-loop equations may be regarded as a cascaded nonlinear time-varying system with state $\zeta:=\left[e^{\top} \eta^{\top}\right]^{\top}$. More precisely, in place of (16b) we write

$$
\dot{\eta}=\tilde{F}_{c l}(t, \tilde{v}, \tilde{\omega})
$$

where $\tilde{F}_{c l}(t, \tilde{v}, \tilde{\omega})=F_{c l}(t, \tilde{v}, \tilde{\omega}, e(t))-c f$. [17]. Then, using arguments for cascaded systems we can establish our second contribution:

Proposition 2: Consider the system (16) with initial conditions $\left(t_{0}, \zeta_{\circ}\right) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^{5}$. Assume that $k_{x}$, $k_{y}$, and $k_{\theta}$ are positive and that Inequalities (8) and (12) hold. In addition, assume that the solutions are complete and the origin of (16b) is globally asymptotically stable, uniformly in the initial times $t_{\circ} \in \mathbb{R}_{\geq 0}$ and in the error trajectories $t \mapsto e$. Assume further that the trajectories $t \mapsto \eta$ are uniformly integrable, that is, there exists $\phi \in \mathcal{K}$ such that

$$
\begin{equation*}
\int_{t_{\circ}}^{\infty}|\eta(\tau)| d \tau \leq \phi\left(\left|\zeta_{\circ}\right|\right) \quad \forall t \geq t_{\circ} \geq 0 \tag{18}
\end{equation*}
$$

Then, the origin is uniformly globally asymptotically stable.
Proof: From Proposition 1, the origin $\{e=0\}$ is uniformly globally asymptotically stable for (9). By assumption the same property holds for (16b). Since, moreover, $B$ is linear in $e$, the result follows from the main results in [18, Theorem 2].

Remark 1: Technically, the function $\tilde{F}_{c l}$ exists only on the interval of existence of $e(t)$, whence the assumption that the solutions exist on $\left[t_{0}, \infty\right)$. Nevertheless, this assumption may be dropped if we impose that $\eta \rightarrow 0$ uniformly in $e(t)$ only on the interval of existence. This is considered in our main result later on -see Proposition 3.

## B. Formation tracking control

Let us consider now $n$ mobile robots that are required to advance in formation. It is assumed that the $i$ th robot follows a leader, indexed $i-1$, thereby forming a spanning-tree graph communication topology. As previously explained, the geometry of the formation may be defined via the relative distances between any pair of leader-follower robots, $d_{x i}, d_{y i}$ and it is independent of the communications graph (two robots may communicate independently of their relative positions). Then, the relative position error dynamics is given by a set of equations similar to (4), that is,

$$
\begin{align*}
\dot{e}_{\theta i} & =\omega_{i-1}(t)-\omega_{i}  \tag{19a}\\
\dot{e}_{x i} & =\omega_{i} e_{y i}-v_{i}+v_{i-1}(t) \cos \left(e_{\theta i}\right)  \tag{19b}\\
\dot{e}_{y i} & =-\omega_{i} e_{x i}+v_{i-1}(t) \sin \left(e_{\theta i}\right) \tag{19c}
\end{align*}
$$

For $i=1$ we recover the tracking error dynamics for the case of one robot following a virtual leader that is, by definition, $v_{0}:=v_{r}$ and $\omega_{0}:=\omega_{r}$. Then, similarly to (6) we introduce the virtual control inputs

$$
\begin{align*}
v_{i}^{*} & :=v_{i-1} \cos \left(e_{\theta i}\right)+k_{x i} e_{x i}  \tag{20}\\
\omega_{i}^{*} & :=\omega_{i-1}+k_{\theta i} e_{\theta_{i}}+v_{i-1} k_{y i} e_{y i} \phi\left(e_{\theta i}\right) \tag{21}
\end{align*}
$$

which serve as references for the actual controls $u_{1 i}$ and $u_{2 i}$ in

$$
\begin{align*}
\dot{v}_{i} & =f_{1 i}\left(t, v_{i}, \omega_{i}, e_{i}\right)+u_{1 i}  \tag{22a}\\
\dot{\omega}_{i} & =f_{2 i}\left(t, v_{i}, \omega_{i}, e_{i}\right)+u_{2 i}, \quad i \leq n . \tag{22b}
\end{align*}
$$

Next, let the velocity errors be defined as

$$
\tilde{\omega}_{i}:=\omega_{i}-\omega_{i}^{*}, \quad \tilde{v}_{i}:=v_{i}-v_{i}^{*}
$$

and let us define $\Delta v_{j}:=v_{j}-v_{r}$ and $\Delta \omega_{j}:=\omega_{j}-\omega_{r}$ for all $j \leq n$ (by definition, $\Delta \omega_{0}=\Delta v_{0}=0$ ). Then, we replace $\omega_{i}$ with $\tilde{\omega}_{i}+\omega_{i}^{*}$ and, respectively, $v_{i}$ with $\tilde{v}_{i}+v_{i}^{*}$ in (19), and we use

$$
\begin{align*}
v_{i}^{*} & =\left[\Delta v_{i-1}+v_{r}\right] \cos \left(e_{\theta i}\right)+k_{x i} e_{x i}  \tag{23}\\
\omega_{i}^{*} & =\Delta \omega_{i-1}+\omega_{r}+k_{\theta i} e_{\theta_{i}}+\left[\Delta v_{i-1}+v_{r}\right] k_{y i} e_{y i} \phi\left(e_{\theta i}\right) . \tag{24}
\end{align*}
$$

It follows that, for each pair of nodes, the error system takes the form

$$
\begin{equation*}
\dot{e}_{i}=A_{v_{r}}\left(t, e_{i}\right) e_{i}+G\left(t, e_{i}, \xi_{i}\right) e_{i}+B\left(e_{i}\right) \eta_{i} \tag{25}
\end{equation*}
$$

$-c f$. (16a), where

$$
\begin{aligned}
e_{i} & :=\left[\begin{array}{lll}
e_{\theta_{i}} & e_{x_{i}} & e_{y_{i}}
\end{array}\right]^{\top}, \quad \eta_{i}:=\left[\tilde{v}_{i} \tilde{\omega}_{i}\right]^{\top} \\
\xi_{i} & :=\left[\Delta \omega_{i-1} \Delta v_{i-1}\right]^{\top} \\
G & :=\left[\begin{array}{ccc}
0 & 0 & -k_{y} g_{1} \\
0 & 0 & g_{2} \\
g_{1} & -g_{2} & 0
\end{array}\right] \\
g_{1} & :=\Delta v_{i-1} e_{y_{i}} \phi\left(e_{\theta_{i}}\right) \\
g_{2} & :=\Delta \omega_{i-1}+k_{y} \Delta v_{i-1} e_{y_{i}} \phi\left(e_{\theta_{i}}\right)
\end{aligned}
$$

and $B$ is defined in (17). Thus, the overall closed-loop system has the convenient cascaded form (in reverse order):

$$
\begin{align*}
\dot{e}_{n} & =A_{v_{r}}\left(t, e_{n}\right) e_{n}+G\left(t, e_{n}, \xi_{n}\right) e_{n}+B\left(e_{n}\right) \eta_{n}  \tag{26a}\\
& \vdots \\
\dot{e}_{2} & =A_{v_{r}}\left(t, e_{2}\right) e_{2}+G\left(t, e_{2}, \xi_{2}\right) e_{2}+B\left(e_{2}\right) \eta_{2}  \tag{26b}\\
\dot{e}_{1} & =A_{v_{r}}\left(t, e_{1}\right) e_{1}+B\left(e_{1}\right) \eta_{1} \tag{26c}
\end{align*}
$$

