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# MODERATE DEVIATIONS FOR BIPOWER VARIATION OF GENERAL FUNCTION AND HAYASHI-YOSHIDA ESTIMATORS

HACÈNE DJELLOUT, HUI JIANG, ARNAUD GULLIN, AND YACOUBA SAMOURA

ABSTRACT. We consider the moderate deviations behaviors for two (co-) volatility estimators: generalised bipower variation, Hayashi-Yoshida estimator. The results are obtained by using a new result about the moderate deviations principle for  $m$ -dependent random variables based on the Chen-Ledoux type condition.

*AMS 2000 subject classifications: 60F10, 62J05, 60J05.*

## 1. MOTIVATION AND CONTEXT

In the last decade there has been a considerable development of the asymptotic theory for processes observed at a high frequency. This was mainly motivated by financial applications, where the data, such as stock prices or currencies, are observed very frequently. As under the no-arbitrage assumptions price processes must follow a semimartingale, there was a need for probabilistic tools for functionals of semimartingales based on high frequency observations. Inspired by potential applications, probabilists started to develop limit theorems for semimartingales. Statisticians applied the asymptotic theory to analyze the path properties of discretely observed semimartingales: for the estimation of certain volatility functionals and realised jumps, or for performing various test procedures.

We consider  $X_t = (X_{1,t}, X_{2,t})_{t \in [0, T]}$  a 2-dimensional semimartingale, defined on the filtered probability space  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{[0, T]}, \mathbb{P})$ , of the form

$$\begin{cases} dX_{1,t} = b_1(t, X_t)dt + \sigma_{1,t}dW_{1,t} \\ dX_{2,t} = b_2(t, X_t)dt + \sigma_{2,t}dW_{2,t} \end{cases} \quad (1.1)$$

where  $W_1 = (W_{1,t})_{t \in [0, T]}$  and  $W_2 = (W_{2,t})_{t \in [0, T]}$  are two correlated Wiener processes with  $\rho_t = \text{Cov}(W_{1,t}, W_{2,t})$ ,  $t \in [0, T]$ . Moreover,  $\rho_t \in [0, 1]$  and  $\sigma_{\ell,t}$ ,  $\ell = 1, 2$  are both unknown deterministic and measurable functions of  $t$ ,  $b_\ell(\cdot, \cdot)$ ,  $\ell = 1, 2$  are progressively measurable (possibly unknown) functions.

Models of the type (1.1) and their extensions are widely used in mathematical finance to capture the dynamics of stock prices or interest rates.

This work aims to provide analysis of moderate deviations principle for two estimators: generalised bipower estimator and Hayashi-Yoshida estimator. To do that we prove a new result about moderate deviations for  $m$ -dependent random variables using the Chen-Ledoux type condition. This condition links the speed of the moderate deviation principle to the queue of the distribution of the random variables. It is known that the Chen-Ledoux type condition is a necessary and sufficient condition for the independent and identically distributed random variables. Djellout [18] using a similar condition have obtained the

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moderate deviation for random variables. Djellout and Guillin [20] have also obtained the moderate deviation for Markov chain using this conditions. See also the work of Djellout Bitseki Penda and Proia [9] for the moderate deviation of the Durbin Watson statistics using this condition.

**Generalised Bipower estimator** Recently, the concept of realised bipower variation has built a non-parametric framework for backing out several variational measures of volatility, which has led to a new development in econometrics. Given the observations of the processes  $(X_{\ell,t_i})_{i=0,\dots,n}$  with  $0 = t_0 < t_1 < \dots < t_n = T, n \in \mathbb{N}$ , the realised Bipower variation which is given by

$$V_{\ell,1}^n(r, q) = n^{\frac{r+q}{2}-1} \sum_{i=1}^{n-1} |\Delta X_{\ell}(I^i)|^r |\Delta X_{\ell}(I^{i+1})|^q, \quad (1.2)$$

where  $I^i = (t_{i-1}, t_i]$   $\Delta X_{\ell}(I^i) = X_{\ell,t_i} - X_{\ell,t_{i-1}}$ , provides a whole variety of estimators for different (integrated) powers of volatility. For practical convenience, it is standard to take equal spacing, i.e.,  $t_i - t_{i-1} = \frac{T}{n} := h$ . An important special case of the class (1.2) is the realised volatility

$$\sum_{i=1}^{[nt]} |\Delta X_{\ell}(I^i)|^2, \quad (1.3)$$

which is a consistent estimator of the quadratic variation of  $X_{\ell}$ , i.e.

$$IV_t = \int_0^t \sigma_{\ell,s}^2 ds,$$

which is often referred to as integrated volatility in the econometric literature.

In the last years, Nielsen et al.([5],[7],[6]) showed the consistency of  $V_{\ell,1}^n(r, q)$  and introduced the (stable) Central Limit Theorem (CLT, in short) for standardised version. Vetter [39] extend the results from Jacod [31] to the case of bipower variation and he prove the CLT of the bipower variation for continuous semimartingales. In [4] they consider the same problems as here where  $X$  have jumps and the CLT has been solved. It was demonstrated that realized bipower variation can estimate integrated power volatility in stochastic volatility models and moreover, under some conditions, it can be a good measure to integrated variance in the presence of jumps.

S. Kinnebrock and M. Podolskij [33] extended the CLT to the bipower variation of general functions (called generalised bipower variation):

$$V_{\ell,1}^n(g, h) = \frac{1}{n} \sum_{i=1}^n g(\sqrt{n}\Delta X_{\ell}(I^i)) h(\sqrt{n}\Delta X_{\ell}(I^{i+1})), \quad (1.4)$$

where  $g, h$  are two maps on  $\mathbb{R}^d$ , taking values in  $\mathbb{R} \times \mathbb{R}$  and given some examples from the litterature to which his theory can be applied.

We know from [5] and [7] that if  $g$  and  $h$  are continuously differentiable with  $g, h, g'$  and  $h'$  being of at most polynomial growth

$$V_{\ell,1}^n(g, h) \xrightarrow{\mathbb{P}} V_{\ell,1}(g, h) := \int_0^1 \Sigma_{\sigma_{\ell,u}}(g) \Sigma_{\sigma_{\ell,u}}(h) du,$$

where

$$\Sigma_{\sigma}(f) := \mathbb{E}(f(\sigma Z)), \quad Z \sim \mathcal{N}(0, 1), \quad f \text{ is a real-valued function on } \mathbb{R}.$$

In addition, if  $h$  and  $g$  are even (i.e.  $g(x) = g(-x)$ ,  $h(x) = h(-x)$  for all  $x \in \mathbb{R}^d$ ), we have the CLT

$$\sqrt{n} (V_{\ell,1}^n(g, h) - V_{\ell,1}(g, h)) \rightarrow \mathcal{N}(0, \Sigma_\ell(g, h))$$

where

$$\Sigma_\ell(g, h) = \int_0^1 \left( \Sigma_{\sigma_{\ell,s}}(g^2) \Sigma_{\sigma_{\ell,s}}(h^2) + 2 \Sigma_{\sigma_{\ell,s}}(g) \Sigma_{\sigma_{\ell,s}}(h) \Sigma_{\sigma_{\ell,s}}(gh) - 3 \Sigma_{\sigma_{\ell,s}}^2(g) \Sigma_{\sigma_{\ell,s}}^2(h) \right) ds.$$

