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Rigidity of the Laughlin liquid

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We consider general N -particle wave functions which have the form of a product of the Laughlin state with filling factor $1/\ell$ and an analytic function of the N variables. This is the most general form of a wave function that can arise through a perturbation of the Laughlin state by external potentials or impurities, while staying in the lowest Landau level and maintaining the strong correlations of the original state. We show that the perturbation can only shift or lower the 1-particle density but nowhere increase it above a maximum value. Regardless of the analytic prefactor, the density satisfies the same bound as the Laughlin function itself, i.e. $(\pi\ell)^{-1}$ in the limit of large particle number. Consequences of this incompressibility bound for the response of the Laughlin state to external fields are discussed. Our theorems apply equally to bosonic and fermionic states.

In theoretical studies of the fractional quantum Hall effect (FQHE) [8–10, 29] Laughlin’s wave function(s) [12–14] play a fundamental role. There is such a function for every positive integer ℓ and it can be written, in appropriate units, as

$$\Psi_{\text{Lau}} = c_{\text{Lau}} \prod_{i < j} (z_i - z_j)^\ell e^{-\sum_{i=1}^N |z_i|^2/2} \quad (1)$$

where the $z_i \in \mathbb{C}$ are the positions of N particles moving in \mathbb{R}^2 , identified with the complex plane, and the constant c_{Lau} is a normalization factor (depending on N and ℓ). For fermions, ℓ is odd and ≥ 3 (the case $\ell = 1$ corresponds to noninteracting fermions), while for bosons $\ell \geq 2$ is even. Bosonic wave functions of this type are potentially relevant for atomic gases in artificial magnetic fields [5, 7, 16, 22, 23, 30]. The analysis below applies equally to Laughlin states of all symmetry types.

The Laughlin state (1) is a special kind of wave function in the Lowest Landau level (LLL) of a Hamiltonian with a strong magnetic field perpendicular to the plane where the particles move. The general form of such functions is

$$\Psi(z_1, \dots, z_N) = A(z_1, \dots, z_N) e^{-\sum_{i=1}^N |z_i|^2/2} \quad (2)$$

with A analytic, antisymmetric for fermions and symmetric for bosons. If there is strong repulsive two-body interaction between the particles, (1) is a natural variational ansatz for low energy states because the factors $(z_i - z_j)^\ell$ suppress the interactions. At the same time they produce strong correlations between the particle positions.

The success of Laughlin’s theory of the FQHE fractions $1/\ell$, based on (1), depends on two crucial properties:

- 1.** The Laughlin wave function is an approximate ground state for the many-body Hamiltonian, separated from the rest of the spectrum by an energy gap.
- 2.** The Laughlin wave function behaves as an incompressible liquid, whose response to perturbations and external fields is extremely rigid.

Up to now, the main evidence for these facts has been experimental and numerical. Property **1** ensures that the Laughlin wave-function indeed emerges as a phase of matter proper (the incompressible quantum fluid [13]). Property **2** is somewhat more subtle, but also crucial for Laughlin’s original argument supporting quantization of the Hall conductivity [11, 14].

In this Letter, we report on recent, mathematically rigorous, estimates that demonstrate the validity of Property **2**. These are related to earlier partial results proved in [27, 28]. The full proofs are lengthy and shall be presented elsewhere [19] but we sketch the main arguments below, and discuss physical applications.

We first have to be more precise as to what, exactly, is a reasonable perturbation of the Laughlin state. By this we mean a state vector made only of LLL orbitals and containing at least the same correlations as the Laughlin state. These demands still leave considerable freedom for the choice of trial states. Indeed, to allow for a response of the Laughlin state to perturbations generated by impurities or external potentials it is natural to consider normalized wave functions of the general form

$$\Psi_F(z_1, \dots, z_N) = F(z_1, \dots, z_N) \Psi_{\text{Lau}}(z_1, \dots, z_N) \quad (3)$$

with F analytic and symmetric under exchange of the z_i . This form exhausts the class of functions that minimize the magnetic kinetic energy and at the same time avoid repulsive interactions by vanishing at least as $(z_i - z_j)^\ell$ as z_i and z_j come together. Assuming that the interaction operator has a gap, it makes sense, for low energy perturbations, to restrict attention to the form (3), which we shall refer to as *fully correlated Quantum Hall states*. Our goal is to prove general density bounds within this class. That is, roughly speaking, we *assume* Property **1** above and wish to argue for Property **2**.