and these closed-loop equations are complemented by the equations that stem from applying the actual control inputs in (22), that is,

$$
\begin{equation*}
\dot{\eta}_{i}=F_{i_{c l}}\left(t, \tilde{v}_{i}, \tilde{\omega}_{i}, e_{i}\right), \quad F_{i_{c l}}:=\left[f_{i_{1 c l}} f_{i_{2 c l}}\right] \tag{27}
\end{equation*}
$$

for all $i \leq n$.
To underline the good structural properties of the system (26)-(27) and to explain the rationale of our main result, let us argue as follows (precise proofs are given in Section III). By assumption, the control inputs $u_{1 i}$ and $u_{2 i}$ are such that $\eta_{i} \rightarrow 0$, independently of the behaviour of $e_{i}$. Furthermore, we see from Equation (26c) that, as $\eta_{1} \rightarrow 0$, we recover the system (9). Hence, using Proposition 1, we may conclude that $\eta_{1} \rightarrow 0$ implies that $e_{1} \rightarrow 0$. With this in mind, let us observe (26b). We have $\xi_{2}:=\left[\Delta \omega_{1} \Delta v_{1}\right]^{\top}$ where $\Delta \omega_{1}=\omega_{1}-\omega_{r}$ and $\Delta v_{1}=v_{1}-v_{r}$. On the other hand, by virtue of the control design, $e_{1}=0$
implies that $\omega_{1}^{*}=\omega_{r}$ and $v_{1}^{*}=v_{r}$, in which case we have $\Delta \omega_{1}=\tilde{\omega}_{1}$ and $\Delta v_{1}=\tilde{v}_{1}$. It follows that $e_{1} \rightarrow 0$ and $\eta_{1} \rightarrow 0$ imply that $\xi_{2} \rightarrow 0$. In addition, as $\eta_{2} \rightarrow 0$ (by the action of the controller at the force level), the terms $G\left(t, e_{2}, \xi_{2}\right) e_{2}+B\left(e_{2}\right) \eta_{2}$ in (26b) vanish and (26b) becomes $\dot{e}_{2}=A_{v_{r}}\left(t, e_{2}\right) e_{2}$. By Proposition 1 we conclude that $e_{2}$ also tends to zero. Carrying on by induction, we conclude that $e \rightarrow 0$.

Although intuitive, the previous arguments implicitly rely on the robustness of $\dot{e}_{i}=A_{v_{r}}\left(t, e_{i}\right)$ (1.e., of the system (9)) with respect to the inputs $\eta_{i}$ and $\xi_{i}$. More precisely, on the condition that the solutions exist on $\left[t_{0}, \infty\right)$ and, moreover, that they remain uniformly bounded during the transient. In our main result, which is presented next, we relax these (technical) assumptions.

Proposition 3: For each $i \leq n$, consider the system (19), (22) with control inputs $u_{1 i}$ and $u_{2 i}$ which are functions of $\left(t, v_{i}, \omega_{i}, e_{i}, v_{i}^{*}, \omega_{i}^{*}\right)$ and $v_{i}^{*}, \omega_{i}^{*}$ are defined in (20) and (21) respectively. Let conditions (8) and (12) hold. Let $\zeta_{i}:=\left[\begin{array}{ll}e_{i}^{\top} & \eta_{i}^{\top}\end{array}\right]^{\top}$. In addition, assume that:
[A1] for each $i$, there exists a function $\beta_{i} \in \mathcal{K} \mathcal{L}$ such that, on the maximal interval of existence ${ }^{1}$ of $t \rightarrow e_{i}$,

$$
\begin{equation*}
\left|\eta_{i}\left(t, t_{\circ}, \eta_{1_{\circ}}, e_{i o}\right)\right| \leq \beta\left(\left|\zeta_{i \circ}\right|, t-t_{\circ}\right) \tag{28}
\end{equation*}
$$

and (18) holds for some $\phi_{i} \in \mathcal{K}$.
Then, $\{\zeta=0\}$, where $\zeta:=\left[\zeta_{1}^{\top} \cdots \zeta_{n}^{\top}\right]^{\top}$, is uniformly globally asymptotically stable.
Assumption A1 means that $\eta_{i}(t)$ converge uniformly to zero while the trajectories $e_{i}(t)$ exist. In particular, if the system is forward complete A1 imposes uniform global asymptotic stability of (27). Even though this may be a strong hypothesis in a general context of nonlinear systems -see [17], it may be easily met in the case of formation tracking control, as we illustrate below.

## C. Example

After [10], a dynamic model of a wheeled mobile robot is given by

$$
\begin{align*}
& \dot{z}=J(z) \nu  \tag{29a}\\
& M \dot{\nu}+C(\dot{z}) \nu+D \nu=\tau \tag{29b}
\end{align*}
$$

where $\tau$ is the torque control input; the variable $\nu:=\left[\begin{array}{ll}\nu_{1} & \nu_{2}\end{array}\right]$ denotes the angular velocities of the two wheels, $M$ is an inertia matrix (hence positive definite, symmetric), $C$ is the matrix of Coriolis forces (which is skew-symmetric) and $D$ denotes natural damping (hence, $D=D^{\top} \geq 0$ ), and

$$
J(z)=\frac{r}{2}\left[\begin{array}{cc}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta \\
1 / b & -1 / b
\end{array}\right]
$$

where $r$ and $b$ are positive constant parameters of the system. The relation between the wheels' velocities, $\nu$, and the robot's velocities in the fixed frame, $\dot{z}$, is given by

$$
\left[\begin{array}{c}
v  \tag{30}\\
\omega
\end{array}\right]=\frac{r}{2 b}\left[\begin{array}{cc}
b & b \\
1 & -1
\end{array}\right]\left[\begin{array}{l}
\nu_{1} \\
\nu_{2}
\end{array}\right] \Leftrightarrow\left[\begin{array}{l}
\nu_{1} \\
\nu_{2}
\end{array}\right]=\frac{1}{r}\left[\begin{array}{cc}
1 & b \\
1 & -b
\end{array}\right]\left[\begin{array}{c}
v \\
\omega
\end{array}\right]
$$

which may be used in (29) to obtain the model (1), (2) with

$$
\left[\begin{array}{l}
u_{1} \\
u_{2}
\end{array}\right]=\frac{r}{2 b}\left[\begin{array}{cc}
b & b \\
1 & -1
\end{array}\right] M^{-1} \tau
$$