All the elements of  $f$  on  $\mathbb{R}^d$  are continuous with at most polynomial growth. This amounts to there being suitable constants  $C > 0$  and  $p \geq 2$  such that

$$\forall x \in \mathbb{R}^d, \|f(x)\| \leq C(1 + \|x\|^p). \quad (1.5)$$

**Hayashi-Yoshida estimator** We concentrate on the estimation of the co-volatility of  $X_1$  and  $X_2$

$$\langle X_1, X_2 \rangle_T = \int_0^T \sigma_{1,t} \sigma_{2,t} \rho_t dt.$$

Given the synchronous observations of the processes  $(X_{1,t_i}, X_{2,t_i})_{i=0,\dots,n}$  a popular statistic to estimate the co-volatility is

$$CV_n := \sum_{i=1}^n \Delta X_1(I^i) \Delta X_2(I^i),$$

which is often called the realized co-volatility estimator ([2]).

The asymptotic distribution of  $CV_n$  was formulated by Barndorff et al. ([5]). Djellout and Samoura ([23]), Djellout et al. ([21]) obtained the large and moderate deviations for the realized covolatility  $CV_n$ . For more references, one can see ([22],[30],[32],[37]) and the references therein.

However, in financial applications, actual transaction data are recorded at irregular times in a nonsynchronous manner, i.e. two transaction prices are usually not observed at the same time. This fact requires one who adopt  $CV_n$  to synchronize the original data a priori, choose a common interval length  $h$  first, then impute missing observations by some interpolation scheme such as previous-tick interpolation of linear interpolation. Unfortunately, those procedures may result in synchronization bias ([28]).

Recently, Hayashi and Yoshida ([28]) postulated a new estimator which is free of synchronization and hence of any bias due to it. To be explicitly, for the nonsynchronous observations  $(X_{1,t_i}, X_{2,s_j})_{i=0,\dots,n; j=0,\dots,m}$  with  $0 = s_0 < s_1 < \dots < s_m = T$ ,  $0 = t_0 < t_1 < \dots < t_n = T$ ,  $m, n \in \mathbb{N}$ , the Hayashi-Yoshida estimator is defined as

$$U_{n,m} := \sum_{i=1}^n \sum_{j=1}^m \Delta X_1(I^i) \Delta X_2(J^j) I_{\{I^i \cap J^j \neq \emptyset\}}, \quad (1.6)$$

where  $I^i = (t_{i-1}, t_i]$ ,  $J^j = (s_{j-1}, s_j]$ . Under some assumptions on the equation (1.1), Hayashi and Yoshida ([28],[29]) showed the consistency and asymptotic normality of  $U_{n,m}$  respectively.

This article is structured as follows. In Section 2 we present the main theoretical results and we state the proofs in the Section 3.

## 2. MAIN RESULTS

**2.1. Moderate deviations for the Hayashi-Yoshida estimator.** Firstly, a reduced design with respect to  $(I^i)_{i=1,\dots,n}$  will be constructed in the following manner. We collect all  $I^i$ s such that  $I^i \subset J^j$  and combine them into a new interval; if such  $I^i$  does not exist, do nothing. Then collecting all such intervals and re-labeling them from left to right yields a new partition of  $(0, T]$ , denoted  $(\hat{I}^i)_{i=1,\dots,\hat{n}}$ . Due to the bilinearity of  $U_{n,m}$ ,

$$U_{n,m} = \sum_{i=1}^{\hat{n}} \sum_{j=1}^m \Delta X_1(\hat{I}^i) \Delta X_2(J^j) K_{ij}^{\hat{I}} = \sum_{i=1}^{\hat{n}} \Delta X_1(\hat{I}^i) \Delta X_2 \left( \bigcup_{j \in \hat{J}(i)} J^j \right), \quad (2.1)$$

where  $\hat{J}(i) := \{1 \leq j \leq m : K_{ij}^{\hat{I}} \neq 0\}$  with  $K_{ij}^{\hat{I}} = I_{\{\hat{I}^i \cap J^j \neq \emptyset\}}$ . Then, we can find that each  $J^j$  contains at most one  $\hat{I}^i$ , which implies that the random variable sequence

$$\left\{ 1 \leq i \leq \hat{n} : \Delta X_1(\hat{I}^i) \Delta X_2 \left( \bigcup_{j \in \hat{J}(i)} J^j \right) \right\}$$

are 2-dependent. Now, the number  $\hat{n}$  can be formulated as follows. Define

$$\tau_1 = \inf\{1 \leq i \leq n : I^i \not\subset J^1\}, \quad \varsigma_1 = \sup\{1 \leq j \leq m : I^1 \cap J^j \neq \emptyset\}$$

and

$$\tau_k = \inf\{\tau_{k-1} < i \leq n : I^i \not\subset J^{s_{k-1}}\}, \quad \varsigma_k = \sup\{\varsigma_{k-1} < j \leq m : I^{\tau_k} \cap J^j \neq \emptyset\},$$

with  $\tau_0 = 0$ ,  $\varsigma_0 = 1$  and  $\inf \emptyset = +\infty$ ,  $\sup \emptyset = 0$ . Let  $n_0 = \sup\{k : \tau_k < +\infty\}$ . Then, one can conclude that

$$n_0 \leq \hat{n} \leq n_0 + \sum_{k=1}^{n_0} I_{\{\tau_k - \tau_{k-1} > 1\}} + I_{\{\tau_{n_0} < n\}} \leq 2n_0 + 1. \quad (2.2)$$

Let  $A_{\ell,t} = \int_0^t b_\ell(s, \omega) ds$ ,  $t \in [0, T]$  and  $\ell = 1, 2$ . When the drift  $b_\ell(t, \omega)$  is known, we can consider the following estimator

$$\begin{aligned} V_{n,m} &:= \sum_{i=1}^n \sum_{j=1}^m \Delta X_1^0(I^i) \Delta X_2^0(J^j) I_{\{I^i \cap J^j \neq \emptyset\}} \\ &= \sum_{i=1}^{\hat{n}} \Delta X_1^0(\hat{I}^i) \Delta X_2^0 \left( \bigcup_{j \in \hat{J}(i)} J^j \right). \end{aligned} \quad (2.3)$$

where  $X_{\ell,t}^0 = X_{\ell,t} - A_{\ell,t}$ . For any Borel set  $I \subset [0, T]$ , define

$$\nu(I) = \int_I \sigma_{1,t} \sigma_{2,t} \rho_t dt, \quad \nu_\ell(I) = \int_I \sigma_{\ell,t}^2 dt, \quad \ell = 1, 2.$$

Let  $r_{n,m} := \max_{1 \leq i \leq n} |I^i| \vee \max_{1 \leq j \leq m} |J^j|$ , the largest interval size. We always assume that as  $n, m \rightarrow +\infty$ ,  $r_{n,m} \rightarrow 0$ , which implies that  $n_0 \rightarrow +\infty$ . For simplicity, let  $m = m(n)$  and as  $n \rightarrow +\infty$ ,  $m \rightarrow +\infty$ . Moreover, write

$$U_{n,m} = U_n, \quad V_{n,m} = V_n, \quad r_{n,m} = r_n.$$

To establish the moderate deviations for  $V_n$ , we introduce the following conditions.