In his pioneering paper [12], Laughlin already argued that the one-particle density of the state (1) has, for large N , the form of a circular droplet of radius $\sqrt{\ell N}$ where the

density takes the constant value $(\pi\ell)^{-1}$. The argument was based on the plasma analogy, where the absolute square of the wave function is written as the Gibbs distribution of a classical 2D Coulomb gas, and subsequently treated by a mean-field approximation.

Previous rigorous mathematical analysis has confirmed the validity of this approximation for the Laughlin states and extended it to more general states of the form (3), including Laughlin's 'quasi hole' states where the prefactor F is a product where each factor depends on a single variable [25, 26]. This analysis was extended to other prefactors of a special kind in [27]. A common feature that emerged was an upper bound on the one-particle density of magnitude $(\pi\ell)^{-1}$ which is the density of the Laughlin state itself. Such a bound was called an *incompressibility estimate* in [27] because it is a manifestation of the resistance of the Laughlin state against attempts to compress its density. As long as one stays in the LLL and maintains the correlation factors $(z_i - z_j)^\ell$, the perturbation can increase or decrease the density profile, but only in such a way that the new density is never greater than $(\pi\ell)^{-1}$ anywhere.

The mean-field methods of [25–27] are not applicable to general prefactors F which, in the plasma analogy, may correspond to genuine N particle interactions. The question of a bound on the density for the general case was treated in [28] with an entirely different technique, rooted in 2D potential theory. The bound obtained was, however, four times the expected optimal value $(\pi\ell)^{-1}$. In this Letter we show that an improved version of the potential theoretic method leads to the correct optimal bound for *completely arbitrary* F .

As indicated by numerical studies [6], the incompressibility bound for the density cannot be expected to hold pointwise for finite N . We prove, however, that for $N \rightarrow \infty$ it holds, at least in the weak sense of averages, over length scales larger than the magnetic length, which is 1 in our units. Such a scale is much smaller than the extension of the wave function, which is at least $O(N^{1/2})$.

In order to study the $N \rightarrow \infty$ limit it is convenient to change variables and consider the scaled N -particle probability density

$$\mu_F(Z_N) := N^N \left| \Psi_F \left(\sqrt{N} Z_N \right) \right|^2 \quad (4)$$

corresponding to the wave-function (3). This has an extension $O(1)$ for the Laughlin states, i.e. for $F = 1$. Here Z_N stands for (z_1, \dots, z_N) . Integrating over the $N - 1$ variables $Z'_{N-1} = (z_2, \dots, z_N)$ we obtain the scaled 1-particle probability density

$$\mu_F^{(1)}(z) = \int \mu_F(z, Z'_N) dZ'_{N-1}. \quad (5)$$

The 1-particle density in the original variables is then

$$\rho_F^{(1)}(z) = \mu_F^{(1)}(N^{-1/2} z).$$

To state our density bound precisely we pick an external potential V , and define the energy $E_V(\ell, N)$ as the infimum of the potential energy over all fully correlated states (3):

$$E_V(\ell, N) := \inf \{ \mathcal{E}_V[\Psi_F], \Psi_F \text{ of the form (3)} \} \quad (6)$$

where

$$\mathcal{E}_V[\Psi_F] = \int_{\mathbb{R}^2} V \mu_F^{(1)}.$$

Note that, implicit in the choice of units above, is that we consider the response of functions of the form (3) to potentials living on the scale of the Laughlin state, that is $O(N^{1/2})$ in the physical variables.