${ }^{1}$ If necessary, we consider the shortest maximal interval of existence among all the trajectories $e_{i}(t)$, with $i \leq n$.
-see [10] for more details on this coordinate transformation.
Then, using (30), for any given virtual control inputs $v^{*}$ and $\omega^{*}$, we can compute $\nu^{*}:=\left[\nu_{1}^{*} \nu_{2}^{*}\right]^{\top}$ and define the torque control input

$$
\tau=M \dot{\nu}^{*}+C(J(z) \nu) \nu^{*}+D \nu^{*}-k_{d} \tilde{\nu}, \quad k_{d}>0
$$

where $\tilde{\nu}:=\nu-\nu^{*}$. We see that the force error equations yields

$$
\begin{equation*}
M \dot{\tilde{\nu}}+\left[C(\dot{z}(t))+D+k_{d} I\right] \tilde{\nu}=0 \tag{31}
\end{equation*}
$$

in which we have replaced $\dot{z}$ with the trajectories $\dot{z}(t)$ to regard this system as linear time-varying, with state $\tilde{\nu}$. Now, due to the skew-symmetry of $C(\cdot)$ the total derivative of

$$
V(\tilde{\nu})=\frac{1}{2} \tilde{\nu}^{\top} M \tilde{\nu}
$$

yields

$$
\dot{V}(\tilde{\nu}) \leq-k_{d}|\tilde{\nu}|^{2}
$$

Although this inequality holds independently of $\dot{z}(t)$, Eq. (31) is valid only on the interval of existence of $\dot{z}(t)$, denoted $\left[t_{0}, t^{\max }\right), t^{\max } \leq \infty$. Hence,

$$
|\tilde{\nu}(t)| \leq \kappa\left|\tilde{\nu}\left(t_{0}\right)\right| e^{-\lambda\left(t-t_{o}\right)} \quad \forall t \in\left[t_{0}, t^{\max }\right)
$$

for some $\kappa$ and $\lambda>0$. From (30) it is clear that a similar bound holds for $\eta(t)=[\tilde{v}(t) \tilde{\omega}(t)]$. In other words, the velocity errors tend exponentially to zero uniformly in the initial conditions and in the position error trajectories, so condition A1 of Proposition 3 holds.

## III. Stability analysis

## A. Proof of Proposition 1

The proof follows via Lyapunov's direct method; it relies on the construction of a Lyapunov function of polynomial type, and it is greatly inspired by the methods in [16].

Firstly, for any locally integrable function $\varphi: \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$, such that $\sup _{t \geq 0}|\varphi(t)| \leq \bar{\varphi}$, let us introduce

$$
\begin{equation*}
Q_{\varphi}(t):=1+2 \bar{\varphi} T-\frac{2}{T} \int_{t}^{t+T} \int_{t}^{m} \varphi(s) d s d m \tag{32}
\end{equation*}
$$

Note that this function satisfies:

$$
\begin{gather*}
1 \leq Q_{\varphi}(t)<\bar{Q}_{\varphi}:=1+2 \bar{\varphi} T \\
\dot{Q}_{\varphi}(t)=-\frac{2}{T} \int_{t}^{t+T} \varphi(s) d s+2 \varphi(t) \tag{33}
\end{gather*}
$$

For the sequel, we will introduce, as needed, several polynomial functions denoted $\rho_{i}: \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$, which will be defined later on in a manner that the derivative of

$$
\begin{align*}
V_{2}(t, e):= & \rho_{1}\left(V_{1}\right) V_{1}+\left[Q_{v_{r}^{2}}(t)+Q_{\omega_{r}^{2}}(t)\right] V_{1}-\omega_{r}(t) e_{x} e_{y} \\
& +v_{r} \rho_{2}\left(V_{1}\right) e_{\theta} e_{y}+\rho_{3}\left(V_{1}\right) V_{1}, \tag{34}
\end{align*}
$$

where $V_{1}$ defined in (10), is positive definite.

Note that, in addition,

$$
V_{2}(t, e) \geq \frac{1}{2}\left[\begin{array}{l}
e_{\theta} \\
e_{x} \\
e_{y}
\end{array}\right]^{\top}\left[\begin{array}{ccc}
\rho_{3}\left(V_{1}\right) & v_{r} \rho_{2}\left(V_{1}\right) & 0 \\
v_{r} \rho_{2}\left(V_{1}\right) & \rho_{3}\left(V_{1}\right) & -\omega_{r} \\
0 & -\omega_{r} & \rho_{3}\left(V_{1}\right)
\end{array}\right]\left[\begin{array}{c}
e_{\theta} \\
e_{x} \\
e_{y}
\end{array}\right]
$$

so $V_{2}$ is positive definite and radially unbounded if the matrix in this inequality is positive semidefinite. The latter holds if $\rho_{3}$ satisfies

$$
\rho_{3}\left(V_{1}\right) \geq \frac{\sqrt{\bar{v}_{r}^{2} \rho_{2}\left(V_{1}\right)^{2}+\bar{\omega}_{r}^{2}}}{\sqrt{2}}
$$

Finally, we introduce

$$
\begin{equation*}
V_{3}(t, e)=V_{2}(t, e)+V_{1} \rho_{4}\left(V_{1}\right) \tag{35}
\end{equation*}
$$

which is also positive definite. We shall show that for an appropriate choice of the polynomials $\rho_{i}$, the total derivative of $V_{3}$ along the trajectories of (9) yields

$$
\begin{equation*}
\dot{V}_{3}(t, e) \leq-\frac{\mu}{T} V_{1}(e)-k_{x} e_{x}^{2}-k_{\theta} e_{\theta}^{2}, \quad \forall(t, e) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^{3} \tag{36}
\end{equation*}
$$

To that end, we start by rewriting (9) in the output-injection form

$$
\begin{align*}
\dot{e} & =A_{v_{r}}^{\circ}(t, e) e+v_{r}\left[\phi\left(e_{\theta}\right)-1\right] B^{\circ}\left(e_{y}\right) e  \tag{37}\\
A_{v_{r}}^{\circ}(t, e) & :=\left[\begin{array}{ccc}
-k_{\theta} & 0 & -v_{r} k_{y} \\
0 & -k_{x} & \varpi_{v_{r}}^{\circ} \\
v_{r} & -\varpi_{v_{r}}^{\circ} & 0
\end{array}\right]  \tag{38}\\
\varpi_{v_{r}}^{\circ}(t, e) & =\omega_{r}(t)+k_{\theta} e_{\theta}+v_{r} k_{y} e_{y}  \tag{39}\\
B^{\circ}(e) & :=\left[\begin{array}{ccc}
0 & 0 & -k_{y} \\
0 & 0 & k_{y} e_{y} \\
1 & -k_{y} e_{y} & 0
\end{array}\right] \tag{40}
\end{align*}
$$

This partition, which facilitates the analysis, is motivated by the fact that $v_{r}\left[\phi\left(e_{\theta}\right)-1\right] B^{\circ}\left(e_{y}\right) e=0$ if $e_{\theta}=0$.

Now and we show that the total derivative of $V_{2}$ along the trajectories of $\dot{e}=A_{v_{r}}^{\circ}(t, e) e$ is negative definite. Firstly, since $\rho_{1}$ is a polynomial that maps $\mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ and $V_{1}$ satisfies (11),

$$
\begin{equation*}
\frac{d}{d t}\left\{\rho_{1}\left(V_{1}\right) V_{1}\right\} \leq-\rho_{1}\left(V_{1}\right)\left[k_{x} e_{x}^{2}+k_{\theta} e_{\theta}^{2}\right] \tag{41}
\end{equation*}
$$

Next, we use (33), as well as $|e| \geq\left|e_{y}\right|$ and $Q_{\varphi}>0$, to obtain

$$
\left.\left.\left.\begin{array}{rl}
\frac{d}{d t}\left\{\left[Q_{v_{r}^{2}}+Q_{\omega_{r}^{2}}\right] V_{1}\right\} \leq- & \frac{2}{T}
\end{array}\right] \int_{t}^{t+T}\left[\omega_{r}(s)^{2}+v_{r}(s)^{2}\right] d s\right] V_{1}\right] \text { }+\left[\omega_{r}^{2}+v_{r}^{2}\right]\left[e_{x}^{2}+\frac{1}{k_{y}} e_{\theta}^{2}+e_{y}^{2}\right] ~ \$
$$