(C1) There exist a sequence of positive numbers  $\{v_n, n \geq 1\}$  and some constant  $\Sigma \in (0, +\infty)$  such that, as  $n \rightarrow +\infty$ ,  $v_n \rightarrow 0$  and

$$v_n^{-1} \left( \sum_{i=1}^{\hat{n}} \sum_{j=1}^m \nu_1(\hat{I}^i) \nu_2(J^j) K_{ij}^{\hat{I}} + \sum_{i=1}^{\hat{n}} \nu^2(I^i) + \sum_{j=1}^m \nu^2(J^j) - \sum_{i=1}^{\hat{n}} \sum_{j=1}^m \nu^2(I^i \cap J^j) \right) \rightarrow \Sigma.$$

(C2) There exist a sequence of positive numbers  $\{b_n, n \geq 1\}$  such that, as  $n \rightarrow +\infty$ ,

$$b_n \rightarrow +\infty, \quad b_n v_n^{1/2} \rightarrow 0 \quad \text{and} \quad b_n v_n^{-1/2} \nu_n \rightarrow 0,$$

where  $\nu_n = \max_{\ell=1,2} \max_{1 \leq i \leq n} \max_{1 \leq j \leq m} (\nu_\ell(I^i) \vee \nu_\ell(J^j))$ .

**Theorem 2.1.** *Under conditions (C1) and (C2),*

$$\left\{ \frac{1}{b_n v_n^{1/2}} \left( V_n - \int_0^T \sigma_{1,t} \sigma_{2,t} \rho_t dt \right), n \geq 1 \right\}$$

satisfies the large deviations with speed  $b_n^2$  and rate function  $L(x) = \frac{x^2}{2\Sigma}$ .  $\square$

**Remark 2.1.** *By the construction of partition  $(\hat{I}^i)_{i=1, \dots, \hat{n}}$ , we obtain immediately that*

$$\begin{aligned} & \sum_{i=1}^{\hat{n}} \sum_{j=1}^m \nu_1(\hat{I}^i) \nu_2(J^j) K_{ij}^{\hat{I}} + \sum_{i=1}^{\hat{n}} \nu^2(\hat{I}^i) + \sum_{j=1}^m \nu^2(J^j) - \sum_{i=1}^{\hat{n}} \sum_{j=1}^m \nu^2(\hat{I}^i \cap J^j) \\ &= \sum_{i=1}^n \sum_{j=1}^m \nu_1(I^i) \nu_2(J^j) K_{ij}^I + \sum_{i=1}^n \nu^2(I^i) + \sum_{j=1}^m \nu^2(J^j) - \sum_{i=1}^n \sum_{j=1}^m \nu^2(I^i \cap J^j). \end{aligned}$$

**Remark 2.2.** *Similar to  $(\hat{I}^i)_{i=1, \dots, \hat{n}}$ , a reduced design with respect to  $(J^j)_{j=1, \dots, m}$  can also be constructed as follows. We collect all  $J^j$ s such that  $J^j \subset I^i$  and combine them into a new interval; if such  $J^j$  does not exist, do nothing. Then collecting all such intervals and re-labeling them from left to right yields a new partition of  $(0, T]$ , denoted  $(\hat{J}^j)_{j=1, \dots, \hat{n}}$ . Then  $V_n$  has the formula,*

$$V_n = \sum_{j=1}^{\hat{n}} \Delta X_1^0 \left( \cup_{i \in \hat{I}(j)} I^i \right) \Delta X_2^0(\hat{J}^j), \quad (2.4)$$

where  $\hat{I}(j) := \{1 \leq i \leq n : K_{ij}^{\hat{I}} \neq \emptyset\}$  with  $K_{ij}^{\hat{I}} = I_{\{I^i \cap \hat{J}^j \neq \emptyset\}}$ .

Letting  $\tilde{n}_0 = \sup\{k : \varsigma_k > 0\}$ , then  $\tilde{n}_0 = n_0 - 1$  and

$$n_0 \leq \tilde{n} \leq n_0 + \sum_{k=1}^{n_0} I_{\{\varsigma_k - \varsigma_{k-1} > 1\}} + 1 \leq 2n_0 + 1.$$

Therefore, we can conclude that the moderate deviations of  $V_n$  are independent of the different partitions  $(\hat{I}^i)_{i=1, \dots, \hat{n}}$  and  $(\hat{J}^j)_{j=1, \dots, \hat{n}}$  in (2.1) and (2.4).

Now we turn to the moderate deviatons for  $U_n$ . We need the following three additional conditions.

(C3) For  $\ell = 1, 2$ ,  $b_\ell(\cdot, \cdot) \in L^\infty(dt \otimes dP)$

(C4) As  $n \rightarrow +\infty$ ,  $\frac{r_n^2}{v_n} \rightarrow 0$ .

**Theorem 2.2.** *Under conditions (C1) through (C4),*

$$\left\{ \frac{1}{b_n v_n^{1/2}} \left( U_n - \int_0^T \sigma_{1,t} \sigma_{2,t} \rho_t dt \right), n \geq 1 \right\}$$

*satisfies the large deviations with speed  $b_n^2$  and rate function  $L(x) = \frac{x^2}{2\Sigma}$ .  $\square$*

**Remark 2.3.** *Suppose synchronous and equidistant sampling as in (3) of Remark 2.1. We have  $r_n = 1/n$ , and condition (C4) holds automatically.*

## 2.2. Examples.

**Perfectly synchronous sampling** Suppose synchronous and equidistant sampling, i.e.  $I^i \equiv J^i$  and  $|I^i| \equiv T/n$ . Then, we can see that

$$n_0 = n, \quad v_n = 1/n, \quad \Sigma = \int_0^T \sigma_{1,t}^2 \sigma_{2,t}^2 (1 + \rho_t^2) dt.$$

Moreover, if  $\sigma_{\ell,\cdot} \in L^\infty(dt)$  for  $\ell = 1, 2$ , condition (C2) equivalent to

$$b_n \rightarrow +\infty, \quad b_n n^{-1/2} \rightarrow 0, \quad n \rightarrow +\infty. \quad (2.5)$$

Under (2.5), from our Theorem 2.1, it follows that

$$\left\{ \frac{n^{1/2}}{b_n} \left( V_n - \int_0^T \sigma_{1,t} \sigma_{2,t} \rho_t dt \right), n \geq 1 \right\} \quad (2.6)$$

satisfies the large deviations with speed  $b_n^2$  and rate function

$$L_0(x) = \frac{x^2}{2 \int_0^T \sigma_{1,t}^2 \sigma_{2,t}^2 (1 + \rho_t^2) dt}.$$

This result can also be obtained by Theorem 2.5 in Djellout et al. ([21]).

