iISS formation tracking control of autonomous vehicles

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Abstract

We present a cascades-based controller for the problem of formation-tracking control in a group of mobile robots. We consider general models composed of a kinematics equation and a generic dynamics equation at the force level. For each robot, a local controller ensures a follow-the-leader task. Then, only one robot possesses the information of the reference trajectory. We establish uniform global asymptotic stability in closed-loop. Our analysis relies on the construction of an original strict Lyapunov function for the position tracking error dynamics and an inductive argument based on cascades-systems theory.

I. Introduction

For a group of autonomous vehicles, the formation-tracking control problem roughly consists in making them form a spatial configuration and move along a reference path while keeping the pattern [1]. It naturally stems from the well-studied leader-follower tracking control problem, in which one robot follows the trajectory described by a virtual leader –see *e.g.*, [2].

For the purpose of control design, autonomous vehicles are often modelled as unicycle systems, having two Cartesian coordinates to determine translation and one for orientation, this is the so-called kinematic model. More complete models include an additional forces-balance equation, expressed, *e.g.*, in Lagrangian form [3] or in Hamiltonian one [4]. Yet, even at the kinematic level alone, tracking control imposes certain difficulties that stem from the nonholonomy of the robot; see, *e.g.*, [5] where a general framework for consensus of nonholonomic systems in chain form is presented.

When the full model is considered, a common approach used in the literature is backstepping control -see *e.g.*, [6], [7], [8] to mention only a few. Notably, the controllers in the last two references use partial-state or output feedback. In [7] an observer is designed based on a transformed model that is linear in the unmeasured velocities, In [8] a similar trend is followed along with the virtual structure and path-tracking approaches. See also [9], in which the output-feedback controller employs observer-generated estimates of relative (Cartesian) positions between the leader and the follower. Under parametric uncertainty, an adaptive state-feedback controller that guarantees the convergence of tracking errors is proposed in [10].

Other problems in formation control focus on *achieving* or maintaining a configuration while following a trajectory. In [11] necessary and sufficient conditions for the solubility of this problem, under distributed control, are given. Under the assumption that robot is modelled as a point-mass (second-order integrators), time-varying configurations are considered in [12].

In this note we address the problem of formation tracking control in a leader-follower topological configuration. This means that each robot follows one leader and communicates its coordinates to one or several followers that stand not necessarily physically close. The reference trajectory is generated by a virtual robot, which may be known by only one, or by several robots. Our controllers are based on a recursive repetition of a time-varying nonlinear tracking controller for the kinematics model, based

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on [2]. In this case, we establish via Lyapunov's direct method that the origin is uniformly globally asymptotically stable. For the complete model, we show that *any* controller ensuring the stabilization at the force level with sufficiently fast rate of convergence, leads to uniform global asymptotic stability.

Our main contributions lie in the construction of a strict Lyapunov function for the kinematics model and the proofs of uniform global asymptotic stability. Even though Lyapunov's direct method is also used, *e.g.*, in [8], our main condition is that either the forward or the angular *reference* velocity is persistently exciting. This is a weaker condition than what is considered in the literature, including the references cited previously. For instance, neither velocity is required to be strictly positive and either one can equal to zero while the other is time-varying non-negative. On the other hand, although our controllers use state feedback (in contrast to [9], [7], [8]) they are mathematically very simple, hence easy to implement.

Our main results are stated in Section II; firstly for leader-follower tracking control and then, for formation tracking of large groups. The constructive stability proofs are provided in Section III, before concluding with some remarks in Section IV.

II. PROBLEM FORMULATION AND ITS SOLUTION

A. Tracking control

Let us consider the force-controlled model of a mobile robot:

$$\begin{cases} \dot{x} &= v\cos\theta \\ \dot{y} &= v\sin\theta \\ \dot{\theta} &= \omega \end{cases} \tag{1}$$

$$\begin{cases} \dot{v} &= f_1(t,v,\omega,z) + u_1 \\ \dot{\omega} &= f_2(t,v,\omega,z) + u_2 \end{cases} \tag{2}$$

where v and ω denote the forward and angular velocities respectively, the first two elements of $z := [x\ y\ \theta]^\top$ correspond to the Cartesian coordinates of a point on the robot with respect to a fixed reference frame, and θ denotes the robot's orientation with respect to the same frame. The two control inputs are the torques u_1, u_2.

Equations (1) correspond to the kinematic model while the last two correspond to the force-balance equations which may be expressed, *e.g.*, in Lagrangian form –see [3], [7]. The control strategy in this paper consists in decoupling the stabilisation task at both levels. For (1) we establish robustness results hence, our main statements are valid for any controller that guarantees the stabilisation of (2) with "fast" convergence (for instance, but not only, local exponential).

The tracking-control problem consists in making the robot to follow a fictitious reference vehicle modelled by

$$\begin{aligned} \dot{x}_r &= v_r\cos\theta_r \\ \dot{y}_r &= v_r\sin\theta_r \\ \dot{\theta}_r &= \omega_r, \end{aligned}$$

and which moves about with reference velocities $v_r(t)$ and $\omega_r(t)$. More precisely, it is desired to steer the differences between the Cartesian coordinates to some values d_x, d_y, and to zero the orientation angles and the velocities of the two robots, that is, the quantities

$$p_\theta = \theta_r - \theta, \quad p_x = x_r - x - d_x, \quad p_y = y_r - y - d_y.$$

The distances d_x, d_y define the position of the robot with respect to the (virtual) leader. In general, these may be functions that depend on time and the state or may be assumed to be constant, depending on the desired path to be followed. In this paper, we consider these distances to be defined as piece-wise constant functions –*cf.* [13].

Then, as it is customary, we transform the error coordinates $[p_\theta,\ p_x,\ p_y]$ of the leader robot from the global coordinate frame to local coordinates fixed on the robot, that is, we define

$$\begin{bmatrix} e_\theta \\ e_x \\ e_y \end{bmatrix} := \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} p_\theta \\ p_x \\ p_y \end{bmatrix}. \tag{3}$$

In these new coordinates, the error dynamics between the virtual reference vehicle and the follower becomes

$$\dot{e}_\theta = \omega_r(t) - \omega \tag{4a}$$
$$\dot{e}_x = \omega e_y - v + v_r(t)\cos(e_\theta) \tag{4b}$$
$$\dot{e}_y = -\omega e_x + v_r(t)\sin(e_\theta) \tag{4c}$$

which is to be completed with Eqs (2). Hence, the control problem reduces to steering the trajectories of (4) to zero via the inputs u_1 and u_2 in (2). A natural method consists in designing virtual control laws at the kinematic level, that is, w^* and v^*, and control inputs u_1 and u_2, depending on the latter, such that the origin $(e, \tilde{v}, \tilde{w}) = (0, 0, 0)$ with

$$\tilde{v} := v - v^*, \quad \tilde{\omega} := \omega - \omega^*, \quad e = [e_\theta\ e_x\ e_y]^\top, \tag{5}$$

is uniformly globally asymptotically stable.

The stabilization problem for (4), that is neglecting the dynamics (2) so that $\omega \equiv \omega^*$, $v \equiv v^*$, has been broadly and long studied in the literature. For instance, in [2] the authors proposed the controller

$$v^* := v_r(t)\cos(e_\theta) + k_x e_x \tag{6a}$$
$$\omega^* := \omega_r(t) + k_\theta e_\theta + v_r(t) k_y e_y \phi(e_\theta) \tag{6b}$$

where ϕ is the so-called 'sync' function defined by

$$\phi(e_\theta) := \frac{\sin(e_\theta)}{e_\theta} \tag{7}$$

and established (non-uniform) convergence of the tracking errors. Then, for the same controller but under slightly relaxed conditions, uniform global asymptotic stability for the closed-loop system is established in [14]. In this paper, we establish the same property and, in addition, for the first time we provide a strict Lyapunov function. As in [14], our standing assumption is that *either* the forward or the angular reference velocities are persistently exciting that is, that there exist positive numbers μ and T such that

$$\int_t^{t+T} [\omega_r(s)^2 + v_r(s)^2] ds \geq \mu \quad \forall\, t \geq 0. \tag{8}$$