Then, using (38) and (39), we obtain

$$
\begin{gather*}
-\frac{d}{d t}\left\{\omega_{r} e_{x} e_{y}\right\}=-\dot{\omega}_{r} e_{x} e_{y}-\omega_{r}\left[-k_{x} e_{x} e_{y}+\omega_{r} e_{y}^{2}+k_{\theta} e_{\theta} e_{y}^{2}\right. \\
\left.+k_{y} v_{r} e_{y}^{3}-\omega_{r} e_{x}^{2}-k_{\theta} e_{\theta} e_{x}^{2}-k_{y} v_{r} e_{y} e_{x}^{2}+v_{r} e_{\theta} e_{x}\right] \tag{43}
\end{gather*}
$$

Now, for the cross-terms we use the inequalities $2 e_{x} e_{y} \leq \epsilon e_{x}^{2}+(1 / \epsilon) e_{y}^{2}$ and $2 e_{\theta} e_{y}^{2} \leq \epsilon V_{1} e_{\theta}^{2}+(1 / \epsilon) e_{y}^{2}$, which hold for any $\epsilon>0$, and we regroup some terms to obtain (see [19] for details)

$$
\begin{align*}
& -\frac{d}{d t}\left\{\omega_{r} e_{x} e_{y}\right\} \leq \frac{\epsilon}{2} v_{r}^{2} V_{1} e_{y}^{2}+\rho_{5}\left(V_{1}\right) e_{x}^{2}+\rho_{6}\left(V_{1}\right) e_{\theta}^{2} \\
& \quad+\frac{1}{2 \epsilon}\left[\bar{\omega}_{r}^{2} k_{y}^{2}+\left(k_{x}+k_{\theta}\right) \bar{\omega}_{r}+\bar{\omega}_{r} k_{y} \bar{v}_{r}+\bar{\omega}_{r}\right] e_{y}^{2}-\omega_{r}^{2} e_{y}^{2} \tag{44}
\end{align*}
$$

where $\rho_{5}$ and $\rho_{6}$ are first-order polynomials of $V_{1}$ defined as

$$
\begin{aligned}
& \rho_{5}\left(V_{1}\right)=\frac{\bar{\omega}_{r}}{2}\left[\left(\epsilon k_{y} \bar{v}_{r}+2 k_{\theta}\right) V_{1}+\left(k_{x}+\frac{\bar{\omega}_{r}}{\bar{\omega}_{r}}\right) \epsilon+2 \bar{\omega}_{r}+\bar{v}_{r}\right] \\
& \rho_{6}\left(V_{1}\right)=\bar{\omega}_{r}\left[k_{\theta}\left(\epsilon V_{1}+1\right)+\frac{\bar{v}_{r}}{2}\right] .
\end{aligned}
$$

Next, we have

$$
\begin{align*}
& \frac{d}{d t}\left\{v_{r} \rho_{2}\left(V_{1}\right) e_{\theta} e_{y}\right\}=-\rho_{2}\left(V_{1}\right) v_{r}^{2} e_{y}^{2}-v_{r} \rho_{2}\left(V_{1}\right)\left[k_{\theta} e_{\theta} e_{y}+\omega_{r} e_{x} e_{\theta}\right. \\
& \left.\quad+k_{\theta} e_{\theta}^{2} e_{x}+k_{y} v_{r} e_{y} e_{x} e_{\theta}+v_{r} e_{\theta}^{2}\right]+\rho_{2}\left(V_{1}\right) \dot{v}_{r} e_{\theta} e_{y} \\
& \quad-v_{r} \nabla \rho_{2}\left(V_{1}\right) e_{\theta} e_{y}\left[k_{x} e_{x}^{2}+k_{\theta} e_{\theta}^{2}\right] . \tag{45}
\end{align*}
$$

Hence, using again the triangle inequality to bound the cross-terms and regrouping them, we obtain

$$
\begin{align*}
\frac{d}{d t}\left\{v_{r} \rho_{2} e_{\theta} e_{y}\right\} \leq & -k_{y} v_{r}^{2} \rho_{2}\left(V_{1}\right) e_{y}^{2}+\rho_{7}\left(V_{1}\right) e_{x}^{2} \\
& +\rho_{8}\left(V_{1}\right) e_{\theta}^{2}+\frac{k_{\theta} \bar{v}_{r}+\overline{\dot{v}}_{r}}{2 \epsilon} e_{y}^{2} \tag{46}
\end{align*}
$$

where $\rho_{7}$ and $\rho_{8}$ are second-order polynomials of $V_{1}$ satisfying

$$
\begin{aligned}
\rho_{7}\left(V_{1}\right) \geq & \rho_{2} \bar{v}_{r}\left[\frac{\bar{\omega}_{r}}{2}+\left(k_{\theta}+k_{y} \bar{v}_{r}\right) V_{1}\right] \\
& \quad+\max \left\{k_{y}, 1\right\} k_{x} \bar{v}_{r} V_{1}\left|\frac{\partial \rho_{2}}{\partial V_{1}}\right| \\
& \quad \bar{v}_{r} \rho_{2}\left(V_{1}\right) \\
\rho_{8}\left(V_{1}\right) \geq & {\left[\bar{\omega}_{r}+k_{\theta}\left(\epsilon \rho_{2}\left(V_{1}\right)+1\right)+\left(k_{y}+2\right) \bar{v}_{r}\right] } \\
& \bar{v}_{r} \frac{\epsilon}{2} \rho_{2}\left(V_{1}\right)^{2}+\bar{v}_{r}\left|\frac{\partial \rho_{2}}{\partial V_{1}}\right| \max \left\{k_{y}, 1\right\} k_{\theta} V_{1} .
\end{aligned}
$$

Now we put all the previous bounds together. Using (8) in (42), we obtain, in view of (44) and (46),

$$
\begin{align*}
\frac{\partial V_{2}}{\partial t} & +\frac{\partial V_{2}}{\partial e} A_{v_{r}}^{\circ}(t, e) e \leq-\frac{2 \mu}{T} V_{1}(e)-\left[k_{y} \rho_{2}\left(V_{1}\right)-1-\frac{\epsilon}{2} V_{1}\right] v_{r}^{2} e_{y}^{2} \\
& +\frac{1}{2 \epsilon}\left[\bar{\omega}_{r}\left[\bar{\omega}_{r} k_{y}^{2}+k_{x}+k_{\theta}+k_{y} \bar{v}_{r}\right]+\bar{\omega}_{r}+k_{\theta} \bar{v}_{r}+\overline{\dot{v}}_{r}\right] e_{y}^{2} \\
& -e_{x}^{2}\left[k_{x} \rho_{1}-\rho_{7}-\rho_{5}-v_{r}^{2}-\omega_{r}^{2}\right] \\
& -e_{\theta}^{2}\left[k_{\theta} \rho_{1}-\rho_{8}-\rho_{6}-\frac{1}{k_{y}}\left(v_{r}^{2}+\omega_{r}^{2}\right)\right] . \tag{47}
\end{align*}
$$