**Nonsynchronous alternating sampling at odd/even times** We now consider the following deterministic, regularly spaced sampling scheme.  $X_1$  is sampled at 'odd' times, i.e.,  $t = \frac{2k-1}{n}T, k = 1, 2, \dots, n$  while  $X_2$  is at 'even' times,  $t = \frac{2k}{2n}T$ . Hence  $X_1$  and  $X_2$  are sampled in a nonsynchronous, alternating way. so we have that the sequence given in (2.6) satisfies the LDP with speed  $b_n^2$  and rate function with

$$\Sigma = \int_0^1 (\sigma_{1,t} \sigma_{2,t})^2 (2 + \frac{3}{2} \rho_t^2) dt.$$

## 2.3. Moderate deviations for bipower variation of general function.

**Proposition 2.3.** *Let  $(b_n)$  be a sequence of positive numbers such that*

$$b_n \rightarrow \infty, \quad b_n/\sqrt{n} \rightarrow 0, \quad \text{and} \quad \nu_n b_n \sqrt{n} \rightarrow 0, \quad (2.7)$$

*where  $\nu_n = \max_{1 \leq i \leq n} \int_{I^i} \Sigma_{\sigma_{\ell,s}}(g) \Sigma_{\sigma_{\ell,s}}(h) ds$ . Assume that  $g$  and  $h$  are two evens, continuous differentiable with at most polynomial growth such that*

$$\limsup_{n \rightarrow \infty} \frac{n}{b_n^2} \log n \max_{j=1}^n \mathbb{P} \left( \left| g(\sqrt{n} \Delta X_\ell(I^j)) h(\sqrt{n} \Delta X_\ell(I^{j+1})) \right. \right. \\ \left. \left. - \mathbb{E}(g(\sqrt{n} \Delta X_\ell(I^i))) \mathbb{E}(h(\sqrt{n} \Delta X_\ell(I^{i+1}))) \right| > n b_n \right) = -\infty. \quad (2.8)$$

So, we have that

$$\frac{\sqrt{n}}{b_n} (V_{\ell,1}^n(g, h) - V_{\ell,1}(g, h))$$

satisfies the large deviations principle with speed  $b_n^2$  and rate function  $I_{g,h}^\ell(x) = x^2/2\Sigma_\ell(g, h)$ .

#### 2.4. Examples.

**Realised Bipower Variation.** Realised bipower variation, which is probably the most important subclass of our model, corresponds to the functions  $g(x) = |x|^r$  and  $h(x) = |x|^q$ . In this case and under the condition that

$$\limsup_{n \rightarrow \infty} \frac{n}{b_n^2} \log n \max_{j=1}^n \mathbb{P} \left( n^{\frac{r+q}{2}} |\Delta X_\ell(I^j)|^r |\Delta X_\ell(I^{j+1})|^q > b_n \right) = -\infty. \quad (2.9)$$

we have the moderate deviation principle for

$$\frac{\sqrt{n}}{b_n} \left( V_{\ell,1}^n(r, q) - \mu_r \mu_q \int_0^1 |\sigma_{\ell,s}|^{r+q} ds \right)$$

$$\Sigma_\ell(g, h) := \Sigma_\ell(r, q) = (\mu_{2r} \mu_{2q} + 2\mu_r \mu_q \mu_{r+q} - 3\mu_r^2 \mu_q^2) \int_0^1 |\sigma_{\ell,u}|^{2(r+q)} du.$$

where  $\mu_r = \mathbb{E}|z|^r$  with  $z \sim \mathcal{N}(0, 1)$ .

If  $r = 2$  and  $q = 0$  we are in the quadratic case and the condition (2.9) is satisfied for all the sequence  $(b_n)$ . The result was already obtained in Djellout-Guillin-Wu [22].

If  $r + q < 2$  the condition (2.9) is satisfied for every sequence  $b_n$ .

**The Cubic Power of Returns.** The Cubic Power of Returns is a special case of the generalised bipower variation, corresponds to the functions  $g(x) = |x|^3$  and  $h(x) = 1$ .

Then, under the condition

$$\limsup_{n \rightarrow \infty} \frac{n}{b_n^2} \log n \max_{j=1}^n \mathbb{P} \left( n^{\frac{3}{2}} |\Delta X_\ell(I^j)|^3 > b_n \right) = -\infty,$$

the sequence  $\frac{\sqrt{n}}{b_n} (V_{\ell,1}^n(|x|^3, 1))$  satisfies the large deviations principle with speed  $b_n^2$  and rate function  $I_{|x|^3}(u) = 24x^2 / \int_0^1 \sigma_s^6 ds$ .

### 3. PROOFS

In this section, we will use the appendix's Proposition 4.1 about the the moderate deviations for  $m$ -dependent random variables sequences which are not necessarily stationary under the Chen-Ledoux type condition.

#### *Proof of Theorem 2.1*

We start this section by calculating the variance of  $V_n$ . Different from the method used in Hayashi and Yoshida ([29]), our approach relies on (2.1), the 2-dependent representation of  $U_n$ . Moreover, the analysis of moderate deviations will benefit from this straightforward thoughts.

**Lemma 3.1.** *For the estimator  $V_n$  defined by (2.3), we have*

$$\text{Var}(V_n) = \sum_{i=1}^{\hat{n}} \sum_{j=1}^m \nu_1(\hat{I}^i) \nu_2(J^j) K_{ij}^{\hat{I}} + \sum_{i=1}^{\hat{n}} \nu^2(\hat{I}^i) + \sum_{j=1}^m \nu^2(J^j) - \sum_{i=1}^{\hat{n}} \sum_{j=1}^m \nu^2(\hat{I}^i \cap J^j).$$



*Proof.* For  $\ell = 1, 2$ , letting

$$M_{\ell,t} = \int_0^t \sigma_{\ell,s} dW_{\ell,s}, \quad t \in [0, T],$$

then

$$V_n = \sum_{i=1}^{\hat{n}} \Delta M_1(\hat{I}^i) \Delta M_2 \left( \cup_{j \in \hat{J}(i)} J^j \right)$$

Since

$$\left\{ \Delta M_1(\hat{I}^i) \Delta M_2 \left( \cup_{j \in \hat{J}(i)} J^j \right) : 1 \leq i \leq \hat{n} \right\}$$

are 2-dependent, one can write that

$$\begin{aligned} \text{Var}(V_n) &= \sum_{i=1}^{\hat{n}} \text{Var} \left( \Delta M_1(\hat{I}^i) \Delta M_2 \left( \cup_{j \in \hat{J}(i)} J^j \right) \right) \\ &\quad + \sum_{i=1}^{\hat{n}} \text{Cov} \left( \Delta M_1(\hat{I}^i) \Delta M_2 \left( \cup_{j \in \hat{J}(i)} J^j \right), \Delta M_1(\hat{I}^{i+1}) \Delta M_2 \left( \cup_{j \in \hat{J}(i+1)} J^j \right) \right) \\ &\quad + \sum_{i=1}^{\hat{n}} \text{Cov} \left( \Delta M_1(\hat{I}^i) \Delta M_2 \left( \cup_{j \in \hat{J}(i)} J^j \right), \Delta M_1(\hat{I}^{i+2}) \Delta M_2 \left( \cup_{j \in \hat{J}(i+2)} J^j \right) \right) \\ &\quad + \sum_{i=1}^{\hat{n}} \text{Cov} \left( \Delta M_1(\hat{I}^i) \Delta M_2 \left( \cup_{j \in \hat{J}(i)} J^j \right), \Delta M_1(\hat{I}^{i-1}) \Delta M_2 \left( \cup_{j \in \hat{J}(i-1)} J^j \right) \right) \\ &\quad + \sum_{i=1}^{\hat{n}} \text{Cov} \left( \Delta M_1(\hat{I}^i) \Delta M_2 \left( \cup_{j \in \hat{J}(i)} J^j \right), \Delta M_1(\hat{I}^{i-2}) \Delta M_2 \left( \cup_{j \in \hat{J}(i-2)} J^j \right) \right) \\ &:= D_1 + D_2 + D_3 + D_4 + D_5, \end{aligned}$$

where  $\hat{I}^i = \emptyset$  for  $i < 1$  or  $i > \hat{n}$ .