The design of the controller (6) is motivated by the resulting structure of the error dynamics for the tracking errors, which is reminiscent of nonlinear adaptive control systems. Indeed, by setting $\omega = \omega^*$ and $v = v^*$, we obtain

$$\begin{bmatrix} \dot{e}_\theta \\ \dot{e}_x \\ \dot{e}_y \end{bmatrix} = \underbrace{\begin{bmatrix} -k_\theta & 0 & -v_r(t) k_y \phi(e_\theta) \\ 0 & -k_x & \omega^*(t,e) \\ v_r(t)\phi(e_\theta) & -\omega^*(t,e) & 0 \end{bmatrix}}_{A_{v_r}(t,e)} \begin{bmatrix} e_\theta \\ e_x \\ e_y \end{bmatrix}. \tag{9}$$

Then, we obtain the crucial property that the trivial solution for this system is uniformly globally stable (it is uniformly stable and all solutions are uniformly globally bounded). To see this, note that the total derivative of $V_1 : \mathbb{R}^3 \to \mathbb{R}_{\geq 0}$, defined as

$$V_1(e) = \frac{1}{2}\left[e_x^2 + e_y^2 + \frac{1}{k_y}e_\theta^2\right] \tag{10}$$

corresponds to

$$\dot{V}_1(e) = -k_x e_x^2 - k_\theta e_\theta^2 \leq 0. \tag{11}$$

Furthermore, after [15], it may be concluded that the origin of this system is uniformly globally asymptotically stable provided that the vector $[-v_r(t)k_y\phi(e_\theta) \quad \omega^*(t,e)]$, subject to $e_\theta = 0$, is δ-persistently exciting with respect to e_y. Roughly, this holds provided that this vector is persistently exciting for any $e_y \neq 0$; condition which, actually, reduces to (8). Thus, our first statement is the following.

Proposition 1 (Kinematic model): For the system (9) assume that (8) holds and there exist $\bar{\omega}_r$, $\bar{\dot{\omega}}_r$, $\bar{\nu}$, $\bar{\dot{\nu}} > 0$ such that

$$|\omega_r|_\infty \leq \bar{\omega}_r, \quad |\dot{\omega}_r|_\infty \leq \bar{\dot{\omega}}_r, \quad |v_r|_\infty \leq \bar{v}_r, \quad |\dot{v}_r|_\infty \leq \bar{\dot{v}}_r. \tag{12}$$

Then, the origin is uniformly globally asymptotically stable and locally exponentially stable, for any positive values of the control gains k_x, k_y, and k_θ. □

Beyond the statement itself, our (first) contribution lies in the original proof of Proposition 1 (see Section III) which is based on Lyapunov's direct method. Concretely, following the methods of [16], we show that there exists a positive definite radially unbounded function $V : \mathbb{R}_{\geq 0} \times \mathbb{R}^3 \to \mathbb{R}_{\geq 0}$ defined as the functional

$$V(t,e) := P_{[3]}(t,V_1)V_1(e) - \omega_r(t)e_x e_y + v_r(t)P_{[1]}(t,V_1)e_\theta e_y \tag{13}$$

where $P_{[k]} : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a smooth function such that $P_{[k]}(\cdot, V_1)$ is uniformly bounded and $P_{[k]}(t,\cdot)$ is a polynomial of degree k, designed such that the total derivative of V along the trajectories of (9) satisfies

$$\dot{V}(t,e) \leq -\frac{\mu}{T}V_1(e) - k_x e_x^2 - k_\theta e_\theta^2 \tag{14}$$

The value of having a strict Lyapunov function for (9) may not be overestimated. Notably, this allows to carry on with a robustness analysis vis-a-vis of the dynamics (2). In other words, to solve the tracking control problem for (1), (2) it is only left to design u_1 and u_2 such that, given the references v^* and ω^*, the origin of the closed-loop dynamics

$$\dot{\tilde{v}} = f_{1cl}(t,\tilde{v},\tilde{\omega},e) \tag{15a}$$

$$\dot{\tilde{\omega}} = f_{2cl}(t,\tilde{v},\tilde{\omega},e) \tag{15b}$$

is globally asymptotically stable uniformly in the initial conditions and in e. The most obvious choice corresponds to the linearizing feedbacks $u_1 := \dot{v}^* - f_1(t,v,\omega,e) - k_1\tilde{v}$ and $u_2 = \dot{\omega}^* - f_1(t,v,\omega,e) - k_2\tilde{\omega}$. In Section II-C we present another example of an effective controller at force level. However, in general, the design of the control inputs u_1 and u_2 depends on the problem setting and is beyond the scope of this technical note.

On the other hand, it is remarkable that the overall error dynamics takes the convenient form

$$\dot{e} = A_{v_r}(t,e)e + B(e)\eta, \tag{16a}$$

$$\dot{\eta} = F_{cl}(t,\tilde{v},\tilde{\omega},e), \quad F_{cl} := [f_{1cl} \ f_{2cl}], \tag{16b}$$

where

$$B(e) := \begin{bmatrix} 0 & -1 \\ -1 & e_y \\ 0 & -e_x \end{bmatrix}, \quad \eta := \begin{bmatrix} \tilde{v} \\ \tilde{\omega} \end{bmatrix}. \tag{17}$$

Now, for the purpose of analysis, we replace e with $e(t)$ in (16b) so the closed-loop equations may be regarded as a cascaded nonlinear time-varying system with state $\zeta := [e^\top \; \eta^\top]^\top$. More precisely, in place of (16b) we write

$$\dot{\eta} = \tilde{F}_{cl}(t, \tilde{v}, \tilde{\omega})$$

where $\tilde{F}_{cl}(t, \tilde{v}, \tilde{\omega}) = F_{cl}(t, \tilde{v}, \tilde{\omega}, e(t))$ –*cf.* [17]. Then, using arguments for cascaded systems we can establish our second contribution:

Proposition 2: Consider the system (16) with initial conditions $(t_\circ, \zeta_\circ) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^5$. Assume that k_x, k_y, and k_θ are positive and that Inequalities (8) and (12) hold. In addition, assume that the solutions are complete and the origin of (16b) is globally asymptotically stable, uniformly in the initial times $t_\circ \in \mathbb{R}_{\geq 0}$ and in the error trajectories $t \mapsto e$. Assume further that the trajectories $t \mapsto \eta$ are uniformly integrable, that is, there exists $\phi \in \mathcal{K}$ such that

$$\int_{t_\circ}^{\infty} |\eta(\tau)| d\tau \leq \phi(|\zeta_\circ|) \quad \forall\, t \geq t_\circ \geq 0. \tag{18}$$

Then, the origin is uniformly globally asymptotically stable. □

Proof: From Proposition 1, the origin $\{e = 0\}$ is uniformly globally asymptotically stable for (9). By assumption the same property holds for (16b). Since, moreover, B is linear in e, the result follows from the main results in [18, Theorem 2]. ■

Remark 1: Technically, the function $\tilde{F}_{cl}$ exists only on the interval of existence of $e(t)$, whence the assumption that the solutions exist on $[t_\circ, \infty)$. Nevertheless, this assumption may be dropped if we impose that $\eta \to 0$ uniformly in $e(t)$ *only* on the interval of existence. This is considered in our main result later on –see Proposition 3. •

B. *Formation tracking control*

Let us consider now n mobile robots that are required to advance in formation. It is assumed that the ith robot follows a leader, indexed $i-1$, thereby forming a spanning-tree graph communication topology. As previously explained, the geometry of the formation may be defined via the relative distances between any pair of leader-follower robots, d_{xi}, d_{yi} and it is independent of the communications graph (two robots may communicate independently of their relative positions). Then, the relative position error dynamics is given by a set of equations similar to (4), that is,

$$\dot{e}_{\theta i} = \omega_{i-1}(t) - \omega_i \tag{19a}$$

$$\dot{e}_{xi} = \omega_i e_{yi} - v_i + v_{i-1}(t)\cos(e_{\theta i}) \tag{19b}$$

$$\dot{e}_{yi} = -\omega_i e_{xi} + v_{i-1}(t)\sin(e_{\theta i}) \tag{19c}$$

For $i = 1$ we recover the tracking error dynamics for the case of one robot following a virtual leader that is, by definition, $v_0 := v_r$ and $\omega_0 := \omega_r$. Then, similarly to (6) we introduce the virtual control inputs

$$v_i^* := v_{i-1}\cos(e_{\theta i}) + k_{xi} e_{xi} \tag{20}$$

$$\omega_i^* := \omega_{i-1} + k_{\theta i} e_{\theta_i} + v_{i-1} k_{yi} e_{yi} \phi(e_{\theta i}) \tag{21}$$

which serve as references for the actual controls u_{1i} and u_{2i} in

$$\begin{aligned} \dot{v}_i &= f_{1i}(t, v_i, \omega_i, e_i) + u_{1i} && \text{(22a)} \\ \dot{\omega}_i &= f_{2i}(t, v_i, \omega_i, e_i) + u_{2i}, \qquad i \le n. && \text{(22b)} \end{aligned}$$