Hence, defining

$$
\begin{aligned}
\epsilon & :=\frac{T}{\mu}\left[\bar{\omega}_{r}\left[\bar{\omega}_{r} k_{y}^{2}+k_{x}+k_{\theta}+k_{y} \bar{v}_{r}\right]+\overline{\dot{\omega}}_{r}+k_{\theta} \bar{v}_{r}+\overline{\dot{v}}_{r}\right] \\
\rho_{1}\left(V_{1}\right) & :=1+\frac{1}{\min \left\{k_{x}, k_{\theta}\right\}}\left[\rho_{5}+\rho_{6}+\rho_{7}+\rho_{8}\right.
\end{aligned}
$$

$$
\left.+\left[1+\frac{1}{k_{y}}\right]\left[\omega_{r}^{2}+v_{r}^{2}\right]\right]
$$

$$
\rho_{2}\left(V_{1}\right):=\frac{1}{k_{y}}\left[1+\frac{\epsilon}{2} V_{1}\right]
$$

we obtain

$$
\begin{equation*}
\frac{\partial V_{2}}{\partial t}+\frac{\partial V_{2}}{\partial e} A_{v_{r}}^{\circ}(t, e) e \leq-\frac{\mu}{T} V_{1}(e)-k_{x} e_{x}^{2}-k_{\theta} e_{\theta}^{2} \tag{48}
\end{equation*}
$$

That is, $V_{2}$ is a strong Lyapunov function for the nominal dynamics $\dot{e}=A_{v_{r}}^{\circ}(t, e) e$.
Using the latter, we evaluate the total derivative of $V_{3}$ along the trajectories of (37) that is, including the output injection term. Hence, we obtain

$$
\begin{align*}
\dot{V}_{3}(t, e) & \leq \frac{\partial V_{2}}{\partial t}+\frac{\partial V_{2}}{\partial e} A_{v_{r}}^{\circ}(t, e) e+W(t, e)  \tag{49}\\
W(t, e) & :=-k_{\theta} \rho_{4}\left(V_{1}\right) e_{\theta}^{2}+v_{r}\left[\phi\left(e_{\theta}\right)-1\right] \frac{\partial V_{2}}{\partial e} B^{\circ}\left(e_{y}\right) e \tag{50}
\end{align*}
$$

for which we used (11), as well as the possitivity of $\rho_{4}\left(V_{1}\right)$ and $\nabla \rho_{4}$, to obtain

$$
\begin{aligned}
\frac{d}{d t}\left\{V_{1} \rho_{4}\left(V_{1}\right)\right\} & =\dot{V}_{1} \rho_{4}\left(V_{1}\right)+V_{1} \nabla \rho_{4} \dot{V}_{1} \\
& \leq-k_{\theta} \rho_{4}\left(V_{1}\right) e_{\theta}^{2}
\end{aligned}
$$

We show that $W(t, e)$, defined in (50), is non-positive. To that end, note that

$$
\begin{equation*}
\left[\phi\left(e_{\theta}\right)-1\right] \leq e_{\theta}^{2} \tag{51}
\end{equation*}
$$

and, in view of the structure of $B^{\circ}$, we have

$$
\frac{\partial V_{1}}{\partial e} B^{\circ}(e) e=0
$$

hence,

$$
\frac{\partial V_{2}}{\partial e}=v_{r} \rho_{2}\left(V_{1}\right)\left[\begin{array}{lll}
e_{y} & 0 & e_{\theta}
\end{array}\right]-\omega_{r}\left[\begin{array}{lll}
0 & e_{y} & e_{x}
\end{array}\right]
$$

and, moreover,

$$
\begin{aligned}
\left|\left[\begin{array}{lll}
e_{y} & 0 & e_{\theta}
\end{array}\right] B^{\circ}(e) e\right| & =\left|-k_{y} e_{y}^{2}+e_{\theta}^{2}-k_{y} e_{y} e_{x} e_{\theta}\right| \\
& \leq\left|e_{\theta}^{2}-\frac{k_{y}}{2} e_{y}^{2}+\frac{k_{y}}{2} e_{x}^{2} e_{\theta}^{2}\right| \\
& \leq 2 k_{y} V_{1}+2 k_{y}^{2} V_{1}^{2} \\
\left.\left\lvert\, \begin{array}{lll}
0 & e_{y} & e_{x}
\end{array}\right.\right] B^{\circ}(e) e \mid & =\left|k_{y} e_{y}^{3}+e_{\theta} e_{x}-k_{y} e_{y} e_{x}^{2}\right| \\
& \leq 2 k_{y} V_{1}^{2}+\max \left\{k_{y}, 1\right\} V_{1} .
\end{aligned}
$$

Thus, $W(t, e) \leq 0$ if

$$
\rho_{4}\left(V_{1}\right) \geq \frac{2 \bar{v}_{r} \max \left\{k_{y}, 1\right\}}{k_{\theta}}\left[\left[k_{y} \rho_{2} \bar{v}_{r}+\bar{\omega}_{r}\right] V_{1}^{2}+\left[\rho_{2} \bar{v}_{r}+\bar{\omega}_{r}\right] V_{1}\right]
$$

and (36) follows from (48) and (49).
It is worth stressing that, based on the previous computations, one can also establish that each subsystem in (26) is integral-input-to-state stable.

Proposition 4: Consider the system (25) with $k_{x}, k_{y}$, and $k_{\theta}$ arbitrary positive gains; assume, moreover, that the references satisfy (8) and (12). If, in addition, the solutions exist over $\left[t_{0}, \infty\right)$, the system is integral input to state stable with respect to the "input" $\left[\xi_{i}^{\top}, \eta_{i}^{\top}\right]^{\top}$.

Proof: Consider the function $W: \mathbb{R}_{\geq 0} \times \mathbb{R}^{3} \rightarrow \mathbb{R}_{\geq 0}$ defined by

$$
\begin{equation*}
W\left(t, e_{i}\right):=\ln \left[1+V\left(t, e_{i}\right)\right] \tag{52}
\end{equation*}
$$

where $V: \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^{3} \rightarrow \mathbb{R}_{\geq 0}$ is the continuously differentiable function defined in (13).
The total derivative of $W$ along the trajectories of (25) yields

$$
\dot{W}\left(t, e_{i}\right) \leq \frac{\dot{V}\left(t, e_{i}\right)}{1+V\left(t, e_{i}\right)}
$$

which, in virtue of (14) implies that

$$
\begin{equation*}
\dot{W}\left(t, e_{i}\right) \leq-\alpha\left(\left|e_{i}\right|\right)+\frac{\partial V}{\partial e_{i}} \frac{G\left(t, e_{i}, \xi_{i}\right) e_{i}+B\left(e_{i}\right) \eta_{i}}{1+V\left(t, e_{i}\right)} \tag{53}
\end{equation*}
$$

where

$$
\alpha\left(\left|e_{i}\right|\right)=\frac{\mu}{T} \frac{V_{1}\left(e_{i}\right)}{1+V\left(t, e_{i}\right)} .
$$

To establish the statement of the proposition we show that the second term on the right hand side of (53) is bounded from above by $\gamma\left[\left|\xi_{i}\right|+\left|\eta_{i}\right|\right]$ with $\gamma>0$. For the sake of argument, remark that $V\left(t, e_{i}\right)=\mathcal{V}\left(t, e_{i}, V_{1}\right)$ where

$$
\mathcal{V}\left(t, e_{i}, V_{1}\right):=P_{[3]}\left(t, V_{1}\right) V_{1}-\omega_{r}(t) e_{x_{i}} e_{y_{i}}+v_{r}(t) P_{[1]}\left(t, V_{1}\right) e_{\theta_{i}} e_{y_{i}}
$$

and, in addition, note that there exists a fourth-order polynomial $\tilde{P}_{4}\left(V_{1}\right)$ such that

$$
\begin{equation*}
\mathcal{V}\left(t, e_{i}, V_{1}\right) \geq \tilde{P}_{4}\left(V_{1}\right), \quad \forall\left(t, e_{i}, V_{1}\right) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^{3} \times \mathbb{R}_{\geq 0} \tag{54}
\end{equation*}
$$