We will only consider  $D_1$  and  $D_2$ , the other terms can be dealt with in the same way. A simple calculation gives us

$$\begin{aligned} D_1 &= \sum_{i=1}^{\hat{n}} \left( \nu_1(\hat{I}^i) \nu_2 \left( \cup_{j \in \hat{J}(i)} J^j \right) + \nu^2(\hat{I}^i) \right) \\ &= \sum_{i=1}^{\hat{n}} \sum_{j=1}^m \nu_1(\hat{I}^i) \nu_2(J^j) K_{ij}^{\hat{I}} + \sum_{i=1}^{\hat{n}} \nu^2(\hat{I}^i). \end{aligned} \tag{3.1}$$

Moreover, from the fact

$$\begin{aligned} \Delta M_2 \left( \cup_{j \in \hat{J}(i)} J^j \right) &= \Delta M_2(\hat{I}^i) + \Delta M_2 \left( \cup_{j \in \hat{J}(i)} J^j \cap \hat{I}^{i+1} \right) + \Delta M_2 \left( \cup_{j \in \hat{J}(i)} J^j \cap \hat{I}^{i+2} \right) \\ &\quad + \Delta M_2 \left( \cup_{j \in \hat{J}(i)} J^j \cap \hat{I}^{i-1} \right) + \Delta M_2 \left( \cup_{j \in \hat{J}(i)} J^j \cap \hat{I}^{i-2} \right), \end{aligned}$$

it follows that

$$\begin{aligned}
& E \left( \Delta M_1(\hat{I}^i) \Delta M_1(\hat{I}^{i+1}) \Delta M_2 \left( \cup_{j \in \hat{J}(i)} J^j \right) \Delta M_2 \left( \cup_{j \in \hat{J}(i+1)} J^j \right) \right) \\
&= \nu \left( \hat{I}^i \right) \nu \left( \hat{I}^{i+1} \right) + \nu \left( \cup_{j \in \hat{J}(i)} J^j \cap \hat{I}^{i+1} \right) \nu \left( \cup_{j \in \hat{J}(i+1)} J^j \cap \hat{I}^i \right) \\
&= \nu \left( \hat{I}^i \right) \nu \left( \hat{I}^{i+1} \right) + \sum_{j_1=1}^m \sum_{j_2=1}^m \nu \left( J^{j_1} \cap \hat{I}^{i+1} \right) \nu \left( J^{j_2} \cap \hat{I}^i \right) K_{ij_1}^{\hat{I}} K_{(i+1)j_2}^{\hat{I}}.
\end{aligned}$$

We can write the second term in the above equality as

$$\begin{aligned}
& \nu \left( J^{j_1} \cap \hat{I}^{i+1} \right) \nu \left( J^{j_2} \cap \hat{I}^i \right) K_{ij_1}^{\hat{I}} K_{(i+1)j_2}^{\hat{I}} \\
&= \nu \left( J^{j_1} \cap \hat{I}^{i+1} \right) \nu \left( J^{j_2} \cap \hat{I}^i \right) K_{ij_1}^{\hat{I}} K_{(i+1)j_2}^{\hat{I}} K_{(i+1)j_1}^{\hat{I}} K_{ij_2}^{\hat{I}}.
\end{aligned}$$

By the following fact,

$$K_{ij_1}^{\hat{I}} K_{(i+1)j_2}^{\hat{I}} K_{(i+1)j_1}^{\hat{I}} K_{ij_2}^{\hat{I}} \neq 0 \Leftrightarrow j_1 = j_2,$$

we can obtain immediately that

$$\begin{aligned}
& E \left( \Delta M_1(\hat{I}^i) \Delta M_1(\hat{I}^{i+1}) \Delta M_2 \left( \cup_{j \in \hat{J}(i)} J^j \right) \Delta M_2 \left( \cup_{j \in \hat{J}(i+1)} J^j \right) \right) \\
&= \nu \left( \hat{I}^i \right) \nu \left( \hat{I}^{i+1} \right) + \sum_{j=1}^m \nu \left( \hat{I}^{i+1} \cap J^j \right) \nu \left( \hat{I}^i \cap J^j \right),
\end{aligned}$$

which implies that

$$D_2 = \sum_{i=1}^{\hat{n}} \sum_{j=1}^m \nu \left( \hat{I}^{i+1} \cap J^j \right) \nu \left( \hat{I}^i \cap J^j \right). \quad (3.2)$$

Applying the same arguments to  $D_3, D_4, D_5$ ,

$$D_3 = \sum_{i=1}^{\hat{n}} \sum_{j=1}^m \nu \left( \hat{I}^{i+2} \cap J^j \right) \nu \left( \hat{I}^i \cap J^j \right) \quad (3.3)$$

$$D_4 = \sum_{i=1}^{\hat{n}} \sum_{j=1}^m \nu \left( \hat{I}^{i-1} \cap J^j \right) \nu \left( \hat{I}^i \cap J^j \right) \quad (3.4)$$

$$D_5 = \sum_{i=1}^{\hat{n}} \sum_{j=1}^m \nu \left( \hat{I}^{i-2} \cap J^j \right) \nu \left( \hat{I}^i \cap J^j \right) \quad (3.5)$$

Putting (3.2) through (3.5) together, and noting fact that if  $K_{ij}^{\hat{I}} \neq 0$ , then

$$J^j = \cup_{\ell=-2}^2 \left( \hat{I}^{i+\ell} \cap J^j \right),$$

we have

$$\begin{aligned}
& D_2 + D_3 + D_4 + D_5 \\
&= \sum_{i=1}^{\hat{n}} \sum_{j=1}^m \left( \nu(J^j) - \nu(\hat{I}^i \cap J^j) \right) \nu(\hat{I}^i \cap J^j) \\
&= \sum_{j=1}^m \nu^2(J^j) - \sum_{i=1}^{\hat{n}} \sum_{j=1}^m \nu^2(\hat{I}^i \cap J^j)
\end{aligned}$$

Together with (3.1), we can conclude

$$\text{Var}(V_n) = \sum_{i=1}^{\hat{n}} \sum_{j=1}^m \nu_1(\hat{I}^i) \nu_2(J^j) K_{ij}^{\hat{I}} + \sum_{i=1}^{\hat{n}} \nu^2(\hat{I}^i) + \sum_{j=1}^m \nu^2(J^j) - \sum_{i=1}^{\hat{n}} \sum_{j=1}^m \nu^2(\hat{I}^i \cap J^j).$$