Next, let the velocity errors be defined as

$$\tilde{\omega}_i := \omega_i - \omega_i^*, \quad \tilde{v}_i := v_i - v_i^*$$

and let us define $\Delta v_j := v_j - v_r$ and $\Delta\omega_j := \omega_j - \omega_r$ for all $j \le n$ (by definition, $\Delta\omega_0 = \Delta v_0 = 0$). Then, we replace ω_i with $\tilde{\omega}_i + \omega_i^*$ and, respectively, v_i with $\tilde{v}_i + v_i^*$ in (19), and we use

$$\begin{aligned} v_i^* &= [\Delta v_{i-1} + v_r]\cos(e_{\theta i}) + k_{xi} e_{xi} && \text{(23)} \\ \omega_i^* &= \Delta\omega_{i-1} + \omega_r + k_{\theta i} e_{\theta_i} + [\Delta v_{i-1} + v_r] k_{yi} e_{yi} \phi(e_{\theta i}). && \text{(24)} \end{aligned}$$

It follows that, for each pair of nodes, the error system takes the form

$$\dot{e}_i = A_{v_r}(t, e_i) e_i + G(t, e_i, \xi_i) e_i + B(e_i)\eta_i \tag{25}$$

–*cf.* (16a), where

$$\begin{aligned} e_i &:= [e_{\theta_i}\ e_{x_i}\ e_{y_i}]^\top, \quad \eta_i \ := \ [\tilde{v}_i\ \tilde{\omega}_i]^\top \\ \xi_i &:= [\Delta\omega_{i-1}\ \Delta v_{i-1}]^\top \\ G &:= \begin{bmatrix} 0 & 0 & -k_y g_1 \\ 0 & 0 & g_2 \\ g_1 & -g_2 & 0 \end{bmatrix} \\ g_1 &:= \Delta v_{i-1} e_{y_i} \phi(e_{\theta_i}) \\ g_2 &:= \Delta\omega_{i-1} + k_y \Delta v_{i-1} e_{y_i} \phi(e_{\theta_i}) \end{aligned}$$

and B is defined in (17). Thus, the overall closed-loop system has the convenient cascaded form (in reverse order):

$$\begin{aligned} \dot{e}_n &= A_{v_r}(t, e_n) e_n + G(t, e_n, \xi_n) e_n + B(e_n)\eta_n && \text{(26a)} \\ &\vdots \\ \dot{e}_2 &= A_{v_r}(t, e_2) e_2 + G(t, e_2, \xi_2) e_2 + B(e_2)\eta_2 && \text{(26b)} \\ \dot{e}_1 &= A_{v_r}(t, e_1) e_1 + B(e_1)\eta_1 && \text{(26c)} \end{aligned}$$

and these closed-loop equations are complemented by the equations that stem from applying the actual control inputs in (22), that is,

$$\dot{\eta}_i = F_{i_{cl}}(t, \tilde{v}_i, \tilde{\omega}_i, e_i), \quad F_{i_{cl}} := [f_{i_{1cl}}\ f_{i_{2cl}}] \tag{27}$$

for all $i \le n$.

To underline the good structural properties of the system (26)–(27) and to explain the rationale of our main result, let us argue as follows (precise proofs are given in Section III). By assumption, the control inputs u_{1i} and u_{2i} are such that $\eta_i \to 0$, independently of the behaviour of e_i. Furthermore, we see from Equation (26c) that, as $\eta_1 \to 0$, we recover the system (9). Hence, using Proposition 1, we may conclude that $\eta_1 \to 0$ implies that $e_1 \to 0$. With this in mind, let us observe (26b). We have $\xi_2 := [\Delta\omega_1\ \Delta v_1]^\top$ where $\Delta\omega_1 = \omega_1 - \omega_r$ and $\Delta v_1 = v_1 - v_r$. On the other hand, by virtue of the control design, $e_1 = 0$

implies that $\omega_1^* = \omega_r$ and $v_1^* = v_r$, in which case we have $\Delta\omega_1 = \tilde{\omega}_1$ and $\Delta v_1 = \tilde{v}_1$. It follows that $e_1 \to 0$ and $\eta_1 \to 0$ imply that $\xi_2 \to 0$. In addition, as $\eta_2 \to 0$ (by the action of the controller at the force level), the terms $G(t, e_2, \xi_2)e_2 + B(e_2)\eta_2$ in (26b) vanish and (26b) becomes $\dot{e}_2 = A_{v_r}(t, e_2)e_2$. By Proposition 1 we conclude that e_2 also tends to zero. Carrying on by induction, we conclude that $e \to 0$.

Although intuitive, the previous arguments implicitly rely on the robustness of $\dot{e}_i = A_{v_r}(t, e_i)$ (i.e., of the system (9)) with respect to the inputs η_i and ξ_i. More precisely, on the condition that the solutions exist on $[t_\circ, \infty)$ and, moreover, that they remain uniformly bounded during the transient. In our main result, which is presented next, we relax these (technical) assumptions.

Proposition 3: For each $i \le n$, consider the system (19), (22) with control inputs u_{1i} and u_{2i} which are functions of $(t, v_i, \omega_i, e_i, v_i^*, \omega_i^*)$ and v_i^*, ω_i^* are defined in (20) and (21) respectively. Let conditions (8) and (12) hold. Let $\zeta_i := [e_i^\top \ \eta_i^\top]^\top$. In addition, assume that:

[A1] for each i, there exists a function $\beta_i \in \mathcal{KL}$ such that, on the maximal interval of existence[1] of $t \to e_i$,

$$|\eta_i(t, t_\circ, \eta_{1\circ}, e_{i\circ})| \le \beta(|\zeta_{i\circ}|, t - t_\circ) \tag{28}$$

and (18) holds for some $\phi_i \in \mathcal{K}$.

Then, $\{\zeta = 0\}$, where $\zeta := [\zeta_1^\top \cdots \zeta_n^\top]^\top$, is uniformly globally asymptotically stable. □

Assumption A1 means that $\eta_i(t)$ converge uniformly to zero while the trajectories $e_i(t)$ exist. In particular, if the system is forward complete A1 imposes uniform global asymptotic stability of (27). Even though this may be a strong hypothesis in a general context of nonlinear systems –see [17], it may be easily met in the case of formation tracking control, as we illustrate below.

C. Example

After [10], a dynamic model of a wheeled mobile robot is given by

$$\dot{z} = J(z)\nu \tag{29a}$$

$$M\dot{\nu} + C(\dot{z})\nu + D\nu = \tau \tag{29b}$$

where τ is the torque control input; the variable $\nu := [\nu_1 \ \nu_2]$ denotes the angular velocities of the two wheels, M is an inertia matrix (hence positive definite, symmetric), C is the matrix of Coriolis forces (which is skew-symmetric) and D denotes natural damping (hence, $D = D^\top \ge 0$), and

$$J(z) = \frac{r}{2}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \\ 1/b & -1/b \end{bmatrix}$$

where r and b are positive constant parameters of the system. The relation between the wheels' velocities, ν, and the robot's velocities in the fixed frame, $\dot{z}$, is given by

$$\begin{bmatrix} v \\ \omega \end{bmatrix} = \frac{r}{2b}\begin{bmatrix} b & b \\ 1 & -1 \end{bmatrix}\begin{bmatrix} \nu_1 \\ \nu_2 \end{bmatrix} \quad \Leftrightarrow \quad \begin{bmatrix} \nu_1 \\ \nu_2 \end{bmatrix} = \frac{1}{r}\begin{bmatrix} 1 & b \\ 1 & -b \end{bmatrix}\begin{bmatrix} v \\ \omega \end{bmatrix} \tag{30}$$

which may be used in (29) to obtain the model (1), (2) with

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \frac{r}{2b}\begin{bmatrix} b & b \\ 1 & -1 \end{bmatrix} M^{-1}\tau$$

[1]If necessary, we consider the shortest maximal interval of existence among all the trajectories $e_i(t)$, with $i \le n$.