We also define the 'bathtub energy' [18, Theorem 1.14]

$$E_V^{\text{bt}}(\ell) := \inf \left\{ \int_{\mathbb{R}^2} V \rho \mid 0 \leq \rho \leq \frac{1}{\pi\ell}, \int_{\mathbb{R}^2} \rho = 1 \right\}. \quad (7)$$

This is the infimum of the potential energy over all densities ρ that satisfy the conjectured incompressibility bound. We formulate our weak density bound on (3) as a bound on the potential energy in terms of the bathtub energy:

Theorem 1 (Potential energy of fully correlated states). *For any twice continuously differentiable potential V growing at infinity, we have,*

$$\liminf_{N \rightarrow \infty} E_V(\ell, N) \geq E_V^{\text{bt}}(\ell). \quad (8)$$

Thus, any compression of the particle density above the "magic value" $(\pi\ell)^{-1}$ that one could imagine to accommodate the variations of the external potential would make us leave the class of fully correlated states, with corresponding increase in either the magnetic kinetic energy or the interaction energy. Assuming the values of the latter are frozen (i.e. Property 1), no such density bump is allowed. This property justifies two things *a posteriori*:

- That it is legitimate to neglect disorder in the sample and/or small external electric fields, as is done as a first approximation in the derivation of FQHE wave-functions.
- Laughlin's argument [11, 13] (see also [10, Sections 4.4, 9.3 and 9.5]) that switching on an electric current moves electrons transversally without creating any charge accumulation, and generates a Hall conductivity of value $1/\ell$.

It has been proposed (see [5, 7, 30] for reviews) that Laughlin wave functions could be created in cold atomic gases, either by rapid rotation or by applying artificial magnetic fields. In this context, the potential V can model, for instance, the magneto-optical trap that confines the gas. Some recent proposals to reach the Laughlin state [22, 23] involve some non-trivial engineering of the latter. How the Laughlin state responds to this is therefore of importance for the experimental set-up.

Moreover, the precursor of FQHE states in a rapidly rotating Bose gas is a Bose-Einstein condensate (see [16, 21] and references therein). Observing the distinctively flat profile of the Laughlin state, by time-of-flight techniques, would already be a strong indication of the transition to the FQHE regime. A more complete probe could be the response of the gas to variations of the trapping potential: the Bose condensate follows the trap by taking a Thomas-Fermi-like shape (see [1, 2, 4] and references therein). The Laughlin state essentially does not respond to such variations, as exemplified by our main theorem.

In this direction, we point out that a combination of Theorem 1 and the methods of [25, 26] yields the following improvement of [27, Corollary 2.3]:

Corollary 2 (Optimization of the energy in radial traps). *Let V be as in Theorem 1. Assume further that it is radial increasing and has at most polynomial growth at infinity. Then*

$$\lim_{N \rightarrow \infty} E_V(\ell, N) = E_V^{\text{bt}}(\ell) = \lim_{N \rightarrow \infty} \mathcal{E}_V[\Psi_{\text{Lau}}]. \quad (9)$$

It is remarkable that the Laughlin state stays an approximate minimizer in any radially increasing trap. No matter how steep and narrow a potential well one imposes, it is impossible to compress the Laughlin state while keeping the form (3). Extensions of Corollary 2 to more general radial traps as in [27, Corollary 2.3] are also possible.