Furthermore,

$$
\frac{\partial V}{\partial e_{i}}=\frac{\partial \mathcal{V}}{\partial V_{1}} \frac{\partial V_{1}}{\partial e_{i}}+\frac{\partial \mathcal{V}}{\partial e_{i}}
$$

and, due to the structure of $G$, we have

$$
\begin{equation*}
\frac{\partial V_{1}}{\partial e_{i}} G\left(t, e_{i}, \xi_{i}\right) e_{i}=0 \tag{55}
\end{equation*}
$$

Therefore

$$
\begin{aligned}
\frac{\partial V}{\partial e_{i}}\left[G\left(t, e_{i}, \xi_{i}\right) e_{i}+B\left(e_{i}\right) \eta_{i}\right]= & \frac{\partial \mathcal{V}}{\partial V_{1}} \frac{\partial V_{1}}{\partial e_{i}} B\left(e_{i}\right) \eta_{i} \\
& +\frac{\partial \mathcal{V}}{\partial e_{i}}\left[G\left(t, e_{i}, \xi_{i}\right) e_{i}+B\left(e_{i}\right) \eta_{i}\right]
\end{aligned}
$$

Now, since $P_{[3]}$ is a polynomial of 3 rd order, we have

$$
\frac{\partial \mathcal{V}}{\partial V_{1}}=P_{[3]}^{\prime}\left(V_{1}\right)+v_{r}(t) \frac{\partial P_{[1]}}{\partial V_{1}} e_{\theta_{i}} e_{y_{i}}
$$

where $P_{[3]}^{\prime}: \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is the polynomial function of 3rd order defined as

$$
P_{[3]}^{\prime}\left(V_{1}\right):=\frac{\partial P_{[3]}}{\partial V_{1}} V_{1}+P_{[3]}\left(V_{1}\right) .
$$

Then, since $P_{[1]}$ is a polynomial of 1 st order and $e_{\theta_{i}} e_{y_{i}} \leq V_{1}\left(e_{i}\right)$, there exists $c>0$ such that

$$
\left|\frac{\partial \mathcal{V}}{\partial V_{1}}\right| \leq P_{[3]}^{\prime}\left(V_{1}\right)+c \bar{c}_{r} V_{1} .
$$

Furthermore, $B\left(e_{i}\right)$ is linear in $e_{i}$ therefore, there exists $c>0$ such that

$$
\left|\frac{\partial V_{1}}{\partial e_{i}} B\left(e_{i}\right) \eta_{i}\right| \leq c V_{1}\left|\eta_{i}\right|
$$

and, on the other hand,

$$
\frac{\partial \mathcal{V}^{\top}}{\partial e_{i}}=\left[\begin{array}{c}
v_{r}(t) P_{[1]}\left(t, V_{1}\right) e_{y_{i}}  \tag{56}\\
-\omega_{r}(t) e_{y_{i}} \\
v_{r}(t) P_{[1]}\left(t, V_{1}\right) e_{\theta_{i}}-\omega_{r}(t) e_{x_{i}}
\end{array}\right]
$$

while $\left|G\left(t, e_{i}, \xi_{i}\right) e_{i}\right|=\mathcal{O}\left(\left|\xi_{i}\right|\right) P_{1}^{\prime}\left(V_{1}\right)$ where $P_{1}^{\prime}$ is a polynomial of first order.
Putting all these bounds together, we conclude that there exists a polynomial of fourth order $P_{4}^{\prime}\left(V_{1}\right)$ such that

$$
\left|\frac{\partial V}{\partial e_{i}}\left[G\left(t, e_{i}, \xi_{i}\right) e_{i}+B\left(e_{i}\right) \eta_{i}\right]\right| \leq P_{4}^{\prime}\left(V_{1}\right)\left|\xi_{i} \eta_{i}\right|
$$

and, therefore,

$$
\dot{W}\left(t, e_{i}\right) \leq-\alpha\left(\left|e_{i}\right|\right)+c\left|\xi_{i} \eta_{i}\right|
$$

where

$$
c:=\limsup _{V_{1} \geq 0} \frac{P_{4}^{\prime}\left(V_{1}\right)}{1+\tilde{P}_{4}\left(V_{1}\right)}
$$

and the claim follows.

## B. Proof of Proposition 3

The proof follows along the arguments developed below (27). For $i=1$ the closed-loop dynamics, composed of (26c) and

$$
\begin{equation*}
\dot{\eta}_{1}=F_{1_{c l}}\left(t, \tilde{v}_{1}, \tilde{\omega}_{1}, e_{1}(t)\right) \tag{57}
\end{equation*}
$$

is defined on the interval of existence of $e_{1}(t)$, denoted $\left[t_{0}, t_{\max }\right)$, and has a cascaded form. By assumption, $\eta_{1}$ satisfies the bound (28) for all $t \in\left[t_{0}, t_{\max }\right.$ ) hence, on this interval,

$$
\begin{align*}
\dot{V}_{1}\left(e_{1}(t)\right) & \leq \frac{\partial V_{1}}{\partial e_{1}}\left(e_{1}(t)\right) B\left(e_{1}(t)\right) \eta_{1}(t) \\
& \leq c V_{1}\left(e_{1}(t)\right) \eta_{1}\left(t_{\max }\right) \leq c^{\prime} V_{1}\left(e_{1}(t)\right) \tag{58}
\end{align*}
$$

where $c$ is a positive number of innocuous value and $c^{\prime}>\left|\eta_{1}\left(t_{\max }\right)\right|$; both are independent of the initial time. Integrating on both sides of the latter from $t_{0}$ to $t_{\max }$ we see that, by continuity of the solutions on the initial conditions, this interval of integration may be stretched to infinity. By the definition of $V_{1}\left(e_{1}\right)$ we obtain that $e_{1}(t)$ exists on $\left[t_{0}, \infty\right)$. Moreover, since by definition $\Delta v_{0}=\Delta \omega_{0}=0$, we conclude from (23) and (24), that $v_{1}^{*}$ and $\omega_{1}^{*}$ exist along trajectories on $\left[t_{0}, \infty\right)$. It follows that the same property holds for $v_{1}(t)$ and $\omega_{1}(t)$ and, consequently, for $\xi_{2}(t)$-recall that

$$
\xi_{2}:=\left[\begin{array}{c}
v_{1}-v_{r} \\
\omega_{1}-\omega_{r}
\end{array}\right]
$$

From forward completeness and condition A1 it follows, in turn, that $\eta_{1}=0$ is uniformly globally asymptotically stable for (57).