□

**Remark 3.1.** From the proof of Lemma 3.1, each  $\Delta M_1(\hat{I}^i) \Delta M_2(\cup_{j \in \hat{J}(i)} J^j)$ ,  $1 \leq i \leq \hat{n}$  contributes to the variance of  $V_n$  with

$$\sum_{j=1}^m \nu_1(\hat{I}^i) \nu_2(J^j) K_{ij}^{\hat{I}} + \nu^2(\hat{I}^i) + \sum_{j=1}^m \left( \nu(J^j) - \nu(\hat{I}^i \cap J^j) \right) \nu(\hat{I}^i \cap J^j). \quad \square$$

**Proof of Theorem 2.1** For the expectation of  $V_n$ , we have

$$EV_n = \sum_{i=1}^{\hat{n}} \nu(\hat{I}^i) = \int_0^T \sigma_{1,t} \sigma_{2,t} \rho_t dt.$$

Therefore,

$$V_n - \int_0^T \sigma_{1,t} \sigma_{2,t} \rho_t dt = \sum_{i=1}^{\hat{n}} \left( \Delta M_1(\hat{I}^i) \Delta M_2(\cup_{j \in \hat{J}(i)} J^j) - \nu(\hat{I}^i) \right) := \sum_{i=1}^{\hat{n}} Y_i, \quad (3.6)$$

where  $Y_i = \Delta M_1(\hat{I}^i) \Delta M_2(\cup_{j \in \hat{J}(i)} J^j) - \nu(\hat{I}^i)$ . Consequently, the random variable sequence  $\{Y_i : 1 \leq i \leq \hat{n}\}$  are 2-dependent. We have to prove that

$$\limsup_{n \rightarrow \infty} \frac{1}{b_n^2} \log \left( \hat{n} \max_{i=1}^{\hat{n}} \mathbb{P}(|Y_i| > b_n v_n^{1/2}) \right) = -\infty. \quad (3.7)$$

By the condition (C4), there exist a sequence  $\varepsilon_n > 0$  satisfying as  $n \rightarrow +\infty$ ,  $\frac{\varepsilon_n^2}{v_n} \rightarrow 0$ . Now, we can write that

$$\begin{aligned}
|Y_i| &= \left| \Delta M_1(\hat{I}^i) \Delta M_2(\cup_{j \in \hat{J}(i)} J^j) - \nu(\hat{I}^i) \right| \\
&\leq \varepsilon_n |\Delta M_1(\hat{I}^i)|^2 + \frac{3}{\varepsilon_n} |\Delta M_2(\hat{I}^i)|^2 + |\nu(\hat{I}^i)|
\end{aligned}$$

Using condition (C2), we have as  $n \rightarrow \infty$

$$b_n v_n^{1/2} |\nu(\hat{I}^i)| \rightarrow 0.$$

Hence, to prove (3.7), we have to show that

$$\limsup_{n \rightarrow \infty} \frac{1}{b_n^2} \log \left( \hat{n} \max_{i=1}^{\hat{n}} \mathbb{P}(\varepsilon_n |\Delta M_1(\hat{I}^i)|^2 > b_n v_n^{1/2}) \right) = -\infty. \quad (3.8)$$

and

$$\limsup_{n \rightarrow \infty} \frac{1}{b_n^2} \log \left( \hat{n} \max_{i=1}^{\hat{n}} \mathbb{P}\left(\frac{3}{\varepsilon_n} |\Delta M_2(\hat{I}^i)|^2 > b_n v_n^{1/2}\right) \right) = -\infty. \quad (3.9)$$

In fact, by virtue of Chebyshev's inequality, for (3.8),

$$\begin{aligned} & \mathbb{P}\left(\frac{3}{\varepsilon_n} |\Delta M_2(\hat{I}^i)|^2 > b_n v_n^{1/2}\right) \\ & \leq \inf_{\lambda > 0} \exp \left\{ -\lambda + \frac{\lambda^2 \varepsilon_n^2 K}{2b_n^2 v_n} \right\} \\ & \leq \exp \left\{ -\frac{b_n^2 v_n}{2\varepsilon_n^2 K} \right\}, \end{aligned}$$

where  $K = \int_0^T \sigma_{1,t}^4 dt + \left(\int_0^T \sigma_{1,t}^2 dt\right)^2$ . From our choice of  $\varepsilon_n$ , (3.8) follows. Therefore, (3.9) can be established similarly.

Hence, from lemma 3.1 and (3.7) together with appendix Proposition 4.1, the Theorem 2.1 holds.  $\square$

**Proof of Theorem 2.2** From Theorem 2.1, we only need to show the exponential equivalence of  $U_n$  and  $V_n$ , i.e. for any  $\delta > 0$

$$\lim_{n \rightarrow +\infty} \frac{1}{b_n^2} \log P \left( \frac{1}{b_n v_n^{1/2}} |U_n - V_n| \geq \delta \right) = -\infty. \quad (3.10)$$

In fact, by simple calculation, we can write that

$$\begin{aligned} & |U_n - V_n| \\ & = \sum_{i=1}^{\hat{n}} \left| \Delta A_1(\hat{I}_i) \Delta A_2(\cup_{j \in \hat{J}(i)} J^j) \right| + \sum_{i=1}^{\hat{n}} \left| \Delta A_1(\hat{I}_i) \Delta M_2(\cup_{j \in \hat{J}(i)} J^j) \right| \\ & \quad + \sum_{i=1}^{\hat{n}} \left| \Delta M_1(\hat{I}_i) \Delta A_2(\cup_{j \in \hat{J}(i)} J^j) \right| \\ & \leq 3T r_n \max_{\ell=1,2} \|b_\ell\|_\infty + r_n \|b_1\|_\infty \sum_{i=1}^{\hat{n}} \left| \Delta M_2(\cup_{j \in \hat{J}(i)} J^j) \right| \\ & \quad + 3r_n \|b_2\|_\infty \sum_{i=1}^{\hat{n}} \left| \Delta M_1(\hat{I}_i) \right| \\ & \leq 3T r_n \max_{\ell=1,2} \|b_\ell\|_\infty + 3r_n \|b_1\|_\infty \sum_{i=1}^{\hat{n}} \left| \Delta M_2(\hat{I}_i) \right| + 3r_n \|b_2\|_\infty \sum_{i=1}^{\hat{n}} \left| \Delta M_1(\hat{I}_i) \right| \end{aligned}$$

Using condition (C4), we have as  $n \rightarrow \infty$

$$\frac{3T r_n \max_{\ell=1,2} \|b_\ell\|_\infty}{b_n v_n^{1/2}} \rightarrow 0.$$

Therefore, to prove (3.10), we only need to show that for any  $\delta > 0$

$$\lim_{n \rightarrow +\infty} \frac{1}{b_n^2} \log P \left( \frac{r_n}{b_n v_n^{1/2}} \sum_{i=1}^{\hat{n}} \left| \Delta M_1(\hat{I}_i) \right| \geq \delta \right) = -\infty \quad (3.11)$$

and

$$\lim_{n \rightarrow +\infty} \frac{1}{b_n^2} \log P \left( \frac{r_n}{b_n v_n^{1/2}} \sum_{i=1}^{\hat{n}} \left| \Delta M_2(\hat{I}_i) \right| \geq \delta \right) = -\infty. \quad (3.12)$$