—see [10] for more details on this coordinate transformation.

Then, using (30), for any given virtual control inputs v^* and ω^*, we can compute $\nu^* := [\nu_1^* \ \nu_2^*]^\top$ and define the torque control input

$$\tau = M\dot{\nu}^* + C(J(z)\nu)\nu^* + D\nu^* - k_d\tilde{\nu}, \quad k_d > 0$$

where $\tilde{\nu} := \nu - \nu^*$. We see that the force error equations yields

$$M\dot{\tilde{\nu}} + \big[C(\dot{z}(t)) + D + k_d I\big]\tilde{\nu} = 0 \tag{31}$$

in which we have replaced $\dot{z}$ with the trajectories $\dot{z}(t)$ to regard this system as linear time-varying, with state $\tilde{\nu}$. Now, due to the skew-symmetry of $C(\cdot)$ the total derivative of

$$V(\tilde{\nu}) = \frac{1}{2}\tilde{\nu}^\top M\tilde{\nu}$$

yields

$$\dot{V}(\tilde{\nu}) \leq -k_d|\tilde{\nu}|^2.$$

Although this inequality holds independently of $\dot{z}(t)$, Eq. (31) is valid only on the interval of existence of $\dot{z}(t)$, denoted $[t_\circ, t^{\max})$, $t^{\max} \leq \infty$. Hence,

$$|\tilde{\nu}(t)| \leq \kappa|\tilde{\nu}(t_\circ)|e^{-\lambda(t-t_\circ)} \qquad \forall\, t \in [t_\circ, t^{\max})$$

for some κ and $\lambda > 0$. From (30) it is clear that a similar bound holds for $\eta(t) = [\tilde{v}(t) \ \tilde{\omega}(t)]$. In other words, the velocity errors tend exponentially to zero uniformly in the initial conditions and in the position error trajectories, so condition A1 of Proposition 3 holds.

III. Stability analysis

A. *Proof of Proposition 1*

The proof follows via Lyapunov's direct method; it relies on the construction of a Lyapunov function of polynomial type, and it is greatly inspired by the methods in [16].

Firstly, for any locally integrable function $\varphi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, such that $\sup_{t\geq 0}|\varphi(t)| \leq \bar{\varphi}$, let us introduce

$$Q_\varphi(t) := 1 + 2\bar{\varphi}T - \frac{2}{T}\int_t^{t+T}\int_t^m \varphi(s)ds dm. \tag{32}$$

Note that this function satisfies:

$$\begin{gathered} 1 \leq Q_\varphi(t) < \bar{Q}_\varphi := 1 + 2\bar{\varphi}T \\ \dot{Q}_\varphi(t) = -\frac{2}{T}\int_t^{t+T}\varphi(s)ds + 2\varphi(t). \end{gathered} \tag{33}$$

For the sequel, we will introduce, as needed, several polynomial functions denoted $\rho_i : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, which will be defined later on in a manner that the derivative of

$$\begin{aligned} V_2(t,e) \quad := \quad & \rho_1(V_1)V_1 + \big[Q_{v_r^2}(t) + Q_{\omega_r^2}(t)\big]V_1 - \omega_r(t)e_x e_y \\ & +v_r\rho_2(V_1)e_\theta e_y + \rho_3(V_1)V_1, \end{aligned} \tag{34}$$

where V_1 defined in (10), is positive definite.

Note that, in addition,

$$V_2(t,e) \geq \frac{1}{2} \begin{bmatrix} e_\theta \\ e_x \\ e_y \end{bmatrix}^\top \begin{bmatrix} \rho_3(V_1) & v_r\rho_2(V_1) & 0 \\ v_r\rho_2(V_1) & \rho_3(V_1) & -\omega_r \\ 0 & -\omega_r & \rho_3(V_1) \end{bmatrix} \begin{bmatrix} e_\theta \\ e_x \\ e_y \end{bmatrix}$$

so V_2 is positive definite and radially unbounded if the matrix in this inequality is positive semidefinite. The latter holds if ρ_3 satisfies

$$\rho_3(V_1) \geq \frac{\sqrt{\bar{v}_r^2 \rho_2(V_1)^2 + \bar{\omega}_r^2}}{\sqrt{2}}.$$

Finally, we introduce

$$V_3(t,e) = V_2(t,e) + V_1\rho_4(V_1) \tag{35}$$

which is also positive definite. We shall show that for an appropriate choice of the polynomials ρ_i, the total derivative of V_3 along the trajectories of (9) yields

$$\dot{V}_3(t,e) \leq -\frac{\mu}{T} V_1(e) - k_x e_x^2 - k_\theta e_\theta^2, \quad \forall\, (t,e) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^3 \tag{36}$$

To that end, we start by rewriting (9) in the output-injection form

$$\dot{e} \;=\; A^\circ_{v_r}(t,e)e + v_r[\phi(e_\theta) - 1]B^\circ(e_y)e \tag{37}$$

$$A^\circ_{v_r}(t,e) \;:=\; \begin{bmatrix} -k_\theta & 0 & -v_r k_y \\ 0 & -k_x & \varpi^\circ_{v_r} \\ v_r & -\varpi^\circ_{v_r} & 0 \end{bmatrix} \tag{38}$$

$$\varpi^\circ_{v_r}(t,e) \;=\; \omega_r(t) + k_\theta e_\theta + v_r k_y e_y \tag{39}$$

$$B^\circ(e) \;:=\; \begin{bmatrix} 0 & 0 & -k_y \\ 0 & 0 & k_y e_y \\ 1 & -k_y e_y & 0 \end{bmatrix}. \tag{40}$$

This partition, which facilitates the analysis, is motivated by the fact that $v_r[\phi(e_\theta) - 1]B^\circ(e_y)e = 0$ if $e_\theta = 0$.

Now and we show that the total derivative of V_2 along the trajectories of $\dot{e} = A^\circ_{v_r}(t,e)e$ is negative definite. Firstly, since ρ_1 is a polynomial that maps $\mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ and V_1 satisfies (11),

$$\frac{d}{dt}\left\{\rho_1(V_1)V_1\right\} \leq -\rho_1(V_1)\big[k_x e_x^2 + k_\theta e_\theta^2\big]. \tag{41}$$

Next, we use (33), as well as $|e| \geq |e_y|$ and $Q_\varphi > 0$, to obtain

$$\begin{aligned} \frac{d}{dt}\left\{\big[Q_{v_r^2} + Q_{\omega_r^2}\big]V_1\right\} \leq &-\frac{2}{T}\left[\int_t^{t+T} \big[\omega_r(s)^2 + v_r(s)^2\big]ds\right] V_1 \\ &+ [\omega_r^2 + v_r^2]\big[e_x^2 + \frac{1}{k_y}e_\theta^2 + e_y^2\big] \end{aligned} \tag{42}$$

Then, using (38) and (39), we obtain

$$\begin{aligned} -\frac{d}{dt}\left\{\omega_r e_x e_y\right\} = &-\dot{\omega}_r e_x e_y - \omega_r\big[-k_x e_x e_y + \omega_r e_y^2 + k_\theta e_\theta e_y^2 \\ &+ k_y v_r e_y^3 - \omega_r e_x^2 - k_\theta e_\theta e_x^2 - k_y v_r e_y e_x^2 + v_r e_\theta e_x\big]. \end{aligned} \tag{43}$$