We now turn to sketching the proof of Theorem 1, postponing the details to a future publication [19]. The first step is to write the N -particle density as a Gibbs factor (plasma analogy),

$$\mu_F(Z_N) = \mathcal{Z}_N^{-1} \exp\left(-\frac{1}{T} H_N(Z_N)\right), \quad (10)$$

with temperature $T = N^{-1}$ and the Hamiltonian

$$H_N(Z_N) = \sum_{j=1}^N |z_j|^2 - \frac{2\ell}{N} \sum_{i < j} \log |z_i - z_j| + W(Z_N) \quad (11)$$

with

$$W(Z_N) = -\frac{2}{N} \log \left| F\left(\sqrt{N} Z_N\right) \right|. \quad (12)$$

The term $W(Z_N)$ has the important property of being superharmonic in each variable:

$$-\Delta_{z_i} W(Z_N) \geq 0 \text{ for all } i. \quad (13)$$

This holds because F is analytic and is, in fact, the only property of W that is used in our method.

A precursor of the desired bound for $\mu_N^{(1)}$ is the fact that the local density of the points z_i^0 in a minimizing configuration Z_N^0 for $H_N(Z)$ is everywhere bounded above by $N(\pi\ell)^{-1}$ for large N . To establish this fact,

which is the core of the proof of the theorem, we introduce and study an auxiliary Thomas-Fermi (TF) model of a special kind.

For K fixed points $x_i \in \mathbb{R}^2$ (“nuclei”) we define an energy for functions σ on \mathbb{R}^2 (“electron density”) by

$$\mathcal{E}^{\text{TF}}[\sigma] = - \int_{\mathbb{R}^2} V(x) \sigma(x) dx + D(\sigma, \sigma) \quad (14)$$

with

$$V_{\text{nuc}}(x) = - \sum_{i=1}^K \log |x - x_i| \quad (15)$$

and

$$D(\sigma, \sigma') = -\frac{1}{2} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \sigma(x) \log |x - x'| \sigma'(x') dx dx'. \quad (16)$$

This functional is minimized under the subsidiary conditions

$$0 \leq \sigma(x) \leq 1, \quad \int \sigma(x) dx = K. \quad (17)$$

In physical terms, this model describes a neutral 2D molecule consisting of fixed nuclei and mobile electrons, with Coulomb interactions. The interpretation of the constraint $0 \leq \sigma \leq 1$ is that the kinetic energy of the electrons is zero for densities ≤ 1 and ∞ for densities > 1 .

The basic facts about this TF model are: (1) There exists a unique minimizer, σ^{TF} . (2) The minimizer has compact support. (3) Apart of a set of measure zero, σ^{TF} takes only the values 0 or 1. (4) The Thomas-Fermi equation holds:

$$\Phi^{\text{TF}}(x) = \begin{cases} \geq 0 & \text{if } \sigma^{\text{TF}}(x) = 1 \\ 0 & \text{if } \sigma^{\text{TF}}(x) = 0 \end{cases} \quad (18)$$

where

$$\Phi^{\text{TF}}(x) = V_{\text{nuc}}(x) + \int_{\mathbb{R}^2} \log |x - x'| \sigma^{\text{TF}}(x') dx'$$

is the total electrostatic potential of the molecule.

The derivation of these properties requires some effort because the TF model is of a singular type and standard methods have to be modified, see [19].

According to the TF equation the support of σ^{TF} is the same as the support of the potential Φ^{TF} , which is continuous away from the “nuclei”. Denote by $\Sigma^{\text{TF}}(x_1, \dots, x_K)$ the open set where Φ^{TF} is strictly larger than 0. Important properties of these sets are:

(1) The area of $\Sigma^{\text{TF}}(x_1, \dots, x_K)$ is equal to K . (2) $\Sigma^{\text{TF}}(x_1, \dots, x_{K-1}) \subset \Sigma^{\text{TF}}(x_1, \dots, x_K)$. (3) For a single nucleus at x_1 , $\Sigma^{\text{TF}}(x_1)$ is the disc with center x_1 and radius $\pi^{-1/2}$.