Now we can apply a cascades argument for the system (26c), (57). Since $B$ in (26c) is linear in $e_{1}$ and the origin of $\dot{e}_{1}=A_{v_{r}}\left(t, e_{1}\right)$ is uniformly globally asymptotically stable, the same property holds
for the origin $\left(e_{1}, \eta_{1}\right)=(0,0)$-see [18, Theorem 2]. This means that there exists a class $\mathcal{K} \mathcal{L}$ function $\beta$ such that

$$
\begin{equation*}
\left|\zeta_{1}\left(t, t_{\circ}, \zeta_{1 \circ}\right)\right| \leq \beta\left(\left|\zeta_{1 \circ}\right|, t-t_{\circ}\right) \quad \forall t \geq t_{\circ} \tag{59}
\end{equation*}
$$

where we recall that $\zeta_{i}=\left[e_{i}^{\top} \eta_{i}^{\top}\right]$ for all $i \leq n$. In particular, $e_{1}(t), \eta_{1}(t)$ and, consequently, $\xi_{2}(t)$, are uniformly globally bounded. To see this more clearly, we recall that, by definition, $\xi_{2}$ is a continuous function of the state $\zeta_{1}$ and time and equals to zero if $\zeta_{1}=0$. Indeed, $\xi_{2}=\psi\left(t, \zeta_{1}\right)$ where

$$
\begin{align*}
\psi_{1}\left(t, \zeta_{1}\right) & =\left[\begin{array}{c}
\tilde{v}_{1}+v_{1}^{*}-v_{r} \\
\tilde{\omega}_{1}+\omega_{1}^{*}-\omega_{r}
\end{array}\right]  \tag{60}\\
& =\left[\begin{array}{c}
\tilde{v}_{1}+v_{r}(t)\left[\cos \left(e_{\theta 1}\right)-1\right]+k_{x 1} e_{x 1} \\
\tilde{\omega}_{1}+k_{\theta 1} e_{\theta_{1}}+v_{r}(t) k_{y 1} e_{y 1} \phi\left(e_{\theta 1}\right)
\end{array}\right] \tag{61}
\end{align*}
$$

Next, let $i=2$ and consider the closed-loop equations:

$$
\begin{align*}
\dot{e}_{2} & =A_{v_{r}}\left(t, e_{2}\right) e_{2}+G\left(t, e_{2}, \psi_{1}\left(t, \zeta_{1}\right)\right) e_{2}+B\left(e_{2}\right) \eta_{2}  \tag{62a}\\
\dot{\zeta}_{1} & =F_{\zeta_{1}}\left(t, \zeta_{1}\right)  \tag{62b}\\
\dot{\eta}_{2} & =F_{2_{c l}}\left(t, \tilde{v}_{2}, \tilde{\omega}_{2}, e_{2}(t)\right) \tag{62c}
\end{align*}
$$

Note that we replaced $e_{2}$ with $e_{2}(t)$ in (27) to obtain the "decoupled" dynamics equation (62c). Then, $\eta_{2}$ is regarded as a perturbation to the system

$$
\begin{align*}
\dot{e}_{2} & =A_{v_{r}}\left(t, e_{2}\right) e_{2}+G\left(t, e_{2}, \psi_{1}\left(t, \zeta_{1}\right)\right) e_{2}  \tag{63a}\\
\dot{\zeta}_{1} & =F_{\zeta_{1}}\left(t, \zeta_{1}\right) . \tag{63b}
\end{align*}
$$

which, in turn, is also in cascaded form. Now, in view of the structure of $G$, we have

$$
\begin{equation*}
\frac{\partial V_{1}}{\partial e_{i}} G\left(t, e_{i}, \xi_{i}\right) e_{i}=0, \quad \forall i \leq n \tag{64}
\end{equation*}
$$

hence, the total derivative of $V_{1}$ along the trajectories of (62a) yields

$$
\dot{V}_{1}\left(e_{2}(t)\right) \leq c V_{1}\left(e_{2}(t)\right) \eta_{2}\left(t_{\max }\right) \leq c^{\prime} V_{1}\left(e_{2}(t)\right)
$$

with an appropriate redefinition of $c$ and $c^{\prime}-c f$. Ineq. (58). Completeness of $e_{2}(t)$, and therefore of $\eta_{2}(t)$, follows using similar arguments as for the case when $i=1$. Consequently, by Assumption A1, the origin of (62c) is uniformly globally asymptotically stable.

To analyze the stability of the origin for (62) we invoke again [18, Theorem 2]. To that end, we only need to establish uniform global asymptotic stability for the system (63) (since $B$ is linear and the origin of (62c) is uniformly globally asymptotically stable). For this, we invoke [20, Theorem 4] as follows: first, we remark that the respective origins of $\dot{e}_{2}=A_{v_{r}}\left(t, e_{2}\right)$ and (63b) are uniformly globally asymptotically stable. Second, note that condition A4 in [20, Theorem 4] is not needed here since we already established uniform forward completeness. Finally, [20, Ineq. (24)] holds trivially with $V=V_{1}$, in view of (64). We conclude that $\left(e_{2}, \zeta_{1}, \eta_{2}\right)=(0,0,0)$ is a uniformly globally asymptotically stable equilibrium of (62).

For $i=3$ the closed-loop dynamics is

$$
\begin{align*}
\dot{e}_{3} & =A_{v_{r}}\left(t, e_{3}\right) e_{3}+G\left(t, e_{3}, \psi_{2}\left(t, \zeta_{12}\right) e_{3}+B\left(e_{3}\right) \eta_{3}\right.  \tag{65a}\\
\dot{\zeta}_{12} & =: F_{\zeta_{12}}\left(t, \zeta_{12}\right)  \tag{65b}\\
\dot{\eta}_{3} & =F_{3_{c l}}\left(t, \tilde{v}_{3}, \tilde{\omega}_{3}, e_{3}(t)\right) \tag{65c}
\end{align*}
$$

where $\zeta_{12}:=\left[\begin{array}{ll}\zeta_{1}^{\top} & \zeta_{2}^{\top}\end{array}\right]^{\top}, \zeta_{2}:=\left[\begin{array}{ll}e_{2}^{\top} & \eta_{2}^{\top}\end{array}\right]$, and

$$
\psi_{2}\left(t, \zeta_{12}\right):=\left[\begin{array}{c}
\tilde{v}_{2}+\left[\xi_{21}+v_{r}(t)\right]\left[\cos \left(e_{\theta 1}\right)+k_{x 1} e_{x 1}-v_{r}\right. \\
\tilde{\omega}_{2}+\xi_{22}+k_{\theta 1} e_{\theta_{1}}+v_{r}(t) k_{y 1} e_{y 1} \phi\left(e_{\theta 1}\right)
\end{array}\right]
$$

which corresponds to $\xi_{3}-c f$. (61). The previous arguments, as for the case $i=2$, apply now to (65) so the result follows by induction.

## IV. Conclusions

We presented a formation-tracking controller for autonomous vehicles that ensures uniform global asymptotic stability of the closed-loop system, under the sole assumption that either the angular or the forward reference velocity is persistently exciting. Moreover, a strict Lyapunov function is provided for the kinematic error dynamics. Because we decouple the tracking problems at kinematic and dynamic levels, our results apply to a range of controllers at the dynamic level. Thus, one can use a variety of control schemes for Lagrangian and Hamiltonian systems, including adaptive and output feedback control designs. Further research in such directions is being carried out.