Now, for the cross-terms we use the inequalities $2e_xe_y \le \epsilon e_x^2 + (1/\epsilon)e_y^2$ and $2e_\theta e_y^2 \le \epsilon V_1 e_\theta^2 + (1/\epsilon)e_y^2$, which hold for any $\epsilon > 0$, and we regroup some terms to obtain (see [19] for details)

$$-\frac{d}{dt}\{\omega_r e_x e_y\} \le \frac{\epsilon}{2} v_r^2 V_1 e_y^2 + \rho_5(V_1)e_x^2 + \rho_6(V_1)e_\theta^2 \\ + \frac{1}{2\epsilon}\big[\bar{\omega}_r^2 k_y^2 + (k_x + k_\theta)\bar{\omega}_r + \bar{\omega}_r k_y \bar{v}_r + \bar{\dot{\omega}}_r\big]e_y^2 - \omega_r^2 e_y^2 \tag{44}$$

where ρ_5 and ρ_6 are first-order polynomials of V_1 defined as

$$\begin{aligned} \rho_5(V_1) &= \frac{\bar{\omega}_r}{2}\Big[\big(\epsilon k_y \bar{v}_r + 2k_\theta\big)V_1 + \big(k_x + \frac{\bar{\dot{\omega}}_r}{\bar{\omega}_r}\big)\epsilon + 2\bar{\omega}_r + \bar{v}_r\Big] \\ \rho_6(V_1) &= \bar{\omega}_r\Big[k_\theta(\epsilon V_1 + 1) + \frac{\bar{v}_r}{2}\Big]. \end{aligned}$$

Next, we have

$$\frac{d}{dt}\{v_r \rho_2(V_1)e_\theta e_y\} = -\rho_2(V_1)v_r^2 e_y^2 - v_r\rho_2(V_1)\big[k_\theta e_\theta e_y + \omega_r e_x e_\theta \\ + k_\theta e_\theta^2 e_x + k_y v_r e_y e_x e_\theta + v_r e_\theta^2\big] + \rho_2(V_1)\dot{v}_r e_\theta e_y \\ - v_r \nabla\rho_2(V_1)e_\theta e_y\big[k_x e_x^2 + k_\theta e_\theta^2\big]. \tag{45}$$

Hence, using again the triangle inequality to bound the cross-terms and regrouping them, we obtain

$$\begin{aligned} \frac{d}{dt}\{v_r\rho_2 e_\theta e_y\} \;\; &\le \;\; -k_y v_r^2 \rho_2(V_1)e_y^2 + \rho_7(V_1)e_x^2 \\ &\qquad + \rho_8(V_1)e_\theta^2 + \frac{k_\theta \bar{v}_r + \bar{\dot{v}}_r}{2\epsilon}e_y^2 \end{aligned} \tag{46}$$

where ρ_7 and ρ_8 are second-order polynomials of V_1 satisfying

$$\begin{aligned} \rho_7(V_1) \;\; &\ge \;\; \rho_2\bar{v}_r\big[\frac{\bar{\omega}_r}{2} + (k_\theta + k_y\bar{v}_r)V_1\big] \\ &\qquad + \max\{k_y, 1\}k_x\bar{v}_r V_1\left|\frac{\partial\rho_2}{\partial V_1}\right| \\ \rho_8(V_1) \;\; &\ge \;\; \frac{\bar{v}_r\rho_2(V_1)}{2}\big[\bar{\omega}_r + k_\theta(\epsilon\rho_2(V_1) + 1) + (k_y + 2)\bar{v}_r\big] \\ &\qquad \bar{\dot{v}}_r\frac{\epsilon}{2}\rho_2(V_1)^2 + \bar{v}_r\left|\frac{\partial\rho_2}{\partial V_1}\right|\max\{k_y, 1\}k_\theta V_1. \end{aligned}$$

Now we put all the previous bounds together. Using (8) in (42), we obtain, in view of (44) and (46),

$$\frac{\partial V_2}{\partial t} + \frac{\partial V_2}{\partial e}A_{v_r}^{\circ}(t,e)e \le -\frac{2\mu}{T}V_1(e) - \Big[k_y\rho_2(V_1) - 1 - \frac{\epsilon}{2}V_1\Big]v_r^2 e_y^2 \\ + \frac{1}{2\epsilon}\Big[\bar{\omega}_r\big[\bar{\omega}_r k_y^2 + k_x + k_\theta + k_y\bar{v}_r\big] + \bar{\dot{\omega}}_r + k_\theta\bar{v}_r + \bar{\dot{v}}_r\Big]e_y^2 \\ - e_x^2\big[k_x\rho_1 - \rho_7 - \rho_5 - v_r^2 - \omega_r^2\big] \\ - e_\theta^2\Big[k_\theta\rho_1 - \rho_8 - \rho_6 - \frac{1}{k_y}\big(v_r^2 + \omega_r^2\big)\Big]. \tag{47}$$

Hence, defining

$$\begin{aligned} \epsilon \; &:= \; \frac{T}{\mu}\Big[\bar{\omega}_r\big[\bar{\omega}_r k_y^2 + k_x + k_\theta + k_y\bar{v}_r\big] + \bar{\dot{\omega}}_r + k_\theta\bar{v}_r + \bar{\dot{v}}_r\Big] \\ \rho_1(V_1) \; &:= \; 1 + \frac{1}{\min\{k_x, k_\theta\}}\Big[\rho_5 + \rho_6 + \rho_7 + \rho_8 \end{aligned}$$

$$+\big[1+\frac{1}{k_y}\big]\big[\omega_r^2+v_r^2\big]\Big].$$

$$\rho_2(V_1) := \frac{1}{k_y}\Big[1+\frac{\epsilon}{2}V_1\Big]$$

we obtain

$$\frac{\partial V_2}{\partial t}+\frac{\partial V_2}{\partial e}A^\circ_{v_r}(t,e)e \;\le\; -\frac{\mu}{T}V_1(e)-k_xe_x^2-k_\theta e_\theta^2. \tag{48}$$

That is, V_2 is a strong Lyapunov function for the nominal dynamics $\dot e = A^\circ_{v_r}(t,e)e$.

Using the latter, we evaluate the total derivative of V_3 along the trajectories of (37) that is, including the output injection term. Hence, we obtain

$$\dot V_3(t,e) \quad\le\quad \frac{\partial V_2}{\partial t}+\frac{\partial V_2}{\partial e}A^\circ_{v_r}(t,e)e+W(t,e) \tag{49}$$

$$W(t,e) \quad:=\quad -k_\theta\rho_4(V_1)e_\theta^2+v_r[\phi(e_\theta)-1]\frac{\partial V_2}{\partial e}B^\circ(e_y)e \tag{50}$$

for which we used (11), as well as the possitivity of $\rho_4(V_1)$ and $\nabla\rho_4$, to obtain

$$\begin{aligned}\frac{d}{dt}\{V_1\rho_4(V_1)\} \quad&=\quad \dot V_1\rho_4(V_1)+V_1\nabla\rho_4\dot V_1\\ &\le\quad -k_\theta\rho_4(V_1)e_\theta^2.\end{aligned}$$

We show that $W(t,e)$, defined in (50), is non-positive. To that end, note that

$$[\phi(e_\theta)-1]\le e_\theta^2 \tag{51}$$

and, in view of the structure of B°, we have

$$\frac{\partial V_1}{\partial e}B^\circ(e)e=0$$

hence,

$$\frac{\partial V_2}{\partial e}=v_r\rho_2(V_1)[e_y\ 0\ e_\theta]-\omega_r[0\ e_y\ e_x]$$

and, moreover,

$$\begin{aligned}\big|\,[e_y\ 0\ e_\theta]B^\circ(e)e\,\big| \quad&=\quad \big|-k_ye_y^2+e_\theta^2-k_ye_ye_xe_\theta\big|\\ &\le\quad \big|e_\theta^2-\frac{k_y}{2}e_y^2+\frac{k_y}{2}e_x^2e_\theta^2\big|\\ &\le\quad 2k_yV_1+2k_y^2V_1^2\\ \big|\,[0\ e_y\ e_x]B^\circ(e)e\,\big| \quad&=\quad \big|k_ye_y^3+e_\theta e_x-k_ye_ye_x^2\big|\\ &\le\quad 2k_yV_1^2+\max\{k_y,1\}V_1.\end{aligned}$$

Thus, $W(t,e)\le 0$ if

$$\rho_4(V_1)\ge\frac{2\bar v_r\max\{k_y,1\}}{k_\theta}\Big[\,\big[k_y\rho_2\bar v_r+\bar\omega_r\big]V_1^2+\big[\rho_2\bar v_r+\bar\omega_r\big]V_1\,\Big]$$

and (36) follows from (48) and (49). ■

It is worth stressing that, based on the previous computations, one can also establish that each subsystem in (26) is integral-input-to-state stable.