Consider now a scaled version of (11),

$$\mathcal{H}(X_N) = \frac{\pi}{2} \sum_{i=1}^N |x_i|^2 - \sum_{1 \leq i < j \leq N} \log |x_i - x_j| + \mathcal{W}(X_N), \quad (19)$$

with \mathcal{W} symmetric and superharmonic in each variable $x_i \in \mathbb{R}^2$ and $X_N = (x_1, \dots, x_N)$. A key property of minimizing configurations of \mathcal{H} is stated in the following:

Lemma 3 (Exclusion rule). *Let $X_N^0 = (x_1^0, \dots, x_N^0)$ be a minimizing configuration for \mathcal{H} . For any subset $y_1, \dots, y_K, y_{K+1} \in X_N^0$,*

$$y_{K+1} \notin \Sigma^{\text{TF}}(y_1, \dots, y_K). \quad (20)$$

Proof. By symmetry of the Hamiltonian we may, without loss, choose $y_i = x_i^0$, $1 \leq i \leq K+1$. Consider then fixing all points but x_{K+1}^0 . The energy to consider is then

$$G(x) = \mathcal{H}(x_1^0, \dots, x_K^0, x, x_{K+2}^0, \dots, x_N^0). \quad (21)$$

We claim that if $x \in \Sigma^{\text{TF}}(x_1^0, \dots, x_K^0) \equiv \Sigma^{\text{TF}}$ then there is an $\tilde{x} \in \partial \Sigma^{\text{TF}}$, the boundary of Σ^{TF} , such that $G(\tilde{x}) < G(x)$. Thus the minimizing point x_{K+1}^0 cannot lie in Σ^{TF} .

To prove the claim, we add and subtract a term $-\int_{\Sigma^{\text{TF}}} \log |x - x'| dx'$ to write $G(x) = \Phi(x) + R(x)$ with

$$\Phi(x) = - \sum_{i=1}^K \log |x - x_i^0| + \int_{\Sigma^{\text{TF}}} \log |x - x'| dx' \quad (22)$$

and

$$R(x) = \frac{\pi}{2} |x|^2 - \int_{\Sigma^{\text{TF}}} \log |x - x'| dx' - \sum_{i=K+2}^N \log |x - x_i^0| + W(x) + \text{const}. \quad (23)$$

Now, Φ is precisely the TF potential corresponding to ‘nuclear charges’ at x_i^0, \dots, x_K^0 . Hence, using (18), $\Phi > 0$ on Σ^{TF} and zero on the boundary $\partial \Sigma^{\text{TF}}$. The first two terms in R are harmonic on Σ^{TF} when taken together. (The Laplacian applied to the first term gives 2π and to the second term -2π on Σ^{TF} .) The other terms are superharmonic on Σ^{TF} . Thus, R takes its minimum on the boundary, so there is a $\tilde{x} \in \partial \Sigma^{\text{TF}}$ with $R(x) \geq R(\tilde{x})$. On the other hand, $\Phi(x) > 0 = \Phi(\tilde{x})$ so $G(x) > G(\tilde{x})$. \square

The particular case $K = 1$ goes back to an unpublished theorem due to Lieb, used in [28]: the minimal distance between points in a minimizing configuration of \mathcal{H} is not less than $1/\sqrt{\pi}$. This property shows that the density of the points is in any case bounded above by 4. The general exclusion rule for all K implies more. The density is, in fact, asymptotically bounded above by 1:

Lemma 4 (Exclusion rule implies density bound).

For $R > 0$ let $n(R)$ denote the maximum number of any points $\{y_1, \dots, y_n\}$ that a ball $B(R)$ of radius R can accommodate while respecting the exclusion rule (20). Then

$$\limsup_{R \rightarrow \infty} \frac{n(R)}{\pi R^2} \leq 1. \quad (24)$$

Proof. The proof is indirect, assuming that for some $\delta > 0$ there are arbitrary large radii with the property that the ball $B(R)$ contains at least $(1 + \delta)\pi R^2$ points. We will lead this to a contradiction with (20). Full details are given in [19] and we present here only a sketch. By taking the maximal δ (which is in any case ≤ 3) we may, without restriction, assume that the density is also at least $(1 + \delta)$ in the annulus \mathcal{A} of width $\delta \cdot R$ around $B(R)$.