In fact, according to Chebyshev's inequality, for (3.11),

$$\begin{aligned} & P \left( \frac{r_n}{b_n v_n^{1/2}} \sum_{i=1}^{\hat{n}} \left| \Delta M_1(\hat{I}_i) \right| \geq \delta \right) \\ & \leq \inf_{\lambda > 0} \exp \left\{ -\lambda \delta + \frac{\lambda^2 r_n^2 \int_0^T \sigma_{1,t}^2 dt}{2b_n^2 v_n} \right\} \\ & \leq \exp \left\{ -\frac{b_n^2 v_n \delta^2}{2r_n^2 \int_0^T \sigma_{1,t}^2 dt} \right\}, \end{aligned}$$

which implies that (3.11) by condition (C4). Moreover, (3.12) can be established similarly.

□

**Proof of Proposition 2.3** Our proof is an application of Proposition 4.1.

In fact, by a straightforward calculation, one can see that

$$\begin{aligned} V_{\ell,1}^n(g, h) - V_{\ell,1}(g, h) &= \frac{1}{n} \sum_{i=1}^n g(\sqrt{n} \Delta X_\ell(I^i)) h(\sqrt{n} \Delta X_\ell(I^{i+1})) - \int_0^1 \Sigma_{\sigma_{1,u}}(g) \Sigma_{\sigma_{1,u}}(h) du \\ &= \frac{1}{n} \sum_{i=1}^n \left[ g(\sqrt{n} \Delta X_\ell(I^i)) h(\sqrt{n} \Delta X_\ell(I^{i+1})) - \mathbb{E} \left( g(\sqrt{n} \Delta X_\ell(I^i)) \right) \mathbb{E} \left( h(\sqrt{n} \Delta X_\ell(I^{i+1})) \right) \right] \\ &= \sum_{i=1}^n Y_i \end{aligned}$$

where  $Y_i = \frac{1}{n} \left[ g(\sqrt{n} \Delta X_\ell(I^i)) h(\sqrt{n} \Delta X_\ell(I^{i+1})) - \mathbb{E} \left( g(\sqrt{n} \Delta X_\ell(I^i)) \right) \mathbb{E} \left( h(\sqrt{n} \Delta X_\ell(I^{i+1})) \right) \right]$  are 1-dependent random variables.

Hence, by the 1-dependence of  $(Y_i)$  and from (2.8) together the Proposition 4.1, the Proposition 2.3 holds. □

#### 4. APPENDIX A

Let us begin with some few bibliographical notes on the MDP. Borovkov and Mogulskii ([10], [11]) considered the MDP for Banach valued i.i.d.r.v. sequences  $(Z_n)_{n \geq 1}$ . Under the condition that  $\mathbb{E} e^{\delta |Z_1|} < +\infty$ , for some  $\delta > 0$ , they proved the MDP for  $\sum_{i=1}^n Z_i$ .

For  $b_n = n^\alpha$  with  $\frac{1}{2} < \alpha < 1$ , Chen [12] found the necessary and sufficient condition for the MDP in a Banach space, and he obtained the lower bound for general  $b_n$  under very weak condition. Using the isoperimetry techniques, Ledoux [34] (see also [24]) obtained the necessary and sufficient condition for the general sequence  $b_n$  satisfying:

$$\frac{b_n}{\sqrt{n}} \longrightarrow \infty \quad \text{and} \quad \frac{b_n}{n} \longrightarrow 0. \quad (4.1)$$

The results of Ledoux [34] is extended to functional empirical processes (in the setting of non parametrical statistics) by Wu [36]. The further developments are given by Dembo and Zeitouni [17]. Arcones [3] obtains the MDP of functional type with the following condition

$$\limsup_{n \rightarrow \infty} \frac{n}{b_n^2} \log \left( n \mathbb{P}(|Z_1| > b_n) \right) = -\infty. \quad (4.2)$$

How to extend the MDP to the dependent situations has recently attracted much attention and remarkable works. The Markovian case has been studied under successively less restrictive conditions (see Wu [36], Chen [12] for the relevant references) and very recently under weak conditions by de Acosta [14] and Chen [12] for the lower bound (under different and non comparable conditions) and by de Acosta-Chen [15] and Chen [12] (under the same condition but different proof) for the upper bound. Guillin [26] obtained uniform (in time) MDP for functional empirical processes. Using regeneration split chain method, Djellout and Guillin ([20], [19]) extend the characterization of MDP for i.i.d.r.v. case of Ledoux [34] to Markov chains. The geometric ergodicity is substituted by a tail on the first time of return to the atom. Their conditions are weaker than Theorem 1 in [15], which allow them to obtain MDP for empirical measures and functional empirical processes.

For the studies of the MDP of martingale see Puhalskii [38], Dembo [16], Gao [25], Worms [41] and Djellout [18].

Those works motivate directly the studies here. Our main aim is to prove the Chen-Ledoux type theorem for the MDP of a sequence of  $m$ -dependent random variables.

**4.1. Moderate deviations for  $m$ -dependent random variables.** Let  $(b_n)$  a sequence of positive real numbers satisfying (4.1).

Let  $X_n$  be an  $m$ -dependent random variables, namely, for every  $k \geq 1$  the following two collections

$$\{X_1, \dots, X_k\} \quad \text{and} \quad \{X_{k+m+1}, X_{k+m+2}, \dots\}$$

are independent. In this terminology, an independent identically distributed (i.i.d.) random variables sequence is 0- dependent and, for any non-negative intergers  $m_1 < m_2$ ,  $m_1$  dependence implies  $m_2$ -dependence. The aim of this section is to consider the moderate deviations for  $m$  dependent random variables sequences which are not necessarily stationnary under the Chen-Ledoux type condition. This condition links the speed of the moderate deviation principle to the queue of the distribution of the random variable.