Proposition 4: Consider the system (25) with k_x, k_y, and k_θ arbitrary positive gains; assume, moreover, that the references satisfy (8) and (12). If, in addition, the solutions exist over $[t_\circ,\infty)$, the system is integral input to state stable with respect to the "input" $[\xi_i^\top,\ \eta_i^\top]^\top$. □

Proof: Consider the function $W : \mathbb{R}_{\geq 0} \times \mathbb{R}^3 \to \mathbb{R}_{\geq 0}$ defined by

$$W(t, e_i) := \ln \left[1 + V(t, e_i)\right] \tag{52}$$

where $V : \mathbb{R}_{\geq 0} \to \mathbb{R}^3 \to \mathbb{R}_{\geq 0}$ is the continuously differentiable function defined in (13).

The total derivative of W along the trajectories of (25) yields

$$\dot{W}(t, e_i) \leq \frac{\dot{V}(t, e_i)}{1 + V(t, e_i)}$$

which, in virtue of (14) implies that

$$\dot{W}(t, e_i) \leq -\alpha(|e_i|) + \frac{\partial V}{\partial e_i} \frac{G(t, e_i, \xi_i)e_i + B(e_i)\eta_i}{1 + V(t, e_i)} \tag{53}$$

where

$$\alpha(|e_i|) = \frac{\mu}{T} \frac{V_1(e_i)}{1 + V(t, e_i)}.$$

To establish the statement of the proposition we show that the second term on the right hand side of (53) is bounded from above by $\gamma[\,|\xi_i| + |\eta_i|\,]$ with $\gamma > 0$. For the sake of argument, remark that $V(t, e_i) = \mathcal{V}(t, e_i, V_1)$ where

$$\mathcal{V}(t, e_i, V_1) := P_{[3]}(t, V_1)V_1 - \omega_r(t)e_{x_i}e_{y_i} + v_r(t)P_{[1]}(t, V_1)e_{\theta_i}e_{y_i}$$

and, in addition, note that there exists a fourth-order polynomial $\tilde{P}_4(V_1)$ such that

$$\mathcal{V}(t, e_i, V_1) \geq \tilde{P}_4(V_1), \qquad \forall\, (t, e_i, V_1) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^3 \times \mathbb{R}_{\geq 0}. \tag{54}$$

Furthermore,

$$\frac{\partial V}{\partial e_i} = \frac{\partial \mathcal{V}}{\partial V_1} \frac{\partial V_1}{\partial e_i} + \frac{\partial \mathcal{V}}{\partial e_i}$$

and, due to the structure of G, we have

$$\frac{\partial V_1}{\partial e_i} G(t, e_i, \xi_i)e_i = 0. \tag{55}$$

Therefore

$$\begin{aligned} \frac{\partial V}{\partial e_i}\left[G(t, e_i, \xi_i)e_i + B(e_i)\eta_i\right] = \; & \frac{\partial \mathcal{V}}{\partial V_1} \frac{\partial V_1}{\partial e_i} B(e_i)\eta_i \\ & + \frac{\partial \mathcal{V}}{\partial e_i}\left[G(t, e_i, \xi_i)e_i + B(e_i)\eta_i\right] \end{aligned}$$

Now, since $P_{[3]}$ is a polynomial of 3rd order, we have

$$\frac{\partial \mathcal{V}}{\partial V_1} = P'_{[3]}(V_1) + v_r(t) \frac{\partial P_{[1]}}{\partial V_1} e_{\theta_i}e_{y_i}$$

where $P'_{[3]} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is the polynomial function of 3rd order defined as

$$P'_{[3]}(V_1) := \frac{\partial P_{[3]}}{\partial V_1} V_1 + P_{[3]}(V_1).$$

Then, since $P_{[1]}$ is a polynomial of 1st order and $e_{\theta_i}e_{y_i} \leq V_1(e_i)$, there exists $c > 0$ such that

$$\left|\frac{\partial \mathcal{V}}{\partial V_1}\right| \leq P'_{[3]}(V_1) + c\bar{v}_r V_1.$$

Furthermore, $B(e_i)$ is linear in e_i therefore, there exists $c > 0$ such that

$$\left|\frac{\partial V_1}{\partial e_i} B(e_i)\eta_i\right| \le cV_1|\eta_i|$$

and, on the other hand,

$$\frac{\partial \mathcal{V}}{\partial e_i}^\top = \begin{bmatrix} v_r(t)P_{[1]}(t,V_1)e_{y_i} \\ -\omega_r(t)e_{y_i} \\ v_r(t)P_{[1]}(t,V_1)e_{\theta_i} - \omega_r(t)e_{x_i} \end{bmatrix} \tag{56}$$

while $|G(t,e_i,\xi_i)e_i| = \mathcal{O}(|\xi_i|)P_1'(V_1)$ where P_1' is a polynomial of first order.

Putting all these bounds together, we conclude that there exists a polynomial of fourth order $P_4'(V_1)$ such that

$$\left|\frac{\partial V}{\partial e_i}\big[G(t,e_i,\xi_i)e_i + B(e_i)\eta_i\big]\right| \le P_4'(V_1)|\xi_i \ \eta_i|.$$

and, therefore,

$$\dot{W}(t,e_i) \le -\alpha(|e_i|) + c|\xi_i \ \eta_i|$$

where

$$c := \limsup_{V_1\ge 0} \frac{P_4'(V_1)}{1+\tilde{P}_4(V_1)}$$

and the claim follows. ■

B. *Proof of Proposition 3*

The proof follows along the arguments developed below (27). For $i = 1$ the closed-loop dynamics, composed of (26c) and

$$\dot{\eta}_1 = F_{1_{cl}}(t,\tilde{v}_1,\tilde{\omega}_1,e_1(t)), \tag{57}$$

is defined on the interval of existence of $e_1(t)$, denoted $[t_\circ, t_{\max})$, and has a cascaded form. By assumption, η_1 satisfies the bound (28) for all $t \in [t_\circ, t_{\max})$ hence, on this interval,

$$\begin{aligned} \dot{V}_1(e_1(t)) &\le \frac{\partial V_1}{\partial e_1}(e_1(t))B(e_1(t))\eta_1(t) \\ &\le cV_1(e_1(t))\eta_1(t_{\max}) \le c'V_1(e_1(t)) \end{aligned} \tag{58}$$

where c is a positive number of innocuous value and $c' > |\eta_1(t_{\max})|$; both are independent of the initial time. Integrating on both sides of the latter from $t_\circ$ to $t_{\max}$ we see that, by continuity of the solutions on the initial conditions, this interval of integration may be stretched to infinity. By the definition of $V_1(e_1)$ we obtain that $e_1(t)$ exists on $[t_\circ,\infty)$. Moreover, since by definition $\Delta v_0 = \Delta\omega_0 = 0$, we conclude from (23) and (24), that v_1^* and ω_1^* exist along trajectories on $[t_\circ,\infty)$. It follows that the same property holds for $v_1(t)$ and $\omega_1(t)$ and, consequently, for $\xi_2(t)$ —recall that

$$\xi_2 := \begin{bmatrix} v_1 - v_r \\ \omega_1 - \omega_r \end{bmatrix}.$$

From forward completeness and condition A1 it follows, in turn, that $\eta_1 = 0$ is uniformly globally asymptotically stable for (57).