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## APPENDIX

We provide below some details on the computation of (43) and (45):

$$
\begin{align*}
& -\frac{d}{d t}\left\{\omega_{r} e_{x} e_{y}\right\} \leq-\omega_{r}\left[-k_{x} e_{x} e_{y}+\omega_{r} e_{y}^{2}+k_{\theta} e_{\theta} e_{y}^{2}+k_{y} v_{r} e_{y}^{3}\right] \\
& -\omega_{r}\left[-\omega_{r} e_{x}^{2}-k_{\theta} e_{\theta} e_{x}^{2}-k_{y} v_{r} e_{y} e_{x}^{2}+v_{r} e_{\theta} e_{x}\right]-\dot{\omega}_{r} e_{x} e_{y} \\
& \leq-\omega_{r}^{2} e_{y}^{2}+\bar{\omega}_{r} k_{x}\left(\frac{\epsilon}{2} e_{x}^{2}+\frac{1}{2 \epsilon} e_{y}^{2}\right)+\bar{\omega}_{r} k_{\theta}\left(\epsilon V_{1} e_{\theta}^{2}+\frac{1}{2 \epsilon} e_{y}^{2}\right) \\
& +\bar{\omega}_{r} \bar{v}_{r} k_{y} e_{y}^{3}+\bar{\omega}_{r}^{2} e_{x}^{2}+\bar{\omega}_{r} \frac{k_{\theta}}{2}\left(e_{\theta}^{2}+2 V_{1} e_{x}^{2}\right) \\
& +\bar{\omega}_{r} k_{y} \bar{v}_{r}\left(\frac{1}{2 \epsilon} e_{y}^{2}+\frac{\epsilon}{2} V_{1} e_{x}^{2}\right)+\frac{\bar{\omega}_{r} \bar{v}_{r}}{2}\left(e_{\theta}^{2}+e_{x}^{2}\right)+\bar{\omega}_{r}\left(\frac{\epsilon}{2} e_{x}^{2}+\frac{1}{2 \epsilon} e_{y}^{2}\right) \\
& \leq-\omega_{r}^{2} e_{y}^{2}+\bar{\omega}_{r} \bar{v}_{r} k_{y} e_{y}^{3}+\frac{1}{2 \epsilon}\left[\left(k_{x}+k_{\theta}\right) \bar{\omega}_{r}+\bar{\omega}_{r} k_{y} \bar{v}_{r}+\bar{\omega}_{r}\right] e_{y}^{2} \\
& +\left[\bar{\omega}_{r} k_{x} \frac{\epsilon}{2}+\bar{\omega}_{r}^{2}+\bar{\omega}_{r} k_{y} \bar{v}_{r} \frac{\epsilon}{2} V_{1}+\bar{\omega}_{r} \frac{\epsilon}{2}+\frac{\bar{\omega}_{r} \bar{v}_{r}}{2}+\bar{\omega}_{r} k_{\theta} V_{1}\right] e_{x}^{2} \\
& +\left[\bar{\omega}_{r} k_{\theta} \epsilon V_{1}+\bar{\omega}_{r} k_{\theta}+\frac{\bar{\omega}_{r} \bar{v}_{r}}{2}\right] e_{\theta}^{2} \\
& \leq-\omega_{r}^{2} e_{y}^{2}+\frac{\epsilon}{2} v_{r}^{2} V_{1} e_{y}^{2}+\rho_{5}\left(V_{1}\right) e_{x}^{2}+\rho_{6}\left(V_{1}\right) e_{\theta}^{2} \\
& +\frac{1}{2 \epsilon}\left[\bar{\omega}_{r}^{2} k_{y}^{2}+\left(k_{x}+k_{\theta}\right) \bar{\omega}_{r}+\bar{\omega}_{r} k_{y} \bar{v}_{r}+\bar{\omega}_{r}\right] e_{y}^{2}  \tag{66}\\
& \frac{d}{d t}\left\{v_{r} \rho_{2}\left(V_{1}\right) e_{\theta} e_{y}\right\} \leq-k_{y} v_{r}^{2} \rho_{2} e_{y}^{2}-\rho_{2} k_{\theta} v_{r} e_{\theta} e_{y}-\rho_{2} v_{r}\left[\omega_{r} e_{x} e_{\theta}+k_{\theta} e_{\theta}^{2} e_{x}+k_{y} v_{r} e_{y} e_{x} e_{\theta}\right] \\
& +\rho_{2} v_{r}^{2} e_{\theta}^{2}+\rho_{2} \dot{v}_{r} e_{\theta} e_{y}-v_{r} \frac{\partial \rho_{2}\left(V_{1}\right)}{\partial V_{1}} e_{\theta} e_{y}\left(k_{x} e_{x}^{2}+k_{\theta} e_{\theta}^{2}\right) \\
& \leq-k_{y} v_{r}^{2} \rho_{2} e_{y}^{2}+k_{\theta} \bar{v}_{r}\left(\frac{\epsilon}{2} \rho_{2}^{2} e_{\theta}^{2}+\frac{1}{2 \epsilon} e_{y}^{2}\right)+\rho_{2} \bar{v}_{r} \frac{\bar{\omega}_{r}}{2}\left(e_{x}^{2}+e_{\theta}^{2}\right) \\
& +\rho_{2} \bar{v}_{r} \frac{k_{\theta}}{2}\left(e_{\theta}^{2}+V_{1} e_{x}^{2}\right)+\rho_{2} k_{y} \bar{v}_{r}^{2}\left(V_{1} e_{x}^{2}+\frac{e_{\theta}^{2}}{2}\right)+\bar{v}_{r}^{2} \rho_{2} e_{\theta}^{2}+ \\
& \overline{\dot{v}}_{r}\left(\frac{\epsilon}{2} \rho_{2}^{2} e_{\theta}^{2}+\frac{1}{2 \epsilon} e_{y}^{2}\right)+\bar{v}_{r}\left|\frac{\partial \rho_{2}\left(V_{1}\right)}{\partial V_{1}}\right| \max \left\{k_{y}, 1\right\} V_{1}\left(k_{x} e_{x}^{2}+k_{\theta} e_{\theta}^{2}\right) \\
& \leq-k_{y} v_{r}^{2} \rho_{2} e_{y}^{2}+\frac{1}{2 \epsilon}\left(k_{\theta} \bar{v}_{r}+\overline{\dot{v}}_{r}\right) e_{y}^{2}+\left[\rho_{2} \bar{v}_{r} \frac{\bar{\omega}_{r}}{2}+k_{\theta} V_{1} \rho_{2} \bar{v}_{r}\right. \\
& \left.+\rho_{2} k_{y} \bar{v}_{r}^{2} V_{1}+\bar{v}_{r}\left|\frac{\partial \rho_{2}\left(V_{1}\right)}{\partial V_{1}}\right| \max \left\{k_{y}, 1\right\} V_{1} k_{x}\right] e_{x}^{2} \\
& +\left[k_{\theta} \bar{v}_{r} \frac{\epsilon}{2} \rho_{2}^{2}+\rho_{2} \bar{v}_{r} \frac{\bar{\omega}_{r}}{2}+\rho_{2} \bar{v}_{r} \frac{k_{\theta}}{2}+\rho_{2} \frac{k_{y}}{2} \bar{v}_{r}^{2}+\bar{v}_{r}^{2} \rho_{2}\right.
\end{align*}
$$

$$
\begin{gather*}
\left.+\overline{\dot{v}}_{r} \frac{\epsilon}{2} \rho_{2}^{2}+\bar{v}_{r}\left|\frac{\partial \rho_{2}\left(V_{1}\right)}{\partial V_{1}}\right| \max \left\{k_{y}, 1\right\} V_{1} k_{\theta}\right] e_{\theta}^{2} \\
\leq-k_{y} v_{r}^{2} \rho_{2} e_{y}^{2}+\rho_{7}\left(V_{1}\right) e_{x}^{2}+\rho_{8}\left(V_{1}\right) e_{\theta}^{2}+\frac{1}{2 \epsilon}\left(k_{\theta} \bar{v}_{r}+\overline{\dot{v}}_{r}\right) e_{y}^{2} \tag{67}
\end{gather*}
$$