**Proposition 4.1.** *Let  $(X_k)$  be a sequence of  $m$ -dependent random variables with zero means such that*

$$\limsup_{n \rightarrow \infty} \frac{n}{b_n^2} \log \left( n \max_{i=1}^n \mathbb{P}(|X_i| > b_n) \right) = -\infty. \quad (4.3)$$

*Write  $S_n = \sum_{j=1}^n X_j$  and let  $(b_n)$  a sequence of positive numbers satisfying (4.1). Then  $S_n/b_n$  satisfies the moderate deviation principle on  $\mathbb{R}$  with speed  $b_n^2/n$  and rate fonction  $I(x) = x^2/2\sigma^2$ , where  $\sigma^2 := \lim_{n \rightarrow \infty} \frac{1}{n} \text{Var}(S_n) < \infty$ .*

**Remark 4.1.** *Remark that the condition (4.3) implies that*

$$\lim_{n \rightarrow \infty} \sup_{1 \leq \ell \leq n} \mathbb{E}(X_\ell^2) < \infty.$$

**Remark 4.2.** *If there exists some  $\delta > 0$  such that*

$$\sup_{1 \leq \ell \leq n} \mathbb{E}(e^{\delta |X_\ell|}) < \infty,$$

*so the condition (4.3) is satisfied for every sequence  $b_n$ .*

**Remark 4.3.** *If  $b_n = n^{1/p}$  with  $1 < p < 2$  so the condition (4.3) is equivalent to*

$$\sup_{1 \leq \ell \leq n} \mathbb{E}(e^{\delta |X_\ell|^{2-p}}) < \infty.$$

**Lemma 4.2.** *Let  $(Y_k)$  be a sequence of independent random variables with zero means such that the condition (4.3) is satisfied. Write  $S'_n = \sum_{j=1}^n Y_j$  and let  $(b_n)$  a sequence of positive numbers satisfying (4.1). Then  $S'_n/b_n$  satisfies the moderate deviation principle on  $\mathbb{R}$  with speed  $b_n^2/n$  and rate fonction  $I(x) = x^2/2\sigma^2$ , where  $\sigma^2 := \lim_{n \rightarrow \infty} \frac{1}{n} \text{Var}(S_n) < \infty$ .*

**Proof of Lemma 4.2** The proof of this result is a little adaptation of the proof of the independent and identically distributed random variables given in [20].  $\square$

**Proof of Proposition 4.1** We will do the proof in the case  $m = 1$  for simplicity.

Fix the integer  $p > 1$  and write, for each  $n \geq 1$ ,

$$n = k_n p + r_n$$

where  $k_n$  and  $r_n$  are non-negative integers with  $0 \leq r_n \leq p - 1$ . Define

$$Y_k = \sum_{(k-1)p < j < kp} X_j, \quad k = 1, 2, \dots$$

Then  $(Y_k)_{k \geq 1}$  is an independent random variables and

Notice that

$$S_n = \sum_{\ell=1}^{k_n} Y_\ell + \sum_{\ell=1}^{k_n} X_{\ell p} + \sum_{\ell=k_n p+1}^n X_j \quad (n \geq 1).$$

We have to prove that

$$\limsup_{n \rightarrow \infty} \frac{n}{b_n^2} \log \mathbb{P} \left( \left| \sum_{\ell=1}^{k_n} X_{\ell p} \right| > \delta b_n \right) = -\infty. \quad (4.4)$$

Since the random variables  $(X_{\ell p})_{\ell \geq 1}$  are independent, we use the moderate deviation principle for  $(\frac{1}{b_n} \sum_{\ell=1}^{k_n} X_{\ell p})_{\ell \geq 1}$ , to obtain that for  $n$  large

$$\frac{n}{b_n^2} \log \mathbb{P} \left( \left| \sum_{\ell=1}^{k_n} X_{\ell p} \right| > \delta b_n \right) \leq -\frac{\delta^2}{2\sigma_0^2}$$

where

$$\sigma_0^2 = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{\ell=1}^{k_n} \text{Var}(X_{\ell p}) \leq C \lim_{n \rightarrow \infty} \frac{k_n}{n} = \frac{C}{p}.$$

So

$$\limsup_{n \rightarrow \infty} \frac{n}{b_n^2} \log \mathbb{P} \left( \left| \sum_{\ell=1}^{k_n} X_{\ell p} \right| > \delta b_n \right) \leq -\frac{p \delta^2}{2C}$$

Letting  $p$  goes to infinity, we get (4.4).

Now we have to prove that

$$\limsup_{n \rightarrow \infty} \frac{n}{b_n^2} \log \mathbb{P} \left( \left| \sum_{\ell=k_n p+1}^n X_\ell \right| > \delta b_n \right) = -\infty. \quad (4.5)$$

If  $r_n = 0$  there is no thing to prove, so we suppose that  $r_n > 0$

$$\begin{aligned} \mathbb{P} \left( \left| \sum_{\ell=k_n p+1}^n X_\ell \right| > \delta b_n \right) &\leq \sum_{\ell=k_n p+1}^n \mathbb{P} \left( |X_\ell| > \frac{\delta b_n}{r_n} \right) \\ &\leq r_n \max_{\ell=k_n p+1}^n \mathbb{P} \left( |X_\ell| > \delta \frac{b_n}{r_n} \right) \\ &\leq n \max_{\ell=1}^n \mathbb{P} \left( |X_\ell| > \delta \frac{b_n}{p} \right). \end{aligned}$$

Taking the log and normalizing by  $n/b_n^2$ , we obtain (4.5).

Now, we have to prove that

$$\limsup_{n \rightarrow \infty} \frac{n}{b_n^2} \log \left[ k_n \max_{\ell=1}^{k_n} \mathbb{P} (|Y_\ell| > b_n) \right] = -\infty. \quad (4.6)$$

We have

$$\begin{aligned} k_n \max_{\ell=1}^{k_n} \mathbb{P} (|Y_\ell| > b_n) &\leq k_n \max_{\ell=1}^{k_n} \sum_{j=(\ell-1)p+1}^{\ell p-1} \mathbb{P} \left( |X_j| > \frac{1}{p} b_n \right) \\ &\leq k_n p \max_{\ell=1}^{k_n} \max_{j=(\ell-1)p+1}^{\ell p-1} \mathbb{P} \left( |X_j| > \frac{1}{p} b_n \right) \\ &\leq n \max_{\ell=1}^n \mathbb{P} \left( |X_\ell| > \frac{1}{p} b_n \right). \end{aligned}$$

Taking the log and normalizing by  $n/b_n^2$  and letting  $n$  goes to infinity, we obtain (4.6).

So we deduce that the sequence  $1/b_n \sum_{\ell=1}^{k_n} Y_\ell$  satisfies the moderate deviations principle with speed  $b_n^2/n$  and rate function  $I(x) = x^2/2\sigma_Y^2$ , where

$$\sigma_Y^2 := \lim_{n \rightarrow \infty} \frac{1}{n} \text{Var} \left( \sum_{\ell=1}^{k_n} Y_\ell \right). \quad (4.7)$$

Now we have to prove that  $\sigma^2 = \sigma_Y^2$ . For that we have

$$\begin{aligned} \Delta_n := \text{Var} S_n - \text{Var} \left( \sum_{\ell=1}^{k_n} Y_\ell \right) &= \sum_{\ell=1}^{k_n} \mathbb{E}(X_{\ell p}^2) + 2 \sum_{\ell=1}^{k_n} (\mathbb{E} X_{\ell p} X_{\ell p+1} + \mathbb{E} X_{\ell p} X_{\ell p-1}) \\ &\quad + \sum_{\ell=p k_n+1}^{p k_n+r_n} \mathbb{E}(X_\ell^2 + X_\ell X_{\ell+1} + X_\ell X_{\ell-1}). \end{aligned}$$



Now since  $\sup_{1 \leq \ell \leq n} \mathbb{E}X_\ell^2 < \infty$ , we have that  $\Delta_n/n \leq (k_n C + 4Ck_n + 3r_n C)/n$ . So  $\lim_{n \rightarrow \infty} \Delta_n/n = C/p$ , which goes to 0 when  $p$  goes to infinity.  $\square$

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