Now we can apply a cascades argument for the system (26c), (57). Since B in (26c) is linear in e_1 and the origin of $\dot{e}_1 = A_{v_r}(t,e_1)$ is uniformly globally asymptotically stable, the same property holds for the origin $(e_1, \eta_1) = (0, 0)$ —see [18, Theorem 2]. This means that there exists a class $\mathcal{KL}$ function β such that

$$|\zeta_1(t, t_\circ, \zeta_{1\circ})| \leq \beta(|\zeta_{1\circ}|, t - t_\circ) \quad \forall\, t \geq t_\circ \tag{59}$$

where we recall that $\zeta_i = [e_i^\top \ \eta_i^\top]$ for all $i \leq n$. In particular, $e_1(t)$, $\eta_1(t)$ and, consequently, $\xi_2(t)$, are uniformly globally bounded. To see this more clearly, we recall that, by definition, ξ_2 is a continuous function of the state ζ_1 and time and equals to zero if $\zeta_1 = 0$. Indeed, $\xi_2 = \psi(t, \zeta_1)$ where

$$\begin{aligned} \psi_1(t, \zeta_1) &= \begin{bmatrix} \tilde{v}_1 + v_1^* - v_r \\ \tilde{\omega}_1 + \omega_1^* - \omega_r \end{bmatrix} && (60) \\ &= \begin{bmatrix} \tilde{v}_1 + v_r(t)[\cos(e_{\theta 1}) - 1] + k_{x1} e_{x1} \\ \tilde{\omega}_1 + k_{\theta 1} e_{\theta_1} + v_r(t) k_{y1} e_{y1} \phi(e_{\theta 1}) \end{bmatrix} && (61) \end{aligned}$$

Next, let $i = 2$ and consider the closed-loop equations:

$$\begin{aligned} \dot{e}_2 &= A_{v_r}(t, e_2) e_2 + G(t, e_2, \psi_1(t, \zeta_1)) e_2 + B(e_2) \eta_2 && (62a) \\ \dot{\zeta}_1 &= F_{\zeta_1}(t, \zeta_1) && (62b) \\ \dot{\eta}_2 &= F_{2_{cl}}(t, \tilde{v}_2, \tilde{\omega}_2, e_2(t)) && (62c) \end{aligned}$$

Note that we replaced e_2 with $e_2(t)$ in (27) to obtain the "decoupled" dynamics equation (62c). Then, η_2 is regarded as a perturbation to the system

$$\begin{aligned} \dot{e}_2 &= A_{v_r}(t, e_2) e_2 + G(t, e_2, \psi_1(t, \zeta_1)) e_2 && (63a) \\ \dot{\zeta}_1 &= F_{\zeta_1}(t, \zeta_1). && (63b) \end{aligned}$$

which, in turn, is also in cascaded form. Now, in view of the structure of G, we have

$$\frac{\partial V_1}{\partial e_i} G(t, e_i, \xi_i) e_i = 0, \qquad \forall\, i \leq n \tag{64}$$

hence, the total derivative of V_1 along the trajectories of (62a) yields

$$\dot{V}_1(e_2(t)) \leq c V_1(e_2(t)) \eta_2(t_{\max}) \leq c' V_1(e_2(t))$$

with an appropriate redefinition of c and c' –*cf.* Ineq. (58). Completeness of $e_2(t)$, and therefore of $\eta_2(t)$, follows using similar arguments as for the case when $i = 1$. Consequently, by Assumption A1, the origin of (62c) is uniformly globally asymptotically stable.

To analyze the stability of the origin for (62) we invoke again [18, Theorem 2]. To that end, we only need to establish uniform global asymptotic stability for the system (63) (since B is linear and the origin of (62c) is uniformly globally asymptotically stable). For this, we invoke [20, Theorem 4] as follows: first, we remark that the respective origins of $\dot{e}_2 = A_{v_r}(t, e_2)$ and (63b) are uniformly globally asymptotically stable. Second, note that condition A4 in [20, Theorem 4] is not needed here since we already established uniform forward completeness. Finally, [20, Ineq. (24)] holds trivially with $V = V_1$, in view of (64). We conclude that $(e_2, \zeta_1, \eta_2) = (0, 0, 0)$ is a uniformly globally asymptotically stable equilibrium of (62).

For $i = 3$ the closed-loop dynamics is

$$\begin{aligned} \dot{e}_3 &= A_{v_r}(t, e_3) e_3 + G(t, e_3, \psi_2(t, \zeta_{12}) e_3 + B(e_3) \eta_3 && (65a) \\ \dot{\zeta}_{12} &=: F_{\zeta_{12}}(t, \zeta_{12}) && (65b) \\ \dot{\eta}_3 &= F_{3_{cl}}(t, \tilde{v}_3, \tilde{\omega}_3, e_3(t)) && (65c) \end{aligned}$$

where $\zeta_{12} := [\zeta_1^\top \ \zeta_2^\top]^\top$, $\zeta_2 := [e_2^\top \ \eta_2^\top]$, and

$$\psi_2(t, \zeta_{12}) \quad := \quad \begin{bmatrix} \tilde{v}_2 + [\xi_{21} + v_r(t)][\cos(e_{\theta 1}) + k_{x1}e_{x1} - v_r \\ \tilde{\omega}_2 + \xi_{22} + k_{\theta 1}e_{\theta_1} + v_r(t)k_{y1}e_{y1}\phi(e_{\theta 1}) \end{bmatrix}$$

which corresponds to ξ_3 —*cf.* (61). The previous arguments, as for the case $i = 2$, apply now to (65) so the result follows by induction.

IV. Conclusions

We presented a formation-tracking controller for autonomous vehicles that ensures uniform global asymptotic stability of the closed-loop system, under the sole assumption that either the angular or the forward reference velocity is persistently exciting. Moreover, a strict Lyapunov function is provided for the kinematic error dynamics. Because we decouple the tracking problems at kinematic and dynamic levels, our results apply to a range of controllers at the dynamic level. Thus, one can use a variety of control schemes for Lagrangian and Hamiltonian systems, including adaptive and output feedback control designs. Further research in such directions is being carried out.

References

[1] Y. Q. Chen and Z. Wang, “Formation control: a review and a new consideration,” in *2005 IEEE/RSJ International Conference on Intelligent Robots and Systems*, pp. 3181–3186, Aug 2005.

[2] C. C. de Wit, H. Khennouf, C. Samson, and O. J. Sørdalen, ”Nonlinear control design for mobile robots”, vol. 11 of *Robotics and Automated Systems*, ch. Recent Trends in Mobile Robots. Y. F. Zheng, ed., London: World Scientific, 1993.

[3] J. Huang, C. Wen, W. Wang, and Z.-P. Jiang, “Adaptive output feedback tracking control of a nonholonomic mobile robot,” *Automatica*, vol. 50, no. 3, pp. 821–831, 2014.

[4] E. Vos, A. J. van der Schaft, and J. M. A. Scherpen, “Formation control and velocity tracking for a group of nonholonomic wheeled robots,” *IEEE Transactions on Automatic Control*, vol. 61, pp. 2702–2707, Sept 2016.

[5] A. Dong and J. A. Farrell, “Cooperative control of multiple nonholonomic mobile agents,” *IEEE Trans. on Automat. Contr.*, vol. 53, no. 6, pp. 1434–1447, 2008.

[6] T. Dierks and S. Jagannathan, “Control of nonholonomic mobile robot formations: Backstepping kinematics into dynamics,” in *2007 IEEE International Conference on Control Applications*, pp. 94–99, Oct 2007.

[7] K. D. Do, Z.-P. Jiang, and J. Pan, “A global output-feedback controller for simultaneous tracking and stabilization of unicycle-type mobile robots,” *IEEE Trans. on Robotics Automat.*, vol. 20, no. 3, pp. 589–594, 2004.

[8] K. D. Do and J. Pan, “Nonlinear formation control of unicycle-type mobile robots,” *J. Rob. and Aut. Syst.*, vol. 55, no. 3, pp. 191–204, 2007.

[9] X. Liang, Y. H. Liu, H. Wang, W. Chen, K. Xing, and T. Liu, “Leader-following formation tracking control of mobile robots without direct position measurements,” *IEEE Transactions on Automatic Control*, vol. PP, no. 99, pp. 1–1, 2016.

[10] K. D. Do, “Formation tracking control of unicycle-type mobile robots,” in *Proceedings 2007 IEEE International Conference on Robotics and Automation*, pp. 2391–2396, April 2007.

[11] Z. Lin, B. Francis, and M. Maggiore, “Necessary and sufficient graphical conditions for formation control of unicycles,” *IEEE Trans. on Automat. Contr.*, vol. 50, no. 1, pp. 121–127, 2005.

[12] D. Sun, C. Wang, W. Shang, and G. Feng, “A synchronization approach to trajectory tracking of multiple mobile robots while maintaining time-varying formations,” *IEEE Transactions on Robotics*, vol. 25, pp. 1074–1086, Oct 2009.