Since the density is everywhere bounded above by 4, apart from the points $y_i \in B(R)$, $i = 1, \dots, n$ there must be points $y_j \in \mathcal{A}$, $j = n + 1, \dots, m$ in the configuration such that every point in \mathcal{A} is at most a distance $O(1)$ from one of the y_j . The TF potential Φ^{TF} generated by the y_i and the corresponding exclusion set $\Sigma^{\text{TF}}(y_1, \dots, y_n)$ must vanish at the y_j by the exclusion rule and (18).

On the other hand, after scaling the variables by R^{-1} and extracting a factor R^2 one can show that the gradient of the TF potential is uniformly bounded in the scaled annulus $R^{-1}\mathcal{A}$. The distance between the scaled y_j is now R^{-1} so the scaled potential goes to zero uniformly on $R^{-1}\mathcal{A}$ as $R \rightarrow \infty$. The same holds then for the circular average of the scaled potential.

We claim, however, that the latter is strictly bounded away from zero close to the radius 1 (corresponding to radius R in the unscaled annulus). This follows from Newton’s theorem, because the nuclear charge in $B(R)$, which is $(1 + \delta)\pi R^2$ by assumption, is not fully screened by the part of the negative charge density σ^{TF} lying in $B(R)$, which is at most equal to the area πR^2 because $\sigma^{\text{TF}} \leq 1$. There is thus a contradiction for R large enough. \square

After scaling, $x \rightarrow z = \sqrt{\frac{\pi \ell}{N}} x$, Lemma 2 applies to the Hamiltonian (11) and implies that in any minimizing configuration $Z_N^0 = (z_1^0, \dots, z_N^0)$ of (11) the number of points z_i^0 contained in any disc of radius $\gg N^{-1/2}$ is no larger than $N(\pi \ell)^{-1}$ times the area of the disc. This is the gist of the proof and already hints at a strong incompressibility property. From there, two main steps are left to conclude the proof of Theorem 1:

- Cover any reasonable region of side-length $\gg N^{-1/2}$ efficiently with balls in which the previous statement applies. This uses the ‘cheese theorem’ (see [20, Section 14.4] or [17, Theorem 14]) which asserts that this can be accomplished up to a residual set of extremely small area. The upshot is that the empirical measure

$$\rho^0(z) = \frac{1}{N} \sum_{j=1}^N \delta(z - z_j^0)$$

can be approximated in the weak sense by an absolutely continuous distribution $\tilde{\rho}^0$ of integral 1 satisfying for large N the bound

$$\tilde{\rho}^0(z) \leq (\pi\ell)^{-1}(1 + o(1)).$$

• Show that the above bound for ground states of (11) applies also to the Gibbs state (10). Here we use crucially that the temperature T in the latter scales as N^{-1} so that the Gibbs measure charges mostly ground state configurations for large N . Turning this intuition into a proof follows the lines of [28, Section 3]. To access the 1-particle density we prove free energy upper and lower bounds for a perturbed version of the Hamiltonian. We show that the latter satisfies approximately the same bounds as the unperturbed (11) and then translate the free-energy estimates into density estimates using a Feynman-Hellmann-type argument.

We conclude by pointing out that our density upper bound holds down to the finest possible scale for ground states of (11), i.e. on length scales $\gg N^{-1/2}$, the typical interparticle distance. Recently, it was proved that, for the purely Coulombic Hamiltonian where $F = 1$ (i.e. $W = 0$), a corresponding density lower bound also holds for ground states [24] and low temperature Gibbs states [3, 15]. We believe that, for the pure Coulomb Hamiltonian, an extension of the method sketched herein can yield a new proof of the results of [3, 15, 24]. This remains a question for future investigations, as does extending Corollary 2 to more general potentials V .

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