[13] A. Loría, J. Dasdemir, and N. Alvarez-Jarquin, “Leader-follower formation control of mobile robots on straight paths,” *IEEE Trans. on Contr. Syst. Techn.*, vol. 24, no. 2, pp. 727–732, 2016.

[14] M. Maghenem, A. Loría, and E. Panteley, “Lyapunov-based formation-tracking control of nonholonomic systems under persistency of excitation,” in *IFAC NOLCOS 2016*, (Monterey, CA, USA), 2016. To appear.

[15] E. Panteley, A. Loría, and A. Teel, “Relaxed persistency of excitation for uniform asymptotic stability,” *IEEE Trans. on Automat. Contr.*, vol. 46, no. 12, pp. 1874–1886, 2001.

[16] M. Malisoff and F. Mazenc, *Constructions of Strict Lyapunov functions*. London: Springer Verlag, 2009.

[17] A. Loría, “From feedback to cascade-interconnected systems: Breaking the loop,” in *Proc. 47th. IEEE Conf. Decision Contr.*, (Cancun, Mex.), pp. 4109–4114, 2008.

[18] E. Panteley and A. Loría, "On global uniform asymptotic stability of non linear time-varying non autonomous systems in cascade," *Syst. & Contr. Letters*, vol. 33, no. 2, pp. 131–138, 1998.
[19] M. Maghenem, A. Loría, and E. Panteley, "iISS formation tracking control of autonomous vehicles," tech. rep., CentraleSupelec, 2016.
[20] E. Panteley and A. Loría, "Growth rate conditions for stability of cascaded time-varying systems," *Automatica*, vol. 37, no. 3, pp. 453–460, 2001.

APPENDIX

We provide below some details on the computation of (43) and (45):

$$\begin{aligned}
-\frac{d}{dt}\left\{\omega_r e_x e_y\right\} \;\le\;& -\omega_r\Big[-k_x e_x e_y + \omega_r e_y^2 + k_\theta e_\theta e_y^2 + k_y v_r e_y^3\Big] \\
& -\omega_r\Big[-\omega_r e_x^2 - k_\theta e_\theta e_x^2 - k_y v_r e_y e_x^2 + v_r e_\theta e_x\Big] - \dot{\omega}_r e_x e_y \\
\le\;& -\omega_r^2 e_y^2 + \bar{\omega}_r k_x\left(\frac{\epsilon}{2}e_x^2 + \frac{1}{2\epsilon}e_y^2\right) + \bar{\omega}_r k_\theta\left(\epsilon V_1 e_\theta^2 + \frac{1}{2\epsilon}e_y^2\right) \\
& + \bar{\omega}_r\bar{v}_r k_y e_y^3 + \bar{\omega}_r^2 e_x^2 + \bar{\omega}_r\frac{k_\theta}{2}\left(e_\theta^2 + 2V_1 e_x^2\right) \\
& + \bar{\omega}_r k_y \bar{v}_r\left(\frac{1}{2\epsilon}e_y^2 + \frac{\epsilon}{2}V_1 e_x^2\right) + \frac{\bar{\omega}_r\bar{v}_r}{2}\left(e_\theta^2 + e_x^2\right) + \bar{\dot{\omega}}_r\left(\frac{\epsilon}{2}e_x^2 + \frac{1}{2\epsilon}e_y^2\right) \\
\le\;& -\omega_r^2 e_y^2 + \bar{\omega}_r\bar{v}_r k_y e_y^3 + \frac{1}{2\epsilon}\Big[(k_x + k_\theta)\bar{\omega}_r + \bar{\omega}_r k_y \bar{v}_r + \bar{\dot{\omega}}_r\Big]e_y^2 \\
& + \Big[\bar{\omega}_r k_x\frac{\epsilon}{2} + \bar{\omega}_r^2 + \bar{\omega}_r k_y\bar{v}_r\frac{\epsilon}{2}V_1 + \bar{\dot{\omega}}_r\frac{\epsilon}{2} + \frac{\bar{\omega}_r\bar{v}_r}{2} + \bar{\omega}_r k_\theta V_1\Big]e_x^2 \\
& + \Big[\bar{\omega}_r k_\theta\epsilon V_1 + \bar{\omega}_r k_\theta + \frac{\bar{\omega}_r\bar{v}_r}{2}\Big]e_\theta^2 \\
\le\;& -\omega_r^2 e_y^2 + \frac{\epsilon}{2}v_r^2 V_1 e_y^2 + \rho_5(V_1)e_x^2 + \rho_6(V_1)e_\theta^2 \\
& + \frac{1}{2\epsilon}\Big[\bar{\omega}_r^2 k_y^2 + (k_x + k_\theta)\bar{\omega}_r + \bar{\omega}_r k_y\bar{v}_r + \bar{\dot{\omega}}_r\Big]e_y^2
\end{aligned} \tag{66}$$

$$\begin{aligned}
\frac{d}{dt}\left\{v_r\rho_2(V_1)e_\theta e_y\right\} \;\le\;& -k_y v_r^2\rho_2 e_y^2 - \rho_2 k_\theta v_r e_\theta e_y - \rho_2 v_r\big[\omega_r e_x e_\theta + k_\theta e_\theta^2 e_x + k_y v_r e_y e_x e_\theta\big] \\
& + \rho_2 v_r^2 e_\theta^2 + \rho_2\dot{v}_r e_\theta e_y - v_r\frac{\partial\rho_2(V_1)}{\partial V_1}e_\theta e_y\left(k_x e_x^2 + k_\theta e_\theta^2\right) \\
\le\;& -k_y v_r^2\rho_2 e_y^2 + k_\theta\bar{v}_r\left(\frac{\epsilon}{2}\rho_2^2 e_\theta^2 + \frac{1}{2\epsilon}e_y^2\right) + \rho_2\bar{v}_r\frac{\bar{\omega}_r}{2}\left(e_x^2 + e_\theta^2\right) \\
& + \rho_2\bar{v}_r\frac{k_\theta}{2}\left(e_\theta^2 + V_1 e_x^2\right) + \rho_2 k_y\bar{v}_r^2\left(V_1 e_x^2 + \frac{e_\theta^2}{2}\right) + \bar{v}_r^2\rho_2 e_\theta^2 + \\
& \bar{\dot{v}}_r\left(\frac{\epsilon}{2}\rho_2^2 e_\theta^2 + \frac{1}{2\epsilon}e_y^2\right) + \bar{v}_r\left|\frac{\partial\rho_2(V_1)}{\partial V_1}\right|\max\{k_y, 1\}V_1\left(k_x e_x^2 + k_\theta e_\theta^2\right) \\
\le\;& -k_y v_r^2\rho_2 e_y^2 + \frac{1}{2\epsilon}\left(k_\theta\bar{v}_r + \bar{\dot{v}}_r\right)e_y^2 + \Bigg[\rho_2\bar{v}_r\frac{\bar{\omega}_r}{2} + k_\theta V_1\rho_2\bar{v}_r \\
& + \rho_2 k_y\bar{v}_r^2 V_1 + \bar{v}_r\left|\frac{\partial\rho_2(V_1)}{\partial V_1}\right|\max\{k_y, 1\}V_1 k_x\Bigg]e_x^2 \\
& + \Bigg[k_\theta\bar{v}_r\frac{\epsilon}{2}\rho_2^2 + \rho_2\bar{v}_r\frac{\bar{\omega}_r}{2} + \rho_2\bar{v}_r\frac{k_\theta}{2} + \rho_2\frac{k_y}{2}\bar{v}_r^2 + \bar{v}_r^2\rho_2
\end{aligned}$$

$$
\begin{aligned}
&+\bar{\dot{v}}_r \frac{\epsilon}{2}\rho_2^2 + \bar{v}_r \left|\frac{\partial \rho_2(V_1)}{\partial V_1}\right| \max\{k_y, 1\} V_1 k_\theta \Bigg] e_\theta^2 \\
\leq\ & -k_y v_r^2 \rho_2 e_y^2 + \rho_7(V_1) e_x^2 + \rho_8(V_1) e_\theta^2 + \frac{1}{2\epsilon}\left(k_\theta \bar{v}_r + \bar{\dot{v}}_r\right) e_y^2 \qquad (67)
\end{aligned}
$$